ON SPECTRUM OF κ–RESPLENDENT MODELS

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ABSTRACT. We prove that some natural "outside" property of counting models up to isomorphism is equivalent (for a first order class) to being stable.

For a model, being resplendent is a strengthening of being κ-saturated. Restricting ourselves to the case $\kappa > |T|$ for transparency, to say a model M is κ-resplendent means:

> when we expand M by $< \kappa$ individual constants $\langle c_i : i < \alpha \rangle$, if $(M, c_i)_{<\alpha}$ has an elementary extension expandable to be a model of T' where $\mathrm{Th}((M, c_i)_{i<\alpha}) \subseteq T'$, $|T'| < \kappa$ then already $(M, c_i)_{i<\alpha}$ can be expanded to a model of T' .

Trivially, any saturated model of cardinality λ is λ-resplendent. We ask: how many κ-resplendent models of a (first order complete) theory T of cardinality λ are there? We restrict ourselves to cardinals $\lambda = \lambda^\kappa + 2^{|T|}$ and ignore the case $\lambda = \lambda^{<\kappa} + |T| < \lambda^\kappa$. Then we get a complete and satisfying answer: this depends only on T being stable or unstable. In this case proving that for stable T we get few, is not hard; in fact, every resplendent model of T is saturated hence it is determined by its cardinality up to isomorphism. The inverse is more problematic because naturally we have to use Skolem functions with any $\alpha < \kappa$ places. Normally we use relevant partition theorems (Ramsey theorem or Erdős-Rado theorem), but in our case the relevant partitions theorems fail so we have to be careful.

§ 0. INTRODUCTION

Our main conclusion speaks on stability of first order theories, but the major (and the interesting) part of the proof has little to do with it and can be read without knowledge of classification theory (only the short proof of 1.8 uses it), except the meaning of $\kappa < \kappa(T)$ which we can take as the property we use, see inside 2.1(1) here (or see [Sheb, 1.5(2)] or [She90]). The point is to construct a model in which, for some infinite sequences of elements, we have appropriate automorphisms, so we need to use "Skolem" functions with infinitely many places. Now having functions with infinite arity makes obtaining models generated by indiscernibles harder.

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References like [Sh:950, Th0.2=Ly5] mean that the internal label of Th0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number 363 in Saharon Shelah's list.

On the old versions, the author expresses gratitude for the partial support of the Binational Science Foundation in this research and thanks Alice Leonhardt for her careful and beautiful typing. In new versions, the author thanks an individual who wishes to remain anonymous for funding typing services, and thanks Matt Grimes for the careful and beautiful typing.

More specifically, the theory of the Skolemizing functions witnessing resplendence for $(M, \bar{b})$ is not continuous in $\mathrm{Th}(M, \bar{b})$. Hence we use a weaker version of indiscernibility though having a linear order is usually a very strong requirement (see [Sheb, §3]), in our proof we use it as if we only have trees (with κ levels).

In [She78] or [She90, VI 5.3–5.6] we characterized first order T and cardinals λ such that for some first order complete T_1, $T \subseteq T_1$, $|T_1| = \lambda$ and any $\tau(T)$–reduct of a model of T_1 is saturated.

In [She87a] we find the spectrum of strongly $\aleph_\varepsilon$–saturated models, but have nothing comparable for strongly $\aleph_1$–saturated ones (on better computation of the numbers see [She88], and more in [Shea, 3.2]). Our interest was:

(A) an instance of complete classification for an "outside" question: the question here is the function giving the number, up to isomorphism, of κ-resplendent models of a (first order complete) theory T as a function of the cardinality; we concentrate on the case $\lambda = \lambda^\kappa + 2^{|T|}$.
(B) an "external" definition of stability which happens to be the dividing line.

Earlier we had such an equivalent "external" definition of stability by saturation of ultra-powers, i.e. Keisler order, see [She90]. Baldwin had told me he was writing a paper on resplendent models: for $\aleph_0$–stable one there are few ($\leq 2^{\aleph_0}$) such models in any cardinality; and for T not superstable — there are 2^λ models of cardinality λ (up to isomorphism).

Note that resplendent models are strongly $\aleph_0$–homogeneous and really the non-structure are related. The reader may thank Rami Grossberg for urging the author to add more explanation to 1.9.

Notation 0.1. *1) For a model M and $\bar{c} \in {}^\alpha M$, let $(M, \bar{c})$ (or $(M, c_i)_{i<\alpha}$) be M expanded by the individual constants c_i for $i < \alpha$.*
2) $\bar{x}_{[u]} = \langle x_\varepsilon : \varepsilon \in u \rangle$

Definition 0.2. *For a complete first order theory we defined $\kappa(T)$, an infinite cardinal or ∞, minimal such that:*

- *$\kappa < \kappa(T)$ iff some $\bar{\varphi}$ witnesses it, which means:*
  - *(1) $\bar{\varphi} = \langle \varphi_\alpha(x, \bar{y}_\alpha) : \alpha < \kappa \rangle$*
  - *(2) for any λ, for some model M of T and sequence $\langle \bar{a}_\eta : \eta \in {}^{\kappa \geq}\lambda \rangle$ with $\bar{a}_\eta \in {}^{\ell g(\bar{y}_\alpha)}M$ we have: if $\varepsilon < \kappa$, $\eta \in {}^\kappa\lambda$, $\alpha < \lambda$ then*
$$M \models \varphi_\varepsilon[a_\eta, \bar{a}_{\eta \restriction \varepsilon \frown \langle \alpha \rangle}]^{\mathrm{if}(\alpha = \eta(\varepsilon))}.$$

Claim 0.3. *In 0.2, for our purposes, we can demand $\bar{y}_\alpha = \langle y \rangle$.*

Proof. Letting $\mathfrak{C}$ be a model of T, use $\mathrm{Th}(\mathfrak{C}^{\mathrm{cf}})$ (see [She90, Ch.III]).

If $\mathrm{cf}(\kappa) > \aleph_0$, without loss of generality $\langle \ell g(\bar{y}_\alpha) : \alpha < \kappa \rangle$ is constant (call this value n) so we can use essentially ${}^n\mathfrak{C}$. This holds for the main lemma 1.9 except when $\kappa = \aleph_0$, in which case the results are easy. $\square_{0.3}$

Definition 0.4. *The sequences $\bar{\eta}$, $\bar{\nu}$ from ${}^{\kappa \geq}\mu$ are called* similar *when:*

(1) $\ell g(\bar{\eta}) = \ell g(\bar{\nu})$ (call this n)
(2) if $\ell < n$ then $\ell g(\eta_\ell) = \ell g(\nu_\ell)$
(3) if $k, \ell < n$ <u>then</u>
  - *(a) $\eta_k \triangleleft \eta_\ell$ iff $\nu_k \triangleleft \nu_\ell$*
  - *(b) $\ell g(\eta_k \cap \eta_\ell) = \ell g(\nu_k \cap \nu_\ell)$*

(c) if $\varepsilon = \ell g(\eta_k \cap \eta_\ell)$ is $<$ both $\ell g(\eta_k)$ and $\ell g(\eta_\ell)$ then

$$\eta_k(\varepsilon) < \eta_\ell(\varepsilon) \Leftrightarrow \nu_k(\varepsilon) < \nu_\ell(\varepsilon)$$

§ 1. Resplendency

Our aim is to prove 1.2 below ("κ–resplendent" is defined in 1.4).

Convention 1.1. T is a fixed first order complete theory; recall that $\tau(T) = \tau_T$, $\tau(M) = \tau_M$ is the vocabulary of T, M respectively and $\mathbb{L}$ is first order logic, so $\mathbb{L}_\tau \equiv \mathbb{L}(\tau)$ is the first order language with vocabulary τ.

We show here

Theorem 1.2. *The following are equivalent (see Definition 1.4 below) for a regular uncountable κ:*

- *(i) $\kappa < \kappa(T)$, see e.g. 0.2 or 2.1(1),*
- *(ii) there is a non-saturated κ–resplendent model of T (see Definition 1.4 below),*
- *(iii) for every $\lambda = \lambda^\kappa \geq 2^{|T|}$, T has $> \lambda$ non-isomorphic κ–resplendent models,*
- *(iv) for every $\lambda = \lambda^\kappa \geq 2^{|T|}$, T has 2^λ non-isomorphic κ–resplendent models.*

Proof. The implication (**i**) $\Rightarrow$ (**iii**) follows from the main Lemma 1.9 below; the implication (**iii**) $\Rightarrow$ (**ii**) is trivial (as any two saturated models of T of the same cardinality are isomorphic), and (**ii**) $\Rightarrow$ (**i**) follows from 1.8 below. Trivially, (**iv**) $\Rightarrow$ (**iii**), and lastly (**i**) $\Rightarrow$ (**iv**) by 3.1+2.23. $\square_{1.2}$

Remark 1.3. (1) If we omit condition (iv) we save §3 as well as the dependency on a theorem from [She22] using only an easy relative. This is sufficient for having an outside characterization of T be stable.
(2) In the proof the main point is (**i**) $\Rightarrow$ (**iii**) (and (**i**) $\Rightarrow$ (**iv**), i.e., the non-structure part).
(3) Remember: T is unstable iff $\kappa(T) = \infty$.
(4) Notice that every saturated model M is $\|M\|$–resplendent (see 1.4(2) below). Actually a little more.

Definition 1.4. *(1) A model M is (κ, ℓ)–resplendent (where $\ell = 0, 1, 2, 3$) if: <u>for every</u> elementary extension N of M and expansion N_1 of N satisfying $|\tau(N_1) \setminus \tau(N)| < \kappa$ and $\alpha < \kappa$, $c_i \in M$ for $i < \alpha$ and $T_1 \subseteq \mathrm{Th}(N_1, c_i)_{i<\alpha}$ satisfying $(*)^\ell_{T_1}$ below, <u>there is</u> an expansion $(M_1, c_i)_{i<\alpha}$ of $(M, c_i)_{i<\alpha}$ to a model of T_1, when:*

$(*)^\ell_{T_1}$ **<u>Case 0</u>:** *$\ell = 0$: $|T_1| < \kappa$,*
<u>Case 1</u>: *$\ell = 1$: for some $\tau' \subseteq \tau(N_1)$, $|\tau'| < \kappa$ and*

$$T_1 \subseteq \mathbb{L}(\tau' \cup \{c_i : i < \alpha\})$$

<u>Case 2</u>: *$\ell = 2$: T_1 is κ–recursive (see 1.4(4) below),*
<u>Case 3</u>: *$\ell = 3$: $T_1 = \mathrm{Th}\,(N_1, c_i)_{i<\alpha}$ (but remember that N_1 has only $< \kappa$ relations and functions not of M).*

(2) κ-resplendent means $(\kappa, 3)$-resplendent.
(3) Assume M is a model of T, $\bar{c} \in {}^{\kappa>}|M|$ and $M_{\bar{c}}$ is an expansion of $(M, \bar{c})$. We say that $M_{\bar{c}}$ witnesses (κ, ℓ)–resplendence for $\bar{c}$ in M, <u>when</u>:
for every first order T_1 such that

$$\mathrm{Th}(M, \bar{c}) \subseteq T_1 \quad \& \quad |\tau(T_1) \setminus \tau(T)| < \kappa$$

and $(*)^{\ell}_{T_1}$ *holds, we have:*
$M_{\bar c}$ *is a model of* T_1 *up to renaming the symbols in* $\tau(T_1)\setminus\tau(M,\bar c)$.

(4) *For* M, N_1, $\langle c_i : i<\alpha\rangle$ *and* $T_1 \subseteq \mathrm{Th}(N_1, c_i)_{i<\alpha}$ *as in part (1),* T_1 *is* κ*–recursive* *when:*
  (a) $\kappa = \aleph_0$ *and* T_1 *is recursive (assuming the vocabulary of* T *is represented in a recursive way)* *or*
  (b) $\kappa > \aleph_0$ *and for some* $\tau^* \subseteq \tau(N_1)$, $|\tau^*|<\kappa$ *the following holds:* *if* $\varphi_\ell(x_0,\ldots,x_{n-1}) \in \mathbb{L}(\tau')$ *for* $\ell = 1,2$ *and there is an automorphism* π *of* τ' *(see parts (9), (10)), where* $\tau^* \subseteq \tau' \subseteq \tau(N_1)$ *such that* π *is the identity on* τ^* *and* $\hat\pi(\varphi_1) = \varphi_2$ *and* $\beta_0 < \beta_1 < \ldots < \alpha$ *then*

$$\varphi_1(c_{\beta_0}, c_{\beta_1}, \ldots) \in T_1 \quad \text{iff} \quad \varphi_2(c_{\beta_0}, c_{\beta_1}, \ldots) \in T_1.$$

(5) *We say* f *is an* (M,N)*-elementary mapping* *when* f *is a partial one-to-one function from* M *to* N, $\tau(M)=\tau(N)$ *and for every* $\varphi(x_0,\ldots,x_{n-1}) \in \mathbb{L}(\tau_M)$ *and* $a_0,\ldots,a_{n-1}\in M$ *we have:*

$$M \models \varphi[a_0,\ldots,a_{n-1}] \quad \text{iff} \quad N \models \varphi[f(a_0),\ldots,f(a_{n-1})].$$

(6) f *is an* M*–elementary mapping* *if* *it is an* (M,M)*–elementary mapping.*

(7) M *is* κ*–homogeneous* *if:*
for any M*–elementary mapping* f *with* $|\mathrm{Dom}(f)|<\kappa$ *and* $a\in M$ *there is* *an* M*–elementary mapping* g *such that:*

$$f \subseteq g, \qquad \mathrm{Dom}(g) = \{a\}\cup \mathrm{Dom}(f).$$

(8) M *is strongly* κ*–homogeneous* *if* *for any* M*-elementary mapping* f *with* $|\mathrm{Dom}(f)|<\kappa$ *there is an automorphism* g *of* M *such that* $f\subseteq g$.

(9) *Let* $\tau_1 \subseteq \tau_2$ *be vocabularies. We say that* π *is an automorphism of* τ_2 *over* τ_1 *when:* π *is a permutation of* τ_2, π *maps any predicate* $P\in\tau_2$ *to a predicate of* τ_2 *with the same arity,* π *maps any function symbol* $F\in\tau_2$ *to a function symbol of* τ_2 *of the same arity and* $\pi\restriction\tau_1$ *is the identity.*

(10) *For* π,τ_2 *as in part (9) let* $\hat\pi$ *be the permutation of the set of formulas in the vocabulary* τ_2 *which* π *induces.*

Note:

Fact 1.5. (1) *If* $\tau=\tau(M)$, *and*

$$\Big[\tau'\subseteq\tau \;\&\; |\tau'|<\kappa \;\Rightarrow\; M\restriction\tau' \text{ is saturated}\Big]$$

then M *is* $(\kappa,1)$*–resplendent.*

(2) *If* M *is saturated of cardinality* λ *then* M *is* λ*-resplendent.*

Proof. Easy, e.g., see [She78] and not used here elsewhere. $\square_{1.5}$

Example 1.6. There is, for each regular κ, a theory T_κ such that:

(A) T_κ is superstable of cardinality κ,
(B) for $\lambda\geq\kappa$, T_κ has 2^λ non-isomorphic $(\kappa,1)$–resplendent models.

Proof. Let $A_0 = \{\kappa\setminus(i+1) : i<\kappa\}$ and $A_1 = A_0\cup\{\varnothing\}$. For every linear order I of cardinality $\lambda\geq\kappa$ we define a model M_I:
its universe is

$$I \cup \Big\{\langle s,t,i,x\rangle : s\in I,\ t\in I,\ i<\lambda,\ x\in A_1 \text{ and } \big[I\models s<t \;\Rightarrow\; x\in A_0\big]\Big\},$$

(And of course, without loss of generality, no quadruple $\langle s,t,i,x\rangle$ as above belongs to I.) Its relations are:

$$\begin{array}{lcl} P & = & I, \\ R & = & \big\{\big\langle s,t,\langle s,t,i,x\rangle\big\rangle : s\in I,\ t\in I,\ \langle s,t,i,x\rangle\in |M_I|\setminus P\big\}, \\ Q_\alpha & = & \{\langle s,t,i,x\rangle : \langle s,t,i,x\rangle\in |M_I|\setminus P,\ \alpha\in x\} \qquad \text{for } \alpha<\kappa. \end{array}$$

In order to have the elimination of quantifiers we also have two unary functions F_1, F_2 defined by:

$$\begin{array}{rcl} s\in I & \Rightarrow & F_1(s)=F_2(s)=s, \\ \langle s,t,i,x\rangle\in |M_I|\setminus I & \Rightarrow & F_1(\langle s,t,i,x\rangle)=s \ \&\ F_2(\langle s,t,i,x\rangle)=t. \end{array}$$

It is easy to see that:

(a) In M_I, the formula
$$P(x)\ \&\ P(y)\ \&\ (\exists z)(R(x,y,z)\ \&\ \bigwedge_{\alpha<\kappa}\neg Q_\alpha(z))$$
linearly orders P^{M_I}; in fact, it defines $<_I$.
(b) $\mathrm{Th}(M_I)$ has elimination of quantifiers.
(c) If $\tau\subseteq\tau(M_I)$, $|\tau|<\kappa$ then $M_I\restriction\tau$ is saturated.
(d) $\mathrm{Th}(M_I)$ does not depend on I (as long as it is infinite) and we call it T_κ.
(e) T_κ is superstable.

Hence: $T_\kappa=\mathrm{Th}(M_I)$ is superstable, does not depend on I, and
$$M_I\cong M_J \quad \underline{\text{iff}} \quad I\cong J,$$
and by 1.5 M_I is $(\kappa,1)$–resplendent.

This suffices for part (A) of the claim. By clause (e) above, part (B) of the claim will follow by [She87b, IV,§3] (or better, [Sheb, §3]). $\square_{1.6}$

Fact 1.7. *(1) for $\ell=1,2$, M is $(\kappa,3)$–resplendent implies M is (κ,ℓ)–resplendent, which implies M is $(\kappa,0)$–resplendent.*
(2) M is $(\kappa,0)$–resplendent implies M is κ-compact.
(3) M is $(\kappa,2)$–resplendent implies M is κ-homogeneous, even strongly κ-homogeneous (see Definition 1.4(7),(8)).
(4) If M is $(\kappa,2)$–resplendent $\kappa>\aleph_0$ and $\{\bar a_n : n<\omega\}$ is an indiscernible set in $|M|$, <u>then</u> it can be extended to an indiscernible set of cardinality $\|M\|$ (similarly for sequences).
(5) M is $(\kappa,3)$–resplendent implies M is κ-saturated.
(6) If $\kappa>|T|$ <u>then</u> the notions of 1.4 “(κ,ℓ)–resplendent” for $\ell=0,1,2,3,$ are equivalent.

Proof. Straightforward: for example,
(3) For given $a_i,b_i\in M$ (for $i<\alpha$, where $\alpha<\kappa$) use

$$\begin{array}{rl} T_1 = & \{G(a_i)=b_i : i<\alpha\}\cup \\ & \{(\forall x,y)(G(x)=G(y)\ \Rightarrow\ x=y),\ (\forall x)(\exists y)(G(y)=x)\}\cup \\ & \{(\forall x_0,\dots,x_{n-1})[R(x_0,\dots,x_{n-1})\equiv R(G(x_0),\dots,G(x_{n-1}))] : \\ & \qquad R \text{ an } n\text{-place predicate of } \tau(M)\}\cup \\ & \{(\forall x_0,\dots,x_{n-1})[F(G(x_0),\dots)=G(F(x_0,\dots))] : \\ & \qquad F \text{ an } n\text{-place function symbol of } \tau(M)\}. \end{array}$$

(4) For notational simplicity let $\bar{a}_n = a_n$. Let T_1 be, with P a unary predicate, g a unary function symbol,

$$\{\text{ “}G\text{ is a one-to-one function into }P\text{” }\} \cup \{P(a_n) : n < \omega\} \cup \Big\{(\forall x_0, \ldots, x_{n-1}) \ \Big[\bigwedge_{\ell<n} P(x_\ell) \ \& \bigwedge_{\ell<m<n} x_\ell \neq x_m \ \& \ \varphi[a_0, \ldots, a_{n-1}] \Rightarrow \varphi(x_0, \ldots, x_{n-1})\Big] : \varphi(x_0, \ldots, x_{n-1}) \in \mathbb{L}(\tau(M))\Big\}$$

$\square_{1.7}$

Conclusion 1.8. *If M is κ–resplendent, $\kappa \geq \kappa(T) + \aleph_1$ then M is saturated.*

Proof. By 1.7(5) M is κ–saturated, so without loss of generality $\|M\| \geq \kappa$ (and $\kappa \geq \aleph_1$). Hence, by [She78] or [She90, III,3.10(1),p.107], it is enough to prove: for $\dot{\mathbf{I}}$ an infinite indiscernible $\subseteq M$, $\dim(\dot{\mathbf{I}}, M) = \|M\|$. But this follows by 1.7(4). $\square_{1.8}$

Main Lemma 1.9. *Suppose that $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$ (for example, T unstable, κ regular) and $\lambda = \lambda^\kappa + 2^{|T|}$. then T has $> \lambda$ pairwise non-isomorphic κ–resplendent models of cardinality λ.*

Before embarking on the proof, we give some explanations.

Remark 1.10. (1) We conjecture that we can weaken in 1.9 the hypothesis “$\lambda = \lambda^\kappa + 2^{|T|}$” to “$\lambda = \lambda^{<\kappa} + 2^{|T|}$”. This holds for many λ-s (see [She22, §2]), and probably for all, but we have not looked at this. See §3.

(2) We naturally try to imitate [She78], [She90, VII,§2, VIII,§2] or [Sheb, §3],[Shea]. In the proof of the theorem, the difficulty is that while expanding to take care of resplendence, we naturally will use Skolem functions with infinite arity, and so we cannot use compactness so easily.

If the indiscernibility is not clear, the reader may look again at [She78, VII,§2] or [She90, VII,§2], (tree indiscernibility). We get below first a weaker version of indiscernibility, as it is simpler to get it, and is totally harmless if we would like just to get $> \lambda$ non-isomorphic models by the old version [She87b, III,4.2(2)] or the new [She22, §2]

Explanation 1.11. Note that the problem is having to deal with sequences of $< \kappa$ elements $\bar{b} = \langle b_i : i < \varepsilon\rangle$, ε infinite. The need to deal with such $\bar{b}$ with all theories of small vocabulary is not serious – there is a “universal one” though possibly of larger cardinality, i.e., if $M \models T$, $b_i \in M$ for $i < \varepsilon$, $\varepsilon < \kappa$, we can find a f.o. theory $T_2 = T_2(\bar{b})$ satisfying $\mathrm{Th}(M, b_i)_{i<\varepsilon} \subseteq T_2$, $|T_2| \leq (2^{|T|+|\varepsilon|})^{<\kappa}$ such that:

- if $\mathrm{Th}(M, b_i)_{i<\varepsilon} \subseteq T'$ and $|\tau(T') \setminus \tau(T) \setminus \{b_i : i < \varepsilon\}| < \kappa$ then renaming the predicates and function symbols outside T, we get $T' \subseteq T_2(\bar{b})$.

This is possible by the Robinson consistency lemma. Let us give more details.

Claim 1.12. *(1) Let M_0 be a model, $\tau_0 = \tau(M_0)$, $\bar{b} = \langle b_i : i < \varepsilon\rangle$ where $b_i \in M_0$ for $i < \varepsilon$ and $\theta \geq \aleph_0$ be a cardinal. Let $\tau_1 = \tau_0 \cup \{b_i : i < \varepsilon\}$ so $M_1 = (M_0, b_i)_{i<\varepsilon}$ is a τ_1–model. then there is a theory $T_2 = T_2[\bar{b}] = T_2[\bar{b}, M]$, depending only on τ_0, τ_1 and $\mathrm{Th}(M_1)$, i.e., essentially on $\mathrm{tp}(\langle b_i : i < \varepsilon\rangle, \varnothing, M_0)$ such that:*

(a) $\tau_2 = \tau(T_2) = \tau(\varepsilon, \tau_0)$ extends τ_1 and has cardinality $\leq 2^{|\tau_1|+\theta+|\varepsilon|}$,

(b) for every M_2, T', the model M_2 is expandable to a model of T', when:

(α) *M_2 is a τ_1–model,*
(β) *M_2 can be expanded to a model of T_2,*
(γ) *$\mathrm{Th}(M_2) \subseteq T'$, equivalently some elementary extension of M_2 is expandable to a model of T',*
(δ) *T' is f.o. and $|\tau(T') \setminus \tau(M_2)| \leq \theta$,*

(a)$^+$ if $\theta > |T| + |\varepsilon|$ then $|\tau_2| \leq 2^{<\theta}$ is enough.

(2) If in part (1), sub-clause (δ) of clause (b) is weakened to:
$(\delta)_2$ T' is f.o., and $|\tau(T') \setminus \tau(M_2)| < \theta$,
<u>then</u> we can strengthen (a) to
$(a)_2$ $\tau_2 = \tau(T_2)$ extends τ_1 and has cardinality $\leq \sum\limits_{\mu<\theta} 2^{|\tau_1|+\mu+\aleph_0+|\varepsilon|}$,
$(a)_2^+$ if $\theta > (|T| + |\varepsilon|)^+$ then $|\tau_2| \leq \sum\limits_{\mu<\theta} 2^{<\mu}$ is enough.

Proof. 1) We ignore function symbols and individual constants as we can replace them by predicates. Let

$$\mathbb{T} = \big\{T' : \ T' \text{ f.o. complete theory, } \mathrm{Th}(M_1) \subseteq T' \text{ and } \tau(T') \setminus \tau(M_1) \text{ has cardinality } \leq \theta\big\}.$$

This is a class; we say that T', $T'' \in \mathbb{T}$ are isomorphic over $\mathrm{Th}(M_1)$ (see [She71]) <u>when</u> there is a function $\mathbf{h}$ satisfying:

(a) $\mathbf{h}$ is one-to-one,
(b) $\mathrm{Dom}(\mathbf{h}) = \tau(T')$,
(c) $\mathrm{Rang}(\mathbf{h}) = \tau(T'')$,
(d) $\mathbf{h}$ preserves arity (i.e., the number of places, and of course being predicate/function symbols),
(e) $\mathbf{h} \restriction (\tau(M_1)) = \text{identity}$,
(f) for a f.o. sentence $\psi = \psi(R_1, \dots, R_k) \in \mathbb{L}[\tau(T')]$, where $R_1, \dots, R_k$ are the non-logical symbols occurring in ψ, we have

$$\psi(R_1, \dots, R_k) \in T' \quad \Leftrightarrow \quad \psi(\mathbf{h}(R_1), \dots, \mathbf{h}(R_k)) \in T''.$$

Now note that
$\boxplus_1$ $\mathbb{T}/\cong$ has cardinality $\leq 2^{|\tau_0|+|\varepsilon|+\theta}$.

Now let $\{T'_\alpha : \alpha < 2^{|\tau_0|+|\varepsilon|+\theta}\}$ be a list of members of $\mathbb{T}$ such that every isomorphism equivalence class over $\mathrm{Th}(M_1)$ is represented, and $\langle \tau(T'_\alpha) \setminus \tau_1 : \alpha < 2^{|\tau_0|+|\varepsilon|+\theta}\rangle$ are pairwise disjoint.

Note that $\mathrm{Th}(M_1) \subseteq T'_\alpha$. Let $T'_2 = \bigcup\{T'_\alpha : \alpha < 2^{|\tau_0|+|\varepsilon|+\theta}\}$ and note

$\boxplus_2$ T'_2 is consistent.

[Why? By Robinson consistency theorem.]

Let T_2 be any completion of T'_2. So condition (a) holds; proving (b) should be easy.

Let us prove **(a)**$^+$ of the claim; this is really the proof that a theory T, $|T| < \theta$, has a model in $2^{<\theta}$ universal for models of T of cardinality $\leq \theta$. We shall define by induction on $\alpha < \theta$, a theory T^2_α such that:

(A) $T^2_0 = \mathrm{Th}(M_1)$,
(B) T^2_α a f.o. theory,
(C) $\tau^2_\alpha = \tau(T^2_\alpha)$ has cardinality $\leq 2^{|\tau_0|+|\varepsilon|+|\alpha|+\aleph_0}$,
(D) T^2_α, τ^2_α are increasing continuous in α,

(E) if $\tau_1 \subseteq \tau' \subseteq \tau^2_\alpha$, $|\tau'| \le |\tau^1| + |\alpha|$, $\tau' \subseteq \tau''$, $\tau'' \cap \tau^2_\alpha = \tau'$, $T^2_\alpha \restriction \mathbb{L}_{\tau'} \subseteq T'' \subseteq \mathbb{L}[\tau'']$, T'' complete and $|\tau'' \setminus \tau'| = \{R\}$,
then we can find $R' \in \tau^2_{\alpha+1} \setminus \tau^2_\alpha$ such that of the same arity.

$$T''[\text{replacing } R \text{ by } R'] \subseteq T^2_{\alpha+1}$$

There is no problem to carry out the induction, and $\bigcup\limits_{\alpha<\theta} T^2_\alpha$ is as required.

2) Similar. $\square_{1.12}$

Explanation 1.13. So for $M \models T$, $\bar{b} \in {}^{\kappa>}M$, we can choose $T_2[\bar{b}] \supseteq \mathrm{Th}(M, \bar{b})$ depending on $\mathrm{Th}(M, \bar{b})$ only, such that:

($\otimes$) $M \models$ "T is κ–resplendent if for every $\bar{b} \in {}^{\kappa>}M$, $(M, \bar{b})$ is expandable to a model of $T_2[\bar{b}]$.

W.l.o.g. $\tau(T_2[\bar{b}])$ depends on $\ell g(\bar{b})$ and τ_0 only, so it is $\tau(\ell g(\bar{b}), \tau_M)$.

The things look quite finitary but $T_2[\bar{b}]$ is not continuous in $\mathrm{Th}(M, \bar{b})$. I.e., $(*) \not\Rightarrow (**)$, where

$(*)$ $\bar{b}^\alpha \in {}^{\kappa>}M$, for $\alpha \le \delta$, (δ a limit ordinal) $\ell g(\bar{b}^\alpha) = \varepsilon$, and for every n, $i_1 < \ldots < i_n < \varepsilon$ and a formula $\varphi(x_{i_1}, \ldots, x_{i_n}) \in \mathbb{L}(\tau_M)$ for some $\beta < \delta$:

$$\beta \le \alpha \le \delta \quad \Rightarrow \quad M \models \varphi[b^\beta_{i_1}, \ldots, b^\beta_{i_n}] \equiv \varphi[b^\alpha_{i_1}, \ldots, b^\alpha_{i_n}],$$

$(**)$ for any $\varphi \in \mathbb{L}(\tau_2)$ for some $\beta < \delta$:

$$\beta \le \alpha \le \delta \quad \Rightarrow \quad \big[\varphi \in T_2[\bar{b}^\alpha] \Leftrightarrow \varphi \in T_2[\bar{b}^\beta]\big].$$

[You can make $T_1[\bar{b}]$ a somewhat continuous function of the sequence $\bar{b}$ if we look at sub-sequences as approximations rather than the type, but this is not used.]

For example, consider the case

- $M \models (\exists !^n x) R(x, b_n)$
- $\psi(y)$ says F is a one-to-one function from $\{x : R(x, y)\}$ but not onto it.

This explains why you need "infinitary Skolem functions".

We shall try to construct M such that for every $\bar{b} \in {}^\varepsilon M$, $(M, \bar{b})$ is expandable to a model of $T_2[\bar{b}]$, so if $\tau^\varepsilon_2 = \tau(T_2[\bar{b}]) \setminus \tau(M, \bar{b})$, this means we have to define finitary relations/functions $R_{\bar{b}}$ (for $R \in \tau^\varepsilon_2$). We write here $\bar{b}$ as a sequence of parameters but from another prospective the predicate/function symbol $R_\square(-)$ has $\varepsilon + \mathrm{arity}(R)$–places.

Explaining the first construction 1.14. (i.e., 2.20 below)
Eventually we build a generalization of $\mathrm{GEM}({}^{\kappa\ge}\lambda, \Psi)$, a model with skeleton $\bar{a}_\eta$ ($\eta \in {}^{\kappa\ge}\lambda$) witnessing $\kappa < \kappa(T)$, but the functions have any $\alpha < \kappa$ places but not κ, and the indiscernibility demand is weak. We start as in [Sheb, §2], so for some formulas $\langle \varphi_\alpha(x, \bar{y}, \alpha) : \alpha < \kappa \rangle$ we have (where $\bar{a}_\eta = \langle a_\eta \rangle$ for $\eta \in {}^\kappa\lambda$):

$$\eta \in {}^\kappa\lambda \;\&\; \nu \in {}^{\alpha+1}\lambda \quad \Rightarrow \quad \varphi_\alpha(a_\eta, \bar{a}_\nu)^{\mathrm{if}(\nu \triangleleft \eta)}.$$

Recall that for a formula φ we let $\varphi^0 = \varphi$ and $\varphi^1 = \neg\varphi$, so $\varphi^{\mathrm{if}(\psi)}$ is either φ or $\neg\varphi$ according to the truth value of ψ. Without loss of generality, for any $\alpha < \kappa$ for some sequence $\bar{G}_\alpha = \big\langle G_{\alpha,\ell} : \ell < \ell g(\bar{y}_\alpha)\big\rangle$ of unary function symbols such that for any $\eta \in {}^\kappa\lambda$,

$$\bar{a}_{\eta\restriction\alpha} = \bar{G}_\alpha(a_\eta) := \big\langle G_{\alpha,\ell}(a_\eta) : \ell < \ell g(\bar{y}_\alpha)\big\rangle$$

so we can look at $\{a_\eta : \eta \in {}^\kappa\lambda\}$ as generators. For $W \in [{}^\kappa\lambda]^{<\kappa}$, let $N_W = N[W]$ be the submodel which $\{a_\eta : \eta \in W\}$ generates. So we would like to have:

(α) N_W has the finitary Skolem function (for T), and moreover
N_W has the finitary Skolem function for $T_2[\bar b]$ for each $\bar b \in {}^{\kappa>}(N_W)$,

(β) monotonicity: $W_1 \subseteq W_2 \Rightarrow N_{W_1} \subseteq N_{W_2}$.

So if $\mathscr{U} \subseteq {}^\kappa\lambda$, then $N[\mathscr{U}] = \{N_W : W \in {}^{\kappa>}[\mathscr{U}]\}$ is a κ–resplendent model of cardinality λ.

(γ) Indiscernibility: (We use here very "minimal" requirement (see below) but still enough for the omitting type in (1) below):

(1) $\eta \in {}^\kappa\lambda \setminus \mathscr{U} \quad \Rightarrow \quad N[\mathscr{U}]$ omits $p_\eta =: \{\varphi_\alpha(x, \bar a_{\eta\restriction\alpha}) : \alpha < \kappa\}$; (satisfaction defined in $N[{}^\kappa\lambda]$),

(2) $\eta \in {}^\kappa\lambda \cap \mathscr{U} \quad \Rightarrow \quad N[\mathscr{U}]$ realizes p_η.

Now (2) was already guaranteed: a_η realizes p_η.

For (1) it is enough

(1)$'$ <u>if</u> $W \in {}^{\kappa>}[{}^\kappa\lambda]$, $\eta \in {}^\kappa\lambda \setminus W$ then p_η is omitted by N_W (satisfaction defined in $N[{}^\kappa\lambda]$).

Fix W, η for (1)$'$. A sufficient condition is

(1)$''$ for $\alpha < \kappa$ large enough, $\langle \bar a_{\eta\restriction\alpha\frown\langle i\rangle} : i < \lambda\rangle$ is indiscernible over N_W in $N[{}^\kappa\lambda]$.

[if $\kappa(T) < \infty$, this immediately suffices; in the general case, and avoiding classification theory, use

$$p'_\eta = \{\varphi_\alpha(x, \bar a_{\eta\restriction(\alpha+1)}) \;\&\; \neg\varphi(x, \bar a_{\eta\restriction\alpha\hat{\ }\langle\eta(\alpha)+1\rangle}) : \alpha < \kappa\}$$

so we use

$$\varphi'(x, \bar y'_\alpha) = \varphi_\alpha(x, \bar y'_\alpha \restriction \ell g(\bar y_\alpha)) \wedge \neg\varphi_\alpha(x, \bar y'_\alpha \restriction (\ell g(\bar y_\alpha), 2\ell g(\bar y_\alpha)))$$

in the end].

Note: as $|W| < \kappa$, for some $\alpha(*) < \kappa$, for every $\eta \in {}^\kappa\lambda$

$$W \cap \{\nu : \eta \restriction \alpha(*) \triangleleft \nu \in {}^\kappa\lambda\} \text{ is a singleton}$$

and $W \in \mathbf{W}_{\alpha(*)}$ (see below), this will be enough to omit the type. The actual indiscernibility is somewhat stronger.

<u>Further Explanation:</u> On the one hand, we would like to deal with arbitrary sequences of length $< \kappa$, on the other hand, we would like to retain enough freedom to have the weak indiscernibility. What do we do? We define our "Φ" (not as nice as in [Sheb, §2], i.e., [She78, Ch.VII §3]) by κ approximations indexed for $\alpha \le \kappa$.

For $\alpha \le \kappa$, we essentially have N_W for

$$W \in \mathbf{W}_\alpha =: \{W : \; W \subseteq {}^\kappa\lambda,\ |W| < \kappa \text{ and the function } \eta \mapsto \eta \restriction \alpha \ (\eta \in W) \text{ is one-to-one}\}.$$

Now, $\mathbf{W}_\alpha$ is partially ordered by $\subseteq$ but (for $\alpha < \kappa$) is not directed. For $\alpha < \beta$ we have $\mathbf{W}_\alpha \subseteq \mathbf{W}_\beta$ and $\mathbf{W}_\kappa = \bigcup\limits_{\alpha<\kappa} \mathbf{W}_\alpha$ is equal to $[{}^\kappa\lambda]^{<\kappa}$.

So if we succeed to carry out the induction for $\alpha < \kappa$, arriving at $\alpha = \kappa$ the direct limit works and no new sequence of length $< \kappa$ arises.

§ 2. Proof of the Main Lemma

In this section we get many models using a weak version of indiscernibility.

Hypothesis 2.1. (1) T is a fixed complete first order theory, $\kappa < \kappa(T)$, $\bar\varphi = \langle \varphi_\alpha(x,y) : \alpha < \kappa\rangle$ is a fixed witness for $\kappa < \kappa(T)$, recalling 0.3 and 2.2. This means:

(∗) for any λ, for some model M of T and sequence $\langle a_\eta : \eta \in {}^{\kappa \geq}\lambda\rangle$ with $a_\eta \in M$ we have: if $\varepsilon < \kappa$, $\eta \in {}^\kappa\lambda$, $\alpha < \lambda$ then

$$M \models \varphi_\varepsilon[a_\eta, a_{\eta\restriction\varepsilon\frown\langle\alpha\rangle}]^{\text{if}(\alpha=\eta(\varepsilon))}.$$

(2) Let μ be infinite large enough cardinal; $\mu = \beth_\omega(|T|)$ is O.K.

Remark: Why are we allowed in 2.1(1) to use $\varphi_\alpha(x,y)$ instead $\varphi(x,\bar y)$? We can work in T^{eq}, see [She90, Ch. III] and anyhow this is, in fact, just a notational change.

Definition 2.2. *(1) For $\alpha < \kappa$ and $\rho \in {}^\alpha\mu$, let $I_\rho = I^\alpha_\rho = I^{\alpha,\mu}_\rho$ be the model*

$$\big(\{\nu \in ({}^\kappa\mu) : \nu \restriction \alpha = \rho\}, E_i, <_i\big)_{i<\kappa},$$

where

$$\begin{aligned} E_i &= \{(\eta,\nu) : \eta \in {}^\kappa\mu, \nu \in {}^\kappa\mu, \eta \restriction i = \nu \restriction i\}, \\ <_i &= \{(\eta,\nu) : \eta\ E_i \nu \ \text{ and } \ \eta(i) < \nu(i)\}. \end{aligned}$$

(2) Let $\mathbf{W}_\alpha = \mathbf{W}^\mu_\alpha = \{W \subseteq {}^\kappa\mu : W$ has cardinality $< \kappa$ and for any $\eta \neq \nu$ from W we have $\eta \restriction \alpha \neq \nu \restriction \alpha$ $\}$, and $\mathbf{W}_{<\alpha} = \bigcup\limits_{\beta<\alpha} \mathbf{W}_\beta$.

(3) We say that $\mathbf{W}$ is α–invariant, or (α,μ)–invariant, when $\mathbf{W} \subseteq \mathbf{W}_\alpha$ and: <u>if</u> $W_1, W_2 \in \mathbf{W}_\alpha$, h is a one-to-one function from W_1 onto W_2 and $\eta \restriction \alpha = h(\eta) \restriction \alpha$ for $\eta \in \mathbf{W}_1$, <u>then</u> $W_1 \in \mathbf{W} \Leftrightarrow W_2 \in \mathbf{W}$.

(4) We say $\mathbf{W} \subseteq \mathbf{W}_\alpha$ is hereditary if $W' \subseteq W \in \mathbf{W} \Rightarrow W' \in \mathbf{W}$.

(5) let $\mathbf{W}_{<\alpha} = \bigcup\{\mathbf{W}_\beta : \beta < \alpha\}$, so obviously it is contained in $\mathbf{W}_\alpha$.

Definition 2.3. *(1) Let $\theta = \theta_{T,\kappa}$ be the minimal cardinal satisfying:*

(a) $\theta = \theta^{<\kappa} \geq |T|$,

(b) if M is a model of T, $\bar b \in {}^{\kappa>}M$, <u>then</u> there is a complete (first order) theory T^* of cardinality $\leq \theta$ with Skolem functions extending $\mathrm{Th}(M,\bar b)$ such that:
if $T' \supseteq \mathrm{Th}(M,\bar b)$ and $\tau(T') \setminus \tau_{(M,\bar b)}$ has cardinality $< \kappa$ <u>then</u> there is a one-to-one mapping from $\tau(T')$ into $\tau(T^*)$ over $\tau_{(M,\bar b)}$ preserving arity and being a predicate / function symbol, and mapping T' into T^*.

(2) For $\varepsilon < \kappa$, let $\tau[T,\varepsilon]$ be a vocabulary consisting of τ_T, the individual constants $\underline{b}_\xi$ for $\xi < \varepsilon$, and the n–place predicates $R_{T,j,n}$ for $j < \theta$ and n–place function symbols $F_{T,j,n}$ for $j < \theta$.

For $\varepsilon < \kappa$ and a complete theory $T^\oplus$ in the vocabulary $\tau_T \cup \{\underline{b}_\xi : \xi < \varepsilon\}$ extending T, let $T^*[T^\oplus]$ be a complete first order theory in the vocabulary $\tau[T,\varepsilon]$ such that if $(M,\bar b)$ is a model of $T^\oplus$, then $T^*[T^\oplus]$ is as in clause (b) of part (1).

(3) For $M \models T$ and $\varepsilon < \kappa$ and $\bar b \in {}^\varepsilon M$, let $T^*[\bar b, M] = T^*[\mathrm{Th}(M,\bar b)]$.

Remark 2.4. Note that θ is well defined by 1.12. In fact, $\theta = \Pi\{2^{|T|+\sigma} : \sigma^+ < \kappa\}$ is OK.

Main Definition 2.5. *We say that $\mathfrak{m}$ is an approximation (or an α–approximation, or (α,μ)–approximation) if*

$(*)_1$ $\alpha \le \kappa$ *(so $\alpha = \alpha_{\mathfrak{m}} = \alpha(\mathfrak{m})$),*

$(*)_2$ $\mathfrak{m}$ *consists of the following (so we may give them subscript or superscript $\mathfrak{m}$):*

- (a) *a model $M = M_{\mathfrak{m}}$;*
- (b) *a set $\mathscr{F} = \mathscr{F}_{\mathfrak{m}}$ of symbols of functions, each $f \in \mathscr{F}$ has an interpretation, a function $f_{\mathfrak{m}}$ with range $\subseteq M$, but when no confusion arises we may write f instead of $f_{\mathfrak{m}}$, (or $f^{\mathfrak{m}}$, note that the role of those f-s is close to that of function symbols in vocabularies, but not equal to);*
- (c) *each $f \in \mathscr{F}$ has $\zeta_f < \kappa$ places, to each place ζ (i.e., an ordinal $\zeta < \zeta_f$) a unique $\eta_\zeta \in {}^\alpha\mu$, $\eta_\zeta = \eta^f_\zeta = \eta(f,\zeta)$ is attached such that*
$$[\zeta \ne \xi \quad \Rightarrow \quad \eta_\zeta \ne \eta_\xi],$$
and the ζ-th variable of f varies on I_{η_ζ}, i.e., $f_{\mathfrak{m}}(\dots,x_\zeta,\dots)_{\zeta<\zeta_f}$ is well defined iff $\bigwedge\limits_{\zeta<\zeta_f} x_\zeta \in I_{\eta_\zeta} = I^{\alpha,\mu}_{\eta_\zeta}$;
we may write $f_{\mathfrak{m}}(\dots,\nu_\eta,\dots)_{\eta\in w[f]}$ instead $f_{\mathfrak{m}}(\dots,\nu_{\eta(f,\zeta)},\dots)_{\zeta<\zeta_f}$, where $w[f] = \{\eta(f,\zeta) : \zeta < \zeta_f\}$; and
$$f \in \mathscr{F} \Rightarrow (\exists W \in \mathbf{W})\big[w[f] = \{\eta \restriction \alpha : \eta \in W\}\big]$$
see clause (e) below;
- (d) *for each $\bar b \in {}^{\kappa>}|M|$, an expansion $M_{\bar b}$ of $(M,\bar b)$ to a model of $T^*[\bar b, M]$, (see above in Definition 2.3; so $M_{\bar b}$ has Skolem functions and it witnesses κ–resplendence for this sequence in M);*
- (e) $\mathbf{W} = \mathbf{W}_{\mathfrak{m}} \subseteq \mathbf{W}_\alpha$ *which is α–invariant and hereditary;*
- (f) *for $W \in \mathbf{W}$, N_W which is the submodel of M with universe*
$$\{f(\dots,\eta_\zeta,\dots)_{\zeta<\zeta_f} :\ f \in \mathscr{F},\ f(\dots,\eta_\zeta,\dots)_{\zeta<\zeta_f} \text{ well defined, and } \eta_\zeta \in W \text{ for every } \zeta\},$$
- (g) *a function $\mathbf{f} = \mathbf{f}_{\mathfrak{m}}$,*

such that $\mathfrak{m}$ satisfies the following:

- (A) *M is a model of T,*
- (B) *[witness for $\kappa < \kappa(T)$:] for our fixed sequence of first order formulas $\langle \varphi_\zeta(x,y) : \zeta < \kappa\rangle$ from $\mathbb{L}(\tau_T)$ (depending neither on α nor on $\mathfrak{m}$) we have $f^*_{\rho,\zeta} \in \mathscr{F}$ for $\zeta \le \kappa$, $\rho \in {}^\alpha\mu$ (we also call them $f^{\mathfrak{m}}_{\rho,\zeta}$) such that:*
  - (i) *$f^*_{\rho,\zeta}$ is a one place function, with $\zeta_{f^*_{\rho,\zeta}}, \eta_0^{f^*_{\rho,\zeta}}$ from clause (c) being $1,\rho$ respectively.*
  - (ii) *$f^*_{\rho_1,\zeta}(\nu_1) = f^*_{\rho_2,\zeta}(\nu_2)$ if $\nu_1 \restriction \zeta = \nu_2 \restriction \zeta$ and they are well defined, i.e. $\rho_\ell \triangleleft \nu_\ell \in {}^\kappa\mu$,*
  - (iii) *if $\rho_\ell \in {}^\alpha\mu$, $\nu_\ell \in I^\alpha_{\rho_\ell}$ for $\ell = 1,2$ and $\zeta < \kappa$ then:*
$$M \models \varphi_\zeta\left[f^*_{\rho_1,\kappa}(\nu_1), f^*_{\rho_2,\zeta+1}(\nu_2)\right] \qquad \text{iff} \qquad [\nu_1 \restriction (\zeta+1) = \nu_2 \restriction (\zeta+1)],$$
- (C) *$N_W \prec M$ for $W \in \mathbf{W}$; moreover, $N_W = M_{\mathfrak{m}} \restriction A_W$, where A_W is the minimal subset of $M_{\mathfrak{m}}$ such that: (see below) if $\bar f \in \mathscr{F}$, $\bar\nu \subseteq W$, $\bar\nu \in \mathrm{dom}(\bar f)$, then $|N_{\bar b}| \subseteq A_W$. Therefore (see 2.6) if $\mathfrak{m}$ is full and closed under terms then $M_{\bar b} \restriction A_W \prec M_{\bar b}$.*
- (D) *[$\mathbf{f} = \mathbf{f}_{\mathfrak{m}}$ witnesses an amount of resplendence]*

(α) *the domain of* $\mathbf{f}$ *is a* <u>*subset*</u> *of* $\mathbb{F}_{\mathfrak{m}}$*, where* $\mathbb{F}_{\mathfrak{m}}$ *is the subset of*

$$\{\bar f = \langle f_\varepsilon : \varepsilon < \varepsilon_{\bar f}\rangle : \varepsilon_{\bar f} < \kappa,\ f_\varepsilon \in \mathscr{F}\}$$

for which $\zeta_{f_\varepsilon} =: \zeta_{\bar f}$ *does not depend on* ε*, the sequence* $\eta(f_\varepsilon, \zeta) =: \eta(\bar f, \zeta)$ *does not depend on* ε *for all* $\zeta < \zeta_{\bar f}$*, and* $\{\eta(\bar f, \zeta) : \zeta < \zeta_{\bar f}\} \subseteq W$ *for some* $W \in \mathbf{W}_{\mathfrak{m}}$*.*
[so $\bar f \in \mathbb{F}_{\mathfrak{m}}$ *maps* $\prod\limits_{\zeta < \zeta_f} I_{\eta(\bar f, \zeta)}$ *into* ${}^{\varepsilon(f)}M$*]*

(β) *for* $\bar f \in \mathrm{Dom}(\mathbf{f})$*,* $\mathbf{f}(\bar f)$ *is a function with domain*

$$\{\sigma(\bar x) : \ \sigma(\bar x) \text{ is a } \tau[T, \varepsilon_{\bar f}]\text{–term, and } \bar x = \langle x_\xi : \xi \in u\rangle \text{ for some finite subset } u = u_\sigma \text{ of } \varepsilon_{\bar f}\ \}$$

and if $\sigma(\bar x) \in \mathrm{Dom}(\mathbf{f}(\bar f))$ <u>*then*</u>

$$\mathbf{f}(\bar f)(\sigma(\bar x)) \in \mathscr{F}[\bar f] := \Big\{ f \in \mathscr{F} : \zeta_f = \zeta_{\bar f},\ (\forall \zeta < \zeta_f)\big[\eta(f, \zeta) = \eta(\bar f, \zeta)\big]\Big\},$$

(γ)
- *if* $\bar f \in \mathrm{Dom}(\mathbf{f})$ *and* $\bar b = \big\langle f_\varepsilon(\ldots, \nu_{\eta(\bar f, \zeta)}, \ldots)_{\zeta < \zeta_{\bar f}} : \varepsilon < \varepsilon_{\bar f}\big\rangle$ <u>*then*</u> *the set* $\big\{f(\ldots, \nu_{\eta(\bar f, \zeta)}, \ldots)_{\zeta < \zeta_{\bar f}} : f \in \mathscr{F}[\bar f]\big\}$ *is the universe of an elementary submodel of* $M_{\bar b}$ *called* $N_{\mathfrak{m}, \bar f}$*. This actually follows from the next point:*
- *if* $(\mathbf{f}(\bar f))(\sigma(\bar x)) = f^* \in \mathscr{F}[\bar f]$*,* $W \in \mathbf{W}$*,* $\nu_\zeta \in W$ *and* $\nu_\zeta \restriction \alpha = \eta(\bar f, \zeta)$ *for* $\zeta < \zeta_{\bar f}$*, and* $\bar x = \langle x_\xi : \xi \in u\rangle$ *(where* $u \subseteq \zeta_{\bar f}$*), and* $\bar b = \bar f(\bar\nu) = \langle f_\varepsilon(\bar \nu) : \varepsilon < \varepsilon_{\bar f}\rangle$*,* <u>*then*</u>

$$\sigma^{M_{\bar b}}(\langle f_\xi(\ldots, \nu_\zeta, \ldots)_{\zeta < \zeta_{\bar f}} : \xi \in u\rangle) = f^*(\ldots, \nu_\zeta, \ldots)_{\zeta < \zeta_{\bar f}}.$$

[explaining (γ)*: we may consider* $\bar b \in N_{W_1} \cap N_{W_2}$*, and we better have that the witnesses for resplendence demands, specialized to* $\bar b$*, in* N_{W_1} *and in* N_{W_2} *are compatible so that in the end resplendence holds].*

However, we shall not get far without at least more closure and coherence of the parts of $\mathfrak{m}$.

Definition 2.6. *(1) An approximation* $\mathfrak{m}$ *is called full* <u>*if*</u> $\mathbf{W}_{\mathfrak{m}} = \mathbf{W}_{\alpha(\mathfrak{m})}$*, and is called semi–full if* $\mathbf{W}_{<\alpha(\mathfrak{m})} \subseteq \mathbf{W}_{\mathfrak{m}} \subseteq \mathbf{W}_{\alpha(\mathfrak{m})}$ *and is called almost full if it is semi full when* α *is limit ordinal and full when* α *is a non-limit ordinal.*

(2) An approximation $\mathfrak{m}$ *is* β*–resplendent if* $\beta \le \alpha_{\mathfrak{m}}$ *and: (recalling* $\mathrm{dom}(\mathbf{f}_{\mathfrak{m}}) \subseteq \mathbb{F}_{\mathfrak{m}}$*)*

$$\begin{array}{ll} \textit{if} & W \in \mathbf{W}_\beta \cap \mathbf{W}_{\mathfrak{m}} \text{ and } \bar f \in \mathbb{F}_{\mathfrak{m}}, \text{ and} \\ & \{\eta(\bar f, \zeta) : \zeta < \zeta_{\bar f}\} \subseteq \{\nu \restriction \alpha : \nu \in W\}, \\ \textit{then} & \bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}}). \end{array}$$

(3) In part (2), if we omit β*, we mean* $\beta = \alpha_{\mathfrak{m}}$*, and "*$< \beta^*$*" means for every* $\beta < \beta^*$*.*

(4) An approximation $\mathfrak{m}$ *is called term closed* <u>*if:*</u>

(E) <u>*Closure under terms of* τ*:*</u>
Assume that $u \subseteq {}^\alpha\mu$*,* $|u| < \kappa$*, and for some* $W \in \mathbf{W}_{\mathfrak{m}}$*,* $u \subseteq \{\eta \restriction \alpha : \alpha \in W\}$*, and* $\langle \eta_\zeta : \zeta < \zeta^*\rangle$ *lists* u *with no repetitions and* $f_\ell \in \mathscr{F}_{\mathfrak{m}}$*,* $\ell < n$*, satisfies* $\{\eta(f_\ell, \zeta) : \zeta < \zeta_{f_\ell}\} \subseteq u$*,* σ *is an* n*-place* $\tau(T)$*-term so* $\sigma = \sigma(x_0, \ldots, x_{n-1})$*.* <u>*then*</u> *for some* $f \in \mathscr{F}_{\mathfrak{m}}$ *satisfying* $\zeta_f = \zeta^*$*,*

$\eta(f,\zeta) = \eta_\zeta$, for any choice of $\langle \nu_\eta : \eta \in u\rangle$ such that $\eta \triangleleft \nu_\eta \in {}^\kappa\mu$ for $\eta \in u$, and $\{\nu_\eta : \eta \in u\} \subseteq W' \in \mathbf{W}$ for some W' we have

$$f_{\mathfrak{m}}\left(\ldots,\nu_\eta,\ldots\right)_{\eta\in w[f]} = \sigma\left(\ldots, f^{\mathfrak{m}}_\ell(\ldots,\nu_\eta,\ldots)_{\eta\in w[f_\ell]},\ldots\right)_{\ell<n}$$

(This clause may be empty, but it helps to understand clause **(F)***; note that it is not covered by 2.5(D)(β) as the functions do not necessarily have the same domain, hence this says something even for σ the identity. This implies that in clause (f) of Definition 2.5 we can demand $\{\eta_{\mathfrak{m}}(f,\zeta) : \zeta < \zeta_f\} = W$. In other words, in 2.5(D), given one $\bar f \in \mathrm{dom}(\mathbf{f}_{\mathfrak{m}})$ we can find others; here we claim the existence of $\bar f$ for a given $\langle \eta(f,\zeta) : \zeta < \zeta_{\bar f}\rangle$.)*

(F) *Closure under terms of $\tau(M_{\bar b})$:*
Assume that $u \subseteq {}^\alpha\mu$, $|u| < \kappa$, and $\langle \eta_\zeta : \zeta < \zeta^*\rangle$ lists u with no repetitions, and for some $W \in \mathbf{W}_{\mathfrak{m}}$, $u \subseteq \{\eta \restriction \alpha : \eta \in W\}$. If $n < \omega$ and $f^\ell \in \mathscr{F}_{\mathfrak{m}}$ for $\ell < n$, $\bar f = \langle f_\varepsilon : \varepsilon < \varepsilon(*)\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}})$, and

$$\begin{array}{rl} \eta(f_\varepsilon,\zeta) \in u & \text{for } \zeta < \zeta_{f_\varepsilon}, \text{ and} \\ \eta(f^\ell,\zeta) \in u & \text{for } \zeta < \zeta_{f^\ell}, \text{ and} \\ b_\varepsilon = f_\varepsilon(\ldots,\nu_{\eta(f_\varepsilon,\zeta)},\ldots)_\zeta & \text{for } \varepsilon < \varepsilon(*), \end{array}$$

$\bar b = \langle b_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ and $\sigma(x_0,\ldots,x_{n-1})$ is a $\tau(M_{\bar b})$–term, then for some $f \in \mathscr{F}_{\mathfrak{m}}$ we have $w[f] = u$ and: if $\nu_\eta \in I^\alpha_\eta$ for $\eta \in u$ and $\{\nu_\eta : \eta \in u\} \in \mathbf{W}_{\mathfrak{m}}$, then

$$f\left(\ldots,\nu_\eta,\ldots\right)_{\eta\in u} = \sigma^{M_{\bar b}}\left(\ldots, f^\ell_{\mathfrak{m}}(\ldots,\nu_{\eta(f^\ell,\zeta)},\ldots)_{\zeta<\zeta(f^\ell)},\ldots\right)_{\ell<n}.$$

Observation 2.7. *In Definition 2.6(4) in clauses (E),(F) it suffice to restrict ourselves to the case $n = 1$ and σ is the identity.*

Proof. By 2.5(D)(γ). $\square_{2.7}$

Of course some form of indiscernibility will be needed.

Definition 2.8. *(1) Let $\mathbb{E}$ be the family of equivalence relations $\mathbf{E}$ on*

$$\{\bar\nu \in {}^{\kappa>}({}^\kappa\mu) : \bar\nu \text{ without repetitions}\},$$

or a subset of it, such that

$$\bar\nu^1 \,\mathbf{E}\, \bar\nu^2 \quad\Rightarrow\quad \ell g(\bar\nu^1) = \ell g(\bar\nu^2).$$

(2) Let $\mathbb{E}_\alpha$ be the family of $\mathbf{E} \in \mathbb{E}$ such that

$$\bar\nu \in \mathrm{Dom}(\mathbf{E}) \quad\Rightarrow\quad \langle \nu_\zeta \restriction \alpha : \zeta < \ell g(\bar\nu)\rangle \text{ is without repetitions.}$$

(3) Let $\mathbf{E}^0_\alpha \in \mathbb{E}_\alpha$ be the following equivalence relation:

$$\begin{array}{lll} \bar\nu^1 \,\mathbf{E}^0_\alpha\, \bar\nu^2 \text{ iff} & & \text{for some } \zeta < \kappa \text{ we have} \\ & (i) & \bar\nu^1, \bar\nu^2 \in {}^\zeta({}^\kappa\mu), \\ & (ii) & \nu^1_\varepsilon \restriction \alpha = \nu^2_\varepsilon \restriction \alpha \text{ for } \varepsilon < \zeta, \\ & (iii) & \langle \nu^1_\varepsilon \restriction \alpha : \varepsilon < \zeta\rangle \text{ is with no repetitions}, \\ & (iv) & \text{the set } \{\varepsilon < \zeta : \nu^1_\varepsilon \neq \nu^2_\varepsilon\} \text{ is finite.} \end{array}$$

(3A) We say that $(\bar\nu^1,\bar\nu^2)$ are immediate neighbours if $\ell g(\bar\nu^1) = \ell g(\bar\nu^2)$, and for some $\xi < \ell g(\bar\nu^1)$ we have $(\forall \varepsilon < \zeta)(\varepsilon \neq \xi \;\Leftrightarrow\; \nu^1_\varepsilon = \nu^2_\varepsilon)$; so the difference with (3)(iv) is that "finite" is replaced by "a singleton".

(4) Let $\mathbf{E}^0_{<\alpha}$ be defined like $\mathbf{E}^0_\alpha$ strengthening clause (iii) to

(iii)$^+$ for some $\beta < \alpha$, the sequence $\langle \nu^\ell_\varepsilon \restriction \beta : \varepsilon < \zeta \rangle$ is with no repetitions.

(5) *For $\alpha < \kappa$ and $\mathbf{W} \subseteq \mathbf{W}_\alpha$ let*

$$\mathrm{seq}_\alpha(\mathbf{W}) = \big\{\bar\nu : \begin{array}{l} \bar\nu \in {}^{\kappa>}({}^\kappa\mu) \text{ is with no repetitions,} \\ \text{and for some } W \in \mathbf{W} \text{ we have} \\ \{\nu_\xi : \xi < \ell g(\bar\nu)\} \subseteq W, \text{ and hence} \\ \langle \nu_\zeta \restriction \alpha : \zeta < \ell g(\bar\nu)\rangle \text{ is with no repetitions}\big\}. \end{array}$$

(6) *We define $\mathbf{E}^1_\alpha$ as we define $\mathbf{E}^0_\alpha$ in part (3) above, omitting clause (iv). We define $\mathbf{E}^1_{<\alpha}$ analogously to (4).*

Remark: The reader may concentrate on $\mathbf{E}^0_\alpha$, so the "weakly" version below.

Definition 2.9. *(1) An approximation $\mathfrak{m}$ is called $\mathbf{E}$–indiscernible <u>if</u>*

- (a) *$\mathbf{E} \in \mathbb{E}_{\alpha(\mathfrak{m})}$ refine $\mathbf{E}^1_{\alpha(\mathfrak{m})}$,*
- (b) *if $\bar\nu^1, \bar\nu^2 \in \mathrm{seq}_{\alpha(\mathfrak{m})}(\mathbf{W}_\mathfrak{m})$ and $\bar\nu^1 \mathbf{E} \bar\nu^2$, <u>then</u> there is g (in fact, a unique $g = g^\mathfrak{m}_{\bar\nu^1,\bar\nu^2}$) such that*
  - (α) *g is an $(M_\mathfrak{m}, M_\mathfrak{m})$–elementary mapping,*
  - (β) *$\mathrm{Dom}(g) = \{f(\langle \nu^1_{h(\zeta)} : \zeta < \zeta_f\rangle) : f \in \mathscr{F}_\mathfrak{m}$ and h is a one-to-one function from ζ_f into $\ell g(\bar\nu^\ell)$ such that $\eta(f,\zeta) \lhd \nu^1_\zeta \in {}^\kappa\mu\}$,*
  - (γ) *$g(f(\langle \nu^1_{h(\zeta)} : \zeta < \zeta_f\rangle)) = f(\langle \nu^2_{h(\zeta)} : \zeta < \zeta_f\rangle)$ for f, h as above;*
- (c) *Assume $\bar\nu^1, \bar\nu^2 \in \mathrm{seq}_{\alpha(\mathfrak{m})}(\mathbf{W}_\mathfrak{m})$, $\bar f^1, \bar f^2 \in \mathrm{Dom}(\mathbf{f}_\mathfrak{m})$, $\zeta^* = \zeta_{\bar f^1} = \zeta_{\bar f^2}$, and for some one-to-one function h from ζ^* to $\ell g(\bar\nu^\ell)$ we have $\eta(\bar f^\ell, \zeta) = \nu^m_{h(\zeta)} \restriction \alpha$ for $\ell, m = 1, 2$, and $\bar\nu^1 \mathbf{E} \bar\nu^2$. Let*

$$\bar b^\ell = \big\langle f^\ell_\varepsilon\big(\langle \nu^\ell_{h(\zeta)} : \zeta < \zeta^*\rangle\big) : \varepsilon < \ell g(\bar f^\ell)\big\rangle.$$

  <u>then</u> there is g such that
  - (α) *g is an $(M^\mathfrak{m}_{\bar b_1}, M^\mathfrak{m}_{\bar b_2})$–elementary mapping,*
  - (β) *$g = g^\mathfrak{m}_{\bar\nu^1,\bar\nu^2}$ from clause (b) above.*

(2) *An approximation $\mathfrak{m}$ is strongly indiscernible if it is $\mathbf{E}^1_{\alpha(\mathfrak{m})}$–indiscernible.*

(3)
- (a) *An approximation $\mathfrak{m}$ is weakly indiscernible when it is $\mathbf{E}^0_{\alpha(\mathfrak{m})}$-indiscernibility.*
- (b) *An approximation $\mathfrak{m}$ is weakly/strongly nice <u>if</u> it is term closed and weakly/strongly indiscernible.*
- (c) *An approximation $\mathfrak{m}$ weakly/strongly good <u>if</u> it is weakly/strongly nice and is almost full.*
- (d) *An approximation $\mathfrak{m}$ is weakly/strongly excellent <u>if</u> it is weakly/strongly good, and is resplendent, see Definition 2.6(2),(3).*

Discussion 2.10. *Why do we have the weak and strong version?*

In the proof of the main subclaim 2.20 below the proof for the weak version is easier <u>but</u> we get from it a weaker conclusion: $\geq \lambda^+$ non-isomorphic κ-resplendent of cardinality $\lambda = \lambda^\kappa$, whereas from the strong version we would get 2^λ. But see §3.

Claim 2.11. *Let $\mathfrak{m}$ be an approximation.*

1. *In the definition of "$\mathfrak{m}$ is $\mathbf{E}^0_\alpha$–indiscernible", it is enough to deal with immediate $\mathbf{E}^0_\alpha$–neighbors (see Definition 2.8(3)).*
2. *If $\mathfrak{m}$ is weakly/strongly excellent <u>then</u> $\mathfrak{m}$ is weakly/strongly good.*
3. *If $\mathfrak{m}$ is weakly/strongly good <u>then</u> $\mathfrak{m}$ is weakly/strongly nice.*
4. *If $\alpha_\mathfrak{m} = 0, \mathbf{E} \in \mathbb{E}_\mathfrak{m}$ <u>then</u> $\mathfrak{m}$ is $\mathbf{E}$-indiscernible <u>iff</u> $\mathfrak{m}$ is strongly indiscernible.*

Definition 2.12. *(1) For approximations $\mathfrak{m}_1, \mathfrak{m}_2$ let "$\mathfrak{m}_1 \le_{\mathbf{h}} \mathfrak{m}_2$" or "$\mathfrak{m}_1 \le \mathfrak{m}_2$ as witnessed by $\mathbf{h}$" mean that:*

(a) $\alpha(\mathfrak{m}_1) \le \alpha(\mathfrak{m}_2)$,

(b) $\mathbf{W}_{\mathfrak{m}_1} \subseteq \mathbf{W}_{\mathfrak{m}_2}$,

(c) $\mathbf{h}$ *is a partial function from* $\mathscr{F}_{\mathfrak{m}_2}$ *into* $\mathscr{F}_{\mathfrak{m}_1}$,

(d) if $\mathbf{h}(f_2) = f_1$ *<u>then</u> they have the same arity (i.e.,* $\zeta^{\mathfrak{m}_1}_{f_1} = \zeta^{\mathfrak{m}_2}_{f_2}$*) and*

$$\zeta < \zeta^{\mathfrak{m}_1}_{f_1} \quad \Rightarrow \quad \eta_{\mathfrak{m}_1}(f_1,\zeta) = \eta_{\mathfrak{m}_2}(f_2,\zeta) \restriction \alpha(\mathfrak{m}_1),$$

(e) if $f_1 \in \mathscr{F}_{\mathfrak{m}_1}$ *and* $W \in \mathbf{W}_{\mathfrak{m}_1}$ *and*

$$\{\nu \restriction \alpha(\mathfrak{m}_1) : \nu \in W\} = \{\eta_{\mathfrak{m}_1}(f_1,\zeta) : \zeta < \zeta^{\mathfrak{m}_1}_{f_1}\},$$

<u>then</u> there is one and only one $f_2 \in \mathscr{F}_{\mathfrak{m}_2}$ *satisfying*

$$\mathbf{h}(f_2) = f_1 \quad \text{and} \quad \{\eta_{\mathfrak{m}_1}(f_2,\zeta) : \zeta < \zeta^{\mathfrak{m}_1}_{f_2}\} = \{\nu \restriction \alpha(\mathfrak{m}_2) : \nu \in W\},$$

(f) for $W \in \mathbf{W}_{\mathfrak{m}_1}$*, the mapping* $g^{\mathfrak{m}_2}_{\mathfrak{m}_1}[W,\mathbf{h}]$ *defined below is an elementary embedding from* $N^{\mathfrak{m}_1}_W$ *into* $N^{\mathfrak{m}_2}_W$*, where:*

(∗) if $f_1 \in \mathscr{F}_{\mathfrak{m}_1}$, $f_2 \in \mathscr{F}_{\mathfrak{m}_2}$ *are as in clause (e) (so* $\mathbf{h}(f_2) = f_1$*), and*

$$a = f^{\mathfrak{m}_1}_1(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta_{f_1}}, \qquad \text{and} \qquad \{\nu_\zeta : \zeta < \zeta^{\mathfrak{m}_1}_f\} \subseteq W$$

(so $a \in N^{\mathfrak{m}_1}_W$*), then* $(g^{\mathfrak{m}_2}_{\mathfrak{m}_1}[W,\mathbf{h}])(a) = f^{\mathfrak{m}_2}_2(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta_{f_2}}$,

(g) if $\bar f^1 = \langle f^1_\xi : \xi < \varepsilon\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1})$ *and* $\eta_{\mathfrak{m}_1}(\bar f^1,\zeta) \trianglelefteq \eta_\zeta \in {}^{\alpha(\mathfrak{m}_2)}\mu$ *for* $\zeta < \zeta^{\mathfrak{m}}_{\bar f^1}$, $\bar f^2 = \langle f^2_\xi : \xi < \varepsilon\rangle \in {}^{\varepsilon}(\mathscr{F}_{\mathfrak{m}_2})$*, and* $\zeta^{\mathfrak{m}_2}_{\bar f^2} = \zeta^{\mathfrak{m}_1}_{\bar f^1}$*, and*

$$\xi < \varepsilon \wedge \zeta < \zeta_{\bar f^1} \;\Rightarrow\; \eta(f^2_\xi,\zeta) = \eta_\zeta \wedge \mathbf{h}(f^2_\xi) = f^1_\xi$$

<u>then</u>

(α) $\bar f^2 \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_2})$,

(β) $\mathbf{h}\big((\mathbf{f}_{\mathfrak{m}_2}(\bar f^2))(\sigma(\langle x_\xi : \xi \in u\rangle))\big) = \big(\mathbf{f}_{\mathfrak{m}_1}(\bar f^1)\big)(\sigma(\langle x_\xi : \xi \in u\rangle))$*, when* u *is a finite subset of* ε

(γ) Assume $\nu_\zeta \in I_{\eta_\zeta}$ *for* $\zeta < \zeta^{\mathfrak{m}_1}_{\bar f^1}$*, and* $W = \{\nu_\zeta : \zeta < \zeta^{\mathfrak{m}_1}_{\bar f^1}\}$, $\bar b^\ell = \langle f^\ell_\xi(\ldots,\nu_\zeta,\ldots)_\zeta : \xi < \varepsilon\rangle$*. Then the mapping* $g^{\mathfrak{m}_2}_{\mathfrak{m}_1}[W,h]$ *(see clause (f) above) is a* $(M_{\bar b_1}, M_{\bar b_2})$*-elementary mapping from* $M^{\mathfrak{m}_1}_{\bar b^1} \restriction |N^{\mathfrak{m}_1}_W|$ *onto* $M^{\mathfrak{m}_2}_{\bar b^2} \restriction |N^{\mathfrak{m}_2}_W|$.

(2) We say that $\langle \mathfrak{m}_\beta, \mathbf{h}^\beta_\gamma : \beta < \alpha, \gamma \le \beta\rangle$ *is an inverse system of approximations <u>if</u>*

(a) $\mathfrak{m}_\beta$ *is a* β*–approximation (for* $\beta < \alpha$*),*

(b) $\mathfrak{m}_\gamma \le_{\mathbf{h}^\beta_\gamma} \mathfrak{m}_\beta$ *for* $\gamma \le \beta$,

(c) $\mathbf{h}^\beta_\beta$ *is the identity,*

(d) if $\beta_0 < \beta_1 < \beta_2 < \alpha$ *then* $\mathbf{h}^{\beta_2}_{\beta_0} = \mathbf{h}^{\beta_1}_{\beta_0} \circ \mathbf{h}^{\beta_2}_{\beta_1}$.

(3) We say that an inverse system of approximations $\langle \mathfrak{m}_\beta, \mathbf{h}^\beta_\gamma : \beta < \alpha, \gamma \le \beta\rangle$ *is continuous at* δ *<u>if</u>:*

(a) $\delta < \alpha$ *is a limit ordinal,*

(b) $\mathbf{W}_{\mathfrak{m}_\delta} = \bigcup\{\mathbf{W}_{\mathfrak{m}_\beta} : \beta < \delta\}$,

(c) $\mathscr{F}_{\mathfrak{m}_\delta} = \bigcup\{\mathrm{Dom}(\mathbf{h}^\delta_\beta) : \beta < \delta\}$,

(d) $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_\delta}) = \{\bar f^2 :$ *for some* $\beta < \delta$ *and* $\bar f^1 \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1})$ *of length* $\ell g(\bar f^2)$ *we have* $\mathbf{h}^\delta_\beta(f^2_\xi) = f^1_\xi$ *for* $\xi < \ell g(\bar f^2)\}$.

Discussion 2.13. *Having chosen above our order, when can we get the appropriate indiscernibility? As we are using finitary partition theorem (with finitely many colours), we cannot make the type of candidates for fixed $\bar{b}$. However, we may have,* a priori, *enough indiscernibility to fix the type of enough $\bar{b}'$-s and then use the existence of indiscernibles to uniformize the related $M_{\bar{b}}$-s.*

Claim 2.14. *1) There is an excellent 0–approximation.*
2) Moreover, there is a Φ such that:

(a) Φ is a template proper for the tree $I_{\langle\ \rangle}$,
(b) Φ is nice (see [Sheb, 1.8]*), that for ℓ the logic $\mathbb{L}$, first order logic, the default value,*
(c) $M_* = \mathrm{GEM}(I_{\langle\ \rangle}, \Phi)$ is a model of T,
(d) if $\eta \in {}^{\kappa}\mu$ then $\mathrm{GEM}(\{\eta\}, \Phi)$ is κ-resplendent,
(e) $\langle a_\eta : \eta \in {}^{\kappa\geq}\mu\rangle$, $\langle F_\varepsilon^{M_*} : \varepsilon < \kappa\rangle$ are as above.

Proof. Recall that the sequence $\langle\varphi_\alpha(x,y) : \alpha < \kappa\rangle$ exemplifies $\kappa < \kappa(T)$, see 2.1 above. Hence by clause (b) of [Sheb, 1.10(3)], we can find a template Φ proper for the tree $I_{\langle\ \rangle}$, i.e., ${}^{\kappa\geq}\mu$, with skeleton $\langle a_\eta : \eta \in {}^{\kappa\geq}\mu\rangle$ such that for $\nu \in {}^{\kappa}\mu$ and $\rho \in {}^{\alpha+1}\mu$ we have

$$\mathrm{GEM}({}^{\kappa\geq}\mu, \Phi) \models \varphi_\alpha(a_\nu, a_\rho) \quad \text{iff} \quad \rho \lhd \nu.$$

Without loss of generality, for some unary function symbols $F_\varepsilon^* \in \tau(\Phi)$, we have $\mathrm{GEM}({}^{\kappa\geq}\mu, \Phi) \models$ "$F_\varepsilon(a_\eta) = a_{\eta\restriction\varepsilon}$" for $\eta \in {}^{\kappa}\mu$. Now, by induction on $\varepsilon < \kappa$ we choose Φ_ε such that:

(a) Φ_ε is a template proper for ${}^{\kappa\geq}\mu$ which is nice (see [Sheb, 1.7] + [Sheb, 1.8(3),(4)]),
(b) $\tau(\Phi_\varepsilon)$ has cardinality $\leq \theta$ (see Definition 2.3),
(c) $\Phi_0 = \Phi$,
(d) the sequence $\langle\Phi_\varepsilon : \varepsilon < \kappa\rangle$ is increasing with ε, that is (see [Sheb, 1.8(1B)]:

$$\zeta < \varepsilon \quad \Rightarrow \quad \tau(\Phi_\zeta) \subseteq \tau(\Phi_\varepsilon) \ \text{ and } \ \mathrm{GEM}_{\tau(T)}({}^{\kappa\geq}\mu, \Phi_\zeta) \prec \mathrm{GEM}_{\tau(T)}({}^{\kappa\geq}\mu, \Phi_\varepsilon),$$

(e) the sequence $\langle\Phi_\varepsilon : \varepsilon < \kappa\rangle$ is continuous, i.e., if ε is a limit ordinal then $\tau(\Phi_\varepsilon) = \bigcup\limits_{\zeta<\varepsilon} \tau(\Phi_\zeta)$,
(f) if $\bar{\sigma} = \langle\sigma_i(x) : i < i^*\rangle$ is a sequence of length $< \kappa$ of unary terms in $\tau(\Phi_\varepsilon)$, and $M^{\varepsilon+1} = \mathrm{GEM}_{\tau(T)}({}^{\kappa\geq}\mu, \Phi_{\varepsilon+1})$, and for $\nu \in {}^{\kappa}\mu$ we define $\bar{b} = \bar{b}_{\bar{\sigma},\nu}$ as

$$\langle\sigma_i^{M^{\varepsilon+1}}(a_\nu) : i < i^*\rangle \in {}^{i^*}(\mathrm{GEM}_{\tau(T)}(\{\nu\}, \Phi_{\varepsilon+1})),$$

<u>then</u> we can interpret a model $M_{\bar{b}}^{\varepsilon+1}$ of $T^*[\bar{b}, M^{\varepsilon+1} \restriction \tau_T]$ in $M^{\varepsilon+1}$, which means:
(α) if $R \in \tau_T[\bar{b}, M^{\varepsilon+1} \restriction \tau(T)] \setminus \tau_T$ is a k–place predicate, then there is a $(k+1)$–place predicate $R_* \in \tau(\Phi_{\varepsilon+1}) \setminus \tau(\Phi_\varepsilon)$ such that

$$M_{\bar{b}}^{\varepsilon+1} \models R[c_0, \ldots, c_{k-1}] \quad \text{iff} \quad M^{\varepsilon+1} \models R_*[c_0, \ldots, c_{k-1}, a_\nu],$$

(β) if $F \in \tau_{T[\bar{b}, M^{\varepsilon+1}\restriction\tau(T)]} \setminus \tau_T$ is a k–place function symbol, then there is a $(k+1)$–place function symbol $F_* \in \tau(\Phi_{\varepsilon+1}) \setminus \tau(\Phi_\varepsilon)$ such that

$$M_{\bar{b}}^{\varepsilon+1} \models \text{"}F[c_0, \ldots, c_{k-1}] = c\text{"} \quad \text{iff} \quad M^{\varepsilon+1} \models \text{"}F_*[c_0, \ldots, c_{k-1}, a_\nu] = c\text{"}.$$

Let us carry out the induction; note that there is a redundancy in our contraction: each relevant $\bar{b}$ is taken care of in the ε-th stage for every $\varepsilon < \kappa$ large enough, independently, for the different ε-s.

For $\varepsilon = 0$:
Let $\Phi_0 = \Phi$.

For a limit ε:
Let Φ_ε be the direct limit of $\langle \Phi_\zeta : \zeta < \varepsilon \rangle$.

For $\varepsilon = \zeta + 1$:
Let the family of sequences of the form $\bar{\sigma} = \langle \sigma_i(x) : i < i^* \rangle$, where $\sigma_i(x)$ is a unary term in $\tau(\Phi_\zeta)$, $i^* < \kappa$, be listed as $\langle \bar{\sigma}^\gamma(x) : \gamma < \theta \rangle$, with $\bar{\sigma}^\gamma(x) = \langle \sigma^\gamma_i(x) : i < i_\gamma \rangle$. Let M^*_ε be a θ^+–resplendent (hence strongly θ^+-homogeneous and κ-resplendent) elementary extension of $\mathrm{GEM}_{\tau(T)}({}^{\kappa\geq}\mu, \Phi_\zeta)$, and let $M_\varepsilon = M^*_\varepsilon \restriction \tau_T$, and choose $\nu^* \in {}^\kappa\mu$. For each $\gamma < \theta$ let $\bar{b}^\gamma_{\nu^*} =: \langle \sigma^\gamma_i(a_{\nu^*}) : i < i_\gamma \rangle$. Now, $(M_\varepsilon, \bar{b}^\gamma_{\nu^*})$ can be expanded to a model $M^\zeta_{\bar{b}^\gamma_{\nu^*}}$ of $T^*[\bar{b}^\gamma_{\nu^*}, M_\varepsilon]$, and let

$$\tau(T^*[\bar{b}, M_\varepsilon]) \setminus \tau_T = \{R^{\varepsilon,\gamma}_{j,n} : j < \theta, n < \omega\} \cup \{F^{\varepsilon,\gamma}_{j,n} : j < \theta, n < \omega\},$$

where $R^{\varepsilon,\gamma}_{j,n}$ is an n–place predicate and $F^{\varepsilon,\gamma}_{j,n}$ is an n–place function symbol. Next we shall define an expansion M^+_ε of M^*_ε. Its vocabulary is

$$\tau(\Phi_\zeta) \cup \{R_{\varepsilon,\gamma,j,n}, F_{\varepsilon,\gamma,j,n} : j < \theta, n < \omega\},$$

where $R_{\varepsilon,\gamma,j,n}$ is an $(n+1)$–place predicate, $F_{\varepsilon,\gamma,j,n}$ is an $(n+1)$–place function symbol, and no one of them is in $\tau(\Phi_\zeta)$ (and there are no repetitions in their list).

Almost lastly, for $\nu \in {}^\kappa\mu$ let g_ν be an automorphism of M_ε mapping $\mathrm{GEM}_{\tau(T)}(\{\nu^*\}, \Phi_\zeta)$ onto $\mathrm{GEM}_{\tau(T)}(\{\nu\}, \Phi_\zeta)$; moreover such that for any $\tau(\Phi_\zeta)$–term $\sigma(x)$ we have $g_\nu(\sigma(a_{\nu^*})) = \sigma(a_\nu)$ (hence $\xi < \kappa \Rightarrow g_\nu(a_{\nu^* \restriction \xi}) = a_{\nu \restriction \xi}$ using $\sigma(x) = F^*_\xi(x)$).

Now we actually define M^+_ε expanding M^*_ε:

$$R^{M^+_\varepsilon}_{\varepsilon,\gamma,j,n} = \big\{(g_\nu(c_0), g_\nu(c_1), \dots, g_\nu(c_{n-1}), g_\nu(a_{\nu^*})) : \\ M^\zeta_{\bar{b}^\gamma_{\nu^*}} \models R^{\varepsilon,\gamma}_{j,n}(c_0, \dots, c_{n-1})\big\},$$

$F^{M^+_\varepsilon}_{\varepsilon,\gamma,j,n}$ is an $(n+1)$–place function such that

$$M^\zeta_{\bar{b}^\gamma_{\nu^*}} \models F^{\varepsilon,\gamma}_{j,n}(c_0, \dots, c_{n-1}) = c \quad \text{implies}$$

$$F^{M^+_\varepsilon}_{\varepsilon,\gamma,j,n}(g_\nu(c_0), \dots, g_\nu(c_{n-1}), a_\nu) = g_\nu(c).$$

We further expand M^+_ε to M^{++}_ε, with vocabulary of cardinality $\leq \theta$ and with Skolem functions.

Now we apply "${}^{\kappa\geq}\mu$ has the Ramsey property" (see [Sheb, 1.14(4)] see "even" there, [Sheb, 1.18]) to get $\Phi_\varepsilon = \Phi_{\zeta+1}$, $\tau(\Phi_\varepsilon) = \tau(M^{++}_\varepsilon)$, such that for every $n < \omega$, $\nu_1, \dots, \nu_n \in {}^\kappa\mu$, and first order formula $\varphi(x_1, \dots, x_n) \in \mathbb{L}(\tau(\Phi_\varepsilon))$, for some $\eta_1, \dots, \eta_n \in {}^\kappa\mu$ we have

(α) $M^{++}_\varepsilon \models \varphi[a_{\eta_1}, \dots, a_{\eta_n}]$ iff $\mathrm{GEM}_{\tau(T)}({}^{\kappa\geq}\mu, \Phi_\varepsilon) \models \varphi[a_{\nu_1}, \dots, a_{\nu_n}]$,

(β) $\langle \eta_1, \dots, \eta_n \rangle, \langle \nu_1, \dots, \nu_n \rangle$ are similar in ${}^{\kappa\geq}\mu$ (see [She90, VII] or 0.4).

It is easy to check that $\Phi_\varepsilon = \Phi_{\zeta+1}$ is as required.

So we have defined the sequence $\langle \Phi_\varepsilon : \varepsilon < \kappa \rangle$ satisfying the requirements above, and let Φ_κ be its limit. It is as required in the claim. $\square_{2.14}$

Claim 2.15. *Assume $\alpha \le \kappa$ is a limit ordinal and $\langle \mathfrak{m}_\gamma, \mathbf{h}^\beta_\gamma : \gamma < \beta < \alpha\rangle$ is an inverse system of approximations.*

(1) *There are $\mathfrak{m}_\alpha$, $\mathbf{h}^\alpha_\gamma$ (for $\gamma < \alpha$) such that $\langle \mathfrak{m}_\gamma, \mathbf{h}^\beta_\gamma : \gamma < \beta < \alpha + 1\rangle$ is an inverse system of approximations continuous at α.*

(2) *For the following properties, if each $\mathfrak{m}_{\gamma+1}$ (for $\gamma < \alpha$) satisfies the property, then so does $\mathfrak{m}_\alpha$: term closed, semi full, almost full, resplendent, weakly/strongly indiscernible, weakly/strongly nice, $\mathbf{E}$-indiscernible for any $\mathbf{E} \in \mathbb{E}$, weakly/strongly good, weakly/strongly excellent.*

Proof. Let $\mathbf{W}_{\mathfrak{m}_\alpha} = \bigcup\limits_{\beta<\alpha} \mathbf{W}_{\mathfrak{m}_\beta}$, and let $M_\beta = M_{\mathfrak{m}_\beta}$ for $\beta < \alpha$. We shall define $\mathscr{F}_\alpha = \mathscr{F}_{\mathfrak{m}_\alpha}$, $M_\alpha = M_{\mathfrak{m}_\alpha}$ and $N^\alpha_W = N^{\mathfrak{m}_\alpha}_W$ and $M^\alpha_{\bar b} = M^{\mathfrak{m}_\alpha}_{\bar b}$ below.

First let $\mathscr{F}_\alpha$ (formal set, consisting of function symbols not of functions), $\mathbf{h}^\alpha_\beta$ $(\beta < \alpha)$ be the inverse limit of $\langle \mathscr{F}_\beta, \mathbf{h}^\beta_\gamma : \gamma \le \beta < \alpha\rangle$, i.e.,

(α) $\mathbf{h}^\alpha_\beta$ is a partial function from $\mathscr{F}_\alpha$ onto $\mathscr{F}_\beta$ as in Definition 2.12.

(β) $\mathbf{h}^\alpha_\gamma = \mathbf{h}^\beta_\gamma \circ \mathbf{h}^\alpha_\beta$ for $\gamma < \beta < \alpha$,

(γ) $\mathscr{F}_\alpha = \bigcup\limits_{\beta<\alpha} \mathrm{Dom}(\mathbf{h}^\alpha_\beta)$,

(δ) If $\beta_* < \alpha$, $f_\beta \in \mathscr{F}_\beta$, for $\beta \in [\beta_*, \alpha)$, satisfy $\mathbf{h}^\beta_\gamma(f_\beta) = f_\gamma$ when $\beta_* \le \gamma < \beta < \alpha$, then for one and only one $f \in \mathscr{F}_\alpha$ we have:
$$\zeta_f = \zeta_{f_\beta} \text{ for } \beta \in [\beta_*, \alpha) \qquad \text{and} \qquad \eta_{f,\zeta} = \bigcup \left\{\eta_{f_\beta,\zeta} : \beta_* \le \beta < \alpha\right\},$$

(ε) every $f \in \mathscr{F}_\alpha$ has the form of f in (δ),

(ζ) $f^*_{\rho,\zeta}$ are as in (B) of Definition 2.5, i.e., for any $\rho \in {}^\alpha\mu$ and $\zeta < \kappa$ we have $\beta < \alpha \ \Rightarrow\ \mathbf{h}^\alpha_\beta(f^*_{\rho\restriction\beta}, \zeta) = f^{*,\mathfrak{m}_\beta}_{\rho,\zeta}$.

Second, we similarly choose $\mathbf{f}_{\mathfrak{m}_\alpha}$.

Thirdly, we choose M_α and interpretation of f (for $f \in \mathscr{F}_\alpha$) and $M^+_{\bar b}$ when
$$\bar b \in \{\mathrm{Rang}(f) : f \in \mathscr{F}_\alpha \ \&\ (\forall \zeta < \zeta_f)(\exists \nu \in W)(\eta^f_\zeta \lhd \nu)\}$$
for some $W \in \mathbf{W}_{<\alpha}$. Though we can use the compactness theorem, it seems to me more transparent to use ultraproduct . So let D be an ultrafilter on α containing all co-bounded subsets of α. Let $M_\alpha = \prod\limits_{\beta<\alpha} M_\beta/D$. If $f \in \mathscr{F}_\alpha$, let $\beta_f < \alpha$ and $\langle f_\gamma : \gamma \in [\beta_f, \alpha)\rangle$ be such that $\beta_f \le \gamma < \alpha \ \Rightarrow\ \mathbf{h}^\alpha_\gamma(f) = f_\gamma$, so $\langle \eta^f_\zeta \restriction \beta_f : \zeta < \zeta_f\rangle$ has no repetitions. Now, when $\eta^f_\zeta \lhd \nu_\zeta \in {}^\kappa\mu$, let
$$f_{\mathfrak{m}}(\ldots, \nu_\zeta, \ldots) = \langle c_\gamma : \gamma < \alpha\rangle/D,$$
where
$$\begin{aligned} \gamma \in (\beta_f, \alpha) \quad &\Rightarrow \quad c_\gamma = \big(\mathbf{h}^\alpha_\gamma(f)_{\mathfrak{m}_\gamma}\big)(\ldots, \nu_\zeta, \ldots) \in M_\gamma, \\ \gamma \le \beta_f \quad &\Rightarrow \quad c_\gamma \text{ is any member of } M_\gamma. \end{aligned}$$
So M^α_W is well defined for $W \in \mathbf{W}_{\mathfrak{m}(\alpha)}$.

Fourth, if $\bar b = \langle b_\varepsilon : \varepsilon < \varepsilon(*)\rangle \in {}^{\kappa>}(M^\alpha_W)$, $b_\varepsilon = f^{\mathfrak{m}}_\varepsilon(\ldots, \nu_\zeta, \ldots)_{\zeta<\zeta_{f_\varepsilon}}$, and $\beta_* < \alpha$ and for $\gamma \in [\beta_*, \alpha)$: $f_{\gamma,\varepsilon} \in \mathscr{F}_{\mathfrak{m}_\beta}$, $\langle f_{\gamma,\varepsilon} : \varepsilon < \varepsilon(*)\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_\beta})$, and $\mathbf{h}^\alpha_\gamma(f_{\gamma,\varepsilon}) = f_\varepsilon$, then we let $\bar b^\beta = \langle b^\beta_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ where b^β_ε is $f^{\mathfrak{m}_\beta}(\ldots, \nu_\zeta, \ldots)_{\zeta<\zeta_{f_\varepsilon}}$ if $\beta \in [\beta_*, \alpha)$ and b^β_ε is any member of M_γ if $\beta < \beta_*$ and lastly we define $M^\alpha_{\bar b} = \prod\limits_{\beta\in[\beta_*,\alpha)} M^\beta_{\bar b_\beta}/D$.

We still have to check that if for the same $\bar b$ we get two such definitions, then they agree, but this is straightforward.

Fifth, we choose $M^\alpha_{\bar b}$ for other $\bar b \in {}^{\kappa>}(M_\alpha)$ for which $M^\alpha_{\bar b}$ is not yet defined to satisfy clause (d) of Definition 2.5; note that by the choice of $\mathbf{W}_{\mathfrak{m}_\alpha}$ those choices do not influence the preservation of weakly/strongly indiscernible. So $\mathfrak{m}_\alpha$ is well defined and one can easily check that it is as required. $\square_{2.15}$

Claim 2.16. *Assume $\alpha = \beta + 1 < \kappa$, and $\mathfrak{m}_1$ is a β–approximation.*

(1) There are $\mathbf{h}_*$ and an α–approximation $\mathfrak{m}_2$ such that $\mathfrak{m}_1 \le_{\mathbf{h}_*} \mathfrak{m}_2$, $M_{\mathfrak{m}_1} \prec M_{\mathfrak{m}_2}$, $M^{\mathfrak{m}_2}_{\bar b} = M^{\mathfrak{m}_1}_{\bar b}$, and $\mathrm{Dom}(\mathbf{h}_*) = \mathscr{F}_{\mathfrak{m}_2}$.
(2) If $\mathfrak{m}_1$ is weakly/strongly nice, then $\mathfrak{m}_2$ is weakly/strongly nice.
(3) If $\mathfrak{m}_1$ is weakly/strongly indiscernible , <u>then</u> $\mathfrak{m}_2$ is weakly/strongly indiscernible; simply for $\mathbf{E}$-indiscernible, $\mathbf{E} \in \mathbb{E}_\alpha$.

Proof. (1) Should be clear.

Let $\alpha(\mathfrak{m}_2) = \alpha$, $\mathbf{W}_{\mathfrak{m}_2} = \mathbf{W}_{\mathfrak{m}_1}$, $M_{\mathfrak{m}_1} \prec M_{\mathfrak{m}_2}$ and $M^{\mathfrak{m}_2}_{\bar b} = M^{\mathfrak{m}_1}_{\bar b}$ for $\bar b \in {}^{\kappa>}(M_{\mathfrak{m}_1})$. Then let

$$\begin{aligned}\mathscr{F}_{\mathfrak{m}_2} = \{g_{f,h} :\ & f \in \mathscr{F}_\beta, h \text{ is a function with domain } \{\eta_{f,\zeta} : \zeta < \zeta_f\} \\ & \text{satisfying } h(\eta_{f,\zeta}) \in \mathrm{Suc}(\eta_{f,\zeta}) = \{\eta_{f,\zeta}{}^\frown\langle\gamma\rangle : \gamma < \mu\}\},\end{aligned}$$

where for $g = g_{f,h}$ we let $\zeta_g = \zeta_f$ and $\eta_{g,\zeta} = h(\eta_{f,\zeta})$, and if $\nu_\zeta \in I_{\eta_{g,\zeta}}$ for $\zeta < \zeta_g$ $(= \zeta_f)$, then

$$g^{\mathfrak{m}_2}_{f,h}(\dots,\nu_\zeta,\dots) = f^{\mathfrak{m}_1}(\dots,\nu_\zeta,\dots) \in M_{\mathfrak{m}_1} \prec M_{\mathfrak{m}_2}.$$

We define $\mathbf{h}_*$ by:

$$\mathrm{Dom}(\mathbf{h}_*) = \mathscr{F}_{\mathfrak{m}_2} \quad \text{and} \quad \mathbf{h}_*(g_{f,h}) = f.$$

Lastly let

$$\begin{aligned}\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_2}) = \big\{\langle g_\varepsilon : \varepsilon < \varepsilon(*)\rangle :\ & \text{for some } \bar f = \langle f_\varepsilon : \varepsilon < \varepsilon(*)\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}) \\ & \text{and a function } h \text{ with domain} \\ & \{\eta_{f_\varepsilon,\zeta} : \zeta < \zeta_{\bar f}\} \text{ i.e., does not depend on } \varepsilon \\ & \text{we have } \varepsilon < \varepsilon(*) \ \Rightarrow\ g_\varepsilon = g_{f_\varepsilon,h}\ \big\},\end{aligned}$$

and if $\mathbf{h}, \bar f, \bar g = \langle g_{f_\varepsilon,h} : \varepsilon < \zeta_{\bar f}\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_2})$ are as above, $\sigma(\bar x)$ is a $\tau[T,\varepsilon(*)]$–term, $\bar x = \langle x_\xi : \xi \in u\rangle$, and u is a finite subset of $\varepsilon(*)$ and $(\mathbf{f}_{\mathfrak{m}_1}(\bar f))(\sigma(\bar x)) = f$, then $(\mathbf{f}_{\mathfrak{m}_2}(\bar g))(\sigma(\bar x)) = g_{f,h}$.

Now check.

2), 3) Easy. $\square_{2.16}$

Definition 2.17. *(1) For approximations $\mathfrak{m}_1, \mathfrak{m}_2$, let $\mathfrak{m}_1 \le^* \mathfrak{m}_2$ mean that $\alpha(\mathfrak{m}_1) = \alpha(\mathfrak{m}_2)$ and $\mathfrak{m}_1 \le_{\mathbf{h}} \mathfrak{m}_2$ with $\mathbf{h}$ being the identity on $\mathscr{F}_{\mathfrak{m}_1} \subseteq \mathscr{F}_{\mathfrak{m}_2}$, and $\mathbf{W}_{\mathfrak{m}_1} \subseteq \mathbf{W}_{\mathfrak{m}_2}$ and $\mathbf{f}_{\mathfrak{m}_1} \subseteq \mathbf{f}_{\mathfrak{m}_2}$ (the last condition means that if $\bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1})$ then $\bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_2})$ and the function $\mathbf{f}_{\mathfrak{m}_2}(\bar f)$ is equal to the function $\mathbf{f}_{\mathfrak{m}_1}(\bar f)$).*
(2) Let $\mathfrak{m}_1 <^* \mathfrak{m}_2$ mean that
  (a) $\mathfrak{m}_1 \le^* \mathfrak{m}_2$,
  (b) if $\bar f \in \mathbb{F}_{\mathfrak{m}_1}$ then $\bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_2})$.

Observation 2.18. *(1) $\leq^*$ is a partial order, $\mathfrak{m}_1 \leq^* \mathfrak{m}_1$, and*

$$\begin{array}{ll} \mathfrak{m}_1 <^* \mathfrak{m}_2 \;\Rightarrow\; \mathfrak{m}_1 \leq \mathfrak{m}_2, & \textit{and} \\ \mathfrak{m}_1 \leq^* \mathfrak{m}_2 <^* \mathfrak{m}_3 \;\Rightarrow\; \mathfrak{m}_1 <^* \mathfrak{m}_3, & \textit{and} \\ \mathfrak{m}_1 <^* \mathfrak{m}_2 \leq \mathfrak{m}_3 \;\Rightarrow\; \mathfrak{m}_1 <^* \mathfrak{m}_3. & \end{array}$$

(2) Each $\leq^*$–increasing chain of length $< \theta^+$ has a lub (essentially its union). If all members of the chain are weakly/strongly indiscernible, then so is the lub.

(3) If $\langle \mathfrak{m}_\varepsilon : \varepsilon < \kappa \rangle$ is $<^*$–increasing <u>then</u> its lub $\mathfrak{m}$ is resplendent and $\varepsilon < \kappa \Rightarrow \mathfrak{m}_\varepsilon <^* \mathfrak{m}$. So if each $\mathfrak{m}_\varepsilon$ is weakly/strongly good then $\mathfrak{m}$ is weakly/strongly excellent.

Proof. Easy.

$\square_{2.18}$

As a warm up.

Claim 2.19. *(1) For any α–approximation $\mathfrak{m}_0$ there is a full, term closed α–approximation $\mathfrak{m}_1$ such that $\mathfrak{m}_0 \leq^* \mathfrak{m}_1$.*

(2) If $\mathfrak{m}_0$ is an α–approximation, <u>then</u> there is a α–approximation $\mathfrak{m}_1$ such that $\mathfrak{m}_0 <^* \mathfrak{m}_1$ and $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}) = \mathbb{F}_{\mathfrak{m}_0}$.

Proof. 1) Let $M_{\mathfrak{m}_1} = M_{\mathfrak{m}_0}$, and $M^{\mathfrak{m}_1}_{\bar b} = M^{\mathfrak{m}_0}_{\bar b}$ for $\bar b \in {}^{\kappa>}(M_{\mathfrak{m}_0})$. Let $\mathbf{W}_{\mathfrak{m}_1} = \mathbf{W}_\alpha$, and let $\langle \bar\nu_\gamma : \gamma < \gamma^* \rangle$ list the sequences $\bar\nu \in {}^{\kappa>}({}^\kappa\mu)$ such that $\langle \nu_\zeta \restriction \alpha : \zeta < \ell g(\bar\nu) \rangle$ is without repetitions and $\{\nu_\zeta : \zeta < \ell g(\bar\nu)\} \notin \mathbf{W}_{\mathfrak{m}_0}$. Let $\bar\nu_\gamma = \langle \nu_{\gamma,\zeta} : \zeta < \zeta^*_\gamma \rangle$ and define $\bar\rho_\gamma =: \langle \nu_{\gamma,\zeta} \restriction \alpha : \zeta < \ell g(\bar\nu_\gamma) \rangle$, and $W_\gamma =: \{\nu_{\gamma,\zeta} : \zeta < \zeta^*_\gamma\}$ for $\gamma < \gamma^*$. Let $\beta_\gamma = \mathrm{otp}\{\gamma_1 < \gamma : (\forall \gamma_2 < \gamma_1)(\bar\rho_{\gamma_2} \neq \bar\rho_{\gamma_1})\}$.

For each $W \in \mathbf{W}_\alpha \setminus \mathbf{W}_{\mathfrak{m}_0}$, let $M^{\mathfrak{m}_1}_W$ be an elementary submodel of $M_{\mathfrak{m}_1}$ of cardinality θ such that

$$\begin{array}{rcl} W^*_1 \subseteq W \;\wedge\; W^*_1 \in \mathbf{W}_{\mathfrak{m}_0} & \Rightarrow & M^{\mathfrak{m}_1}_{W^*_1} \prec M^{\mathfrak{m}_1}_W \qquad \text{and} \\ \bar b \in {}^{\kappa>}\big(M^{\mathfrak{m}_1}_W\big) & \Rightarrow & M^{\mathfrak{m}_0}_{\bar b} \restriction |M^{\mathfrak{m}_1}_W| \prec M^{\mathfrak{m}_0}_{\bar b}. \end{array}$$

Let $\langle a_{W,i} : i < \theta \rangle$ list the elements of $M^{\mathfrak{m}_1}_W$. For $\beta < \beta_{\gamma^*}$ and $i < \theta$ we choose $f_{\beta,i}$ such that

$$\gamma < \gamma^* \wedge \beta_\gamma = \beta \;\Rightarrow\; \zeta_{f_{\beta,i}} = \ell g(\bar\nu_\gamma) = \ell g(\bar\rho_\gamma) = \ell g(\bar\nu_{\beta_\gamma}),\; \eta(f_{\beta,i}, \zeta) = \rho_{\gamma,\zeta}$$

and we define $f^{\mathfrak{m}_1}_{\beta,i}$ by: if $\nu_\zeta \in I_{\rho_{\gamma,\zeta}}$ for $\zeta < \zeta_{f_{\beta,i}}$, and $\langle \nu_\zeta : \zeta < \zeta^*_{f_{\beta,i}} \rangle = \bar\nu_\gamma$ then $f^{m_1}_{\beta,i}(\ldots, \nu_{\gamma,\zeta}, \ldots) = a_{W_\gamma, i}$.
Next, $\mathscr{F}_{\mathfrak{m}_1}$ is almost $\mathscr{F}_{\mathfrak{m}_0} \cup \{f_{\beta,i} : \beta < \beta_{\gamma^*},\ i < \theta\}$: we just have to term-close it. Lastly $\mathbf{f}_{\mathfrak{m}_1}$ is defined as $\mathbf{f}_{\mathfrak{m}_0}$ recalling that $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1})$ is required just to be a subset of $\mathbb{F}_{\mathfrak{m}_0}$.

2) Also easy. $\square_{2.19}$

Let M^* be a $\|M_{\mathfrak{m}_0}\|^+$–resplendent elementary extension of $M_{\mathfrak{m}_0}$. We define an α–approximation $\mathfrak{m}_1$ as follows:

(a) $\alpha_{\mathfrak{m}_1} = \alpha_{\mathfrak{m}_0}$, $\mathbf{W}_{\mathfrak{m}_1} = \mathbf{W}_{\mathfrak{m}_0}$, $M_{\mathfrak{m}_1} = M^*$,
(b) if $\bar b \in {}^{\kappa>}(M_{\mathfrak{m}_0})$, then $M^{\mathfrak{m}_1}_{\bar b}$ is an elementary extension of $M^{\mathfrak{m}_0}_{\bar b}$,
(c) $\mathbf{f}_{\mathfrak{m}_1} \supseteq \mathbf{f}_{\mathfrak{m}_0}$ and $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}) = \mathbb{F}_{\mathfrak{m}_0}$,
(d) $\mathscr{F}_{\mathfrak{m}_1} = \mathscr{F}_{\mathfrak{m}_0}$

(e) if $(\mathbf{f}_{\mathfrak{m}_1}(\bar f))(\sigma_\xi(\bar x^\xi)) = f$, $\eta(\bar f,\zeta) \lhd \nu_\eta \in {}^\kappa\mu$, and

$$\bar b = \langle f^{\mathfrak{m}_1}_\varepsilon(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta_{\bar f}} : \varepsilon < \varepsilon_{\bar f}\rangle \quad \text{and} \quad \bar x^\xi = \langle x^\xi_i : i \in u\rangle,$$

then

$$f^{\mathfrak{m}_1}(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta_{\bar f}} = \sigma^{M^{\mathfrak{m}_1}_{\bar b}}(\langle f_i(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta_{\bar f}} : i \in u\rangle).$$

Main Claim 2.20. *Assume $\mathfrak{m}_0$ is a weakly nice approximation. then there is a weakly good approximation $\mathfrak{m}_1$ such that $\mathfrak{m}_0 <^* \mathfrak{m}_1$ with $\mathbf{W}_{\mathfrak{m}_1} = \mathbf{W}_{\mathfrak{m}_0}$.*

Proof. By 2.19(1)+(2) there is a full term closed $\mathfrak{m}_1$ such that $\mathfrak{m}_0 <^* \mathfrak{m}_1$ and $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}) = \mathbb{F}_{\mathfrak{m}_0}$. We would like to "correct"$\mathfrak{m}_1$ so that it is weakly indiscernible. Let $\mathfrak{m}_2$ be an $\alpha_{\mathfrak{m}_1}$–approximation as guaranteed in Claim 2.21 below, so it is good and reflecting we clearly see that $\mathfrak{m}_0 \le^* \mathfrak{m}_2$ and even $\mathfrak{m}_0 <^* \mathfrak{m}_2$. $\square_{2.20}$

Main SubClaim 2.21. *(1) Assume $\mathfrak{m}_0$ is a weakly nice α–approximation and $\mathfrak{m}_0 <^* \mathfrak{m}_1$ and $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}) = \mathbb{F}_{\mathfrak{m}_0}$ and $\mathbf{W}_{\mathfrak{m}_1}$ is an ideal (that is closed under finite union). then there is a good α–approximation $\mathfrak{m}_2$ such that:*

(a) $\alpha_{\mathfrak{m}_2} = \alpha_{\mathfrak{m}_1}$, $\mathscr{F}_{\mathfrak{m}_2} = \mathscr{F}_{\mathfrak{m}_1}$, $\mathbf{f}_{\mathfrak{m}_2} = \mathbf{f}_{\mathfrak{m}_1}$, and $\mathbf{W}_{\mathfrak{m}_2} = \mathbf{W}_{\mathfrak{m}_1}$.
(b) $\mathfrak{m}_0 <^* \mathfrak{m}_2$;

We may add
(c) Assume:

(α) $n < \omega$ and $f_\ell \in \mathscr{F}_{\mathfrak{m}_1}$, $\nu^\ell_\zeta \in I_{\eta(f_\ell,\zeta)}$ for $\zeta < \zeta_{f_\ell}$, $\ell < n$, and Δ is a finite set of formulas in $\mathbb{L}(\tau_T)$

(β) $m < \omega$ and for $k < m$ we have $\bar f^k = \langle f^k_\varepsilon : \varepsilon < \varepsilon_k\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1})$ and $n_k < \omega$ and $g_{k,\ell} \in \mathscr{F}_{\mathfrak{m}_2}$ (for $\ell < n_k$) satisfying

$$\langle \eta(g_{k,\ell},\zeta) : \zeta < \zeta_{g_{k,\ell}}\rangle = \langle \eta(\bar f^k,\zeta) : \zeta < \zeta_{\bar f^k}\rangle,$$

and $\nu^\ell_{k,\zeta} \in I_{\eta(\bar f^k,\zeta)}$ for $\ell < n_k$, $\zeta < \zeta_{\bar f^k}$, and Δ_k is a finite set of formulas in $\mathbb{L}(\tau[\zeta_{\bar f},\tau(T)])$.

then we can find ρ^ℓ_ζ for $\ell < n_k$, $\zeta < \zeta_{f_\ell}$ and $\rho^\ell_{k,\zeta}$ for $\ell < n_k$, $\zeta < \zeta_{\bar f^k}$ for $\ell < n_k, k < m$ such that

(i) $\rho^\ell_\varepsilon \in I_{\eta(f_\ell,\zeta)}$ for $\zeta < \zeta_{f_\ell}$ and $\rho_{k,\zeta} \in I_{\eta(\bar f^k,\zeta)}$ for $\zeta < \zeta_{\bar f^k}$ for $\ell < n, k < m$,

(ii) the sequences $\langle \rho^\ell_\zeta : \ell < n,\ \zeta < \zeta_{f_\ell}\rangle{}^\frown\langle \rho_{k,\zeta} : \ell < n_k, k < m,\ \zeta < \zeta_{\bar f^k}\rangle$ and $\langle \nu^\ell_\zeta : \ell < n,\ \zeta < \zeta_{f_\ell}\rangle{}^\frown\langle \nu_{k,\zeta} : \ell < n_k, k < m,\ \zeta < \zeta_{\bar f^k}\rangle$ are similar (see [She90, VII] or 0.4),

(iii) the Δ–type realized by the sequence

$$\langle f^{\mathfrak{m}_2}_\ell(\ldots,\nu^\ell_\zeta,\ldots)_{\zeta<\zeta_{f_\ell}} : \ell < n\rangle$$

in $M_{\mathfrak{m}_2}$ is equal to the Δ–type which the sequence

$$\langle f^{\mathfrak{m}_1}_\ell(\ldots,\rho^\ell_\zeta,\ldots)_{\zeta<\zeta_{f_\ell}} : \ell < n\rangle$$

realizes in $M_{\mathfrak{m}_1}$,

(iv) for $k < m_1$, the Δ_k–type realized by the sequence

$$\langle g^{\mathfrak{m}_2}_{k,\ell}(\ldots,\nu^\ell_{k,\zeta},\ldots)_{\zeta<\zeta_{\bar f^k}} : \ell < n_k\rangle$$

in the model $M^{\mathfrak{m}_2}_{\langle f^{k,\mathfrak{m}_2}_\varepsilon : \varepsilon<\varepsilon_k\rangle}(\ldots\nu^\ell_{k,\zeta}\ldots)_{\zeta<\zeta:\varepsilon<\varepsilon_k\rangle}$ is equal to the Δ_k–type realized by the sequence

$$\langle g^{\mathfrak{m}_1}_{k,\ell}(\ldots,\rho^\ell_{k,\zeta},\ldots)_{\zeta<\zeta_{\bar f^k}} : \ell < n_k\rangle$$

in the model $M^{\mathfrak{m}_1}_{\langle f^{k,\mathfrak{m}_2}_\varepsilon : \varepsilon < \varepsilon_k\rangle}(\ldots \nu^\ell_{k,\zeta} \ldots)_{\zeta < \zeta : \varepsilon < \varepsilon_k\rangle}$

(v) *if* $k_1, k_2 < m$ *then*

$$\langle f^{k_1,\mathfrak{m}_2}_\varepsilon(\ldots, \nu^{k_1}_\zeta, \ldots)_{\zeta < \zeta_{\bar f^{k_1}}} : \varepsilon < \varepsilon_{k_1}\rangle = \langle f^{k_2,\mathfrak{m}_2}_\varepsilon(\ldots, \nu^{k_2}_\zeta, \ldots)_{\zeta < \zeta_{\bar f^{k_2}}} : \varepsilon < \varepsilon_{k_2}\rangle$$

if and only if

$$\langle f^{k_1,\mathfrak{m}_1}_\varepsilon(\ldots, \rho^{k_1}_\zeta, \ldots)_{\zeta < \zeta_{\bar f^{k_1}}} : \varepsilon < \varepsilon_{k_1}\rangle = \langle f^{k_2,\mathfrak{m}_1}_\varepsilon(\ldots, \rho^{k_2}_\zeta, \ldots)_{\zeta < \zeta_{\bar f^{k_2}}} : \varepsilon < \varepsilon_{k_2}\rangle,$$

(vi) *if* $\ell < n_k$, $k < m$, $\ell^* < n$, *then*

$$f^{\mathfrak{m}_2}_\ell(\ldots, \nu^{\ell^*}_\zeta, \ldots)_{\zeta < \zeta_{f_\ell}} = f^{\mathfrak{m}_2}_{k,\ell}(\ldots, \nu^\ell_{k,\zeta}, \ldots)_{\zeta < \zeta_{\bar f^k}}$$

if and only if

$$f^{\mathfrak{m}_1}_\ell(\ldots, \rho^{\ell^*}_\zeta, \ldots)_{\zeta < \zeta_{f_\ell}} = f^{\mathfrak{m}_1}_{k,\ell}(\ldots, \rho^\ell_{k,\zeta}, \ldots)_{\zeta < \zeta_{\bar f^k}}.$$

Discussion 2.22. Now we have to apply the Ramsey theorem to recapture weak indiscernibility. Why do we only promise $\mathfrak{m}_0 <^* \mathfrak{m}_1$ and $\mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}) = \mathbb{F}_{\mathfrak{m}_0}$, not that $\mathfrak{m}_1$ is excellent? Because $T^*[\bar b, M]$ is not a continuous function of $(\bar b, M)$, and more down to earth, during the proof we need to know the type of $\bar b$ whenever we consider types in $M^{\mathfrak{m}_1}_{\bar b}$ in order to know $T^*[\bar b, M_\mathfrak{m}]$.

Usually a partition theorem on what we already have is used at this moment, but partition of infinitary functions tend to contradict ZFC. However, in the set Λ expressing what we need, the formulas are finitary. So using compactness we will reduce our problem to the consistency of the set Λ of first order formulas in the variables

$$\{f(\ldots, \eta_\zeta, \ldots)_{\zeta < \zeta(f)} : f \in \mathscr{F}^{\mathfrak{m}_1} \text{ and } \zeta < \zeta_f \ \Rightarrow\ \eta(f,\zeta) \lhd \eta_\zeta \in {}^\kappa\mu\}.$$

This can be easily reduced to the consistency of a set Λ of formulas in $\mathbb{L}(\tau_T)$ (first order).

We can get Λ because for all relevant $\bar b$ we know $T^*[\bar b, M]$.

Proof. Let $Y = \{y_{f(\ldots,\nu_\eta,\ldots)_{\eta \in w[f]}} : f \in \mathscr{F}_{\mathfrak{m}_1}$ and $\nu_\eta \in I_\eta$ for $\eta \in w[f]\}$ be a set of individual variables with no repetitions, recalling that $w[f] = \{\eta[f,\varepsilon] : \varepsilon < \zeta_f\}$. For each $\bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1})$ and $\bar\nu = \langle \nu_\eta : \eta \in w[\bar f]\rangle$ such that $\nu_\eta \in I_\eta$, let $\tau_{\bar f,\bar\nu}$ be $\tau[T, \ell g\bar f]$ where $w[\bar f] = w[f_\varepsilon]$ for each $\varepsilon < \ell g(\bar f)$; pedantically a copy of it over τ_T so $(\bar f_1, \bar\nu_1) \neq (\bar f_2, \bar\nu_2) \ \Rightarrow\ \tau_{\bar f_1,\nu_1} \cap \tau_{\bar f_2,\bar\nu_2} = \tau_T$. Let $\tau^* = \bigcup\{\tau_{\bar f,\bar\nu} : \bar f, \bar\nu \text{ as above }\} \cup \tau_T$.

Let $\mathbf{g}_{\bar f,\bar\nu}$ be a one to one function from $\tau[T, \ell g(\bar f)]$ onto $\tau_{\bar f,\bar\nu}$ which is the identity on τ_T preserve the arity and being a predicate function symbol, individual constant. Let $\hat{\mathbf{g}}_{\bar f,\bar\nu}$ be the mapping from $\mathbb{L}(\tau[T, \ell g(\bar f)])$ onto $\mathbb{L}(\tau_{\bar f,\bar\nu})$ which $\mathbf{g}_{\bar f,\bar\nu}$ induce.

We now define a set Λ (the explanations are for the use in the proof of $\boxtimes_1$ below).

$\boxtimes_0$ $\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \Lambda_3 \cup \Lambda_4 \cup \Lambda_5 \cup \Lambda_6 \cup \Lambda_7 \cup \Lambda_8$ where

(a) $\Lambda_0 = \{\varphi_\zeta(y_{f^*_{\rho_1,\kappa}(\nu_1)}, y_{f^*_{\rho_2,\zeta+1}(\nu_2)})^{\mathbf{t}}$: where $\mathbf{t} = \text{truth}$ iff $[\nu_1 \restriction (\zeta+1) = \nu_2 \restriction (\zeta+1)]$ and $\zeta < \kappa, \rho_\ell \in {}^\alpha\mu$, and $\nu_\ell \in I^\alpha_{\rho_\ell}$ for $\ell = 1,2\}$
[explanation: to satisfy (iii) in clause (B) of Definition 2.5].

(b) $\Lambda_1 = \{y_{f^*_{\rho_1,\zeta}(\nu_1)} = y_{f^*_{\rho_2,\zeta}(\nu_2)} : \zeta < \kappa, \rho_\ell \in {}^\alpha\mu, \nu_\ell \in I^\alpha_{\rho_\ell}$ for $\ell = 1,2$ and $\nu_1 \restriction \zeta = \nu_2 \restriction \zeta\}$
[explanation:to satisfy (ii) in clause (B) of Definition 2.5].

(c) $\Lambda_2 = \{y_{f(\ldots\nu_\eta,\ldots)_{\eta\in w[f]}} = \sigma(\ldots, y_{f_\ell(\ldots,\nu_\eta,\ldots)_{\eta\in w[f]}}\ldots) : f, \langle f_\ell : \ell < n\rangle$ and $\langle \nu_\eta : \eta \in w[f]\rangle$ are as in clause (E) of Definition 2.6(4) for $\mathfrak{m}_1\}$.

[explanation:this is preservation of the witnesses for closure under terms of τ, in clause (E) of Definition 2.6(4) for $\mathfrak{m}_1$].

(d) $\Lambda_3 = \{y_{f(\ldots,\nu_\eta,\ldots)_{\eta\in w[f]}} = \sigma(\ldots, f^\ell(\ldots,\nu_\eta(f^\ell,\zeta),\ldots)_{\zeta<\zeta(f^\ell)},,\ldots)_{\ell<n} : f$, $\langle f^\ell : \ell < n\rangle$ and $\langle f_\varepsilon : \varepsilon < \varepsilon(*)\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_0})$ are as in clause (F) of Definition 2.6(4)$\}$.

[explanation: this is preservation of the witness for closure under terms of the $\tau(M_{\bar b})$-'s as in clause (F) of Definition 2.6(4) for $\mathfrak{m}_1$).

(e) $\Lambda_4 = \{\varphi(\ldots, y_{f_\ell(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta^*}},\ldots)_{\ell<n} : \varphi(x_0,\ldots,x_{n-1}) \in \mathbb{L}(\tau_T)$ and $f_\ell \in \mathscr{F}_{\mathfrak{m}_0}$ and $\zeta^* = \zeta_{f_\ell}$ for $\ell < n$, and $\nu_\zeta \in I_{\eta(f_\ell,\zeta)}$ for $\zeta < \zeta^*$ and $M_{\mathfrak{m}_0} \models \varphi[\ldots, f^{\mathfrak{m}_0}_\ell(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta^*},\ldots)_{\ell<n}\}$

[explanation: this is for being above $\mathfrak{m}_0$, the $\mathbb{L}(\tau_T)$-formulas].

(f) Λ_5 like Λ_4 for the M_b-'s that is

$\Lambda_5 = \{\varphi(\ldots, y_{f_\ell(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta^*},\ldots)_{\ell<n}}$: for some $\bar f, \bar\nu$ and $\langle f_\ell : \ell < n\rangle$ we have $\varphi(x_0,\ldots,x_{n-1}) \in \mathbb{L}(\tau_{\bar f,\bar\nu})$ and $\bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_0}), \zeta(\bar f) = \zeta^* = \zeta_{f_\ell}, f_\ell \in \mathscr{F}_{\mathfrak{m}_0}, \eta(\bar f,\zeta) = \eta(f_\ell,\zeta)$ for $\zeta < \zeta^*, \ell < n$ and $M_{\mathfrak{m}_0} \models \varphi[\ldots f^{\mathfrak{m}}_\ell(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta^*}\ldots)_{\ell<n\}}$

[explanation: this is for being above $\mathfrak{m}_0$, the formulas from the $M^{\mathfrak{m}_0}_{\bar b}-'s$].

(g) $\Lambda_6 = \{\varphi(\ldots, y_{f_\ell(\ldots\nu^1_\zeta,\ldots)_{\zeta<\zeta^*}},\ldots)_{\ell<n} \equiv \varphi(\ldots, y_{f_\ell(\ldots,\nu^2_\zeta,\ldots)_{\zeta<\zeta^*}},\ldots)_{\ell<n} : \varphi(x_0,\ldots,x_{n-1}) \in \mathbb{L}))\tau_T)$ and $\zeta^* = \zeta(f_\ell)$, $f_\ell \in \mathscr{F}^{\mathfrak{m}_1}, \nu^k_\zeta \in I_{\eta(f_\ell,\zeta)}$ if $k = 1,2$ and $\zeta < \zeta^*$ such that for exactly one $\zeta < \zeta^*$ we have $\nu^1_\zeta \neq \nu^2_\zeta\}$

[explanation: this is for weak indiscernibility, see Definition 2.9(1) clause (b) and Definition 2.9(3).].

(h) $\Lambda_7 = \{(\forall x_1 \ldots x_n)[\varphi_1(x_0,\ldots x_{n-1}) \equiv \varphi_2(x_0,\ldots,x_{n-1})]$: for some $\bar f^1, \bar f^2 \in \mathbb{F}_{\mathfrak{m}}, \ell g(\bar f^1) = \ell g(\bar f^2)$, $\eta(\bar f^\ell,\zeta) \triangleleft \nu^\ell_\zeta \in {}^\kappa\mu$ for $\zeta < \ell g(\bar f^\ell)$ and $\eta(\bar f^1,\zeta_1) = \eta(\bar f^2,\zeta_2) \Rightarrow \nu^1_{\zeta_1} = \nu^2_{\zeta_2}$; $\bar\varphi_\ell \in \mathbb{L}(\tau_{\bar f^\ell,\bar\nu^\ell})$ and $\hat{\mathbf{g}}^{-1}_{\bar f^1,\bar\nu^1}(\varphi_1) = \hat{\mathbf{g}}^{-1}_{\bar f^2,\nu^2}(\varphi_2)\}$.

[Explanation: this has to show the existence of the $M_{\bar b}$: we can avoid this if we change the main definition such that instead $M_{\bar b}$ we have $M_{\bar f,\bar\nu}$]

(i) $\Lambda_8 = \{\varphi(\ldots, y_{f_\ell(\ldots,\nu^1_\zeta,\ldots)_{\zeta<\zeta^*}},\ldots)_{\ell<n} \equiv \varphi(\ldots, y_{f_\ell(\ldots,\nu^2_\zeta,\ldots)_{\zeta<\zeta^*}},\ldots)_{\ell<n}$: for some $\bar f \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_1}), \bar f, \bar\nu, \bar\nu^1, \bar\nu^2$ and $\langle f_\ell : \ell < n\rangle$ we have $\varphi(x_0,\ldots,x_{n-1}) \in \mathbb{L}(\tau_{\bar f,\bar\nu})$, $f_\ell \in \mathscr{F}_{\mathfrak{m}_1}, \zeta_{f_\ell} = \zeta^*, \eta(f_\ell,\zeta) = \eta_\zeta \in {}^\alpha\mu$ for $\zeta < \zeta^*$, and $w(\bar f) \subseteq \{\eta_\zeta : \zeta < \zeta^*\}$, and for $k = 1,2$ we have $\eta_\zeta \triangleleft \nu^k_\zeta \in {}^\kappa\mu$ for $\zeta < \zeta^*$, and $\eta(\bar f,\zeta_1) = \eta_{\zeta_2} \Rightarrow \nu_{\zeta_1} = \nu^k_{\zeta_2}\}$

Clearly (e.g. for the indiscernibility we use term closure)

($\boxtimes_1$) Λ is a set of first order formulas in the free variables from Y and the vocabulary τ^* such that an α–approximation $\mathfrak{m}$ satisfying (i) below is as required if and only if clause (ii) below holds where

(i) $\mathscr{F}_{\mathfrak{m}} = \mathscr{F}_{\mathfrak{m}_1}$, $\mathbf{f}_{\mathfrak{m}} = \mathbf{f}_{\mathfrak{m}_1}$, $\mathbf{W}_{\mathfrak{m}} = \mathbf{W}_{\mathfrak{m}_1}$,

(ii) interpreting $y_{f(\ldots,\nu_n,\ldots)_{\eta\in w[f]}} \in Y$ as $f^{\mathfrak{m}}(\ldots,\nu_\eta,\ldots)_{\eta\in w[f]}$ and the predicates and relation symbols in each $\tau_{\bar f,\bar\nu}$ naturally, $M_{\mathfrak{m}}$ is a model of Λ or more exactly not $M_{\mathfrak{m}}$ but the common expansion of the $M^{\mathfrak{m}}_{\bar b}$-'s for $\bar b \in \{\langle f^{\mathfrak{m}}_\varepsilon(\ldots,\nu_\zeta,\ldots)_{\zeta<\zeta^*} : \varepsilon < \varepsilon^*\rangle : \bar f = \langle f_\varepsilon : \varepsilon < \varepsilon^*\rangle \in \mathrm{Dom}(\mathbf{f}_{\mathfrak{m}_0})$, and $\nu_\zeta \in I_{\eta(\bar f,\zeta)}\}$.

So it is enough to prove

($\boxtimes_2$) Λ has a model.

We use the compactness theorem, so let $\Lambda^a \subseteq \Lambda$ be finite. We say that $\nu \in {}^\kappa\mu$ appears in Λ^a, if for some variable $y_{f(\ldots,\nu_\eta,\ldots)_{\eta \in w[f]}}$ appearing as a free variable in some $\varphi \in \Lambda^a$ we have $\nu \in \{\nu_\eta : \eta \in w[f]\}$ or some formula in Λ^a belongs to $\mathbb{L}(\tau_{\bar f,\bar\nu}) \setminus \mathbb{L}(\tau_T)$. We may also say "$\nu$ appears in φ", and/or "$f(\ldots,\nu_\eta,\ldots)_{\eta \in w[f]}$ appears in Λ^a" (or in φ).

Let $n_0^* = |\Lambda^a|$. Now, for each $\eta \in {}^\alpha\mu$ the set of $\nu \in I^\alpha_\eta$ appearing in Λ^a, which we call J^α_η, is finite but on $\bigcup\limits_{\eta \in {}^\alpha\mu} J^\alpha_\eta$ we know only that its cardinality is $< \kappa$. Note that, moreover, $n_1^* =: \max\{|J^\alpha_\eta| : \eta \in {}^\alpha\mu\}$ is well defined $< \aleph_0$ as well as $m_0^* = |\Lambda^a \cap \Lambda_0|$. For each $\eta \in {}^\alpha\mu$ we can find a finite set $\mathbf{u}_\eta \subseteq \kappa$ such that:

$(\otimes)$ (i) if $\nu_1 \neq \nu_2 \in J^\alpha_\eta$, <u>then</u> $\min\{\zeta : \nu_1(\zeta) \neq \nu_2(\zeta)\} \in \mathbf{u}_\eta$
(ii) if $\varphi_\zeta[f^*_{\rho_1,\kappa}(\nu_1), f^*_{\rho_2,\zeta}(\nu_2)]^{\mathbf{t}}$ from clause (B) appears in $\Lambda^a \cap \Lambda_0$, then $\zeta, \zeta+1 \in \mathbf{u}_\eta$
(iii) $\alpha \in \mathbf{u}_\eta$.
(iv) $|u_\eta| \leq (n_1^*)^2 + 2m_0^* + 1$.

Clearly $n_2^* = \max\{|\mathbf{u}_\eta| : \eta \in {}^\alpha\mu\}$ is well defined $(< \aleph_0)$, so without loss of generality, $\eta \in {}^\alpha\mu \;\Rightarrow\; |\mathbf{u}_\eta| = n_2^*$.

Let $v \subseteq {}^\alpha\mu$ be finite, in fact of size $\leq |\Lambda^a| = n_0^*$ such that:

(I) if $\varphi_\zeta[y_{f^*_{\rho_1,\kappa}(\nu_1)}, y_{f^*_{\rho_2,\zeta}(\nu_2)}]^{\mathbf{t}}$ appears in $\Lambda^a \cap \Lambda_0$, so $\ell \in \{1,2\} \Rightarrow \nu_\ell \in J^\alpha_{\rho_\ell}$, then $\rho_\ell \in v$ for $\ell \in \{1,2\}$,

Now, for all $\eta \in {}^\alpha\mu \setminus v$ we replace in Λ^a all members of J^α_η by one $\nu_\eta \in I^\alpha_\eta$ and we call what we get Λ^b, i.e., we identify some variables. It suffices to prove Λ^b is consistent. Now, by the choice of the set v also Λ^b is of the right kind, i.e., $\subseteq \Lambda$.

[Why? We should check the formulas φ, in $\Lambda^a \cap \Lambda_i$ for each $i \leq 8$; let it be replaced by $\varphi' \in \Lambda^b$. If in $\varphi \in \Lambda_0 \cap \Lambda^a$ by clause (ii) of $\otimes$ this substitution has no affect on φ. If $\varphi \in \Lambda_1$, either $\varphi' = \varphi$ or φ' is trivially true. If $\varphi \in \Lambda_3$, clearly $\varphi' \in \Lambda_3$. If $\varphi \in \Lambda_4$ then $\varphi' \in \Lambda_4$ as $\mathfrak{m}_0$ is nice hence weakly indiscernible, i.e. clause (b) of Definition 2.9(1) (and the demand $f_\ell \in \mathscr{F}_{\mathfrak{m}_0}$). If $\varphi \in \Lambda_5$, similarly using clause (c) of Definition 2.9(1). Lastly if $\varphi \in \Lambda_6$ we just note that similarly is preserved and similarly for $\varphi \in \Lambda_7 \cup \Lambda_8$].

We then transform Λ^b to Λ^c by replacing each φ by φ', gotten by replacing, for each $\rho \in v$, every $\nu \in J^\alpha_\rho$ by $\nu^{[*]} \in {}^\kappa\mu$ where $\nu^{[*]}(\beta) = \nu(\beta)$ if $\beta \in \alpha \cup u_\rho$ and $\nu^{[*]}(\beta) = 0$ otherwise. It suffices to prove the consistency of Λ^c. Now, the effect is renaming variables and again $\Lambda^c \subseteq \Lambda$. Let $\rho^* = \langle \rho^*_k : k < k^* \rangle$ list the $\rho \in {}^\kappa\mu$ which appear in Λ^c such that $\rho \restriction \alpha \in v$. Let $\eta_k = \rho^*_k \restriction \alpha$ so $\eta_k \in v$, and let

$$\Upsilon = \big\{\bar\rho : \begin{array}{l} \bar\rho = \langle \rho_k : k < k^* \rangle,\ \eta_k \triangleleft \rho_k \in {}^\kappa\mu, \\ (\forall \varepsilon)\big[\alpha \leq \varepsilon < \kappa \wedge \varepsilon \notin u_{\eta_k} \Rightarrow \rho_k(\varepsilon) = 0\big] \\ \text{and } \bar\rho \text{ is similar to } \bar\rho^* \big\} \end{array}$$

(i.e., for $k_1, k_2 < k^*$ and $\varepsilon < \kappa$ we have $\rho_{k_1}(\varepsilon) < \rho_{k_2}(\varepsilon) \Rightarrow \rho^*_{k_1}(\varepsilon) < \rho^*_{k_2}(\varepsilon)$.)

For each $\bar\rho \in \Upsilon$ we can try the following model as a candidate to be a model of Λ^c. It expands $M_{\mathfrak{m}_1}$, and if symbols from $\tau_{\bar f,\bar\nu} \setminus \tau_T$ appear they are interpreted as their $\mathbf{g}^{-1}_{\bar f,\bar\nu}$-images are interpreted in $M^{\mathfrak{m}_1}_{\langle f_\varepsilon(\bar\nu) : \varepsilon < \ell g(\bar f) \rangle}$. Lastly we assign to the variable $y_f(\ldots,\nu_\zeta,\ldots)_{\zeta < \zeta_f}$ appearing in Λ^c the element $f_{\mathfrak{m}_1}(\ldots,\nu_\zeta,\ldots)_{\zeta < \zeta_f}$ of $M_{\mathfrak{m}_1}$. Call this the $\bar\rho$-interpretation. Considering the formulas in $\Lambda^c \cap \Lambda_i$ for $i \in \{0,\ldots,5,7\}$ they always holds. For the formulas in $\Lambda^c \cap \Lambda_6, \Lambda_8$ we can use a partition theorem on trees with $|n_2^*| < \aleph_0$ levels (use [Sheb, 1.16](4), which is an overkill, but has the same spirit (or [She90, AP2.6,p.662])). $\square_{2.21}$

Claim 2.23. *There is an increasing continuous inverse system of approximations*

$$\langle \mathfrak{m}_\gamma, \mathbf{h}^\alpha_\beta : \gamma \le \kappa,\ \beta \le \alpha \le \kappa\rangle$$

such that each $\mathfrak{m}_\gamma$ is weakly excellent.

Proof. By induction on $\alpha \le \kappa$ we choose $\mathfrak{m}_\alpha$ and $\langle \mathbf{h}^\alpha_\beta : \beta < \alpha\rangle$ with our inductive hypothesis being

(∗) (a) $\langle \mathfrak{m}_{\beta_1}, \mathbf{h}^\beta_\gamma : \beta_1 \le \alpha, \gamma < \beta \le \alpha\rangle$ is an inverse system of approximations,
(b) $\mathfrak{m}_\beta$ is a weakly excellent β–approximation.

<u>For $\alpha = 0$</u>:
A weakly excellent good 0–approximation exists by 2.14.

<u>For α limit</u>:
Clearly $\langle \mathfrak{m}_{\beta_1}, \mathbf{h}^\beta_\gamma : \beta_1 < \alpha, \gamma < \beta < \alpha\rangle$ is an inverse system of good weakly excellent approximations with $\alpha(\mathfrak{m}_\beta) = \beta$. So by 2.15 we can find $\mathfrak{m}_\alpha, \mathbf{h}^\alpha_\beta$ $(\beta < \alpha)$ as required.

<u>For $\alpha = \beta + 1$</u>:
By 2.16(1+2) there is $\mathfrak{m}^*_{\alpha,0}$ a weakly nice α–approximation such that $\mathfrak{m}_\beta \le^* \mathfrak{m}^*_{\alpha,0}$. By 2.20 there is a full term closed α–approximation $\mathfrak{m}^*_{\alpha,1}$ such that $\mathfrak{m}^*_{\alpha,0} \le^* \mathfrak{m}^*_{\alpha,1}$ and $\mathfrak{m}^*_{\alpha,1}$ is good. We can choose by induction on $\varepsilon \in [1,\kappa]$ good α–approximations $\mathfrak{m}_{\alpha,\varepsilon}$, $\le^*$–increasing continuously, $\mathfrak{m}_{\alpha,\varepsilon} <^* \mathfrak{m}_{\alpha,\varepsilon+1}$.

For $\varepsilon = 1$, $\mathfrak{m}_{\alpha,\varepsilon}$ is defined; for ε limit use 2.18(2), for ε successor use 2.20, and $\mathfrak{m}_\alpha =: \mathfrak{m}_{\alpha,\kappa}$ is good by 2.18(3). $\square_{2.23}$

Claim 2.24. *Assume $\mathfrak{m}_\alpha, \mathbf{h}^\alpha_\gamma$ for $\alpha \le \kappa, \gamma < \alpha$ as in 2.23 with $\mu = \lambda$ and $\lambda = \lambda^\kappa \ge \theta$ (e.g., $\lambda = \lambda^\kappa \ge 2^{|T|}$). <u>then</u> there are $> \lambda$ pairwise non-isomorphic κ–resplendent models of T of cardinality λ.*

Proof. Let $\mathfrak{m} = \mathfrak{m}_\kappa$ and $I \subseteq {}^{\kappa \ge}\lambda$, $|I| = \lambda$ and for simplicity $\{\eta \in {}^\kappa\lambda : \eta(\varepsilon) = 0$ for every large enough $\varepsilon < \kappa\} \cup {}^{\kappa >}\lambda \subseteq I$. Let M_I be the submodel of $M_{\mathfrak{m}}$ with universe

$$\left\{ f\left(\ldots, \nu_{\eta(f,\zeta)}, \ldots\right)_{\zeta < \zeta_f} : f \in \mathscr{F}_{\mathfrak{m}} \text{ and } \eta(f,\zeta) \in I \cap {}^\kappa\lambda \text{ for every } \zeta < \zeta_f \right\}.$$

Trivially, $\|M_I\| \le \lambda^\kappa = \lambda$ and by clause (B) of Definition 2.5 clearly by 2.1(1) it follows that the sequence $\langle a_\eta : \eta \in {}^\varepsilon\lambda\rangle$ is with no repetitions for each $\varepsilon < \lambda$ hence by the indiscernibility the sequence $\langle a_\eta : \eta \in I\rangle$ is with no repetition, so $\|M_I\| \ge |I| \ge \lambda$, so $\|M_I\| = \lambda$.

Now, M_I is a κ-resplendent model of T as $\mathfrak{m}$ being weakly excellent is full and resplendent.

For $\zeta < \kappa$, $\nu \in {}^\zeta\lambda$ let $a_\nu = f^{*,\mathfrak{m}_\kappa}_{\eta,\zeta}(\eta)$ $(\in M_I)$ for any $\eta \in I^\zeta_\nu \cap I$.

The point is:

(⊗) For $\eta \in {}^\kappa\lambda$, $\nu_\gamma \in {}^{\gamma+1}\lambda$, $\nu_\gamma \restriction \gamma = \eta \restriction \gamma$, $\nu_\gamma \ne \eta \restriction (\gamma+1)$, we have:
⊛ the type $\left\{\varphi(x, a_{\eta\restriction(\gamma+1)}) \equiv \neg\varphi(x, a_{\nu_\gamma}) : \gamma < \kappa\right\}$
is realized in M_I <u>iff</u> $\eta \in I$.

[Why? The implication "⇐" holds by clause (B)(iii) of Definition 2.5. For the other direction, if $c \in M_I$, then for some $W \in \mathbf{W}_\kappa$, satisfying $W \subseteq I$, we have $c \in N^{\mathfrak{m}}_W$, and as $\eta \notin I$ and $|W| < \kappa$ clearly for some $\alpha < \kappa$ we have

$$\{\nu : \eta \restriction \alpha \lhd \nu \in {}^\kappa\mu\} \cap W = \varnothing.$$

Let $c = f_{\mathfrak{m}_\kappa}(\dots,\nu_\zeta,\dots)_{\zeta<\zeta_f}$, where $f \in \mathscr{F}_\kappa$, so $\nu_\zeta = \eta(f,\zeta)$. By the continuity of the system, for some $\gamma \in (\alpha,\kappa)$ we have $f \in \mathrm{Dom}(\mathbf{h}^\kappa_\gamma)$, and it suffices to prove that

$$M_I \models \text{“ } \varphi[c, a_{\eta\restriction(\gamma+1)}] \equiv \varphi[c, a_{\nu_\gamma}] \text{ ”}.$$

By the definition of a system, $\mathfrak{m}$ is full. Choose $\nu \in I_{\nu_\zeta}$; recalling $\mathfrak{m}_\gamma \leq_{\mathbf{h}^\kappa_\gamma} \mathfrak{m}_\kappa$ it suffices to prove that

$$\begin{aligned} M_{\mathfrak{m}_\gamma} \models \text{“ } & \varphi[(\mathbf{h}^\kappa_\gamma(f))(\dots,\nu_\zeta,\dots)_{\zeta<\zeta_f}, f^*_{\eta\restriction\gamma,\gamma}(a_\eta)] \equiv \\ & \varphi[(\mathbf{h}^\kappa_\gamma(f))(\dots,\nu_\zeta,\dots)_{\zeta<\zeta_f}, f^*_{\nu\restriction\gamma,\gamma}(a_\nu)] \text{ ”}. \end{aligned}$$

But $\mathfrak{m}_\gamma$ is weakly excellent, hence it is $\mathbf{E}^0_\gamma$–indiscernible, and hence the requirement holds.]

Now use [She22, §2] to get, among those models, $> \lambda$ which are non-isomorphic; putting in the $\eta \in {}^\kappa\lambda$ which are eventually zero does not matter. $\square_{2.24}$

§ 3. Strengthening

Claim 3.1. *If there is strongly excellent κ-approximation $\mathfrak{m}$ and $\mu \geq \lambda = \lambda^\kappa \geq 2^{|T|}$, <u>then</u> T has 2^κ non-isomorphic κ–resplendent models of cardinality λ.*

Proof. This time use Theorem [Shea, 2.3]. For any $I \subseteq {}^{\kappa \geq}\lambda$ which includes ${}^{\kappa >}\lambda$, let $M_I \prec M_{\mathfrak{m}_\kappa}$ be defined as in the proof of 2.24. For $\eta \in {}^\kappa\lambda$ let $a_\eta = f^*_{\eta,\kappa}(\eta)$, and for $\eta \in {}^{\kappa >}\lambda$ of length $\gamma + 1$ let $\eta' = \eta \restriction \gamma{}^\frown\langle \eta(\gamma) + 1\rangle$, and for any $\nu \in I_\eta$, $\nu' \in I_{\eta'}$ let $\bar{a}_\eta = \langle f^*_{\nu,\gamma+1}(\nu), f^*_{\nu',\gamma+1}(\nu')\rangle$; the choice of (ν, ν') is immaterial. Let

$$\varphi(\langle \bar{x}_\alpha : \alpha < \kappa\rangle) = (\exists y)\left(\bigwedge_{\alpha<\kappa}(\varphi(y, x_{\alpha,0}) \equiv \neg\varphi(y, x_{\alpha,1})\right).$$

Now we can choose $f_I : M_I \longrightarrow \mathcal{M}_{\lambda,\kappa}$ such that

(a) if $f_I(b) = \sigma(\langle t_i : i < i^*\rangle)$ such that $t_i \in I \cap {}^\kappa\lambda$ with no repetitions and $\sigma \in \tau[\mathcal{M}_{\lambda,\tau}]$,
<u>then</u> for some $W \in \mathbf{W}_\mathfrak{m}$ and $\gamma < \kappa$ such that $\langle \eta \restriction \gamma : \eta \in W\rangle$ is with no repetition we have $\{t_i : i < i^*\} = W$ and for some $f \in \mathscr{F}_\mathfrak{m}$ with $\zeta_f = \zeta_*$, and $\eta(f, \zeta) = t_\zeta$ for $\zeta < \zeta^f$ we have $b = f_\mathfrak{m}(\ldots, t_\zeta, \ldots)_{\zeta<\zeta_f}$ and
(b) $f_1(b) = \eta \in I$ if $b = a_\eta$ (see above).

The new point is that we have to prove the statement $(*)$ in [Shea, 2.3](c)(β).

So assume that for $\ell = 1, 2$ and $\alpha < \kappa$: $\bar{b}^\ell_\alpha \in {}^2(M_{I_\ell})$, $f_{I_\ell}(\bar{b}^\ell_\alpha) = \bar{\sigma}^\ell_\alpha(\bar{t}^\ell_\alpha)$, and $\bar{t}^1_\alpha = \langle t^2_{\alpha,\varepsilon} : \varepsilon < \varepsilon_\alpha\rangle$. Assume furthermore that $\bar{\sigma}^1_\alpha = \bar{\sigma}^2_\alpha$, $\bar{t}^1_\alpha = \bar{t}^2_\alpha$ for $\alpha < \kappa$ (call it then $\bar{\sigma}^\alpha(\bar{t}_\alpha)$ though possibly $I_1 \neq I_2$), and the truth value of each statement

$$(\exists \nu \in I_\ell \cap {}^\kappa\lambda)\Big(\bigwedge_{i<\kappa} \nu \restriction \varepsilon_i = t^\ell_{\beta_i,\gamma_i} \restriction \varepsilon_i\Big)$$

does not depend on $\ell \in \{1, 2\}$. Assume further that $M_{I_1} \models \varphi(\ldots, \bar{b}^1_\gamma, \ldots)_{\gamma<\kappa}$, and we shall prove that $M_{I_2} \models \varphi(\ldots, \bar{b}^2_\gamma, \ldots)_{\gamma<\kappa}$; this suffices.

First note that, as $f_{I_1}, f_{I_2} \subseteq f_{({}^{\kappa\geq}\lambda)}$, necessarily $\bar{b}^1_\alpha = \bar{b}^2_\alpha$ (so call it $\bar{b}_\alpha$). Now, $M_{I_1} \models \varphi(\ldots, \bar{b}^1_\gamma, \ldots)_{\gamma<\kappa}$ means that for some $c_1 \in M_{I_1}$ we have

$$M_{I_1} \models \bigwedge_{\gamma<\kappa} \varphi[c_1, b_{\gamma,0}] \equiv \neg\varphi[c_1, b_{\gamma,1}],$$

and let $c_1 = f_1(\ldots, \eta, \ldots)_{\eta \in w[f_1]}$. Let

$$\begin{array}{rcl} J & = & \{\eta : \eta \trianglelefteq t_{\alpha,j} \text{ for some } \alpha < \kappa,\ j < \ell g(\bar{t}_\alpha)\,\} \qquad \text{and} \\ J^+_\ell & = & \{\eta : \eta \in I_\ell \text{ or } \ell g(\eta) = \kappa \text{ and } (\forall \alpha < \kappa)(\eta \restriction \alpha \in J)\,\}. \end{array}$$

By the assumption, J is $\triangleleft$–closed, $J \subseteq I_1 \cap I_2$, moreover $J^+_1 = J^+_2$. Let $\gamma < \kappa$ be minimal such that $\eta \in w[f_1] \setminus J^+ \;\Rightarrow\; \eta \restriction \gamma \notin J$, and the sequence $\langle \eta(f_1, \zeta) \restriction \gamma : \zeta < \zeta_f\rangle$ is with no repetitions and $f_1 \in \mathrm{Dom}(\mathbf{h}^\kappa_\gamma)$.

Now we can choose $\nu_\varepsilon \in I_{\eta(f_1,\zeta)\restriction\gamma}$ from I_2 such that $\eta(f_1, \zeta) \in J^+ \;\Rightarrow\; \nu_\varepsilon = \eta(f_1, \zeta)$. Let $f_2 \in \mathscr{F}_{\mathfrak{m}_\kappa}$ be such that $h^\kappa_\gamma(f_2) = h^\kappa_\gamma(f_1)$ and $\eta(f_2, \zeta) = \nu_\zeta$ for $\zeta < \zeta_{f_2} = \zeta_{f_2}$. Easily, $c_2 = f^{\mathfrak{m}_\kappa}(\ldots, \nu_\zeta, \ldots)_{\zeta<\zeta_{f_1}} \in M_{I_2}$ witness that

$$M_{I_2} \models (\exists y)[\bigwedge_{\alpha<\kappa} \varphi_\alpha(x, b_{\alpha,0}) \equiv \neg\varphi(x, b_{\alpha,1})]$$

(recalling $M_{I_1}, M_{I_2} \prec M_{({}^{\kappa\geq}\lambda)}$).

$\square_{3.1}$

Recall and add

Definition 3.2. *(1)* $\mathbf{E}^1_\alpha \in \mathbb{E}$ *(see Definition 2.8) is defined like* $\mathbf{E}^0_\alpha$ *(see Definition 2.8(3)) except that we omit clause (iv) there.*

(2) For $\alpha < \kappa$ *define* $\mathbf{E}^2_\alpha \in \mathbb{E}$ *as the following equivalence relation on* $\{\bar\nu : \bar\nu \in {}^{\kappa>}({}^\kappa\mu), \bar\nu$ *with no repetition*$\}$

$\bar\nu^1 \mathbf{E}^2_\alpha \bar\nu^2$ *iff*

(i) $\bar\nu^1, \bar\nu^2 \in {}^{\kappa>}({}^\kappa\mu)$ *are with no repetition.*

(ii) $\bar\nu^1, \bar\nu^2$ *have the same length, all it* ζ^*.

(iii) $\nu^1_\zeta \restriction \alpha = \nu^2_\zeta \restriction \alpha$ *for* $\zeta < \zeta^*$.

(iv) for every $\zeta \in {}^\alpha\mu$*, the sets* $u^\ell_\eta = \{\zeta < \zeta^* : \eta \triangleleft \nu^\ell_\zeta\}$ *are finite equal and* $\langle \nu^1_\zeta : \zeta \in u^1_\eta\rangle$, $\langle \nu^2_\zeta : \zeta \in u^2_\eta\rangle$ *are similar.*

Claim 3.3. *(1) In 2.23 we can demand that every* $\mathfrak{m}_\gamma$ *is* $\mathbf{E}^1_\gamma$*–indiscernible i.e. get the strong version.*

(2) Moreover we can get even $\mathbf{E}^2_\alpha$*-indiscernibility.*

Proof. (1) Very similar to the proof of 2.23. In fact, we need to repeat §2 with minor changes. One point is that defining "good" we use $\mathbf{E}^1_\gamma$; the second is that we should not that this indiscernibility demand is preserved in limits, this is 2.15, 2.18. In fact this is the "strongly" version which is carried in §2 the until 2.20. From then on we should replace "weakly" by "strongly" and change the definition of Λ_6, Λ_8 appropriately in the proof of 2.21.

(2) Similarly, only we need a stronger partition theorem in the end of the proof of 2.21, but it is there anyhow. $\square_{3.3}$

Remark 3.4. Clearly in many cases in 3.1, $\lambda = \lambda^{<\kappa} \geq \theta$ suffices, and it seems to me that with high probability for all. Similarly for getting many κ–resplendent models no one elementarily embeddable into another.

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