KAPLANSKY TEST PROBLEMS FOR R-MODULES IN ZFC

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Abstract. We conclude a long-standing research program in progress since 1954 by giving negative answers to test problems à la Kaplansky. Among these problems, the first was largely open, but the others were known to be consequences of Jensen's diamond principle and therefore impossible to answer affirmatively. Let $\mathbf{R}$ be a left non-pure semisimple, and let $m > 1$ be a natural number. For example, we construct an $\mathbf{R}$-module $\mathbb{M}$ such that $\mathbb{M}^n \cong \mathbb{M}$ if and only if m divides $n-1$, and thus solving the first test problem in the negative. As an application, we also construct an $\mathbf{R}$-module $\mathbb{M}$ of arbitrary size such that $\mathbb{M}^{n_1} \cong \mathbb{M}^{n_2}$ if and only if m divides $(n_1 - n_2)$, giving a strongly negative answer to the cube problem of whether an $\mathbf{R}$-module $\mathbb{M}$ which is isomorphic to $\mathbb{M}^3$ must be isomorphic to its square $\mathbb{M}^2$? We will treat the other two problems in a similar way. The crux of our method is to construct a ring $\mathbf{S}$ and an $(\mathbf{R}, \mathbf{S})$-bimodule with few endomorphisms, for which we rely heavily on techniques from algebra and set theory, in particular the black box.

Contents

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1. Introduction

Throughout this paper $\mathbf{R}$ is an associative ring with $1 = 1_{\mathbf{R}}$, which is neither necessarily commutative nor Noetherian. An $\mathbf{R}$-module $\mathbb{M}$ is a left $\mathbf{R}$-module, unless stated otherwise. Moreover, for any positive integer n, the notation $\mathbb{M}^n$ stands for $\bigoplus_{i=1}^n \mathbb{M}$. We will prove the following theorem.

Theorem 1.1. *(See Theorem 8.1) Let $\mathbf{R}$ be a ring which is not left pure semisimple. Let $m > 1$ be an integer, and let $\lambda > |\mathbf{R}|$ be a cardinal of the form $\lambda = \left(\mu^{\aleph_0}\right)^+$. Then there is an $\mathbf{R}$-module $\mathbb{M}$ of cardinality λ such that:*

$$\mathbb{M}^n \cong \mathbb{M} \Longleftrightarrow m \text{ divides } n-1.$$

Recall that $\mathbf{R}$ is called left pure semisimple if every left $\mathbf{R}$-module is a direct sum of countably generated indecomposable left $\mathbf{R}$-modules, and furthermore such a representation is unique up to isomorphism. This can be seen as a pure version of semisimple rings.

We will prove Theorem 1.1 in ZFC, the Zermelo-Fraenkel set theory with the axiom of choice. To do so, we will introduce the construction of $(\mathbf{R}, \mathbf{S})$-bimodules, i.e. the structures that are both left **R**-modules and right **S**-modules with appropriate associativity over a commutative ring **T**. Ultimately, we will choose **S** to solve each of the test questions. However, we will first analyze what the smallest endomorphism ring of an $(\mathbf{R}, \mathbf{S})$-bimodule can be. As an application, we will prove the following result.

Corollary 1.2. *(See Corollary 8.2) Assume that* $\mathbf{R}$ *is not pure semisimple. Let* $\lambda = (\mu^{\aleph_0})^+ > |\mathbf{R}|$, *and let* $m > 2$ *be an integer. Then there is an* $\mathbf{R}$*-module* $\mathbb{M}$ *of cardinality* λ *such that:*

$$\mathbb{M}^{n_1} \cong \mathbb{M}^{n_2} \Longleftrightarrow m|(n_1 - n_2).$$

This result extends several well-known theorems that began with Corner [2]. A special case of this, namely when **R** is the ring of integers, was recently reconstructed by Göbel-Herden-Shelah [21, Corollary 9.1(iii)] and by Eklof-Shelah [15].

When an algebraic theory is given, one of the main goals is to find some structure theorems for the objects. In 1954, Kaplansky formulated a list of three test problems (see [28, page 12]), where, in his opinion, a structure theory can only be satisfactory if it can solve these problems. He formulated his problems in the context of abelian groups as follows

Problem 1.3. *(I) If* $\mathbf{G}$ *is isomorphic to a direct summand of* $\mathbf{H}$ *and* $\mathbf{H}$ *is isomorphic to a direct summand of* $\mathbf{G}$*, are* $\mathbf{G}$ *and* $\mathbf{H}$ *necessarily isomorphic?*

(II) If $\mathbf{G} \oplus \mathbf{G}$ *and* $\mathbf{H} \oplus \mathbf{H}$ *are isomorphic, are* $\mathbf{G}$ *and* $\mathbf{H}$ *isomorphic?*

(III) If $\mathbf{F}$ *is finitely generated and* $\mathbf{F} \oplus \mathbf{G}$ *is isomorphic to* $\mathbf{F} \oplus \mathbf{H}$*, are* $\mathbf{G}$ *and* $\mathbf{H}$ *isomorphic?*

He also noted that the problems 1.3 can be formulated for very general mathematical systems, and he also mentioned that problem (I) in set theory has a positive answer, namely the Cantor-Schröder-Bernstein theorem. It is perhaps worth mentioning that problems 1.3 (I) and (II) were posed earlier, in 1948, independently by

Sikorski [48] and Tarski [49] in the context of Boolean algebras, where they gave a positive answer to the problem 1.3 for countably complete Boolean algebras. Sierpinski also posed the *cube problem* (see below for explanation) in the context of linear orders, see [47].

There are numerous publications on the Kaplansky test problems for various algebraic structures. Some of these are summarized below. For Boolean algebras, the first negative result was proved by Kinoshita [30], who gave a negative answer to the problem 1.3 (I) for countable Boolean algebras. [25] gave a negative answer to the problem 1.3 (II) for the same class of structures, and Ketonen [29] solved, among many other interesting things, the Tarski cube problem by producing a countable Boolean algebra which is isomorphic to its cube but not to its square. A solution to the Schroeder-Bernstein problem for Banach spaces is the subject of [24] by Gowers. Recently, Ervin [18] proved that every linear order isomorphic to its cube is also isomorphic to its square, thus solving Sierpinski's cube problem for linear orders.

It is known that Kaplansky's problems have positive answers for many classes of abelian groups, such as finitely generated groups, free groups, divisible groups, and so on, see [17]. Jónsson [26] gave negative answers to problems 1.3 (I) and (II) for countable centerless non-commutative groups and then in [27] he gave a negative answer to problem 1.3 (II) for the class of torsion-free abelian groups of rank 2. In 1961 Sasiada [35] gave a negative answer to the first problem for the class of torsion-free groups of rank $2^{\aleph_0}$. Corner, in his groundbreaking work [2], proved that any countable torsion-free reduced ring can be realized as an endomorphism ring of a torsion-free abelian group and derived a negative answer to Kaplansky's test problems (I) and (II) for countable torsion-free reduced groups. Later, Corner [3] constructed a countable torsion-free abelian group $\mathbf{G}$ which is isomorphic to $\mathbf{G}^3$ but not to $\mathbf{G}^2$, giving a negative answer to the cube problem which asks, if $\mathbf{G}$ is isomorphic to $\mathbf{G}^3$, does it follow that $\mathbf{G}$ is isomorphic to $\mathbf{G}^2$?

Let m and n be positive integers. As a further contribution, Corner proved that for every positive integer r there exists a countable torsion-free abelian group $\mathbf{G}$ such that $\mathbf{G}^m \cong \mathbf{G}^n$ if and only if $m \equiv_r n$. This property of $\mathbf{G}$ is called the *Corner pathology*. In particular, the ring of integers has a pathological module. For countable separable p-groups, Elm's theorem gives a positive solution to the Kaplansky problems (I) and (II), and Crawley [6] showed in 1965 that this does not extend to the uncountable case by giving a negative answer to these problems for the case of uncountable separable p-groups. For any $n \geq 2$, Eklof and Shelah [16] constructed a locally free abelian group $\mathbf{G}$ of cardinality $2^{\aleph_0}$ such that $\mathbf{G} \oplus \mathbb{Z}^n \cong \mathbf{G}$. More recently, Richard [34] has presented more friendly examples of abelian groups endowed with Corner's pathological property. Corner's ideas have been used by many mathematicians to give negative answers to the first two test questions for some classes of abelian groups. See for example [9], [10], [11] [15], [34], and [13].

Shelah's work [46] deals with Kaplansky's first test problem in the category of modules over a general ring, but his results were obtained under set-theoretic assumptions beyond ZFC. In fact, Shelah used Jensen's diamond principle, a prediction principle whose truth is independent of ZFC, to present rigid-like modules that give negative answers to Kaplansky's test problem.

Thomé [50] and Eklof-Shelah [15] constructed a $\aleph_1$-separable abelian group $\mathbb{M}$ in ZFC such that the Corner ring is algebraically closed in $\operatorname{End}(\mathbb{M})$. Consequently, $\mathbb{M}$ is isomorphic to its cube, but not to its square.

In summary, the Kaplansky test problems for the category of modules over a general ring have remained largely open.

In this paper we are interested in the category of modules over a general ring which is neither necessarily commutative nor countable. We work in ZFC and answer the test problems. This continues and even completes the program announced in [46], but can also be read independently. We would like to emphasize that all our results are in ZFC (without additional set-theoretic axioms) and that the results we obtain are even stronger than those proved in [46]. See for example Theorem 1.1. We will do this using a simple case of "*Shelah's black box*", see Lemma 4.47 and Theorem 4.48. Black boxes were introduced by Shelah in [43] and [44], where he shows that they follow from ZFC. We can think of black boxes as a general way to generate a class of diamond-like principles that are provable in ZFC.

In [36] Shelah proved that every ring $\mathbf{R}$ satisfies one of the following two possibilities:

(i) All modules are direct sums of countably generated modules, or
(ii) For every cardinal $\lambda > |\mathbf{R}|$, there exists an $\mathbf{R}$-module of cardinality λ that is not a direct sum of $\mathbf{R}$-modules of cardinality $\leq \mu$ for some $\mu < \lambda$.

Shelah's work in [46] extends (ii) with respect to endomorphism algebras. From this and from $\mathbf{V} = \mathbf{L}$ he constructed an $\mathbf{R}$-module $\mathbb{M}$ with prescribed endomorphisms modulo an ideal of small endomorphisms. Consequently, Shelah found a connection from (ii) to the Kaplansky test problems. Here we remove the additional assumption of $\mathbf{V} = \mathbf{L}$.

Here we lose the λ-freeness (this is unavoidable even for abelian groups, see Magidor and Shelah [31]). In particular, we prove here that for every $m > 1$ there exists an $\mathbf{R}$ module $\mathbb{M}$ such that $\mathbb{M} \cong \mathbb{M}^n$ if and only if m divides $n-1$, and we also answer the other Kaplansky test problems promised in [46]. Furthermore, we explicitly prove that the theorems hold for elementary classes of modules that are not superstable.

In the course of proving Theorem 1.1, we develop general methods that allow us to prove the following results in ZFC. Note that these results give negative solutions to the Kaplansky test problems (I) and (II).

Theorem 1.4. *Let $\mathbf{R}$ be a ring which is not pure semisimple and let $\lambda = \left(\mu^{\aleph_0}\right)^+ > |\mathbf{R}|$ be a regular cardinal. Then there are $\mathbf{R}$-modules $\mathbb{M}$, $\mathbb{M}_1$, $\mathbb{M}_2$ of cardinality λ such that* *i)* $\mathbb{M} \oplus \mathbb{M}_1 \cong \mathbb{M} \oplus \mathbb{M}_2$,

ii) $\mathbb{M}_1 \ncong \mathbb{M}_2$.

iii) $\mathbb{M}_1 \equiv_{\mathcal{L}_{\infty,\lambda}} \mathbb{M}_2$.

Here by $\mathcal{L}_{\infty,\lambda}$, we mean the infinitary language $\mathcal{L}_{\infty,\lambda}(\tau_{\mathbf{R}})$, where $\tau_{\mathbf{R}}$ is the language of modules over the ring $\mathbf{R}$.

Theorem 1.5. *Let $\mathbf{R}$ be a ring which is not pure semisimple and let $\lambda > |\mathbf{R}|$ be a regular cardinal of the form $\left(\mu^{\aleph_0}\right)^+$. Then there are $\mathbf{R}$-modules $\mathbb{M}_1$ and $\mathbb{M}_2$ of cardinality λ such that:*

i) $\mathbb{M}_1, \mathbb{M}_2$ *are not isomorphic,*

ii) $\mathbb{M}_1$ *is isomorphic to a direct summand of* $\mathbb{M}_2$*,*

iii) $\mathbb{M}_2$ *is isomorphic to a direct summand of* $\mathbb{M}_1$*.*

In particular, Theorems 1.4 and 1.5 improve the main results of [46] by removing the use of the diamond principle.

Since the Kaplansky test problems and also our results are related to the indecomposability of modules, we will also look at some background in this direction. Fuchs [17] proved the existence of an indecomposable abelian group in many cardinals λ (e.g. up to the first strongly inaccessible cardinal), and even a rigid system of 2^λ abelian groups of power λ. At the time, it was conjectured that this might fail for some "large cardinals" (e.g., supercompact). Corner [4] reduced the number of primes to five, and later Göbel and May [20] reduced it to four, in the following sense. Suppose $\mathbf{R}$ is an algebra over a commutative ring A, λ is an infinite cardinal, and suppose $\mathbf{R}$ can be generated as an A-algebra by λ or less elements. Let $M = \oplus_\lambda \mathbf{R}$ and also assume that there is an embedding $\mathbf{R} \hookrightarrow \mathrm{End}_A(M)$ by scalar multiplication. Göbel and May introduced four A-submodules M_0, M_1, M_2, M_3 of M such that

$$\mathbf{R} = \left\{\varphi \in \mathrm{End}_A(M) : \varphi(M_i) \subset M_i \textit{ for all } 0 \leq i \leq 3\right\}.$$

Using certain stationary sets, Shelah [39] proved the existence of an indecomposable abelian group of cardinality λ in every λ, and even a rigid system of 2^λ such groups. Recently, Göbel and Ziegler [23] generalized this to $\mathbf{R}$-modules for "$\mathbf{R}$ with five ideals". In response to a question from Pierce, Shelah [40] constructed essentially decomposable[1] abelian p-groups for any cardinal λ, which is strong limit of uncountable cofinality.

Eklof and Mekler [12] have obtained a λ-free indecomposable abelian group of power λ using diamond on some non-reflecting stationary set of ordinals $< \lambda$ of cofinality $\aleph_0$. Shelah [42] continued this and showed that the diamond can be replaced by the weak diamond on a non-reflecting stationary subset of

$$S^\lambda_{\aleph_0} := \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$$

(Thus,for $\lambda = \aleph_1$, $2^{\aleph_0} < 2^{\aleph_1}$ is sufficient).

Dugas [8], continuing [12], proved, assuming $\mathbf{V} = \mathbf{L}$, the existence of a strongly κ-free abelian group with endomorphism ring $\mathbb{Z}$ and then, using p-adic rings, Göbel [19] realized a larger family of rings.

Dugas and Göbel [9], continuing [8], [19], and [42], and working over a Dedekind domain $\mathbf{R}$ which is not a field proved that:

i) there exist arbitrarily large indecomposable $\mathbf{R}$-modules, and
ii) there exist arbitrarily large $\mathbf{R}$-modules that do not satisfy the Krull-Schmidt cancellation property.

Moreover, they related these to the Kaplansky test problems by showing that $\mathbf{R}$ is not a complete discrete valuation ring if and only if there are $\mathbf{R}$-modules of arbitrary high rank which do not satisfy Problem 1.3. In addition, they showed that every torsion-free ring is the endomorphism ring of a suitable abelian group.

In [10], Dugas and Göbel characterized the rings which can be represented as $\mathrm{End}(\mathbf{G})$ modulo "the small endomorphism" for some abelian p-group, but as it

[1] $\mathbf{G}$ is essentially decomposable if $\mathbf{G} = \mathbf{G}_1 \oplus \mathbf{G}_2$ implies that $\mathbf{G}_1$ or $\mathbf{G}_2$ is bounded.

follows [40] (which dealt with the case when we want the smallest such ring), the representation of a ring $\mathbf{R}$ by an abelian group $\mathbf{G}$ of size a strong limit cardinal of cofinality $> |\mathbf{R}|$. The situation is similar in Dugas and Göbel [11], where the results of [9] and more are obtained in such cardinals.

Black box allows us to get the results of [9], [10] in more and smaller cardinals, e.g. $\lambda = \left(|\mathbf{R}|^{\aleph_0}\right)^+$. Corner and Göbel [5] continued this work and, using ideas from Shelah, presented a detailed treatment of the construction of abelian groups and modules with given endomorphism rings and satisfying additional constraints. We also mention the recent work [21].

The organization of the paper and the strategy of the proof of Theorem 1.1 are presented in the next section.

For more information, see the books Eklof-Mekler [14] and Göbel-Trlifaj [22]. The books [32] and [33] by Prest discuss model theory of modules, and also pure semisimple rings.

2. Outline of the proof of Theorem 1.1

This section is divided into two subsections. In the first subsection, we present an introduction to the concept of semi-nice construction, and survey things that we need from [36]. The interested reader is referred to §4, where some definitions and details are given there. In subsection 2.2 we give an overview of the proof of Theorem 1.1.

2.1. **Towards a semi-nice construction.** In Section 4, and more formally, we develop the setting needed for our approach. We will do this by introducing the concept of semi-nice construction and its specialization and generalization. To explain it, let us first present a simple case. Let $\mathbf{R}$ be a ring with unit $1_{\mathbf{R}}$ which is not left purely semisimple. Then we consider formulas $\varphi(x)$ of the form

$$(\exists y_0, \ldots, y_{k_m})\Big[\bigwedge_{m<m(*)} a_m x = \sum_{i<k_m} b_{m,i} y_i\Big]$$

where $\{a_m, b_{m,i}\}$ are elements of the ring $\mathbf{R}$. For any left $\mathbf{R}$–module $\mathbb{M}$ we set

$$\varphi(\mathbb{M}) := \{x : \mathbb{M} \models \varphi(x)\}.$$

This is not necessarily a submodule, because $\mathbf{R}$ is not necessarily commutative, but $\varphi(\mathbb{M})$ is an additive subgroup of $\mathbb{M}$. Now, our assumption implies that for some sequence $\bar{\varphi} = \langle \varphi_n(x) : n < \omega \rangle$, each φ_n is as above. For every n and $\mathbb{M}$ we have $\varphi_{n+1}(\mathbb{M}) \subseteq \varphi_n(\mathbb{M})$, and for some $\mathbb{M}$, the sequence $\langle \varphi_m(\mathbb{M}) : m < \omega \rangle$ is strictly increasing. Without loss of generalitywe may assume that $\varphi_n(x)$ is of the following form:

$$\varphi_n(x) = (\exists\, y_0, \ldots y_{k_{n-1}-1}) \Big[\bigwedge_{m<n} a_m x = \sum_{i<k_m} b_{m,i} y_i \Big],$$

and also $k_n < k_{n+1}$.

Let $\mathbb{N}_n$ be the $\mathbf{R}$–module generated by x^n, y^n_i $(i < k_m)$ freely, except the following relations

$$a_m x^n = \sum_{i<k_n} b_{m,i} y^n_i \quad \forall m < n.$$

Let g_n be the homomorphism from $\mathbb{N}_n$ into $\mathbb{N}_{n+1}$ mapping x^n to x^{n+1} and y^n_i to y^{n+1}_i (for $i < k_n$). Let

$$\mathfrak{e} = \langle \mathbb{N}_n, x_n, g_n : n < \omega \rangle = \langle \mathbb{N}^{\mathfrak{e}}_n, x^{\mathfrak{e}}_n, g^{\mathfrak{e}}_n : n < \omega \rangle.$$

Next, for any n, we define the following subgroup of $\mathbb{M}$:

$$\varphi_n(\mathbb{M}) = \varphi^{\mathfrak{e}}_n(\mathbb{M}) := \{f(x) : f \text{ is a homomorphism from } \mathbb{N}_n \text{ to } \mathbb{M}\}.$$

In Definition 4.36 we will define the notion of "semi-nice construction", which in particular gives a sequence $\bar{\mathbb{M}}$ of $\mathbf{R}$-modules, and in Theorem 4.48 we show that the existence of a semi-nice construction follows from a suitable version of black boxs. This gives us an $\mathbf{R}$–module $\mathbb{M} = \bigcup\limits_{\alpha} \mathbb{M}_\alpha$ such that every endomorphism $\mathbf{f}$ of $\mathbb{M}$ is in a suitable sense trivial. To be more explicit, on general grounds, $\mathbf{f}$ maps $\varphi_n(\mathbb{M})$ into itself. Hence it maps

$$\varphi^{\mathfrak{e}}_\omega(\mathbb{M}) := \bigcap_{n<\omega} \varphi_n(\mathbb{M})$$

into itself. So it induces an additive endomorphism

$$\hat{\mathbf{f}}_n = \mathbf{f} \restriction (\varphi_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}))$$

of the abelian group $\varphi_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$. By the construction, for some $n = n(\mathbf{f})$, the mapping $\hat{\mathbf{f}}_n$ is of the "unavoidable" kind. Here, unavoidable means multiplication by some element a from the center of $\mathbf{R}$.

The point is that for undesirable $\mathbf{f}$, we will be able to find x_n and x such that $x - x_n \in \varphi_n(\mathbb{M})$ but for no $y \in \mathbb{M}$ we have:

$$n < \omega \quad \Rightarrow \quad y - \mathbf{f}(x_n) \in \varphi_n(\mathbb{M}).$$

In other words, omitting countable types helps. But, we would like to have more, namely not only to generalize "the abelian group $\mathbb{M}$ has no endomorphism except multiplication by $a \in \mathbb{Z}$", but also we rather want to generalize "the abelian group $(\mathbb{M}, +)$ has a prescribed endomorphism ring $\mathbf{S}$". For running this propose, we consider a fixed pair $(\mathbf{R}, \mathbf{S})$ of rings and a commutative subring $\mathbf{T}$ of the center of $\mathbf{R}$ and the center of $\mathbf{S}$, and work with $(\mathbf{R}, \mathbf{S})$-bimodules, where by an $(\mathbf{R}, \mathbf{S})$-bimodule, we mean a left $\mathbf{R}$-module $\mathbb{M}$ which is a right $\mathbf{S}$-module and satisfies the following equations:

(1) $\forall r \in \mathbf{R}$, $\forall s \in \mathbf{S}$, $(rx)s = r(xs)$,
(2) $\forall t \in \mathbf{T}$, $tx = xt$.

We would like to build a bimodule $\mathbb{M}$ such that all of its $\mathbf{R}$–endomorphisms (i.e., endomorphisms as an $\mathbf{R}$–module) are, in a sense, equal to left multiplication by a member of $\mathbf{S}$. To be able to construct such an $\mathbb{M}$, we need to have

$$\mathfrak{e} = \langle \mathbb{N}^{\mathfrak{e}}_n, x^{\mathfrak{e}}_n, g^{\mathfrak{e}}_n : n < \omega \rangle,$$

but now $\mathbb{N}^{\mathfrak{e}}_n$ is a bimodule, $x^{\mathfrak{e}}_n \in \mathbb{N}^{\mathfrak{e}}_n$ and $g^{\mathfrak{e}}_n$ is a bimodule homomorphism from $\mathbb{N}^{\mathfrak{e}}_n$ to $\mathbb{N}^{\mathfrak{e}}_{n+1}$, mapping $x^{\mathfrak{e}}_n$ to $x^{\mathfrak{e}}_{n+1}$. As a first approximation, let $\varphi^{\mathfrak{e}}_n(\mathbb{M})$ be

$$\Big\{ x : \text{ there is an } \mathbf{R}\text{-homomorphism from } \mathbb{N}^{\mathfrak{e}}_n \text{ to } \mathbb{M} \text{ mapping } x^{\mathfrak{e}}_n \text{ to } x \Big\}.$$

Of course, $\varphi^{\mathfrak{e}}_n(\mathbb{M})$ is an additive subgroup of $\mathbb{M}$. We define $\psi^{\mathfrak{e}}_n(\mathbb{M})$ similarly but using bimodule homomorphism. In fact, we consider a set $\mathfrak{E}$ of such $\mathfrak{e}$'s.

However, $\mathbb{N}^{\mathfrak{e}}_n$ is not necessarily finitely generated as an $\mathbf{R}$-module. So let us restrict ourselves to bimodules such that, locally they look like direct sums of $\mathbf{R}$-modules from some class $\mathcal{K}$. This property is denoted by $0 \leq_{\aleph_0} \mathbb{M}$. This can be represented as being $\mathcal{L}_{\infty,\aleph_0}$-equivalent to such a sum. More generally, we say $\mathbb{M}_1$ is an almost direct $\mathcal{K}$-summand of $\mathbb{M}_2$ with respect to κ denoted by $\mathbb{M}_1 \leq^{\mathrm{ads}}_{\mathcal{K},\kappa} \mathbb{M}_2$, provided player II has a winning strategy in the following game $\Game^{\mathbb{M}_1,\mathbb{M}_2}_{\mathcal{K},\kappa}$ of length ω: in the n^{th} move player I chooses a subset $A_n \subseteq \mathbb{M}_2$ of cardinality $< \kappa$, and then player II chooses a sub-bimodule $\mathbb{K}_n \subseteq \mathbb{M}_2$. Player II wins iff:

(a) $A_n \subseteq \mathbb{M}_1 + \sum\limits_{\ell \leq n} \mathbb{K}_\ell$,

(b) $\mathbb{K}_n$ is in $c\ell^{\kappa}_{\mathrm{is}}(\mathcal{K})$,

(c) $\mathbb{M}_1 + \sum\limits_{\ell<\omega} \mathbb{K}_\ell = \mathbb{M}_1 \oplus \sum\limits_{\ell<\omega} \mathbb{K}_\ell$.

For more details, see Definition 4.16. Now, if $0 \leq_{\aleph_0} \mathbb{M}$, we let

$$\varphi^{\mathfrak{e}}_n(\mathbb{M}) := \big\{x \in \mathbb{M} : \exists\ \mathbf{R}\text{–homomorphism } \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{N} \text{ mapping } x^{\mathfrak{e}}_n \text{ to } x\big\},$$

and

$$\psi^{\mathfrak{e}}_n(\mathbb{M}) := \big\{x : \exists\ \textit{bimodule homomorphism } \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{N} \text{ mapping } x^{\mathfrak{e}}_n \text{ to } x\big\}.$$

Also, our complicated set $\mathfrak{e}$ enable us to define the set $\mathbb{L}^{\mathfrak{e}}_n$ of elements of $\mathbb{N}^{\mathfrak{e}}_n$ whose image under bimodule homomorphism is determined by the image of $x^{\mathfrak{e}}_n$.

In addition, we would like to include in our framework the class of $\mathbf{R}$-modules of a fix first order complete theory $\mathcal{T}$. This is fine for $\mathcal{T}$ not to be superstable, but we need to replace the requirement $0 \leq_{\aleph_0} \mathbb{M}$ by "$\mathbb{M}^* \leq_{\aleph_0} \mathbb{M}$" and choose $\mathcal{K}$ and $\mathbb{M}^*$ appropriate for $\mathcal{T}$. For example, $\mathbb{M}^*$ can be any $\aleph_1$-saturated model of $\mathcal{T}$ and

$$\mathcal{K} := \big\{\mathbb{N} : \mathbb{N} \text{ is an } \mathbf{R}\text{–module such that } \mathbb{M}^* \prec_{\mathcal{L}(\tau_{\mathbf{R}})} \mathbb{M}^* \oplus \mathbb{N}\big\},$$

where $\mathcal{L}(\tau_{\mathbf{R}})$ is the language of $\mathbf{R}$-modules (see Definition 3.6). Also, $\mathbb{N}$ is not too large, e.g. it has size at most $\|\mathbf{R}\| + \|\mathbf{S}\|$.

2.2. **Test equations for Theorem 1.1.** Let $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ be a semi-nice construction and let $\mathbb{M}_\kappa$ be the module defined by $\bar{\mathbb{M}}$. For any $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$, $\alpha < \kappa$ and $n < \omega$, we consider the following principle: For any bimodule homomorphism $\mathbf{h}$ from $\mathbb{N}_n^{\mathfrak{e}}$ into $\mathbb{M}_\kappa$, and for every $\ell < \omega$ we have

$$\mathbf{f}(\mathbf{h}(x_n^{\mathfrak{e}})) \in \mathbb{M}_\alpha + \varphi_\ell^{\mathfrak{e}}(\mathbb{M}_\kappa) + \mathrm{Rang}(\mathbf{h}).$$

We refer to this property by saying that the statement $(\mathrm{Pr})_\alpha^n[\mathbf{f}, \mathfrak{e}]$ holds. In Lemma 5.2 we prove that every $\mathbf{R}$-endomorphism $\mathbf{f}$ of the module $\mathbb{M}_\kappa$ constructed in Section 4 is "somewhat definable" and specifically satisfies $(\mathrm{Pr})_\alpha^{n(*)}[\mathbf{f}, \mathfrak{e}]$ (for some $\alpha < \kappa$, $n(*) < \omega$ and for all the $\mathfrak{e}$'s we have taken care of). We regard this as a key stone of §4. Despite its importance, $(\mathrm{Pr})_\alpha^{n(*)}[\mathbf{f}, \mathfrak{e}]$ is not enough strong to run our program. However, we show that $(\mathrm{Pr})_\alpha^{n(*)}[\mathbf{f}, \mathfrak{e}]$ implies a stronger version, denoted by $(\mathrm{Pr}\,1)_{\alpha,z}^{n(*)}[\mathbf{f}, \mathfrak{e}]$. As the notation suggests, the new invariant z is involved. More explicitly, if h is a bimodule homomorphism from $\mathbb{N}_n^{\mathfrak{e}}$ into $\mathbb{M}_\kappa$, then $(\mathrm{Pr}\,1)_{\alpha,z}^{n(*)}[\mathbf{f}, \mathfrak{e}]$ implies that

$$\mathbf{f}(h(x_n^{\mathfrak{e}})) - h(z) \in \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\kappa).$$

This is subject of Lemma 5.8. These enable us to connect to subsection 1.1, by defining the concept of strongly semi-nice construction. For more details, see Definition 5.16.

In Section 6 we try to say more. Working with $\mathbb{M}_\kappa$, every endomorphism is in some suitable sense equal to one in a ring dE, whose definition is given in Lemma 6.24(5b). This means, we restrict $\mathbf{f}$ to an additive subgroup $\varphi_n^{\mathfrak{e}}(\mathbb{M}_\kappa)$ closed under $\mathbf{f}$, and divide it by another $\varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\kappa)$, then take direct limit; on the top of this we have an "error term". To handle this, we have to divide by a "small" submodule of $\mathbb{M}_\kappa$, which means of cardinality $< \lambda = \|\mathbb{M}_\kappa\|$. In order to get a stronger result than this, we divide the ring of such endomorphisms by the ideal of those with "small" range and then even "compact ones" which are essentially of cardinality

$\leq \sup\{\|\mathbb{M}\| : \mathbb{M} \in \mathcal{K}\}$. More explicitly, see Lemma 6.25. We close Section 6 by proving the following auxiliary result.

Corollary 2.1. *Suppose* $\mathbf{S}$ *is a ring extending* $\mathbb{Z}$ *such that* $(\mathbf{S}, +)$ *is free, and let* $\mathbf{R}$ *be a ring which is not pure semisimple. Let* $\mathbf{D}$ *be a field such that*

$$p := \operatorname{char}(\mathbf{D}) | \operatorname{char}(\mathbf{R})$$

and set

$$\mathbb{Z}_p := \begin{cases} \mathbb{Z}/p\mathbb{Z} & p > 0, \\ \mathbb{Z} & otherwise. \end{cases}$$

Suppose Σ *is the set of equations over* $\mathbf{S}$ *which is not solvable in* $\mathbf{D} \otimes_{\mathbb{Z}_p} (\mathbf{S}/p\mathbf{S})$. *Finally, let* $\mathbb{M}$ *be strongly nicely constructed. Then* Σ *is not solvable in* $\operatorname{End}(\mathbb{M})$.

In Section 7 we present the proof of theorems 1.5 and 1.4. It may be worth to mention that 1.5 follows directly from Theorem 1.1. In sum, we drive it by applying different test equations.

Let us now explain how the above constructions can be applied to prove Theorem 1.1 (for more details see Section 8). We first introduce a ring $\mathbf{S}_0$, it is incredibly easy compared to $\mathbf{S}$. To this end, let $\mathbf{T}$ be the subring of $\mathbf{R}$ which 1 generates. Let $\mathbf{S}_0$ be the ring extending $\mathbf{T}$ generated by $\{\mathcal{X}_0, \ldots, \mathcal{X}_{m(*)}, \mathcal{W}, \mathcal{Z}\}$ freely except the following list of *test equations:*

$(*)_1$: $\mathcal{X}_\ell^2 = \mathcal{X}_\ell$,

$\mathcal{X}_\ell \mathcal{X}_m = 0$ *($\ell \neq m$),*

$1 = \mathcal{X}_0 + \ldots + \mathcal{X}_{m(*)}$,

$\mathcal{X}_\ell \mathcal{W} \mathcal{X}_m = 0$ *for* $\ell + 1 \neq m \mod m(*) + 1$,

$\mathcal{W}^{m(*)+1} = 1$,

$\mathcal{Z}^2 = 1$,

$\mathcal{X}_0 \mathcal{Z}(1 - \mathcal{X}_0) = \mathcal{X}_0 \mathcal{Z}$,

$(1 - \mathcal{X}_0)\mathcal{Z}\mathcal{X}_0 = (1 - \mathcal{X}_0)\mathcal{Z}$.

For an integer m let $[-\infty, m) := \{n : n$ is an integer and $n < m\}$. To each $\eta \in {}^{[-\infty,m)}\omega$ we assign

$$\eta \restriction k := \eta \restriction [-\infty, \min\{m, k\}).$$

The next object is Γ. It consists all functions with domain of the form $[-\infty, n)$ and the range is a subset of $\{1, \dots, m(*)\}$. Now, we look at

$$W_1 := W_0 \times \{0, \dots, m(*)\},$$

where

$$W_0 := \Big\{\eta \in \Gamma : \eta(m) = 1 \text{ for every small enough } m \in \mathbb{Z}\Big\}.$$

Let $\mathbf{D}$ be a field such that

$$\mathbf{T} \text{ is finite} \quad \Rightarrow \quad \mathrm{char}(\mathbf{D}) \text{ divides } |\mathbf{T}|.$$

So, $\mathbf{D} \otimes \mathbf{S}$ is the ring extending $\mathbf{D}$ by adding $\mathbf{D}, \mathcal{X}_0, \dots, \mathcal{X}_{m(*)}, \mathcal{W}, \mathcal{Z}$ as non commuting variables freely except satisfying the equations in $(*)_1$. Let $\mathbb{M}^* = {}_{\mathbf{D}}\mathbb{M}^*$ be the left $\mathbf{D}$-module freely generated by $\{x_{\eta,\ell} : \eta \in W_0,\ \ell < m(*)+1\}$. We make ${}_{\mathbf{D}}\mathbb{M}^*$ to a right $(\mathbf{D} \otimes \mathbf{S}_0)$-module by defining xz for $x \in {}_{\mathbf{D}}\mathbb{M}^*$ and $z \in \mathbf{S}_0$, so it is enough to deal with

$$z \in \{\mathcal{X}_m : m < m(*) + 1\} \cup \{\mathcal{Z}, \mathcal{W}\}.$$

Let $x = \sum\limits_{\eta,\ell} a_{\eta,\ell} x_{\eta,\ell}$ where

(1) *(η, ℓ) vary on W_0,*

(2) *$a_{\eta,\ell} \in \mathbf{D}$ and $\{(\eta, \ell) : a_{\eta,\ell} \neq 0\}$ is finite.*

It is natural to extend things linearly, that is

$$\big(\sum_{\eta,\ell} a_{\eta,\ell} x_{\eta,\ell}\big) z = \sum_{\eta,\ell} a_{\eta,\ell} (x_{\eta,\ell} z),$$

where

$$x_{\eta,\ell}\mathcal{X}_m \quad := \quad \begin{cases} x_{\eta,\ell} & \text{if } \ell = m, \\ 0 & \text{if } \ell \neq m, \end{cases}$$

$$x_{\eta,\ell}\mathcal{Z} \quad := \quad \begin{cases} x_{\eta^\frown\langle\ell\rangle,0} & \text{if } \ell > 0, \\ x_{\eta\restriction[-\infty,n-1),\eta(n-1)} & \text{if } \ell = 0, \text{ and } (-\infty,n) = \mathrm{Dom}(\eta). \end{cases}$$

Also,

$$x_{\eta,\ell}\mathcal{W} := x_{\eta,m} \quad \text{when } m = \ell+1 \mod m(*)+1.$$

These assignments allow us to deal with an additional structure of the $(\mathbf{D},\mathbf{S})$*-bimodule. Indeed, it is enough to look at*

$${}_{\mathbf{D}}\mathbb{M}^*_\ell := \Big\{ \sum_\eta d_{\eta,\ell} x_{\eta,\ell} : \ \eta \in W_0 \text{ and } d_{\eta,\ell} \in \mathbf{D} \Big\}.$$

In other words,

$${}_{\mathbf{D}}\mathbb{M}^* = \bigoplus_{\ell=0}^{m(*)} {}_{\mathbf{D}}\mathbb{M}^*_\ell.$$

Now, we are ready to define $\mathbf{S}$*, but in addition* $\sigma = 0$ *when* $\mathbb{M}^*_{\mathbf{D}}\sigma = 0$ *for every* $\mathbf{D}$ *and every* $(\mathbf{D},\mathbf{S}_0)$*-bimodule* ${}_{\mathbf{D}}\mathbb{M}^*$ *as defined above.*

We will prove that $\mathbf{S}$ *is a free* $\mathbf{T}$*-module. This allows us to apply Corollary 2.1. Then, we look at* $\mathbb{P} := \mathbb{M}_\kappa$*, the bimodule obtained by the semi-nice construction that equipped with the strong property presented in above. We define* $\mathbb{P}_\ell := \mathbb{P}\mathcal{X}_\ell$*. It follows easily that*

$$\bigoplus_{\ell=1}^{m(*)} {}_{\mathbf{R}}\mathbb{P}_\ell \cong ({}_{\mathbf{R}}\mathbb{P}_0)^{m(*)}.$$

It is enough to show ${}_{\mathbf{R}}\mathbb{P}_0^k \ncong {}_{\mathbf{R}}\mathbb{P}_0$ *for all* $1 < k < m(*)$*. Assume* k *is a counterexample. We apply Corollary 2.1 to find a field* $\mathbf{D}$ *and* $\mathcal{Y} \in \mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}$ *satisfying the following equations:*

$$(*)_2 : \mathcal{Y} \restriction {}_{\mathbf{D}}\mathbb{M}^*_0 \text{ is an isomorphism from } {}_{\mathbf{D}}\mathbb{M}^*_0 \text{ onto } \bigoplus_{\ell=1}^{k} {}_{\mathbf{D}}\mathbb{M}^*_\ell,$$
$$\mathcal{Y} \restriction \bigoplus_{\ell=1}^{k} {}_{\mathbf{D}}\mathbb{M}^*_\ell \text{ is an isomorphism from } \bigoplus_{\ell=1}^{k} {}_{\mathbf{D}}\mathbb{M}^*_\ell \text{ onto } {}_{\mathbf{D}}\mathbb{M}^*_0,$$

$$\mathcal{Y} \restriction \bigoplus_{\ell=k+1}^{m(*)+1} {}_{\mathbf{D}}\mathbb{M}^*_\ell \text{ is the identity,}$$
$$\mathcal{Y}^2 = 1.$$

The next task is to introduce the following sets:

$$\begin{aligned}
w_{\eta,\ell} &:= \big\{(\nu,m) | (\nu,m) = (\eta,\ell) \text{ or } \eta^\frown\langle\ell\rangle \trianglelefteq \nu \text{ and } m < m(*)+1\big\},\\
u_\ell &:= \big\{(\eta,m) | m = \ell,\ \eta \in w_0\big\},\\
w^m_{\eta,\ell} &:= \ w_{\eta,\ell} \cap u_m,\\
w^{[1,n]}_{\eta,\ell} &:= w_{\eta,\ell} \cap \bigcup_{m\in[1,n]} u_m.
\end{aligned}$$

For any

$$u \subseteq \big\{(\eta,\ell) : \eta \in W_0,\ \ell < m(*)+1\big\},$$

we define $\mathbb{N}_u$ *be the* $\mathbf{D}$*-subspace generated by* $\{x_{\eta,\ell} : (\eta,\ell) \in u\}$. *For every large enough finite subset* $v \subseteq w_{\eta,\ell}$, *we show the following is well defined:*

$$\mathbf{n}_{\eta,\ell} := \dim\left(\frac{\mathbb{N}_{w^0_{\eta,\ell}}}{\mathbb{N}_{w^0_{\eta,\ell}\setminus v}}\right) - \dim\left(\frac{\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}}{\mathbb{N}_{w^0_{\eta,\ell}\setminus v}\mathcal{Y}}\right).$$

These integers are independent from the first factor: Suppose η, $\nu \in w_0$ *are given. We show*

$$\mathbf{n}_{\eta,\ell} = \mathbf{n}_{\nu,\ell} \quad \forall \ell \in [0, m(*)],$$

so we shall denote these common quantities by $\mathbf{n}_\ell$. *These numbers are such that they satisfy the following equations:*

$$\mathbf{n}_\ell = \begin{cases} 0 & \text{if } \ell \in [1,k] \\ \mathbf{n}_0 + \mathbf{n}_1 + \ldots + \mathbf{n}_{m(*)} & \text{if } \ell \in [k+1, m(*)+1) \text{ or } \ell = 0 \end{cases}$$

Finally, we use this equation to derive the following contradiction

$$\sum_{\ell=1}^{m(*)} \mathbf{n}_\ell = \frac{k}{m(*)}.$$

3. Preliminaries from algebra and logic

In this section we provide some preliminaries from algebra and logic that are needed for the rest of the paper. Let us start by fixing some notations that we will use through the paper. Recall that $\mathbf{R}$ *is a ring, not necessary commutative, with* $1 = 1_{\mathbf{R}}$.

Notation 3.1. *By* $\mathrm{Cent}(-)$ *we mean the center of a ring* $(-)$.

Convention 3.2. *The rings* $\mathbf{S}$ *and* $\mathbf{R}$ *are with* 1*, and we always assume that* $\mathbf{T} := \mathrm{Cent}(\mathbf{R}) \cap \mathrm{Cent}(\mathbf{S})$ *is a commutative ring.*[2]

Definition 3.3. *An* $(\mathbf{R}, \mathbf{S})$*-bimodule* $\mathbb{M}$ *is a left* $\mathbf{R}$*-module and right* $\mathbf{S}$*-module such that for all* $r \in \mathbf{R}, x \in \mathbb{M}$ *and* $s \in \mathbf{S}$ *we have* $(rx)s = r(xs)$ *and that* $tx = xt$ *for all* $t \in \mathbf{T}$*. When the pair* $(\mathbf{R}, \mathbf{S})$ *is clear from the context, we refer to* $\mathbb{M}$ *as a bimodule.*

Convention 3.4. *We use* $\mathbb{K}$*,* $\mathbb{M}$*,* $\mathbb{N}$ *and* $\mathbb{P}$ *to denote bimodules (or left* $\mathbf{R}$*-modules).*

Definition 3.5. *Let* $\mathbf{f} : \mathbb{M} \to \mathbb{N}$ *be a bimodule homomorphism.*

1. *The kernel,* $\mathrm{Ker}(\mathbf{f}) := \{x \in \mathbb{M} : \mathbf{f}(x) = 0\}$ *is a sub-bimodule of* $\mathbb{M}$.
2. *The image,* $\mathrm{Rang}(\mathbf{f}) := \{\mathbf{f}(x) : x \in \mathbb{M}\}$ *is a sub-bimodule of* $\mathbb{N}$.
3. *If* $\mathbb{N}$ *is a sub-bimodule of* $\mathbb{M}$ *then* $\mathbb{M}/\mathbb{N} := \{x+\mathbb{N} : x \in \mathbb{M}\}$ *is a homomorphic image of* $\mathbb{M}$*. The mapping* $x \mapsto x + \mathbb{N}$ *is a homomorphism with kernel* $\mathbb{N}$.
4. *For a bimodule* $\mathbb{M}, \mathrm{End}_{\mathbf{R}}(\mathbb{M})$ *is the endomorphism ring of* $\mathbb{M}$ *as a left* $\mathbf{R}$*-module.*

In this paper we also consider modules and bimodules as logical structures, so let us fix a language for them. We only sketch the basic definitions and results which are needed here, and refer to [1] *and* [7] *for further information.*

[2] Indeed, in our applications we have $\mathbf{T} = \mathbf{R} \cap \mathbf{S}$ (see Section 6, below).

Definition 3.6. *(1) An $\mathbf{R}$-module $\mathbb{M}$ is considered as a $\tau_{\mathbf{R}}$-structure. In other words,*

$$\tau_{\mathbf{R}} = \{0, +, -\} \cup \{{}_rH : r \in \mathbf{R}\},$$

where the universe is the set of elements of $\mathbb{M}$, and $0, +, -$ are interpreted naturally and ${}_rH$ is interpreted as a multiplication from left by r, i.e., ${}_rH(x) := rx$. Let us recall terms and atomic formulas:

(a) Terms of $\tau_{\mathbf{R}}$ are expressions of the form $\sum_{i<m} r_i x_i$, where $m < \omega$, r_i is in $\mathbf{R}$ and x_i is a variable.

(b) Atomic formulas of $\tau_{\mathbf{R}}$ are equations of the form $t_1 = t_2$, where t_1, t_2 are terms.

(2) An $(\mathbf{R}, \mathbf{S})$-bimodule is similarly interpreted as a $\tau_{(\mathbf{R},\mathbf{S})}$-structure where

$$\tau_{(\mathbf{R},\mathbf{S})} := \{0, +, -\} \cup \{{}_rH : r \in \mathbf{R}\} \cup \{H_s : s \in \mathbf{S}\}$$

and ${}_rH, H_s$ for $r \in \mathbf{R}, s \in \mathbf{S}$ are unary function symbol, which will be interpreted as follows: ${}_rH$ may regard as a left multiplication by r. In other words, ${}_rH(x) := rx$. Similarly, H_s stands for right hand side multiplication by s, i.e., $H_s(x) = xs$. Here, we recall terms and atomic formulas:

(a) Terms of $\tau_{(\mathbf{R},\mathbf{S})}$ are expressions of the form

$$\sum_{i<d} \sum_{h<h_d} r_{i,h} x_i s_{i,h}$$

where $d, h_d < \omega$, $r_{i,h} \in \mathbf{R}$, $s_{i,h} \in \mathbf{S}$ and x_i is a variable.

(b) Atomic formulas of $\tau_{(\mathbf{R},\mathbf{S})}$ are equations of the form $t_1 = t_2$, where t_1, t_2 are terms.

The next lemma is trivial.

Lemma 3.7. *The class of $(\mathbf{R}, \mathbf{S})$-bimodules is a variety, i.e., there are equations in the language of $\tau_{(\mathbf{R},\mathbf{S})}$ such that a $\tau_{(\mathbf{R},\mathbf{S})}$-structure is an $(\mathbf{R}, \mathbf{S})$-bimodule iff it satisfies these equations.*

Let us now introduce the infinitary languages for modules and bimodules. They will play essential roles in the sequel.

Definition 3.8. *Suppose κ and μ are infinite cardinals, which we allow to be ∞, and let τ be one of $\tau_{\mathbf{R}}$ or $\tau_{(\mathbf{R},\mathbf{S})}$. The infinitary language $\mathcal{L}_{\mu,\kappa}(\tau)$ is defined so as its vocabulary is the same as τ, it has the same terms and atomic formulas as in τ, but we also allow conjunction and disjunction of length less than μ, i.e., if ϕ_j, for $j < \beta < \mu$ are formulas, then so are $\bigvee_{j<\beta} \phi_j$ and $\bigwedge_{j<\beta} \phi_j$. Also, quantification over less than κ many variables (i.e., if $\phi = \phi((v_i)_{i<\alpha})$, where $\alpha < \kappa$, is a formula, then so are $\forall_{i<\alpha} v_i \phi$ and $\exists_{i<\alpha} v_i \phi$).*

Note that $\mathcal{L}_{\omega,\omega}(\tau)$ is just the first order logic with vocabulary τ. Given κ, μ and τ as above, we are sometimes interested in some special formulas from $\mathcal{L}_{\mu,\kappa}(\tau)$.

Definition 3.9.
1. *$\mathcal{L}^{cpe}_{\mu,\kappa}(\tau)$, the class of conjunctive positive existential formulas, consists of those formulas of $\mathcal{L}_{\mu,\kappa}(\tau)$ which in their formulation only $\wedge, \bigwedge_{j<\beta}, \exists x$ and $\exists_{i<\alpha} v_i$ are used (where $\beta < \mu$ and $\alpha < \kappa$).*
2. *$\mathcal{L}^{pe}_{\mu,\kappa}(\tau)$, the class of positive existential formulas, is defined similarly where we also allow $\vee$ and $\bigvee_{j<\beta}$*
3. *$\mathcal{L}^{p}_{\mu,\kappa}(\tau)$, the class of positive formulas, is defined similarly where we allow $\vee$, $\bigvee_{j<\beta}$ and also the universal quantifiers $\forall x$ and $\forall_{i<\alpha} v_i$.*
4. *By a simple formula of $\mathcal{L}_{\mu,\kappa}(\tau)$, we mean a formula of the form*
$$\phi = \exists_{i<\alpha} v_i [\bigwedge_{j<\beta} \phi_j],$$
where each $\phi_j = \phi_j((v_i)_{i<\alpha})$ is an atomic formula.

Lemma 3.10. *The following assertions are valid:*

1. *Suppose $\mathbf{f} : \mathbb{M} \to \mathbb{N}$ is an $(\mathbf{R}, \mathbf{S})$-bimodule homomorphism. Then $\mathbf{f}$ preserves $\mathcal{L}^{pe}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$-formulas.*
2. *Suppose in addition to 1) that f is surjective. Then $\mathbf{f}$ preserves $\mathcal{L}^{p}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$-formulas.*

Proof. (1). Suppose $\phi((v_i)_{i<\alpha}) \in \mathcal{L}^{\mathrm{pe}}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$ and let $a_i \in \mathbb{M}$, where $i < \alpha$. We need to show:

$$\mathbb{M} \models \phi((a_i)_{i<\alpha}) \Rightarrow \mathbb{N} \models \phi((\mathbf{f}(a_i))_{i<\alpha}).$$

This can be proved by induction on the complexity of the formulas, and we leave its routine check to the reader.

(2). This can be proved in a similar way; let us only consider the case of the universal formula to show how the surjectivity of the function $\mathbf{f}$ is used. To this end, we assume that

$$\phi((v_i)_{i<\alpha}) = \forall x \psi((v_i)_{i<\alpha}, x) \in \mathcal{L}^{\mathrm{pe}}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$$

and let $a_i \in \mathbb{M}$, where $i < \alpha$. We may assume that the lemma holds for ψ. Suppose $\mathbb{M} \models \phi((a_i)_{i<\alpha})$. We are going to show that

$$\mathbb{N} \models \phi((\mathbf{f}(a_i))_{i<\alpha}).$$

To see this, let $b \in \mathbb{N}$. As $\mathbf{f}$ is surjective, there exists $a \in \mathbb{M}$ such that $\mathbf{f}(a) = b$. Due to the assumption, we know that $\mathbb{M} \models \psi((a_i)_{i<\alpha}, a)$. By the induction hypothesis,

$$\mathbb{N} \models \psi((\mathbf{f}(a_i))_{i<\alpha}, \mathbf{f}(a)),$$

i.e., $\mathbb{N} \models \psi((\mathbf{f}(a_i))_{i<\alpha}, b)$. Since b was arbitrary we have $\mathbb{N} \models \phi((\mathbf{f}(a_i))_{i<\alpha})$. $\square$

Remark 3.11. *Note that there is no need for* $\mathbf{f}$ *to preserve* $\neg$ *formulas. But if* $\mathbf{f}$ *is an isomorphism, then it preserves all formulas.*

The next lemma shows that under suitable conditions, we can replace cpe*-formulas by simple formulas.*

Lemma 3.12. *Let* τ *be either* $\tau_{\mathbf{R}}$ *or* $\tau_{(\mathbf{R},\mathbf{S})}$*, and suppose* $\mu_1 \geq \mu, \kappa$ *is regular. If* $\varphi(\bar{x}) \in \mathcal{L}^{\mathrm{cpe}}_{\mu,\kappa}(\tau)$*, then we can find as equivalent simple formula in* $\mathcal{L}_{\mu_1,\mu_1}(\tau)$*.*

We are also interested in definable subsets of (bi)modules.

Definition 3.13. *Let τ be one of $\tau_{\mathbf{R}}$ or $\tau_{(\mathbf{R},\mathbf{S})}$. Given a τ-structure $\mathbb{M}$ and a formula $\phi(x_0,\cdots,x_{n-1})$ in $\mathcal{L}_{\infty,\infty}(\tau)$, let*

$$\phi(\mathbb{M})=\big\{\langle a_0,\cdots,a_{n-1}\rangle\in {}^{n}\mathbb{M}: M\models\phi(a_0,\cdots,a_{n-1})\big\}.$$

Here, by $\lg(-)$ we mean the length function.

Lemma 3.14. *Let τ be either $\tau_{\mathbf{R}}$ or $\tau_{(\mathbf{R},\mathbf{S})}$, and let $\varphi(\bar{x})\in\mathcal{L}^{\mathrm{p}}_{\mu,\kappa}(\tau)$. Suppose $\bar{z}_\ell\in {}^{\lg(\bar{x})}\mathbb{M}_\ell$, for $\ell=1,2$ and $\bar{z}_\ell=\langle z^\ell_i : i<\lg(\bar{x})\rangle$, $\mathbb{M}=\mathbb{M}_1\oplus\mathbb{M}_2$, and $\bar{z}=\langle z_i : i<\lg(\bar{x})\rangle$ where $\mathbb{M}\models z_i=z^1_i+z^2_i$. Then*

$$\mathbb{M}\models\varphi(\bar{z})\quad\Leftrightarrow\quad\bigwedge_{\ell=1}^{2}\mathbb{M}_\ell\models\varphi(\bar{z}_\ell).$$

Furthermore, if for $\ell=1,2$ and $i<\lg(\bar{x}), z^\ell_i=0_{M_\ell}$, then $\mathbb{M}_\ell\models\varphi(\bar{z}_\ell)$ and $\bar{z}=\bar{0}_{\lg(\bar{z})}$.

Proof. The desired claim follows by an easy induction on the complexity of the formula $\varphi(\bar{x})$. $\square$

The above lemma implies if $\varphi(x)\in\mathcal{L}^{\mathrm{cpe}}_{\mu,\kappa}(\tau)$, where τ is $\tau_{\mathbf{R}}$ or $\tau_{(\mathbf{R},\mathbf{S})}$ and if $\mathbb{M}=\mathbb{M}_1\oplus\mathbb{M}_2$, then $\varphi(\mathbb{M})=\varphi(\mathbb{M}_1)\oplus\varphi(\mathbb{M}_2)$.

Lemma 3.15. *For each bimodule $\mathbb{M}$, the following assertions hold:*

1. *If $\varphi(x)\in\mathcal{L}^{\mathrm{cpe}}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$, then $\varphi(\mathbb{M})$ is a subgroup of $\mathbb{M}$.*
2. *If $\varphi(x)\in\mathcal{L}^{\mathrm{cpe}}_{\infty,\infty}(\tau_{\mathbf{R}})$, then $\varphi(\mathbb{M})$ is a right $\mathbf{S}$-submodule, but not necessarily an $\mathbf{R}$-submodule. Furthermore, if $\mathbf{R}$ is commutative, then it is an $\mathbf{R}$-submodule as well.*

Proof. This follows by induction on the complexity of $\varphi(x)$. The straightforward details are leave to the reader. $\square$

Another notion from model theory which is of importance for us is the notion of omitting types:

Discussion 3.16. *Suppose τ is a countable languages and suppose $\mathcal{M}$ is a τ-structure.*

(1) *Given $A \subseteq \mathcal{M}$, by an n-type over A we mean a set $p(v_1, \cdots, v_n)$ of formulas, whose free variables are $v_1, \cdots, v_n$ such that every finite $p_0 \subseteq p$ is realized, i.e., there are $x_1, \cdots, x_n$ in $\mathcal{M}$ such that for all $\varphi \in p_0$, $\mathcal{M} \models \varphi(x_1, \cdots, x_n)$.*

(2) *A type is complete if it is maximal under inclusion. By the axiom of choice each type can be extended into a complete type.*

(3) *The type p is isolated by some formula $\psi(v_1, \cdots, v_n) \in p$ if for every $\varphi \in p$,*

$$\mathcal{M} \models \forall x_1 \cdots \forall x_n (\psi(x_1, \cdots x_n) \to \varphi(x_1, \cdots, x_n)).$$

It is clear that if p is isolated by some formula $\psi(v_1, \cdots, v_n) \in p$, then it is realized. The omitting types theorem says that the converse is also true.

Lemma 3.17. *Let τ be a first order countable vocabulary and let T be a complete τ-theory. If p is a complete type which is not isolated, then there is a countable τ-structure $\mathcal{M} \models T$ which omits (i.e., does not realize) p.*

Definition 3.18. *Suppose that $\mathbb{M}_0, \mathbb{M}_1$ and $\mathbb{M}_2$ are (bi)modules and $\mathbf{g}_i : \mathbb{M}_0 \to \mathbb{M}_i$ are (bi)module homomorphisms for $i = 1, 2$. The amalgamation of $\mathbb{M}_1$ and $\mathbb{M}_2$ along $\mathbb{M}_0$ is $\mathbb{M} := \frac{\mathbb{M}_1 \oplus \mathbb{M}_2}{(\mathbf{g}_1(m), -\mathbf{g}_2(m) : m \in \mathbb{M}_0)}$. There are natural maps $\mathbf{f}_i : \mathbb{M}_i \to \mathbb{M}$ such that the following diagram is commutative:*

$$\begin{tikzcd}
\mathbb{M}_1 \arrow[r] & \mathbb{M} \\
\mathbb{M}_0 \arrow[u] \arrow[r] & \mathbb{M}_2 \arrow[u]
\end{tikzcd}$$

It may be nice to note that the common notation for this is $\mathbb{M}_1 +_{\mathbb{M}_0} \mathbb{M}_2$.

Definition 3.19. *Let* $\mathbb{M}$ *be an* $\mathbf{R}$*-module and* $\mathbf{S}$ *be any subring of* $\mathrm{End}_{\mathbf{R}}(\mathbb{M})$. *We say that* $\mathbf{S}$ *is algebraically closed in* $\mathrm{End}_{\mathbf{R}}(\mathbb{M})$ *if every finite system of equations in several variables over* $\mathbf{S}$ *which has a solution in* $\mathrm{End}_{\mathbf{R}}(\mathbb{M})$*, also has a solution in* $\mathbf{S}$

Let us recall the definition of pure semisimple rings.

Definition 3.20. *A ring* $\mathbf{R}$ *is left pure semisimple if every left* $\mathbf{R}$*-module is pure-injective.*

The next theorem gives several equivalent formulations for the above notion.

Theorem 3.21. *(See* [33, Theorem 4.5.7]*) The following conditions on a ring* $\mathbf{R}$ *are equivalent:*

- *(a)* $\mathbf{R}$ *is left pure semisimple;*
- *(b) every left* $\mathbf{R}$*-module is a direct sum of indecomposable modules;*
- *(c) there is a cardinal number* κ *such that every left* $\mathbf{R}$*-module is a direct sum of modules, each of which is of cardinality less than* κ*;*
- *(d) there is a cardinal number* κ *such that every left* $\mathbf{R}$*-module is a pure submodule of a direct sum of modules each of cardinality less than* κ.

The following result of Shelah gives us a model theoretic criteria for rings which are not pure semisimple, and plays an important role in this paper.

Theorem 3.22. *(Shelah,* [36, 8.7]*) Let* $\mathbf{R}$ *be a ring which is not left pure semisimple. Then there is a bimodule* $\mathbb{M}$ *and there is a sequence* $\bar{\varphi} = \langle \varphi_n(x) : n < \omega \rangle$ *such that:*

- *(a) each* φ_n *is a conjunctive positive existential formula, and*
- *(b) the sequence* $\langle \varphi_n(\mathbb{M}) : n < \omega \rangle$ *is strictly decreasing.*

4. Developing the framework of the construction

In this section we develop some part of the theory that we need for our construction. The main result of this section is Theorem 4.48, which gives, in ZFC, a semi-nice construction, that plays an essential role in the next sections.

Definition 4.1. *Let* $(\mathbf{R}, \mathbf{S})$ *be as Convention 3.2.*

(1) *Let* $\mathfrak{E}_{(\mathbf{R},\mathbf{S})} = \mathfrak{E}(\mathbf{R}, \mathbf{S})$ *be the class of all*

$$\mathfrak{e} = \langle \mathbb{N}_n, x_n, g_n : n < \omega \rangle$$

where:

(a) $\mathbb{N}_n$ *is an* $(\mathbf{R}, \mathbf{S})$*-bimodule,*

(b) $x_n \in \mathbb{N}_n$,

(c) g_n *is a bimodule homomorphism from* $\mathbb{N}_n$ *to* $\mathbb{N}_{n+1}$ *mapping* x_n *to* x_{n+1}.

Given $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ *we denote it by* $\mathfrak{e} = \langle \mathbb{N}^{\mathfrak{e}}_n, x^{\mathfrak{e}}_n, g^{\mathfrak{e}}_n : n < \omega \rangle$.

(2) *We call* $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ *non-trivial if for every* n *there is no homomorphism from* $\mathbb{N}^{\mathfrak{e}}_{n+1}$ *to* $\mathbb{N}^{\mathfrak{e}}_n$ *as* $\mathbf{R}$*-modules mapping* $x^{\mathfrak{e}}_{n+1}$ *to* $x^{\mathfrak{e}}_n$.

(3) *Let* $\mathfrak{e} = \langle \mathbb{N}_n, x_n, g_n : n < \omega \rangle \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$. *For each* $n \leq m$, *we define:*

$$g_{n,m} := g_n \circ g_{n+1} \circ \ldots \circ g_{m-1}.$$

We set $g_{n,n} := \mathrm{id}_{\mathbb{N}_n}$ *and note that* $n_0 \leq n_1 \leq n_2$ *implies* $g_{n_1,n_2} \circ g_{n_0,n_1} = g_{n_0,n_2}$. *We denote* $g_{n,m}$ *by* $g^{\mathfrak{e}}_{n,m}$.

(4) *For* $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ *and an infinite* $\mathcal{U} \subseteq \omega$, *let* $\mathfrak{e}' =: \mathfrak{e} \restriction \mathcal{U}$, *the restriction of* $\mathfrak{e}$ *to* $\mathcal{U}$, *be defined by*

(a) $\mathbb{N}^{\mathfrak{e}'}_\ell := \mathbb{N}^{\mathfrak{e}}_{m(\ell)}$,

(b) $x^{\mathfrak{e}'}_\ell := x^{\mathfrak{e}}_{m(\ell)}$,

(c) $g^{\mathfrak{e}'}_\ell := g^{\mathfrak{e}}_{m(\ell),m(\ell+1)}$,

where $m(\ell)$ *is the* ℓ*–th member of* $\mathcal{U}$. *Clearly* $\mathfrak{e}'$ *is in* $\mathfrak{E}_{(\mathbf{R},\mathbf{S})}$.

Let us define a special sub-class of $\mathfrak{E}_{(\mathbf{R},\mathbf{S})}$.

Definition 4.2. *Let $\mathfrak{E}_{\mu,\kappa}$ be the class of $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ such that for each $n < \omega$, $\mathbb{N}^{\mathfrak{e}}_n$, as a bimodule, is generated by $< \kappa$ elements freely except satisfying $< \mu$ equations. This means, there are $(x_i)_{i<\alpha<\kappa}$ and atomic formulas $t_j = 0$, for $j < \beta < \mu$, such that $\mathbb{N}^{\mathfrak{e}}_n = (\bigoplus_{i<\alpha} \mathbf{R}x_i\mathbf{S})/\mathbb{K}$, where $\mathbb{K}$ is the bimodule generated by $\langle t_j : j < \beta\rangle$.*

We now define what it means for a sequence of formulas to be adequate with respect to an element of $\mathfrak{E}_{(\mathbf{R},\mathbf{S})}$:

Definition 4.3. *Let $\mathfrak{e} := \langle \mathbb{N}_n, x_n, g_n : n < \omega\rangle$ and $\bar{\varphi} := \langle \varphi_n : n < \omega\rangle$.*

(1) *The sequence $\bar{\varphi}$ is called (μ, κ)-adequate with respect to $\mathfrak{e}$ if the following conditions satisfied:*
   (α) *$\varphi_n = \varphi_n(x)$ is a formula from $\mathcal{L}^{\text{cpe}}_{\mu,\kappa}(\tau_{\mathbf{R}})$,*
   (β) *$\varphi_{n+1}(x) \vdash \varphi_n(x)$ (for the class of $\mathbf{R}$-modules)*[3],
   (γ) *$\mathbb{N}_n \models \varphi_n(x_n)$ & $\neg\varphi_{n+1}(x_n)$.*

   Also, we say $\mathfrak{e}$ is (μ, κ)-adequate with respect to $\mathfrak{e}$.
(2) *We say $\mathfrak{e}$ is explicitly (μ, κ)-adequate with respect to $\mathfrak{e}$ if $\mathfrak{e}$ is (μ, κ)-adequate with respect to $\mathfrak{e}$ and $\mathfrak{e} \in \mathfrak{E}_{\mu,\kappa}$.*
(3) *For simplicity, κ-adequate means (∞, κ)-adequate with respect to $\mathfrak{e}$. Also, adequate is referred to $\aleph_0$-adequate with respect to $\mathfrak{e}$.*
(4) *We say that $\bar{\varphi}$ is (μ, κ)-adequate if $\bar{\varphi}$ is (μ, κ)-adequate with respect to some $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$.*
(5) *We say $\bar{\varphi}$ is adequate with respect to $\mathfrak{E} \subseteq \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ if it is adequate with respect to some $\mathfrak{e} \in \mathfrak{E}$.*

It follows from Lemma 3.10 that any (μ, κ)-adequate $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ is non-trivial.

Definition 4.4. *Let $\mathfrak{e} \in \mathfrak{E}$, $n < \omega$ and κ be given. We define*

(1) $\varphi^{\mathfrak{e},\kappa}_n := \bigwedge \left\{\varphi(x) : \varphi \in \mathcal{L}^{\text{cpe}}_{\infty,\kappa}(\tau_{\mathbf{R}}) \text{ and } \mathbb{N}^{\mathfrak{e}}_n \models \varphi(x^{\mathfrak{e}}_n)\right\}$[4],

[3] we can restrict ourselves to those we actually use, i.e. the ones from $\mathrm{cl}(\mathcal{K})$.

[4] This is not a formula being a conjunction on a class, not a set, but when we deal with an $\mathbf{R}$-module $\mathbb{M}$ it is enough to restrict ourselves to $\mathcal{L}^{\text{cpe}}_{\lambda,\kappa}$ for $\lambda = (||\mathbb{M}||^{<\kappa})^+$.

(2) $\psi_n^{\mathfrak{e},\kappa} := \bigwedge\{\psi(x) : \psi \in \mathcal{L}^{\mathrm{cpe}}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$ and $\mathbb{N}_n^{\mathfrak{e}} \models \psi(x_n^{\mathfrak{e}})\}$,

(3) $\bar{\varphi}^{\mathfrak{e},\kappa} := \langle \varphi_n^{\mathfrak{e},\kappa} : n < \omega\rangle$,

(4) $\bar{\psi}^{\mathfrak{e},\kappa} := \langle \psi_n^{\mathfrak{e},\kappa} : n < \omega\rangle$,

(5) $\varphi_\omega^{\mathfrak{e},\kappa}(x) := \bigwedge\limits_{n<\omega} \varphi_n^{\mathfrak{e},\kappa}(x)$,

(6) $\psi_\omega^{\mathfrak{e},\kappa}(x) := \bigwedge\limits_{n<\omega} \psi_n^{\mathfrak{e},\kappa}(x)$.

Remark 4.5. *In Definition 4.4, we omit the index κ, when it is $\kappa(\mathfrak{e})$ for ψ or $\kappa_{\mathbf{R}}(\mathfrak{e})$ for φ, where $\kappa(\mathfrak{e})$ and $\kappa_{\mathbf{R}}(\mathfrak{e})$ are defined in Definition 4.7, see below.*

The next lemma shows the relation between different $\varphi_n^{\mathfrak{e},\kappa}$'s and $\psi_n^{\mathfrak{e},\kappa}$'s.

Lemma 4.6. *For each $n \geq m$ the following holds:*

(1) $\varphi_n^{\mathfrak{e},\kappa}(x) \vdash \varphi_m^{\mathfrak{e},\kappa}(x)$,

(2) $\psi_n^{\mathfrak{e},\kappa}(x) \vdash \psi_m^{\mathfrak{e},\kappa}(x)$,

(3) $\psi_n^{\mathfrak{e},\kappa}(x) \vdash \varphi_n^{\mathfrak{e},\kappa}(x)$.

Proof. The lemma follows easily by applying Lemma 3.10 to $g^{\mathfrak{e}}_{m,n}$. □

Definition 4.7. (1) *For $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ we define:*

(a) *$\kappa(\mathfrak{e})$ be the first infinite cardinal κ such that for each $n < \omega$, the bimodule $\mathbb{N}_n^{\mathfrak{e}}$ is generated by a set of $< \kappa$ elements.*

(b) *$\kappa_{\mathbf{R}}(\mathfrak{e})$ be the first infinite cardinal κ such that for each $n < \omega$, the bimodule $\mathbb{N}_n^{\mathfrak{e}}$ as an $\mathbf{R}$-module is generated by a set of $< \kappa$ elements.*

(2) *For each $\mathfrak{E} \subseteq \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$, we let*

(a) *$\kappa(\mathfrak{E}) := \sup\{\kappa(\mathfrak{e}) : \mathfrak{e} \in \mathfrak{E}\}$ and*

(b) *$\kappa(\mathfrak{E})_{\mathbf{R}} := \sup\{\kappa_{\mathbf{R}}(\mathfrak{e}) : \mathfrak{e} \in \mathfrak{E}\}$.*

We frequently use the following lemma.

Lemma 4.8. (1) *Assume $\mathbb{N}$ is a bimodule (resp. an $\mathbf{R}$-module) generated by $< \kappa$ members freely except satisfying $< \mu$ equations and $x \in \mathbb{N}$. Then there is a simple formula $\varphi \in \mathcal{L}_{\mu,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$ (resp. $\varphi \in \mathcal{L}_{\mu,\kappa}(\tau_{\mathbf{R}})$) such that* *for any bimodule (resp. an* $\mathbf{R}$*-module)* $\mathbb{M}$ *and* $y \in \mathbb{M}$*, the following are equivalent:*

(a) $\mathbb{M} \models \varphi(y)$,

(b) for some homomorphism $\mathbf{f}$ *from* $\mathbb{N}$ *into* $\mathbb{M}$*,* $\mathbf{f}(x) = y$ *(if* $\mathbb{N}$ *is an* $\mathbf{R}$*-module this means* $\mathbf{f}$ *is an* $\mathbf{R}$*-homomorphism).*

(c) for some $\mathbb{N}'$ *extending* $\mathbb{N}$ *and a homomorphism* $\mathbf{f}$ *from* $\mathbb{N}'$ *into* $\mathbb{M}$ *we have* $\mathbf{f}(x) = y$*.*

(2) *Clause (1) applies if* $\mathbb{N} = \mathbb{N}^{\mathfrak{c}}_n$*,* $x = x^{\mathfrak{c}}_n$ *and* $\varphi = \psi^{\mathfrak{c},\kappa}_n$*, when* $\mathbb{N}^{\mathfrak{c}}_n$ *is generated by* $< \kappa$ *members freely except satisfying* $< \mu$ *equations (as a bimodule). Similarly for* $\varphi = \varphi^{\mathfrak{c},\kappa}_n$ *for* $\mathbf{R}$*-modules.*

Proof. (1). We prove the lemma for the case of bimodules, exactly the same proof works for $\mathbf{R}$-modules. Suppose $\mathbb{N}$ is a bimodule generated by $< \kappa$ members freely except satisfying $< \mu$ equations and $x \in \mathbb{N}$. So it has the form $\mathbb{N} = \bigoplus_{i<\alpha} \mathbf{R}x_i\mathbf{S}/\mathbb{K}$, where $\mathbb{K}$ is the bimodule generated by $\langle t_j : j < \beta\rangle$, where each $t_j = 0$ is an atomic formula, $\alpha < \kappa$ and $\beta < \mu$. Then

- $x = \sum_{i<d}\sum_{h<h_d} r_{i,h}x_i s_{i,h}$, where $d, h_d < \omega$, $r_{i,h} \in \mathbf{R}$ and $s_{i,h} \in \mathbf{S}$. We should remark that for all but finitely many of them we have $r_{i,h}x_i s_{i,h} = 0$.
- $t_j = \sum_{i<d}\sum_{h<h_d} r^j_{i,h}x_i s^j_{i,h}$, where $r^j_{i,h} \in \mathbf{R}$ and $s^j_{i,h} \in \mathbf{S}$. Again, for all but finitely many of them we have $r^j_{i,h}x_i s^j_{i,h} = 0$.

Now consider the formula

$$\varphi(v) = \exists_{i<\alpha} v_i \Big[\bigwedge_{j<\beta} \sum_{i<\alpha}\sum_{h<h_d} r^j_{i,h} v_i s^j_{i,h} = 0 \wedge v = \sum_{i<\alpha}\sum_{h<h_d} r_{i,h} v_i s_{i,h} \Big].$$

It is clearly a simple formula in $\mathcal{L}_{\mu,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$.

$(a) \Rightarrow (b)$: Now suppose $\mathbb{M} \models \varphi(y)$. Thus we can find y_i, for $i < \alpha$, such that in M,

$$\sum_{i<\alpha}\sum_{h<h_d} r^j_{i,h} y_i s^j_{i,h} = 0,$$

and

$$y = \sum_{i<\alpha} \sum_{h<h_d} r_{i,h} y_i s_{i,h}.$$

Define $\mathbf{f} : \mathbb{N} \to \mathbb{M}$ so that for each $i < \alpha, f(x_i) = y_i$. Then f is as required.

$(b) \Rightarrow (c)$: This is easy.

$(c) \Rightarrow (a)$: This is easy.

(2). This follows from the fact that $\psi_n^{\mathfrak{e},\kappa} \vdash \varphi$, because $\mathbb{N}_n^{\mathfrak{e}} \models \varphi(x_n^{\mathfrak{e}})$. $\square$

The next lemma shows that each non-trivial $\mathfrak{e} \in \mathfrak{E}_{\mu,\kappa}$ has a canonical adequate sequence.

Lemma 4.9. *If $\mathfrak{e} \in \mathfrak{E}_{\mu,\kappa}$ is non-trivial and $\mu \geq \kappa \geq \aleph_0$ then $\bar{\varphi}^{\mathfrak{e},\kappa}$ is (μ,κ)-adequate with respect to $\mathfrak{e}$.*

Proof. It is easily seen, by the structure of each $\mathbb{N}_n^{\mathfrak{e}}$ that $\varphi_n^{\mathfrak{e},\kappa}$ is equivalent to a formula of $\mathcal{L}^{\rm cpe}_{\infty,\kappa}(\tau_{\mathbf{R}})$. According to Lemma 4.6 $\varphi_{n+1}^{\mathfrak{e},\kappa} \vdash \varphi_n^{\mathfrak{e},\kappa}$. By the definition of $\varphi_n^{\mathfrak{e},\kappa}$, we have $\mathbb{N}_n^{\mathfrak{e}} \models \varphi_n^{\mathfrak{e},\kappa}(x_n^{\mathfrak{e}})$. It remains to show that $\mathbb{N}_n^{\mathfrak{e}} \models \neg\varphi_{n+1}^{\mathfrak{e},\kappa}(x_n^{\mathfrak{e}})$. Suppose not. It follows from Lemma 4.8(3) that there is a bimodule homomorphism $\mathbf{h} : \mathbb{N}_{n+1}^{\mathfrak{e}} \to \mathbb{N}_n^{\mathfrak{e}}$ with $\mathbf{h}(x_{n+1}^{\mathfrak{e}}) = x_n^{\mathfrak{e}}$, contradicting the non-triviality of $\mathfrak{e}$. $\square$

Definition 4.10. *A* λ-context *is a tuple*

$$\mathfrak{m} := (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T}) = (\mathcal{K}^{\mathfrak{m}}, \mathbb{M}_*^{\mathfrak{m}}, \mathfrak{E}^{\mathfrak{m}}, \mathbf{R}^{\mathfrak{m}}, \mathbf{S}^{\mathfrak{m}}, \mathbf{T}^{\mathfrak{m}})$$

where

(1) $\mathbf{R}, \mathbf{S}$ and $\mathbf{T}$ are as usual, see Convention 3.2.

(2) $\mathcal{K}$ is a set of $(\mathbf{R}, \mathbf{S})$-bimodules.

(3) $\mathfrak{E}$ is a subset of $\mathfrak{E}_{\infty,\lambda}$ closed under restrictions (i.e., if $\mathfrak{e} \in \mathfrak{E}$ and $\mathcal{U} \subseteq \omega$ is infinite then $\mathfrak{e} \restriction \mathcal{U} \in \mathfrak{E}$).

(4) If $\mathfrak{e} \in \mathfrak{E}$, then $\mathbb{N}_n^{\mathfrak{e}} \in \mathcal{K}$ for each $n < \omega$.

(5) $\mathbb{M}_*$ is a bimodule. If $\mathbb{M}_*$ is omitted we mean the zero bimodule.

Notation 4.11. *Let $\chi(\mathcal{K})$ be the minimal cardinal $\chi \geq \aleph_0$ such that every $\mathbb{N} \in \mathcal{K}$ is generated, as a bimodule, by a set of $< \chi$ members. Also, we set*

$$||\mathfrak{m}|| := \sum\{\|\mathbb{N}\| : \mathbb{N} \in \mathcal{K}\} + \|\mathfrak{E}\| + ||\mathbb{M}_*||||\mathbf{R}|| + ||\mathbf{S}|| + \aleph_0.$$

Remark 4.12. *We omit λ from the λ-context, if it is clear from the context and then we mean for the minimal such λ.*

Convention 4.13. *From now on suppose that $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ is a context.*

Definition 4.14. *(1) Given a context $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$, we set:*

i) *$c\ell_{\mathrm{is}}(\mathcal{K})$ be the class of bimodules isomorphic to some $\mathbb{K} \in \mathcal{K}$.*

ii) *$c\ell^{\kappa}_{\mathrm{is}}(\mathcal{K})$ be the class of bimodules of the form $\mathbb{M} = \bigoplus\limits_{i<j} \mathbb{M}_i$, where $j < \kappa$ and $\mathbb{M}_i \in c\ell_{\mathrm{is}}(\mathcal{K})$ for all $i < j$.*

iii) *$c\ell(\mathcal{K}) = c\ell_{\mathrm{ds}}(\mathcal{K})$ be the class of bimodules isomorphic to a direct sums of bimodules from $c\ell_{\mathrm{is}}(\mathcal{K})$.* [5]

iv) *$\kappa(\mathfrak{m}) = \kappa(\mathfrak{E}^{\mathfrak{m}})$.*

v) *$\kappa_{\mathbf{R}}(\mathfrak{m}) = \kappa_{\mathbf{R}}(\mathfrak{E}^{\mathfrak{m}})$.*

(2) Following Definition 4.3, let us say that $\bar{\varphi}$ is adequate for a context $\mathfrak{m}$ if it is adequate with respect to $\mathfrak{E}^{\mathfrak{m}}$.

Definition 4.15. *By a $\mathcal{K}$-bimodule we mean a bimodule from $c\ell_{\mathrm{is}}(\mathcal{K})$. Also, we say $\mathbb{M}_1$ is a $\mathcal{K}$-direct summand of $\mathbb{M}_2$ if $\mathbb{M}_2 = \mathbb{M}_1 \oplus \mathbb{K}$ for some $\mathbb{K} \in c\ell_{\mathrm{ds}}(\mathcal{K})$.*

Definition 4.16. *We say $\mathbb{M}_1$ is an almost direct $\mathcal{K}$-summand of $\mathbb{M}_2$ with respect to κ denoted by $\mathbb{M}_1 \leq^{\mathrm{ads}}_{\mathcal{K},\kappa} \mathbb{M}_2$, provided player II has a winning strategy in the following game $\Game^{\mathbb{M}_1,\mathbb{M}_2}_{\mathcal{K},\kappa}$ of length ω: in the n^{th} move player I chooses a subset $A_n \subseteq \mathbb{M}_2$ of cardinality $< \kappa$, and then player II chooses a sub-bimodule $\mathbb{K}_n \subseteq \mathbb{M}_2$. Player II wins iff:*

(a) $A_n \subseteq \mathbb{M}_1 + \sum\limits_{\ell \leq n} \mathbb{K}_\ell$,

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[5] In particular, $c\ell_{\mathrm{is}}(c\ell(\mathcal{K})) = c\ell(\mathcal{K})$.

(b) $\mathbb{K}_n$ *is in* $c\ell^{\kappa}_{\rm is}(\mathcal{K})$,

(c) $\mathbb{M}_1 + \sum\limits_{\ell<\omega} \mathbb{K}_\ell = \mathbb{M}_1 \oplus \sum\limits_{\ell<\omega} \mathbb{K}_\ell$.

We usually write $\leq_\kappa$ *or* $\leq^{\rm ads}_\kappa$ *instead of* $\leq^{\rm ads}_{\mathcal{K},\kappa}$ *if* $\mathcal{K}$ *is clear. We also may write* $\leq_{\mathfrak{m},\kappa}$ *or* $\leq^{\rm ads}_{\mathfrak{m},\kappa}$, *when* $\mathcal{K} = \mathcal{K}^{\mathfrak{m}}$.

We now define another order between bimodules and then connect it to the above defined notion.

Definition 4.17. *Let* $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *be two bimodules.*

(1) $\mathbb{M}_1 \leq^{\rm pr}_\varphi \mathbb{M}_2$ *means that* $\mathbb{M}_1 \subseteq \mathbb{M}_2$ *and if* $\psi = \psi(\bar{x})$ *is a sub-formula of* φ *and* $\bar{z} \in {}^{\lg(\bar{x})}\mathbb{M}_1$ *then*

$$\mathbb{M}_1 \models \psi(\bar{z}) \iff \mathbb{M}_2 \models \psi(\bar{z}).$$

(2) By $\mathbb{M}_1 \leq^{\rm qr}_\varphi \mathbb{M}_2$ *we mean that* $\mathbb{M}_1 \subseteq \mathbb{M}_2$ *and if* $\psi = \psi(\bar{x})$ *is a sub-formula of* φ *and* $\bar{z} \in {}^{(\lg \bar{x})}\mathbb{M}_1$, *then*

$$\mathbb{M}_2 \models \varphi(\bar{z}) \Rightarrow \mathbb{M}_1 \models \psi(\bar{z}).$$

(3) Assume $\mathcal{F}$ *is a set of formulas. The notation* $\mathbb{M}_1 \leq^{\rm pr}_{\mathcal{F}} \mathbb{M}_2$ *(resp.* $\mathbb{M}_1 \leq^{\rm qr}_{\mathcal{F}} \mathbb{M}_2$*) means* $\mathbb{M}_1 \leq^{\rm pr}_\varphi \mathbb{M}_2$ *(resp.* $\mathbb{M}_1 \leq^{\rm qr}_\varphi \mathbb{M}_2$*) for all* $\varphi \in \mathcal{F}$. *In the special case* $\bar{\varphi} := \langle \varphi_n : n < \omega \rangle$, *the property* $\mathbb{M}_1 \leq^{\rm pr}_{\bar{\varphi}} \mathbb{M}_2$ *holds iff* $\mathbb{M}_1 \leq^{\rm pr}_{\varphi_n} \mathbb{M}_2$ *for each* n.

For instance, let φ *be the following simple formula*

$$\varphi = \varphi(x) = (\exists y_0, \ldots, y_i \ldots)_{i<\alpha} \bigwedge_{j<\beta} [a_j x = \sum_{i<\alpha} b_{j,i} y_i]$$

where $a_j, b_{j,i} \in \mathbf{R}$. *Then* $\mathbb{M}_1 \leq^{\rm pr}_\varphi \mathbb{M}_2$ *means that if for some* $x \in \mathbb{M}_1$ *and* $y_i \in \mathbb{M}_2$ *we have*

$$a_j x - \sum_{i<\alpha} b_{j,i} y_i = 0 \quad \forall j < \beta,$$

then there are $y'_i \in \mathbb{M}_1$ such that

$$a_j x - \sum_{i<\alpha} b_{j,i} y'_i = 0 \quad \forall j < \beta.$$

Remark 4.18. *Note that if φ is existential (e.g. a simple formula), then $\leq^{\mathrm{qr}}_{\varphi}$ is equal to $\leq^{\mathrm{pr}}_{\varphi}$.*

Definition 4.19. *We say $\mathbb{M}_1$ is $\mathfrak{m}$-nice (or $\mathcal{K}$-nice), when some $\mathbb{M}_0$ witnesses it which means the following items are satisfied:*

(1) *$\mathbb{M}_0$ is a bimodule,*
(2) *$\mathbb{M}_0 \leq_{\mathcal{K}} \mathbb{M}_1$,*
(3) *if $\langle (A_\ell, \mathbb{K}_\ell) : \ell < n\rangle$ is a winning position of player II in the game $\Game^{\mathbb{M}_0,\mathbb{M}_1}_{\mathcal{K},\kappa}$, $A_n = \emptyset$ and $\mathbb{K} \in \mathcal{K}$, then for some $\mathbb{K}_n \in \mathcal{K}_{\mathfrak{m}}$ we have*
    (a) *$\mathbb{K}_n \cong \mathbb{K}$,*
    (b) *$\langle (A_\ell, \mathbb{K}_\ell) : \ell \leq n\rangle$ is a winning position of player II in the game.*

The next lemma connects the above defined relations to each other.

Lemma 4.20. *Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two bimodules. The following assertions are true:*

(1) *$\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$ iff $\mathbb{M}_1 \leq^{\mathrm{pr}}_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} \mathbb{M}_2$.*
(2) *If $\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$ and $\varphi(x) \in \mathcal{L}^{\mathrm{cpe}}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$, then $\varphi(\mathbb{M}_1) = \mathbb{M}_1 \cap \varphi(\mathbb{M}_2)$.*

Proof. Clause (2) follows from (1) and the definition of $\leq^{\mathrm{pr}}$, so let us prove (1). Suppose $\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$. We prove, by induction on the complexity of the formula $\varphi \in \mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$, that $\mathbb{M}_1 \leq^{\mathrm{pr}}_{\varphi} \mathbb{M}_2$.

The only non-trivial case where the assumption $\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$ is used is the case of existential quantifier[6], so let us suppose $\varphi(v_i)_{i<\alpha}$ is of the form

$$\exists_{j<\beta} w_j \big[\psi((w_j)_{j<\beta}, (v_i)_{i<\alpha})\big],$$

[6]The case of universal quantifier then follows easily as $\forall \equiv \neg\exists\neg$.

the claim holds for $\psi = \psi((w_j)_{j<\beta}, (v_i)_{i<\alpha})$ and suppose $(x_i)_{i<\alpha} \in \mathbb{M}_1$. We want to show that

$$\mathbb{M}_1 \models \varphi((x_i)_{i<\alpha}) \iff \mathbb{M}_2 \models \varphi((x_i)_{i<\alpha}).$$

As $\mathbb{M}_1$ is a submodule of $\mathbb{M}_2$, by the induction hypothesis, if $\mathbb{M}_1 \models \varphi((x_i)_{i<\alpha})$, then $\mathbb{M}_2 \models \varphi((x_i)_{i<\alpha})$. Conversely suppose that $\mathbb{M}_2 \models \varphi((x_i)_{i<\alpha})$. Then we can find $(y_j)_{j<\beta} \in \mathbb{M}_2$ such that $\mathbb{M}_2 \models \psi((y_j)_{j<\beta}, (x_i)_{i<\alpha})$.

Now consider the game $\Game^{\mathbb{M}_1,\mathbb{M}_2}_{\mathcal{K},\kappa}$, in which player I plays $A_n = \{y_j : j < \beta\}$ at each step n. At the end, we have bimodules $\{\mathbb{K}_n : n < \omega\}$ such that:

- $A_n \subseteq \mathbb{M}_1 + \sum\limits_{\ell \leq n} \mathbb{K}_\ell$,
- $\mathbb{K}_n$ is in $c\ell^\kappa_{\rm is}(\mathcal{K})$,
- $\mathbb{M}_1 + \sum\limits_{\ell<\omega} \mathbb{K}_\ell = \mathbb{M}_1 \oplus \sum\limits_{\ell<\omega} \mathbb{K}_\ell$.

Set $\mathbb{K} = \sum\limits_{\ell<\omega} \mathbb{K}_\ell$. It then follows that $\mathbb{M}_1 \oplus \mathbb{K} \models \varphi((x_i)_{i<\alpha})$. In view of Lemma 3.14, $\mathbb{M}_1 \models \varphi((x_i)_{i<\alpha})$.

Conversely, suppose that $\mathbb{M}_1 \leq^{\rm pr}_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} \mathbb{M}_2$. Let us first state a few simple facts:

(i) Suppose $\alpha < \kappa$ and $\bar{b} \in {}^\alpha\mathbb{M}_2$. Then there exists $\bar{a} \in {}^\alpha\mathbb{M}_1$ such that

$$(\mathbb{M}_1, \bar{a}) \equiv_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} (\mathbb{M}_2, \bar{b}).$$

To see this, let $\alpha < \kappa$ and $\bar{b} \in {}^\alpha\mathbb{M}_2$. Suppose by contradiction that there is no $\bar{a} \in {}^\alpha\mathbb{M}_1$ such that $(\mathbb{M}_1, \bar{a}) \equiv_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} (\mathbb{M}_2, \bar{b})$. Thus for each $\bar{a} \in {}^\alpha\mathbb{M}_1$ we can find $\phi_{\bar{a}}(\bar{\nu}) \in \mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$ such that

$$\mathbb{M}_1 \models \neg\phi_{\bar{a}}(\bar{a}) \ \& \ \mathbb{M}_2 \models \phi_{\bar{a}}(\bar{b}).$$

Let

$$\phi(\bar{\nu}) = \bigwedge\{\phi_{\bar{a}}(\bar{\nu}) : \bar{a} \in {}^\alpha\mathbb{M}_1\} \in \mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})}).$$

Then $\mathbb{M}_1 \models \forall\bar{\nu}\neg\phi(\bar{\nu})$, while $\mathbb{M}_2 \models \phi(\bar{b})$, a contradiction to our assumption.

(ii) Suppose $\bar{a}_1 \in {}^{\lg(\bar{a}_1)}\mathbb{M}_1$ and $\bar{a}_2 \in {}^{\lg(\bar{a}_2)}\mathbb{M}_1$ are so that $\bar{a}_1 \trianglelefteq \bar{a}_2$. Let $\bar{b}_1 \in {}^{\lg(\bar{a}_1)}\mathbb{M}_2$ be such that

$$(\mathbb{M}_1, \bar{a}_1) \equiv_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} (\mathbb{M}_2, \bar{b}_1).$$

There is some $\bar{b}_2 \in {}^{\lg(\bar{a}_2)}\mathbb{M}_2$ such that $\bar{b}_1 \trianglelefteq \bar{b}_2$ and $(\mathbb{M}_1, \bar{a}_2) \equiv_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} (\mathbb{M}_2, \bar{b}_2)$.

(iii) Suppose $\alpha < \kappa, \bar{a} \in {}^{\alpha}\mathbb{M}_1$, $\bar{b} \in {}^{\alpha}\mathbb{M}_2$ and $(\mathbb{M}_1, \bar{a}) \equiv_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} (\mathbb{M}_2, \bar{b})$. Let $\bar{c} = \langle b_i - a_i : i < \alpha \rangle$ and let $\mathbb{K}$ be the submodule of $\mathbb{M}_2$ generated by $\bar{c}$. Then $\bar{b} \in \mathbb{M}_1 + \mathbb{K}$ and $\mathbb{M}_1 + \mathbb{K} = \mathbb{M}_1 \oplus \mathbb{K}$.

(iv) Suppose $\langle K_\ell : \ell < \omega \rangle$ is an increasing sequence of submodules of $\mathbb{M}_2$ such that for each $\ell < \omega, \mathbb{M}_1 + \mathbb{K}_\ell = \mathbb{M}_1 \oplus \mathbb{K}_\ell$. Then

$$\mathbb{M}_1 + \sum_{\ell<\omega} \mathbb{K}_\ell = \mathbb{M}_1 \oplus \sum_{\ell<\omega} \mathbb{K}_\ell.$$

We are now ready to define a winning strategy for the game $\partial^{\mathbb{M}_1,\mathbb{M}_2}_{\mathcal{K},\kappa}$. Fix a well-ordering $<^*$ of $\mathbb{M}_2$. To start set $A_{-1} = \emptyset$ and $\bar{b}_{-1} = \langle\rangle$. At stage n, suppose player I has chosen $A_n \subseteq \mathbb{M}_2$ of size $< \kappa$. We may assume that $A_n \supseteq A_{n-1}$. Let

$$\bar{a}_n := \big\langle a_n(i) : i < \lg(\bar{a}_n) \big\rangle$$

enumerate A_n in the $<^*$-order in such a way that A_{n-1} is an initial segment of the enumeration. By parts (i) and (ii), we choose

$$\bar{b}_n = \big\langle b_n(i) : i < \lg(\bar{a}_n) \big\rangle \in \mathbb{M}_2$$

such that $\bar{b}_n \trianglerighteq \bar{b}_{n-1}$ and

$$(\mathbb{M}_1, \bar{a}_n) \equiv_{\mathcal{L}_{\infty,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})} (\mathbb{M}_2, \bar{b}_n).$$

Let $\mathbb{K}_n$ be the submodule of $\mathbb{M}_2$ generated by $\{b_n(i) - a_n(i) : i < \lg(\bar{a}_n)\}$. Now (iii) and (iv) guarantee that items (a) and (c) of Definition 4.16 are satisfied. This completes the proof. $\square$

Claim 4.21. *Let φ be in $\mathcal{L}^{\mathrm{cpe}}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$. The following holds:*

(1) Assuming $\varphi = \varphi(\bar{x})$ we can find a simple formula $\varphi_*(\bar{z}) \in \mathcal{L}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$ satisfying:

(a) if $\mathbb{M}_1 \leq^{\mathrm{pr}}_{\varphi_*} \mathbb{M}_2$ and $\bar{x} \in \mathbb{M}_2$ then $\langle x_i + \mathbb{M}_1 : i < \lg(\bar{x})\rangle \in \varphi(\mathbb{M}_2/\mathbb{M}_1)$ iff $\bar{x} \in \varphi(\mathbb{M}_2) + \mathbb{M}_1$.

(b) if $\varphi(\bar{x}) \in \mathcal{L}^{\mathrm{cpe}}_{\mu,\mu}(\tau_{(\mathbf{R},\mathbf{S})})$ and μ is regular, then $\varphi_*(\bar{z}) \in \mathcal{L}_{\mu,\mu}(\tau_{(\mathbf{R},\mathbf{S})})$.

(2) Assume $\mathbb{M}_2 \in c\ell_{\mathrm{ds}}(\mathcal{K})$ and $\mathbb{M}_1$ is $\mathcal{K}$-nice. If $\mathbb{M}_1 \oplus \mathbb{M}_2 = \mathbb{M}_3$, then $\mathbb{M}_1 \leq^{\mathrm{pr}}_{\varphi} \mathbb{M}_3$.

(3) If $\langle \mathbb{M}_i : i < \delta\rangle$ is increasing, $\mathbb{M}_i \leq^{\mathrm{pr}}_{\varphi} \mathbb{M}$ for all $i < \delta$ and $\varphi \in \mathcal{L}_{\infty,\mathrm{cf}(\delta)}(\tau_{(\mathbf{R},\mathbf{S})})$, then $\bigcup_{i<\delta} \mathbb{M}_i \leq^{\mathrm{pr}}_{\varphi} \mathbb{M}$ and $\mathbb{M}_j \leq^{\mathrm{pr}}_{\varphi} \bigcup_{i<\delta} \mathbb{M}_i$ for all $j < \delta$.

(4) Assume φ is a simple formula of $\mathcal{L}_{\kappa,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$. Then $\mathbb{M}_1 <^{\mathrm{pr}}_{\varphi} \mathbb{M}_2$ iff for every $< \kappa$-generated submodule $\mathbb{N} \subseteq \mathbb{M}_2$ we have $\mathbb{M}_1 \leq^{\mathrm{pr}}_{\varphi} \mathbb{M}_1 + \mathbb{N}$.

(5) If $\mathbb{M} = \bigoplus_{t\in I} \mathbb{M}_t$ and $\varphi(x) \in \mathcal{L}^{p}_{\infty,\infty}(\tau_{(\mathbf{R},\mathbf{S})})$, then $\varphi(\mathbb{M}) = \bigoplus_{t\in I} \varphi(\mathbb{M}_t)$.

Proof. (1). For simplicity, suppose that $\varphi = \varphi(x)$. By Lemma 3.12, we can assume the formula φ is simple, so let it be of the form

$$\varphi = \exists_{i<\alpha} y_i \bigwedge_{j<\beta} \varphi_j(\bar{y}, x)$$

where each $\varphi_j(\bar{y}, x)$ is an atomic formula. But the only atomic relation is equality, so by moving to one side, without loss of generality $\varphi_j(\bar{y}, x)$ is of the form

$$\sigma_j(\bar{y}, x) = 0$$

for some term σ_i. Let $\bar{z} = \langle z_j : j < \beta\rangle$ and

$$\varphi_*(\bar{z}) = (\exists x, \bar{y})\big[\bigwedge_{j<\beta} \sigma_j(\bar{y}, x) = z_j\big].$$

We show that $\varphi_*(\bar{z})$ is as required. Clause (b) clearly holds. In order to prove clause (a), take $\mathbb{M}_1 \leq^{\mathrm{pr}}_{\varphi_*} \mathbb{M}_2$ and $x \in \mathbb{M}_2$. First, assume that $x + \mathbb{M}_1 \in \varphi(\mathbb{M}_2/\mathbb{M}_1)$. Then

$$\mathbb{M}_2/\mathbb{M}_1 \models \text{“}\exists \bar{y} \bigwedge_{j<\beta} \sigma_j(\bar{y}, x + \mathbb{M}_1) = 0''.$$

We can find $\bar{b} = (b_i)_{i<\alpha} \in \mathbb{M}_2$ such that $\mathbb{M}_2/\mathbb{M}_1 \models \sigma_j(\bar{b} + \mathbb{M}_1, x + \mathbb{M}_1) = 0$ for all $j < \beta$. [7] By definition, $\sigma_j(\bar{b} + \mathbb{M}_1, x + \mathbb{M}_1) = c_j + \mathbb{M}_1$ for some $c_j \in M_1$. We set $\bar{c} := \langle c_j : j < \beta \rangle$. Then $\mathbb{M}_2 \models \varphi_*(\bar{c})$. Hence,

$$M_2 \models \sigma_j(\bar{b}, x) = c_j$$

for all $j < \beta$. We apply our assumption to see $\mathbb{M}_1 \models \varphi_*(\bar{c})$, e.g., there are $\bar{b}' = (b'_i)_{i<\alpha}$ and x' in $\mathbb{M}_1$ such that

$$M_1 \models \sigma_j(\bar{b}', x') = c_j$$

for all $j < \beta$. Then $\mathbb{M}_2$ satisfies the same formulas. In particular,

$$M_2 \models \text{“}\sigma_j((b_i - b'_i)_{i<\alpha}, x - x') = 0'',$$

which implies $M_2 \models \varphi(x - x')$. Thus $x \in \varphi(\mathbb{M}_2) + \mathbb{M}_1$.

Conversely, suppose that $x \in \varphi(\mathbb{M}_2) + \mathbb{M}_1$. Let $y \in \mathbb{M}_1$ be such that $x - y \in \varphi(\mathbb{M}_2)$. Then $\mathbb{M}_2 \models \varphi(x - y)$, and consequently there is $\bar{b} = ((b_i)_{i<\alpha}) \in \mathbb{M}_2$ such that

$$M_2 \models \sigma_j(\bar{b}, x - y) = 0$$

for all $j < \beta$. Set $\bar{z} := \bar{0}$. Then $\mathbb{M}_2 \models \varphi_*(\bar{z})$, and hence by our assumption $\mathbb{M}_1 \models \varphi_*(\bar{z})$. It follows that for some $\bar{b}' = (b'_i)_{i<\alpha}$ and x' in $\mathbb{M}_1$ and

$$M_1 \models \sigma_j(\bar{b}', x') = 0$$

for all $j < \beta$. Thus, $\mathbb{M}_2$ satisfies the same formulas. Therefore,

$$M_2 \models \text{“}\sigma_j((b_i - b'_i)_{i<\alpha}, x - y - x') = 0''$$

for all $j < \beta$. Since $\bar{b}', y + x'$ are in $\mathbb{M}_1$, we have

$$\mathbb{M}_2/\mathbb{M}_1 \models \text{“}\sigma_j(\bar{b} + \mathbb{M}_1, x + \mathbb{M}_1) = 0''$$

for all $j < \beta$. This implies $\mathbb{M}_2/\mathbb{M}_1 \models \varphi(x + \mathbb{M}_1)$. Hence, $x + \mathbb{M}_1 \in \varphi(\mathbb{M}_2/\mathbb{M}_1)$.

[7] By $\bar{b} + \mathbb{M}_1$ we mean $\langle b_i + \mathbb{M}_1 : i < \alpha \rangle$.

(2). This is in Lemma 3.14.

(3). Let us first show that $\bigcup_{i<\delta} \mathbb{M}_i \leq^{\rm pr}_{\varphi} \mathbb{M}$. To see this, let $\psi(\bar{x})$ be a subformula of φ and $\bar{z} \in \bigcup_{i<\delta} \mathbb{M}_i$. There is $i_* < \delta$ such that $\bar{z} \in \mathbb{M}_{i_*}$. Then for all $i_* \leq i < \delta$, $\mathbb{M}_i \models \psi(\bar{z})$ iff $\mathbb{M} \models \psi(\bar{z})$. It follows that

$$\bigcup_{i<\delta} \mathbb{M}_i \models \psi(\bar{z}) \iff \mathbb{M} \models \psi(\bar{z}).$$

Let $j < \delta$, $\psi(\bar{x})$ a subformula of φ and $\bar{z} \in \mathbb{M}_j$. We conclude from the above argument that

$$\mathbb{M}_j \models \psi(\bar{z}) \iff \mathbb{M} \models \psi(\bar{z}) \iff \bigcup_{i<\delta} \mathbb{M}_i \models \psi(\bar{z}).$$

Thus $\mathbb{M}_j \leq^{\rm pr}_{\varphi} \bigcup_{i<\delta} \mathbb{M}_i$.

(4). It is easily seen that if $\mathbb{M}_1 \leq^{\rm pr}_{\varphi} \mathbb{M}_2$ then $\mathbb{M}_1 \leq^{\rm pr}_{\varphi} \mathbb{M}_1 + \mathbb{N}$ for each $< \kappa$-generated submodule $\mathbb{N}$ of $\mathbb{M}_2$.

Conversely, suppose that the above condition holds. Let $\psi(\bar{x})$ be a subformula of φ. Then it is a simple formula as well, thus it is of the form

$$\exists_{i<\alpha} v_i \bigwedge_{j<\beta} [\sigma_j(\bar{x}, \bar{v}) = 0],$$

for some terms σ_j. Let $\bar{x} \in \mathbb{M}_1$. It is obvious that if $\mathbb{M}_1 \models \psi(\bar{x})$, then $\mathbb{M}_2 \models \psi(\bar{x})$. Now let $\mathbb{M}_2 \models \psi(\bar{x})$. Thus we can find $\bar{b} \in \mathbb{M}_2$ such that

$$\mathbb{M}_2 \models \bigwedge_{j<\beta} [\sigma_j(\bar{x}, \bar{b}) = 0].$$

Let $\mathbb{N}$ be the submodule generated by $\bar{b}$. Then $\mathbb{M}_1 + \mathbb{N} \models \bigwedge_{j<\beta} [\sigma_j(\bar{x}, \bar{b}) = 0]$. We combine this along with our assumption to conclude that $\mathbb{M}_1 \models \psi(\bar{x})$.

(5). We proceed by induction on the complexity of the formula φ (see also Lemma 3.14). We leave the routine check to the reader. $\square$

Let κ be an infinite cardinal and let $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{C}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ be a context. Set $\mathrm{Mod}_{\mathfrak{m},\kappa} := \{\mathbb{M} : \mathbb{M}_* \leq^{\rm ads}_{\mathcal{K},\kappa} \mathbb{M}\}$.

Lemma 4.22. *Let $\mathfrak{m}$ be a context as above and let κ be an infinite cardinal.*

(1) *If* $\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$, *then* $\mathbb{M}_1$ *is a submodule of* $\mathbb{M}_2$.

(2) $\leq_\kappa$ *is a partial order on* $\mathrm{Mod}_{\mathfrak{m},\kappa}$.

(3) *If* $\kappa \leq \theta$ *and* $\mathbb{M} \leq_\theta \mathbb{N}$ *then* $\mathbb{M} \leq_\kappa \mathbb{N}$.

(4) *If* $\langle \mathbb{M}_i : i < \delta \rangle$ *is a* $\leq_\kappa$*-increasing and continuous in* $\mathrm{Mod}_{\mathfrak{m},\kappa}$, *then*

   (a) $\bigcup\limits_{i<\delta} \mathbb{M}_i \in \mathrm{Mod}_{\mathfrak{m},\kappa}$,

   (b) *For each* $j < \delta, \mathbb{M}_j \leq_\kappa \bigcup\limits_{i<\delta} \mathbb{M}_i$.

(5) *If* $\mathbb{M}_1, \mathbb{M}_2, \mathbb{M}_3 \in \mathrm{Mod}_{\mathfrak{m},\kappa}$ *are such that* $\mathbb{M}_1, \mathbb{M}_2 \leq_\kappa \mathbb{M}_3$ *and* $\mathbb{M}_1 \subseteq \mathbb{M}_2$, *then* $\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$.

(6) *If* $\mathbb{M}_1 \leq_\kappa \mathbb{M}_2$ *and* $A \subseteq \mathbb{M}_2$ *has cardinality* $< \kappa$ *then for some* $\mathbb{N} \in c\ell^\kappa_{is}(\mathcal{K})$, *we have* $A \subseteq \mathbb{M}_1 + \mathbb{N} = \mathbb{M}_1 \oplus \mathbb{N} \leq_\kappa \mathbb{M}_2$.

(7) *The pair* $(\mathrm{Mod}_{\mathfrak{m},\kappa}, \leq_\kappa)$ *has the amalgamation property.*

(8) *There is a cardinal* χ *such that if* $\mathbb{M}_1 \subseteq \mathbb{N}$ *are in* $\mathrm{Mod}_{\mathfrak{m},\kappa}$, *then there is* $\mathbb{M}_2 \in \mathrm{Mod}_{\mathfrak{m},\kappa}$ *such that* $\mathbb{M}_1 \subseteq \mathbb{M}_2 \leq_\kappa \mathbb{N}$ *and* $|\mathbb{M}_2| \leq |\mathbb{M}_1| + \chi$.

Proof. The lemma follows easily from Lemmas 4.20 and 4.21. □

Definition 4.23. *We say the bimodule* $\mathbb{N}$ *is free as an* $\mathbf{S}$*–module provided the following two items hold:*

(a) *as an* $\mathbf{R}$*-module, it can be written as* $\bigoplus\limits_{i<\alpha} \mathbb{N}_i$ *where each* $\mathbb{N}_i$ *is an* $\mathbf{R}$*-submodule of* $\mathbb{N}$.

(b) *If* $\mathbb{M}$ *is a bimodule,* $i < \alpha$ *and* $g : \mathbb{N}_i \longrightarrow \mathbb{M}$ *is an* $\mathbf{R}$*-homomorphism, then there is a unique bimodule homomorphism* $h : \mathbb{N} \longrightarrow \mathbb{M}$ *extending* g*:*

$$\begin{tikzcd}
\mathbb{N} \arrow[dr, "\exists! h"] & \\
\mathbb{N}_i \arrow[u, "\subseteq"] \arrow[r, "\mathbf{g}"'] & \mathbb{M}
\end{tikzcd}$$

The next lemma can be proved as in Lemma 4.8.

Lemma 4.24. *Let* $\mathfrak{e} \in \mathfrak{E}$.

(1) *There exists a formula* $\varphi_n^{\mathfrak{e}} \in \mathcal{L}^{cpe}_{\infty,\infty}(\tau_{\mathbf{R}})$ *such that for an* $\mathbf{R}$*-module* $\mathbb{M}$, $\mathbb{M} \models \varphi_n^{\mathfrak{e}}(x)$ *iff for some* $\mathbb{M}'$ *with* $\mathbb{M} \leq_{\aleph_0} \mathbb{M}'$, $\mathbb{M}' \models \varphi_n^{\mathfrak{e},\infty}(x)$*; equivalently there is an* $\mathbf{R}$*-module homomorphism from* $\mathbb{N}_n^{\mathfrak{e}}$ *into* $\mathbb{M}'$ *mapping* $x_n^{\mathfrak{e}}$ *to* x.

(2) *There exists* $\psi_n^{\mathfrak{e}} \in \mathcal{L}_{\infty,\infty}(\tau_{\mathbf{R}})$ *such that for a bimodule* $\mathbb{M}$, $\mathbb{M} \models \psi_n^{\mathfrak{e}}(x)$ *iff for some* $\mathbb{M}'$ *with* $\mathbb{M} \leq_{\aleph_0} \mathbb{M}'$, $\mathbb{M}' \models \psi_n^{\mathfrak{e},\infty}(x)$.

The above lemma suggests the following:

Definition 4.25. *By* $\varphi_\omega^{\mathfrak{e}}(x)$ *we mean* $\bigwedge\limits_{n<\omega} \varphi_n^{\mathfrak{e}}(x)$ *and* $\overline{\varphi}^{\mathfrak{e}} := \langle \varphi_n^{\mathfrak{e}} : n < \omega \rangle$. *Similarly, we define* $\psi_\omega^{\mathfrak{e}}(x) := \bigwedge\limits_{n<\omega} \psi_n^{\mathfrak{e}}(x)$ *and let* $\overline{\psi}^{\mathfrak{e}} := \langle \psi_n^{\mathfrak{e}} : n < \omega \rangle$.

Definition 4.26. (1) *A bimodule* $\mathbb{M}$ *is called* $\mathfrak{E}$*-closed if for every* $\mathfrak{e} \in \mathfrak{E}$, $n < \omega$ *and every* $x \in \mathbb{M}$ *we have*

$$\mathbb{M} \models \psi_n^{\mathfrak{e}}(x) \iff \mathbb{M} \models \psi_n^{\mathfrak{e},\infty}(x).$$

(2) *We say that* $\mathfrak{e}$ *is non-trivial for* $\mathfrak{m}$ *if for every* $n < \omega$ *there exists* $\mathbb{M} \in \mathcal{K}^{\mathfrak{m}}$ *such that* $\mathbb{M}_*^{\mathfrak{m}} \oplus \mathbb{M} \models (\exists x)[\varphi_n^{\mathfrak{e}}(x) \wedge \neg\varphi_\omega^{\mathfrak{e}}(x)]$.

Fact 4.27. *It is easily seen that if* $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ *is adequate and each* $\mathbb{N}_n^{\mathfrak{e}}$ *is finitely generated as an* $\mathbf{R}$*-module, then* $\mathfrak{e}$ *is non-trivial for* $\mathfrak{m}$.

Convention 4.28. *From now on we fix a context* $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ *which is non-trivial which means:*

(a) $\mathcal{K} \neq \emptyset$,

(b) $\mathfrak{E} \neq \emptyset$,

(c) *every* $\mathfrak{e} \in \mathfrak{E}_{\infty,\kappa}$ *is non-trivial for* $\mathfrak{m}$.

Definition 4.29. *Let* $\mathfrak{e} \in \mathfrak{E}$ *and* $\bar{\varphi}$ *is adequate with respect to* $\mathfrak{e}$.

(1) *Suppose* $\mathbb{M}$ *is a bimodule and* $\mathbf{h_1}, \mathbf{h_2} : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}$ *are bimodule homomorphisms. We define*

$$\mathbb{L}_n^{\mathfrak{e},\bar{\varphi},\mathbf{h_1},\mathbf{h_2}}(\mathbb{M}) := \{z \in \varphi_n(\mathbb{N}_n^{\mathfrak{e}}) : \mathbf{h_1}(z) = \mathbf{h_2}(z) \mod \varphi_\omega(\mathbb{M})\}.$$

For simplicity, and if there is no danger of confusion, we set

$$\mathbb{L}_n^{\mathfrak{e},\bar{\varphi},\mathbf{h_1},\mathbf{h_2}} := \mathbb{L}_n^{\mathfrak{e},\bar{\varphi},\mathbf{h_1},\mathbf{h_2}}(\mathbb{M}).$$

If $\bar{\varphi} = \bar{\varphi}^{\mathfrak{e},\kappa}$ we may write $\mathbb{L}_n^{\mathfrak{e},\kappa,\mathbf{h}_\ell,\mathbf{h_2}}$ and if $\kappa = \aleph_0$ we may omit it.

(2) *Let Σ be the family of all pairs $(\mathbf{h_1},\mathbf{h_2})$ such that $\mathbf{h_1},\mathbf{h_2} : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}$ that are bimodule homomorphisms, $\mathbb{M} \in \mathcal{K} \cup \{\mathbb{M}_*\}$ and $\mathbf{h_1}(x_n^{\mathfrak{e}}) = \mathbf{h_2}(x_n^{\mathfrak{e}})$. We define*

$$\mathbb{L}_n^{\mathfrak{e}}[\mathfrak{m}] := \bigcap_{(\mathbf{h_1},\mathbf{h_2})\in\Sigma} \mathbb{L}_n^{\mathfrak{e},\mathbf{h_1},\mathbf{h_2}}$$

We may write $\mathbb{L}_n^{\mathfrak{e}} := \mathbb{L}_n^{\mathfrak{e}}[\mathfrak{m}]$, when $\mathfrak{m}$ is clear from the context.

(3) *$\mathbb{L}_n^{\mathfrak{e}}[\mathcal{K}]$, is defined similarly but we only require $\mathbb{M} \in \mathcal{K}$.*

Lemma 4.30. *Let $\mathfrak{e} \in \mathfrak{E}$. Then, there are bimodules $\mathbb{P}^{\mathfrak{e}}$, $\mathbb{P}_n^{\mathfrak{e}}$ and $\mathbb{K}_n$, embeddings $\mathbf{h}_n^{\mathfrak{e}} : \mathbb{N}_n^{\mathfrak{e}} \longrightarrow \mathbb{P}^{\mathfrak{e}}$, an element $x = x_{\mathfrak{e}}$, an embedding $\mathbf{f}^{\mathfrak{e}} : \mathbb{N}_0^{\mathfrak{e}} \longrightarrow \mathbb{P}^{\mathfrak{e}}$ and $x_n := x_{\mathfrak{e},n} \in \mathbb{P}^{\mathfrak{e}}$ furnished with the following four properties:*

(a) $\mathrm{Rang}(\mathbf{h}_n^{\mathfrak{e}}) \cap \sum_{m\neq n} \mathrm{Rang}(\mathbf{h}_m^{\mathfrak{e}}) = \{0\}$,

(b) *For each $\mathbb{K}_n$ we have:*

$$\mathbb{P}^{\mathfrak{e}} = \big(\sum_{\ell<n} \mathrm{Rang}(\mathbf{h}_\ell^{\mathfrak{e}})\big) + \mathbb{K}_n = \bigoplus\nolimits_{\ell<n} \mathrm{Rang}(\mathbf{h}_\ell^{\mathfrak{e}}) \oplus \mathbb{K}_n,$$

and $\mathbb{K}_n$ is a direct sum of copies of $\mathbb{N}_m^{\mathfrak{e}}$'s.

(c) *$\sum_{n<\omega} \mathrm{Rang}(\mathbf{h}_n^{\mathfrak{e}})$ is not a direct summand of $\mathbb{P}^{\mathfrak{e}}$; moreover, for some embeddings $\mathbf{f}_n^{\mathfrak{e}} : \mathbb{N}_n^{\mathfrak{e}} \longrightarrow \mathbb{P}^{\mathfrak{e}}$ satisfying:*

$(*)_1$ $x_n = \sum_{\ell<n} \mathbf{h}_\ell^{\mathfrak{e}}(x_\ell^{\mathfrak{e}}) \in \sum_{\ell<n} \mathrm{Rang}(h_\ell^{\mathfrak{e}})$,

$(*)_2$ $x - x_n \in \varphi_n^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$,

$(*)_3$ $\mathbf{f}^{\mathfrak{e}} = \mathbf{f}_0^{\mathfrak{e}}$,

$(*)_4$ $\mathbf{f}^{\mathfrak{e}}(x_0^{\mathfrak{e}}) = x$,

$(*)_5$ $x \notin \sum_{n<\omega} \mathrm{Rang}(\mathbf{h}_n^{\mathfrak{e}}) + \varphi_\omega^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$.

(d) *$\mathbb{P}^{\mathfrak{e}}$ is the direct sum of the copies $\mathrm{Rang}(\mathbf{f}_n^{\mathfrak{e}})$ of $\mathbb{N}_n^{\mathfrak{e}}$.*

Proof. Set $\mathbb{P}^{\mathfrak{c}} := \bigoplus_{n<\omega} \mathbb{N}^{\mathfrak{c}}_n$ and denote the natural embedding from $\mathbb{N}^{\mathfrak{c}}_n$ into $\mathbb{P}^{\mathfrak{c}}$ by $\mathbf{f}^{\mathfrak{c}}_n$. In particular,

$$\mathbb{P}^{\mathfrak{c}} = \bigoplus_{n<\omega} \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_n),$$

i.e., (d) holds. We define $\mathbf{h}^{\mathfrak{c}}_n : \mathbb{N}^{\mathfrak{c}}_n \longrightarrow \mathbb{M}$, for all $n < \omega$, as follows, where $g^{\mathfrak{c}}_{n,n+1}$ is defined as in Definition 4.1(3):

$$\mathbf{h}^{\mathfrak{c}}_n(y) = \mathbf{f}^{\mathfrak{c}}_n(y) - \mathbf{f}^{\mathfrak{c}}_{n+1}(g^{\mathfrak{c}}_{n,n+1}(y)) \text{ for every } y \in \mathbb{N}^{\mathfrak{c}}_n.$$

As $\mathbb{P}^{\mathfrak{c}} = \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_n) \oplus \bigoplus_{\ell \neq n} \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_\ell)$, we observe that $\mathbf{h}^{\mathfrak{c}}_n$ is a bimodule embedding. Set also

$$\mathbb{P}^{\mathfrak{c}}_n := \sum_{\ell<n} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell).$$

The next claim can be proved easily.

Claim 4.31. *Adopt the above notation. Then, for each n,*

$$\mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_n) \oplus \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_{n+1}) = \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_n) \oplus \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_{n+1}).$$

In particular, $\mathbb{P}^{\mathfrak{c}} = \bigoplus_{\ell<n} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_n) \oplus \bigoplus_{\ell \geq n} \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_\ell)$ for all n.

We set $\mathbb{K}_n := \bigoplus_{\ell \geq n} \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_\ell)$. In view of Claim 4.31, the items (a), (b) and (d) of Lemma 4.30 are hold. Next we shall show that $x := \mathbf{f}^{\mathfrak{c}}_0((x^{\mathfrak{c}}_0))$ and $x_n := \sum_{\ell<n} \mathbf{h}^{\mathfrak{c}}_\ell(x^{\mathfrak{c}}_\ell)$ are as required in Lemma 4.30(c). This implies the first statement of $(*)_1$ in item (c). Trivially $(*)_3$ and $(*)_4$ are true. Now,

$$\begin{aligned} x &= \mathbf{f}^{\mathfrak{c}}_0(x^{\mathfrak{c}}_0) \\ &= \mathbf{h}^{\mathfrak{c}}_0(x^{\mathfrak{c}}_0) + \mathbf{f}^{\mathfrak{c}}_1(g^{\mathfrak{c}}_{0,1}(x^{\mathfrak{c}}_0)) \\ &= \mathbf{h}^{\mathfrak{c}}_0(x^{\mathfrak{c}}_0) + \mathbf{f}^{\mathfrak{c}}_1(x^{\mathfrak{c}}_1) \\ &= \mathbf{h}^{\mathfrak{c}}_0(x^{\mathfrak{c}}_0) + \mathbf{h}^{\mathfrak{c}}_1(x^{\mathfrak{c}}_1) + \mathbf{f}^{\mathfrak{c}}_2(x^{\mathfrak{c}}_2). \end{aligned}$$

By induction on n we have

$$x = \sum_{\ell<n} \mathbf{h}^{\mathfrak{c}}_\ell(x^{\mathfrak{c}}_\ell) + \mathbf{f}^{\mathfrak{c}}_n(x^{\mathfrak{c}}_n) = x_n + \mathbf{f}^{\mathfrak{c}}_n(x^{\mathfrak{c}}_n).$$

Clearly, $x_n \in \bigoplus\limits_{\ell<n+1} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell) \subseteq \bigoplus\limits_{\ell<\omega} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell)$ and by the choice of $\mathbf{f}^{\mathfrak{c}}_n$ the second term is in $\varphi^{\mathfrak{c}}_n(\mathbb{P}^{\mathfrak{c}})$, so $(*)_1 + (*)_2$ of clause (c) holds. We shall now show that $x \notin \sum\limits_{\ell<\omega} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell) + \varphi^{\mathfrak{c}}_\omega(\mathbb{P}^{\mathfrak{c}})$, i.e., $(*)_5$ holds.

Suppose by contradiction that there is an $n \geq 1$ such that $x \in y + \sum\limits_{\ell<n} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell)$, where $y \in \varphi^{\mathfrak{c}}_\omega(\mathbb{P}^{\mathfrak{c}})$.

We now define a bimodule endomorphism $\mathbf{f}_n \in \mathrm{End}(\mathbb{P}^{\mathfrak{c}})$. As $\mathbb{P}^{\mathfrak{c}} = \bigoplus\limits_{\ell<\omega} \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_\ell)$, it is clearly enough to define each $\mathbf{f}_n \restriction \mathrm{Rang}(f^{\mathfrak{c}}_\ell)$, for $n \geq 1$, separately. Recall that $\mathbf{f}^{\mathfrak{c}}_\ell$ is one-to-one. Let $z \in \mathbb{N}^{\mathfrak{c}}_\ell$. We define

$$\mathbf{f}_n(\mathbf{f}^{\mathfrak{c}}_\ell(z)) = \begin{cases} \mathbf{f}^{\mathfrak{c}}_n(g^{\mathfrak{c}}_{\ell,n}(z)) & \text{if } \ell \leq n, \\ 0 & \text{otherwise} \end{cases}$$

Now, we bring the following claim:

Claim 4.32. *1) If $\ell \neq n$, then $\mathbf{f}_n \restriction \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell)$ is identically zero.*

2) If $\ell = n$, then $\mathbf{f}_n(x) = \mathbf{f}^{\mathfrak{c}}_n(x^{\mathfrak{c}}_n)$.

Proof. 1): For $\ell > n$ this is trivial, so suppose that $\ell < n$. Let $z \in \mathbb{N}^{\mathfrak{c}}_\ell$. Then $\mathbf{h}^{\mathfrak{c}}_\ell(z) = \mathbf{f}^{\mathfrak{c}}_\ell(z) - \mathbf{f}^{\mathfrak{c}}_{\ell+1}(g^{\mathfrak{c}}_{\ell,\ell+1}(z))$. So,

$$\begin{aligned} \mathbf{f}_n(\mathbf{h}^{\mathfrak{c}}_\ell(z)) &= \mathbf{f}_n\Big(\mathbf{f}^{\mathfrak{c}}_\ell(z) - \mathbf{f}^{\mathfrak{c}}_{\ell+1}(g^{\mathfrak{c}}_{\ell,\ell+1}(z))\Big) \\ &= \mathbf{f}_n(\mathbf{f}^{\mathfrak{c}}_\ell(z)) - \mathbf{f}_n(\mathbf{f}^{\mathfrak{c}}_{\ell+1}(g^{\mathfrak{c}}_{\ell,\ell+1}(z))) \\ &= \mathbf{f}^{\mathfrak{c}}_n(g^{\mathfrak{c}}_{\ell,n}(z)) - \mathbf{f}^{\mathfrak{c}}_n(g^{\mathfrak{c}}_{\ell,\ell+1}(g^{\mathfrak{c}}_{\ell+1,n}(z))) \\ &= \mathbf{f}^{\mathfrak{c}}_n(g^{\mathfrak{c}}_{\ell,n}(z)) - \mathbf{f}^{\mathfrak{c}}_n(g^{\mathfrak{c}}_{\ell,n}(z)) \\ &= 0. \end{aligned}$$

The third equation holds because of clause (e) of Lemma 4.30.

2): It is enough to recall $\mathbf{f}_n(x) = \mathbf{f}_n(\mathbf{f}^{\mathfrak{c}}_0(x^{\mathfrak{c}}_0)) = \mathbf{f}^{\mathfrak{c}}_n(g^{\mathfrak{c}}_{0,n}(\mathbf{f}^{\mathfrak{c}}_0(x^{\mathfrak{c}}_0)) = \mathbf{f}^{\mathfrak{c}}_n(x^{\mathfrak{c}}_n)$. $\square$

Since $x^{\mathfrak{c}}_n \notin \varphi_\omega(\mathbb{N}^{\mathfrak{c}}_n)$, we have $\mathbf{f}_n(x) = \mathbf{f}^{\mathfrak{c}}_n(x^{\mathfrak{c}}_n) \neq 0$. Indeed, we have

$$x^{\mathfrak{c}}_n \notin \varphi^{\mathfrak{c}}_\omega(\mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_n)) = \varphi^{\mathfrak{c}}_\omega(\mathbb{P}^{\mathfrak{c}}) \cap \mathrm{Rang}(\mathbf{f}^{\mathfrak{c}}_n).$$

But $x - y \in \sum_{\ell<n} \mathrm{Rang}(\mathbf{h}^{\mathfrak{e}}_\ell)$ and $\mathbf{f}_n{\restriction}\mathrm{Rang}(h^{\mathfrak{e}}_n)$ is zero, so $\mathbf{f}_n(x-y) = 0$. Hence $\mathbf{f}_n(x) = \mathbf{f}_n(y)$. However $\mathbf{f}_n(x) \notin \varphi^{\mathfrak{e}}_\omega(\mathbb{P}^{\mathfrak{e}})$. Thus, $\mathbf{f}_n(y) \notin \varphi^{\mathfrak{e}}_\omega(\mathbb{P}^{\mathfrak{e}})$. Recall that $y \in \varphi^{\mathfrak{e}}_\omega(\mathbb{P}^{\mathfrak{e}})$. Therefore, $\mathbf{f}_n(y) \in \varphi^{\mathfrak{e}}_\omega(\mathbb{P}^{\mathfrak{e}})$. This contradiction completes the proof of Lemma 4.30. $\square$

Lemma 4.33. *(1) Let $x \in \mathbb{M}_1$, $\mathfrak{e} \in \mathfrak{E}$ and $\varphi \in \{\tilde{\varphi}^{\mathfrak{e}}_\alpha, \tilde{\psi}^{\mathfrak{e}}_\alpha : \alpha \le \omega\}$. If $\mathbb{M}_1 \le_{\aleph_0} \mathbb{M}_2$, then*

$$\mathbb{M}_1 \models \varphi(x) \iff \mathbb{M}_2 \models \varphi(x).$$

(2) If $\mathbb{M}_* \le_{\aleph_0} \mathbb{M}_1$ and $\mathfrak{E}' \subseteq \mathfrak{E}$, then there is an $\mathbb{M}_2$ with the following properties:

(a) $\mathbb{M}_1 \le_{\aleph_0} \mathbb{M}_2$,

(b) $\|\mathbb{M}_2\| \le \|\mathbb{M}_1\| + \|\mathfrak{m}\| + |\mathfrak{E}'|$,

(c) if $x \in \mathbb{M}_1$, $n < \omega$ and $\mathfrak{e} \in \mathfrak{E}'$ then

$$\mathbb{M}_2 \models \varphi^{\mathfrak{e},\infty}_n(x) \iff \mathbb{M}_1 \models \varphi^{\mathfrak{e}}_n[x],$$

(d) $\mathbb{M}_2$ is the free sum of $\{\mathbb{M}_{2,i} : i < i^*\} \cup \{\mathbb{M}_1\}$ where each $\mathbb{M}_{2,i}$ isomorphic to some member of $\mathcal{K}$,

(e) For each $\mathbb{N} \in \mathcal{K}$, there are $||\mathbb{M}_2||$ many bimodules from $\{\mathbb{M}_{2,i} : i < i^*\}$ each of them being isomorphic to $\mathbb{N}$.

Proof. Clause (1) can be proved easily. To prove (2), let $\kappa := |\mathbb{M}_1\| + \|\mathfrak{m}\| + |\mathfrak{E}'|$ and set

$$\mathbb{M}_2 := \mathbb{M}_1 \oplus \bigoplus_{\mathbb{N}\in\mathcal{K}} \bigoplus_{i<\kappa} \mathbb{M}^{\mathbb{N}}_i,$$

where each $\mathbb{M}^{\mathbb{N}}_i$ is isomorphic to $\mathbb{N}$. It is easily seen that $\mathbb{M}_2$ is as required. $\square$

Notation 4.34. $S^\kappa_{\aleph_0} := \{\alpha < \kappa : \mathrm{cf}(\alpha) = \aleph_0\}$.

We now introduce the notion of (semi) nice construction. This concept plays an important role for the proof of our main results.

Definition 4.35. *Let us first fix the following quadric $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$:*

(1) Let $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ *be a context.*

(2) Let $\lambda \geq \kappa = \mathrm{cf}(\kappa) > \kappa(\mathfrak{E})$*. Also,* $\lambda = \lambda^\kappa$ *and for all* $\alpha < \lambda$, $|\alpha|^{\aleph_0} < \lambda$.

(3) Let $\bar{\gamma}^* = \langle \gamma^*_\alpha : \alpha \leq \kappa \rangle$ *be increasing and continuous.*

(4) Let $S \subseteq S^\kappa_{\aleph_0}$ *be stationary so that* $S^\kappa_{\aleph_0} \setminus S$ *is stationary as well.*

Let also $\langle \mathfrak{e}_\alpha : \alpha < |\mathfrak{E}| \rangle$ *be a fixed enumeration of* $\mathfrak{E}$ *and*

$$\mathrm{rep}(\mathcal{K}) := \left\langle \mathbb{N}^\beta : \beta < |\mathcal{K}/\cong| \right\rangle$$

be an enumeration of $\mathcal{K}$ *up to isomorphism.*

By a weakly semi-nice construction $\mathcal{A}$ *with respect to* $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$ *we mean a sequence* $\bar{\mathbb{M}} := \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ *together with the other objects mentioned below satisfying the following conditions*

(A) $\mathbb{M}_\alpha$ *is a bimodule whose universe is the ordinal* γ^*_α *for* $\alpha \leq \kappa$.

(B) $\alpha < \beta \Rightarrow \mathbb{M}_\alpha \subseteq \mathbb{M}_\beta$.

(C) if $\delta \leq \kappa$ *is a limit ordinal, then* $\mathbb{M}_\delta = \bigcup_{\alpha<\delta} \mathbb{M}_\alpha$.

(D) $\mathbb{M}_0 = \mathbb{M}_*$.

(E) for every $\alpha \in \kappa \setminus S$, $\mathbb{M}_{\alpha+1} = \mathbb{M}_\alpha \oplus \bigoplus_{t \in J_\alpha} \mathbb{N}^\alpha_t$ *where*

(a) $|J_\alpha| \leq \lambda$,

(b) $J_\alpha \cap (\bigcup_{\beta<\alpha} J_\beta) = \emptyset$,

(c) $\mathrm{cf}(\gamma^*_{\alpha+1}) < \lambda \Leftrightarrow |J_\alpha| < \lambda$,

(d) *for every* $t \in J_\alpha$ *there is* $\beta < |\mathcal{K}/\cong|$ *such that* $\mathbb{N}^\alpha_t \cong \mathbb{N}^\beta$, *i.e., every* $\mathbb{N}^\alpha_t$ *is isomorphic to some member of* $\mathcal{K}$,

(e) *for every* $\beta < |\mathcal{K}/\cong|$ *there are* $||\mathbb{M}_\alpha||$*-many* $t \in J_\alpha$ *such that* $\mathbb{N}^\alpha_t \cong \mathbb{N}^\beta$.

By the assumption (b), for each $t \in \bigcup_{\beta\leq\alpha} J_\beta$ *there is a unique* $\beta \leq \alpha$ *such that* $t \in J_\beta$*; so we may replace* $\mathbb{N}^\beta_t$ *by* $\mathbb{N}_t$*. Also given* $t \in J_\alpha$*, there is a unique* $\mathbb{N}^{\mathrm{so}}_t \in \mathrm{rep}(\mathcal{K})$ *such that* $\mathbb{N}^{\mathrm{so}}_t \cong \mathbb{N}^\alpha_t$*. Also, we use the notations* h_t *and* $h_{\alpha,t} : \mathbb{N}^{\mathrm{so}}_t \xrightarrow{\cong} \mathbb{N}^\alpha_t$ *for the mentioned isomorphism.*

(F) for $\delta \in S$, either the demand in clause[8] *(E) holds or else there are $\gamma_\delta^{**} < \delta$ and*

$$\langle(\mathfrak{e}_s, \mathbb{P}_s, \bar{\alpha}_s, \bar{t}_s, \bar{g}_s, h_s, q_s) : s \in J_\delta\rangle = \langle(\mathfrak{e}_s^\delta, \mathbb{P}_s^\delta, \bar{\alpha}_s^\delta, \bar{t}_s^\delta, \bar{g}_s^\delta, h_s^\delta, q_s^\delta) : s \in J_\delta\rangle$$

such that $J_\delta \cap (\bigcup_{\alpha<\delta} J_\alpha) = \emptyset$ and

(a) $\mathfrak{e}_s \in \mathfrak{E}$ *and in fact* $\mathfrak{e}_s \in \{\mathfrak{e}_\beta : \beta < \gamma_\delta^{**}$ *and* $\beta < |\mathfrak{E}|\}$,

(b) $\bar{\alpha}_s = \langle \alpha_{s,n} : n < \omega\rangle$,

(c) $\bar{t} = \langle t_{s,n} : n < \omega\rangle$,

(d) $\langle \alpha_{s,n} : n < \omega\rangle$ *is an increasing sequence of ordinals bigger than* γ_δ^{**} *such that* $\alpha_{s,n} \notin S$,

(e) $\delta = \sup\{\alpha_{s,n} : n < \omega\}$,

(f) $t_{s,n} \in J_{\alpha_{s,n}}$,

(g) $\mathbb{N}_{t_{s,n}}^{so} = \mathbb{N}_n^{\mathfrak{e}_s}$,

(h) if $s_1 \neq s_2$ *are in* J_δ *then the sets*

$$\{t_{s_1,n} : n < \omega\}, \{t_{s_2,n} : n < \omega\}$$

are tree-like, i.e.,

- $\{t_{s_1,n} : n < \omega\} \cap \{t_{s_2,n} : n < \omega\}$ *is finite,*
- *if* $t_{s_1,n_1} = t_{s_2,n_2}$ *then* $n_1 = n_2$ *and* $\bigwedge\limits_{n<n_1} t_{s_1,n} = t_{s_2,n}$,

(i) $\bar{g}_s = \langle g_{s,n} : n < \omega\rangle$, *where each* $g_{s,n}$ *is a (bimodule) homomorphism from* $\mathbb{N}_n^{\mathfrak{e}_s}$ *into* $\mathbb{M}_{\gamma_\delta^{**}}$,

(j) for $s \in J_\delta$, h_s *is a bimodule homomorphism from* $\mathbb{P}^{\mathfrak{e}_s}$[9] *onto* $\mathbb{P}_s^\delta$, *where* $\mathbb{P}_s^\delta$ *is a sub-bimodule of* $\mathbb{M}_{\delta+1}$,

(k) the following diagram commutes for $n < \omega$, $s \in J_\delta$ *and* $\mathfrak{e} = \mathfrak{e}_s$ *(where* $\mathbb{P}^{\mathfrak{e}}, \mathbf{h}_n^{\mathfrak{e}}$ *are from Lemma 4.30 and the* $h_{t_{s,n}}$ *are from clause (E)):*

[8] we can ignore this first possibility as we can just shrink S if the result is stationary.

[9] where $\mathbb{P}^{\mathfrak{e}_s}$ is as in Lemma 4.30.

$$\begin{tikzcd} \mathbb{M}_\delta \cap \mathbb{P}^\delta_s \arrow[r, "\mathrm{id}"] & \mathbb{P}^\delta_s \\ \mathbb{N}^{\mathfrak{e}}_n \arrow[u, "(h_{t_{s,n}} + g^\delta_{s,n})"] \arrow[r, "\mathbf{h}^{\mathfrak{e}}_n"'] & \mathbb{P}^{\mathfrak{e}} \arrow[u, "h^\delta_s"'] \end{tikzcd}$$

(l) for $s \in J_\delta$ we have

(a) $\mathbb{P}^\delta_s \subseteq \mathbb{M}_{\delta+1}$,

(b) $\mathbb{M}_{\delta+1}$ *is generated by* $\mathbb{M}_\delta \cup \bigcup_{s \in J_\delta} \mathbb{P}^\delta_s$,

(c) $\mathbb{P}^\delta_s$, $\langle \mathbb{M}_\delta \cup \bigcup_{s' \neq s} \mathbb{P}^\delta_{s'} \rangle_{\mathbb{M}_{\delta+1}}$ *are amalgamated freely over*

$$\sum_{n<\omega} \mathrm{Rang}(h^\delta_s \circ \mathbf{h}^{\mathfrak{e}_s}_n),$$

(m) for $s \in J_\delta$ the type q_s with the free variable y has the form

$$\{\varphi^{\mathfrak{e}}_n(y - z_{s,n}) : n < \omega\},$$

where $\mathfrak{e} \in \mathfrak{E}$ and $z_{s,n} \in \mathbb{M}_\delta$,

(n) q_s is omitted by $\mathbb{M}_\alpha$ for $\alpha \in [\delta, \kappa]$, i.e. there is no $y \in \mathbb{M}_\alpha$ such that for all $n < \omega$, $\mathbb{M}_\alpha \models \varphi^{\mathfrak{e}}_n(y - z_{s,n})$.

In abuse of notation we sometimes use $\langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ as being $\mathcal{A}$, we of course may write $\mathbb{M}^{\mathcal{A}}_\alpha, J^{\mathcal{A}}_\alpha$ etc. and even $\kappa = \kappa^{\mathcal{A}}, \gamma_\alpha = \gamma^{\mathcal{A}}_\alpha, \lambda = \lambda^{\mathcal{A}}$. We now define various refinements of weakly semi-nice construction:

Definition 4.36. *We adopt the notation of Definition 4.35.*

(1) (a) By a weakly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$ we mean a weakly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$, where one of the following occurs:

- $\kappa = \mathrm{cf}(\lambda)$, $(\forall \alpha < \kappa)(\gamma^*_\alpha < \lambda)$ *and if* $\alpha < \beta < \kappa$ *then*

$$\gamma^*_{\alpha+1} - \gamma^*_\alpha \leq \gamma^*_{\beta+1} - \gamma^*_\beta,$$

- $\kappa \neq \mathrm{cf}(\lambda)$ *and* $(\forall \alpha < \lambda)$ *we have* $\gamma^*_\alpha + \lambda \leq \gamma^*_{\alpha+1} < (\lambda^{\aleph_0})^+$.

(b) if we omit κ*, i.e., write* $(\lambda, \mathfrak{m}, S)$ *we mean* $\kappa = \mathrm{cf}(\lambda)$.

(2) We omit "semi" from "weakly semi-nice construction" if whenever $\delta \in S$ *and* $s_1,\ s_2 \in J_\delta$ *then* $\bar{g}^\delta_{s_1} = \bar{g}^\delta_{s_2}$ *and* $\mathfrak{e}_{s_1} = \mathfrak{e}_{s_2}$.

(3) We omit "weakly" from "weakly semi-nice construction" if we add the following two properties $(G)_1$ *and* $(G)_2$*:*

$(G)_1$ *if* $\mathbf{f}$ *is an endomorphism of* $\mathbb{M}_\kappa$ *as an* $\mathbf{R}$*-module,* $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}, \gamma < \kappa$ *and* $\mathbf{g_n} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\gamma$ *is a bimodule homomorphism for each* $n = 1, 2, \ldots$, *then there are* $y \in \mathbb{M}_\kappa$, $z \in \mathbb{P}^{\mathfrak{e}}$ *and* $\{z_{n,i,\ell} : n, \ell < \omega, i < 2\}$ *with* $z_{n,i,\ell} \in \mathrm{Dom}(\mathbf{g}_\ell) = \mathbb{N}^{\mathfrak{e}}_\ell$ *such that for each* n *we have*

$$z_{n,i,\ell} = 0 \quad \forall \ell \gg 0.$$

Also, for all large enough n*, we have:*

(a) $z \in \sum_{\ell<\omega} \mathbf{h}^{\mathfrak{e}}_\ell(z_{n,1,\ell} - z_{n,0,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{P}^{\mathfrak{e}})$, *and*

(b) $\sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{e}}_\ell)) \in y + \sum_{\ell<\omega} \mathbf{g}_\ell(z_{n,1,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$,

$(G)_2$ *if* $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ *and for* $n < \omega$ *and* $\alpha \in \kappa \setminus S$, $\mathbf{h}_{\alpha,n}$ *is an embedding of* $\mathbb{N}^{\mathfrak{e}}_n$ *into* $\mathbb{M}_\kappa$ *such that*

$$\mathbb{M}_\alpha + \mathrm{Rang}(\mathbf{h}_{\alpha,n}) = \mathbb{M}_\alpha \oplus \mathrm{Rang}(\mathbf{h}_{\alpha,n}) \leq_{\aleph_0} \mathbb{M}_\kappa,$$

then there are an embedding $\mathbf{h}$ *of* $\mathbb{P}^{\mathfrak{e}}$ *into* $\mathbb{M}_\kappa$ *and ordinals* $\alpha_n \in \kappa \setminus S$ *for* $n = 1, 2, \ldots$ *such that* $\mathbf{h}_{\alpha_n,n} = \mathbf{h} \circ \mathbf{h}^{\mathfrak{e}}_n$*:*

$$\begin{tikzcd} & \mathbb{N}^{\mathfrak{e}}_n \arrow[dl, "\mathbf{h}^{\mathfrak{e}}_n"'] \arrow[d, "\mathbf{h}_{\alpha_n,n}"] \\ \mathbb{P}^{\mathfrak{e}} \arrow[r, "\mathbf{h}"'] & \mathbb{M}_\kappa \end{tikzcd}$$

for each such n.

(4) *We say $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \le \kappa \rangle$ is strongly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$ if $\mathbf{f}$ is an $\mathbf{R}$-endomorphism of $\mathbb{M}_\kappa$ and $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$, then there are $n(*) < \omega, \alpha < \lambda$ and $z \in \mathbb{N}^{\mathfrak{e}}_{n(*)}$ such that for every bimodule homomorphism $h : \mathbb{N}^{\mathfrak{e}}_{n(*)} \to \mathbb{M}_\kappa$,*

$$\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

(5) *We say $\mathbb{M}$ is weakly semi-nice (resp. semi-nice) with respect to $(\lambda, S, \mathfrak{m}, \kappa)$ if for some weakly semi-nice (resp. semi-nice) construction $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \le \kappa \rangle$ with respect to $(\lambda, S, \mathfrak{m}, \kappa)$ we have $\mathbb{M} = \mathbb{M}_\kappa$. If we omit S we mean for some $S \subseteq S^\kappa_{\aleph_0}$, similarly for λ and κ.*

Lemma 4.37. *Let $\langle \mathbb{M}_\alpha : \alpha \le \kappa \rangle$ be a semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$ and $\kappa(\mathfrak{E}^{\mathfrak{m}}) = \aleph_0$. The following assertions are valid:*

(1) *If $\alpha \notin S$ and $\alpha \le \beta \le \kappa$, then $\mathbb{M}_\alpha \le_{\aleph_0} \mathbb{M}_\beta$, i.e., $\mathbb{M}_\alpha$ is an almost direct $\mathcal{K}$-summand of $\mathbb{M}$ with respect to $\aleph_0$.*
(2) *If $\alpha \le \beta \le \kappa$ and $n < \omega$ then $\varphi^{\mathfrak{e}}_n(\mathbb{M}_\alpha) = \mathbb{M}_\alpha \cap \varphi^{\mathfrak{e}}_n(\mathbb{M}_\beta)$.*
(3) *Assume $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$, $\delta \in S$ and clause (F) of Definition 4.35 holds for δ. Suppose $s_0, \ldots, s_{k(*)-1}$ are distinct members of J_δ. Then*
    (a) *for each n, the set $\varphi^{\mathfrak{e}}_n\big(\langle \mathbb{M}_\delta \cup \bigcup\{\mathbb{P}^{\mathfrak{e}}_{s_k} : k < k(*)i\}\rangle_{\mathbb{M}_{\delta+1}}\big)$ is equal to*

$$\big\langle \mathbb{M}_\delta \cup \bigcup\{\mathbb{P}^{\mathfrak{e}}_{s_k} : k < k(*)\}\big\rangle_{\mathbb{M}_{\delta+1}} \cap \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa).$$

    (b) *Suppose $z_\ell \in \mathbb{P}^{\mathfrak{e}}$ for $\ell < n, z \in \mathbb{M}_\delta$, and $z + \sum\{h^\delta_{s_k}(z_k) : k < k(*)\} \in \varphi_n(\mathbb{M}_{\delta+1})$. Then, there is*

$$z'_k \in \sum\{\mathrm{Rang}(h^{\mathfrak{e}}_\ell) : \ell < \omega\}$$

equipped with the following two properties:

(b.1) : $z_k - z'_k \in \varphi_n(\mathbb{P}^{\mathfrak{e}})$ and

(b.2) : $z + \sum\{h^\delta_{s_k}(z'_k) : k < k(*)\} \in \varphi_n(\mathbb{M}_{\delta+1})$.

    (c) *$\langle \mathbb{M}_\delta \cup \{\mathbb{P}^{\mathfrak{e}}_{s_k} : k < k(*)\}\rangle_{\mathbb{M}_{\delta+1}} \le^{\mathrm{pr}}_{\varphi^{\mathfrak{e}}_n} \mathbb{M}_{\delta+1}$.*

(d) $\varphi_n^{\mathfrak{e}}(\mathbb{M}_{\delta+1}) = \varphi_n^{\mathfrak{e}}(\mathbb{M}_\delta) + \sum\{\varphi_n^{\mathfrak{e}}(\mathbb{P}_s^{\mathfrak{e}}) : s \in J_\delta\}$.

Proof. (1). We proceed by induction on $\beta \geq \alpha$. The straightforward details leave to reader.

(2). If $\alpha \notin S$, then the conclusion follows from (1) and Lemma 4.20. Now suppose that $\alpha \in S$. By Definition 4.35(F), the result holds if $\beta = \alpha + 1$. But then for any $\beta > \alpha$ (and since $\alpha + 1 \notin S$), we have

$$\mathbb{M}_\alpha \cap \varphi_n^{\mathfrak{e}}(\mathbb{M}_\beta) = \mathbb{M}_\alpha \cap (\varphi_n^{\mathfrak{e}}(\mathbb{M}_\beta) \cap \mathbb{M}_{\alpha+1}) = \varphi_n^{\mathfrak{e}}(\mathbb{M}_{\alpha+1}) \cap \mathbb{M}_\alpha = \varphi_n^{\mathfrak{e}}(\mathbb{M}_\alpha).$$

(3). Items (a), (c) and (d) can be proved easily by Definition 4.35(F). Let us prove (b). To this end, take $z_\ell \in \mathbb{P}^{\mathfrak{e}}$ for $\ell < k(*), z \in \mathbb{M}_\delta$, and

$$z + \sum\{h_{s_k}^\delta(z_k) : k < k(*)\} \in \varphi_n(\mathbb{M}_{\delta+1}).$$

In the light of (c) we observe that

$$z + \sum\{h_{s_k}^\delta(z_k) : k < k(*)\} \in \varphi_n^{\mathfrak{e}}(\langle \mathbb{M}_\delta \cup \{\mathbb{P}_{s_k}^{\mathfrak{e}} : k < k(*)\}\rangle_{\mathbb{M}_{\delta+1}}).$$

Let $k < k(*)$. We now apply the items (a), (c) and (d) to find $y \in \varphi_n^{\mathfrak{e}}(\mathbb{M}_\delta)$ and $y_k \in \varphi_n^{\mathfrak{e}}(\mathbb{P}_{s_k}^{\mathfrak{e}})$ such that

$$z + \sum\{h_{s_k}^\delta(z_k) : k < k(*)\} = y + \sum\{y_k : k < k(*)\}.$$

Since $\varphi_n^{\mathfrak{e}}(\mathbb{P}_{s_k}^{\mathfrak{e}})$ is the image of $\varphi_n^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$ under $h_{s_k}^\delta$, we have $y_k = h_{s_k}^\delta(x_k)$ for some $x_k \in \varphi_n^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$. Then

$$\sum\{h_{s_k}^\delta(z_k - x_k) : k < k(*)\} = y - z \in \varphi_n^{\mathfrak{e}}(\mathbb{M}_\delta).$$

As $s_0, \cdots, s_{k(*)-1}$ are chosen distinct, we deduce that

$$h_{s_k}^\delta(z_k - x_k) \in \mathrm{Rang}(h_{s_k}^\delta) \cap \varphi_n^{\mathfrak{e}}(\mathbb{M}_\delta) = \varphi_n^{\mathfrak{e}}(\mathrm{Rang}(h_{s_k}^\delta)),$$

where $k < k(*)$. Hence, $z_k - x_k \in \varphi_n^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$.

Let $h < \omega$ be such that for each $k < k(*)$,

$$\begin{aligned} x_k &\in \varphi_n^{\mathfrak{c}}\left(\bigoplus_{\ell \leq h}(\mathrm{Rang}(\mathbf{f}_n^{\mathfrak{c}}))\right) \\ &= \bigoplus_{\ell \leq h} \varphi_n^{\mathfrak{c}}(\mathrm{Rang}(\mathbf{f}_\ell^{\mathfrak{c}})) \\ &= \bigoplus_{\ell < h} \varphi_n^{\mathfrak{c}}(\mathrm{Rang}(\mathbf{h}_\ell^{\mathfrak{c}})) \oplus \varphi_n^{\mathfrak{c}}(\mathrm{Rang}(\mathbf{f}_h^{\mathfrak{c}})). \end{aligned}$$

Thus, for some $z'_k \in \bigoplus_{\ell < h} \varphi_n^{\mathfrak{c}}(\mathrm{Rang}(\mathbf{h}_\ell^{\mathfrak{c}}))$, we have

$$x_k - z'_k \in \varphi_n^{\mathfrak{c}}(\mathrm{Rang}(\mathbf{f}_h^{\mathfrak{c}})) \subseteq \varphi_n^{\mathfrak{c}}(\mathbb{P}^{\mathfrak{c}}).$$

It then follows that

$$z_k - z'_k = (z_k - x_k) + (x_k - z'_k) \in \varphi_n^{\mathfrak{c}}(\mathbb{P}^{\mathfrak{c}}) + \varphi_n^{\mathfrak{c}}(\mathbb{P}^{\mathfrak{c}}) = \varphi_n^{\mathfrak{c}}(\mathbb{P}^{\mathfrak{c}}).$$

We now show that $w := z + \sum\{h^\delta_{s_k}(z'_k) : k < k(*)\} \in \varphi_n(\mathbb{M}_{\delta+1})$. Indeed,

$$\begin{aligned} w &= z + \sum\{h^\delta_{s_k}(z'_k - z_k) : k < k(*)\} + \sum\{h^\delta_{s_k}(z_k) : k < k(*)\} \\ &= (z + \sum\{h^\delta_{s_k}(z_k) : k < k(*)\}) + \sum\{h^\delta_{s_k}(z'_k - z_k) : k < k(*)\} \\ &\in \varphi_n(\mathbb{M}_{\delta+1}) + \sum\{\varphi_n^{\mathfrak{c}}(\mathbb{P}^\delta_{s_k}) : k < k(*)\} \\ &= \varphi_n(\mathbb{M}_{\delta+1}). \end{aligned}$$

This completes the proof of (b) and hence of the lemma. $\square$

The main result of this section is that under suitable assumptions on $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$, *there is, in ZFC, a semi-nice construction for it. Before, we state and prove our result, let us show that under extra set theoretic assumptions we can get a stronger result.*

Definition 4.38. *Suppose* $\lambda = \mathrm{cf}(\lambda) > |\mathbf{R}| + |\mathbf{S}| + \aleph_0$, $S \subseteq S^\lambda_{\aleph_0}$ *is stationary such that* $S^\lambda_{\aleph_0} \setminus S$ *is stationary as well. Then* $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \lambda \rangle$ *is called very nicely constructed with respect to* $(\lambda, \mathfrak{m}, S)$ *if the following properties are satisfied:*

1. *Clauses (A)-(E) of Definition 4.35 holds.*
2. *Each* J_α *is singleton.*

(3) *For each $\mathbb{N} \in \mathcal{K}$, for stationary many $\alpha \in \lambda \setminus S$, $\mathbb{N}$ appears as one of the summands of $\mathbb{M}_{\alpha+1}$.*

(4) *Replace clause (F) of Definition 4.35 by the following modification:*

For $\delta \in S$, $\mathbb{M}_{\delta+1}$ is defined either as in (E), or else there are an infinite $\mathcal{U} \subseteq \omega$ and the sequences $\langle \alpha_n : n < \omega \rangle$ and $\langle \beta_m(\alpha_n) : m \in \mathcal{U} \rangle$ and $\langle h_{\alpha_n,n} : n \in \mathcal{U} \rangle$ such that the following items $(a), \ldots, (g)$ are hold:

(a) $\alpha_n \in \lambda \setminus S$.

(b) $\langle \alpha_n : n < \omega \rangle$ *is increasing and cofinal in* δ.

(c) $\mathfrak{e}_\delta = \mathfrak{e}$.

(d) *for each* $n < \omega$ *and* $m \in \mathcal{U}$, $\alpha_n < \beta_m(\alpha_n) < \alpha_{n+1}$ *is in* $\lambda \setminus S$.

(e) *for* $n \in \mathcal{U}$, $h_{\alpha_n,n} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_{\beta_n(\alpha_n)}$ *is a bimodule homomorphism and*

$$\mathbb{M}_{\alpha_n} + \mathrm{Rang}(h_{\alpha_n,n}) = \mathbb{M}_{\alpha_n} \oplus \mathrm{Rang}(h_{\alpha_n,n}) \leq_{\aleph_0} \mathbb{M}_{\beta_m(\alpha_n)}.$$

(f) *Set* $\mathbb{P}_{\mathcal{U}} := \bigoplus_{n\in\mathcal{U}} \mathrm{Rang}(\mathbf{f}^{\mathfrak{e}}_n)$ *and* $\mathbb{P}_{\mathcal{U},n} := \sum\limits_{\ell \in \mathcal{U}\cap n} \mathrm{Rang}(\mathbf{h}^{\mathfrak{e}}_\ell)$. *Recall from Lemma 4.30 that there are bimodule homomorphisms* $\mathbf{h}^{\mathfrak{e}}_n : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{P}^{\mathfrak{e}}$ *and* $\mathbf{f}^{\mathfrak{e}} : \mathbb{N}^{\mathfrak{e}}_0 \to \mathbb{P}^{\mathfrak{e}}$. *We define* $\mathbb{N}^*_\delta := \sum\limits_{n\in\mathcal{U}} h_{\alpha_n,n}(\mathbb{N}_n)$. *Then* $\mathbb{P}^{\mathfrak{e}}$ *is isomorphic to* $\mathbb{P}_{\mathcal{U}}$ *by an isomorphism* h_δ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{N}_n \arrow[r, "h_{\alpha_n,n}"] \arrow[d, "\mathbf{h}^{\mathfrak{e}}_m"'] & h_{\alpha_n,n}(\mathbb{N}_n) \arrow[d, "\mathrm{id}"] \\
\mathbb{P}_{\mathcal{U}} & \mathbb{P}^{\mathfrak{e}} \arrow[l, "h_\delta"]
\end{tikzcd}$$

where the notation $n = i(m)$ *stands for the* m*-th member of* $\mathcal{U}$.

(g) $\mathbb{P}^{\mathfrak{e}}_n = h''_\delta(\mathbb{P}_{\mathcal{U},n})$. *So, taking intersection inside* $\mathbb{M}_{\delta+1}$ *gives us* $\mathbb{P}^{\mathfrak{e}} \cap \mathbb{M}_\delta = \mathbb{N}^*_\delta$.

Discussion 4.39. *Recall that for a stationary set $S \subseteq \lambda$, Jensen's diamond $\diamondsuit_\lambda(S)$ asserts the existence of a sequence $\langle S_\alpha : \alpha \in S \rangle$ such that for every $X \subseteq \lambda$ the set*

$$\{\alpha \in S : X \cap \alpha = S_\alpha\}$$

is stationary. It is easily seen that $\diamondsuit_\lambda(S)$ implies $2^{<\lambda} = \lambda$. One the one hand, due to a well-known theorem of Jensen, $\diamondsuit_\lambda(S)$ holds in Gödel's constructible universe L for all uncountable regular cardinals λ and all stationary sets $S \subseteq \lambda$. On the other hand, if $2^{<\lambda}$ is forced to be above λ, then $\diamondsuit_\lambda(S)$ fails for all stationary sets $S \subseteq \lambda$.

The next result of Shelah shows that under diamond principle, we can get very nicely $\langle \mathbb{M}_\alpha : \alpha \leq \lambda \rangle$.

Lemma 4.40. *Suppose $\mathfrak{m}$ is a non-trivial context, $\lambda = \mathrm{cf}(\lambda) > |\mathbf{R}| + |\mathbf{S}| + \kappa(\mathfrak{m}) + ||\mathfrak{m}||$, $S \subseteq S^\lambda_{\aleph_0}$ is a stationary non-reflecting*[10] *subset of λ and $\diamondsuit_\lambda(S)$ holds. Then there is a very nicely construction with respect to $(\lambda, \mathfrak{m}, S)$ such that*

(1) $(\lambda, \mathfrak{m}, S)$ is strongly very nice.

(2) If $\mathrm{cf}(\alpha) \in \kappa \setminus S$ and $\alpha < \beta \leq \lambda$, then $\mathbb{M}_\alpha$ is a $\mathcal{K}$- directed summand of $\mathbb{M}_\beta$.

Proof. See [46, Section 2]. □

We would like to prove a similar result in ZFC, thus we have to avoid the use of diamond in the above lemma. To this end, we use Shelah's Black Box. This leads us to get a weaker conclusion than Lemma 4.40, but as we will see later, it is sufficient to prove Theorem 1.1.

Notation 4.41. *For the rest of this section we use the following:*

1) *$\lambda > \kappa$ be infinite cardinals.*

[10] S is non-reflecting if for all limit ordinals $\xi < \lambda$ of uncountable cofinality, $S \cap \xi$ is non-stationary.

2) *$\mathcal{H}_{<\theta}(X)$ be the least set Y such that $Y \supseteq X$ and if $y \subseteq Y$ and $|y| < \theta$, then $y \in Y$.*

3) *$\langle \tau_n : n < \omega \rangle$ be an increasing sequence of vocabularies, each of size $\leq \kappa$ such that for each $n < \omega$, there exists some unary predicate P_n in $\tau_{n+1} \setminus \tau_n$.*

Definition 4.42. *For $n < \omega$ let $\mathcal{F}_n$ be the family of sets of the form*

$$\{(\mathcal{N}_\ell, f_\ell) : \ell \leq n\}$$

satisfying the following conditions:

(a) *$f_\ell : \kappa^{\leq \ell} \to \lambda^{\leq \ell}$ is a tree embedding, i.e.,*
   (1) *for each $\eta \in \kappa^{\leq \ell}$, η and $f_\ell(\eta)$ have the same length,*
   (2) *for $\eta \lhd \nu$ in $\kappa^{\leq \ell}$, $f_\ell(\eta) \lhd f_\ell(\nu)$,*
   (3) *f_ℓ is one-to-one,*

(b) *for $\ell + 1 \leq n$, $f_{\ell+1}$ extends f_ℓ,*

(c) *for some $\tau'_\ell \subseteq \tau_\ell$, $\mathcal{N}_\ell$ is a τ'_ℓ-structure of size $\leq \kappa$ and the universe of $\mathcal{N}_\ell$, denoted by N_ℓ, is a subset of $\mathcal{H}_{<\kappa^+}(\lambda)$,*

(d) *$\tau'_{\ell+1} \cap \tau_\ell = \tau'_\ell$ and $\mathcal{N}_{\ell+1} \restriction \tau'_\ell$ extends $\mathcal{N}_\ell$,*

(e) *if $P_m \in \tau'_{m+1}$, then $P_m^{\mathcal{N}_\ell} = N_\ell$,*

(f) *if $x, y \in N_\ell$, then $\{x, y\} \in N_\ell$ and $\emptyset \in N_\ell$,*

(g) *$\mathrm{Rang}(f_\ell) \subseteq \mathcal{N}_\ell$.*

Definition 4.43. *Let $\langle \mathcal{F}_n : n < \omega \rangle$ be as in Definition 4.42. Then $\mathcal{F}_\omega$ is the family consisting of all pairs $(\mathcal{N}, f)$ such that for some sequence $\langle (\mathcal{N}_\ell, f_\ell) : \ell < \omega \rangle$, we have*

(a) *for each $n < \omega$, $\{(\mathcal{N}_\ell, f_\ell) : \ell \leq n\}$ belongs to $\mathcal{F}_n$,*

(b) *$f = \bigcup_{\ell<\omega} f_\ell$ and $\mathcal{N} = \bigcup_{\ell<\omega} \mathcal{N}_\ell$.*

We may note that if for each $m < \omega, P_m \in \tau'_{m+1}$, then for each $(\mathcal{N}, f) \in \mathcal{F}_\omega$, there is a unique sequence $\langle (\mathcal{N}_\ell, f_\ell) : \ell < \omega \rangle$ witnessing this.

Definition 4.44. *Let $(\mathcal{N}, f) \in \mathcal{F}_\omega$. A branch of f is any $\eta \in \lambda^\omega$ such that for each $n < \omega$, $\eta \restriction n \in \mathrm{Rang}(f)$. We use $\lim(f)$ for the set of all branches of f.*

Given $W \subseteq \mathcal{F}_\omega$, we define two games $\Game_W$ and $\Game'_W$ as follows.

Definition 4.45. *Suppose $W \subseteq \mathcal{F}_\omega$.*

(1) The game $\Game_W$ of length ω is defined as follows. In the n^{th} move, player I chooses $\mathcal{N}_n$ such that $\{(\mathcal{N}_\ell, f_\ell) : \ell \leq n\} \in \mathcal{F}_n$ (noting that f_0 is determined), and player II chooses a tree embedding $f_n : \kappa^{\leq n} \to \lambda^{\leq n}$ extending $\bigcup\limits_{\ell<n} f_\ell$ such that $\mathrm{Rang}(f_n) \setminus (\bigcup\limits_{\ell<n} \mathrm{Rang}(f_\ell))$ is disjoint to $\bigcup\limits_{\ell<n} N_\ell$. Player II wins if $(\bigcup\limits_{\ell<\omega} \mathcal{N}_\ell, \bigcup\limits_{\ell<\omega} f_\ell) \in W$.

(2) The game $\Game'_W$ of length ω is defined as follows. In the zero move, player I chooses $k < \omega$, $\{(\mathcal{N}_\ell, f_\ell) : \ell \leq k\} \in \mathcal{F}_k$, and $X_0 \subseteq \lambda^{<\omega}$ of size less than λ. For $n > 0$, in the n^{th} move, player I chooses $\mathcal{N}_{k+n}$ and X_n such that $\{(\mathcal{N}_\ell, f_\ell) : \ell \leq k+n\} \in \mathcal{F}_{k+n}$ and $X_n \subseteq \lambda^{<\omega}$ is of size less than λ. Then player II chooses a tree embedding $f_{k+n} : \kappa^{\leq k+n} \to \lambda^{\leq k+n}$ extending $\bigcup\limits_{\ell<k+n} f_\ell$ such that $\mathrm{Rang}(f_{k+n}) \setminus (\bigcup\limits_{\ell<k+n} \mathrm{Rang}(f_\ell))$ is disjoint to $\bigcup\limits_{\ell<n} N_\ell \cup \bigcup\limits_{\ell<n} X_\ell$. Player II wins if $(\bigcup\limits_{\ell<\omega} \mathcal{N}_\ell, \bigcup\limits_{\ell<\omega} f_\ell) \in W$.

Definition 4.46. *Suppose $W \subseteq \mathcal{F}_\omega$. Recall that*

(1) W is called a barrier if player I does not win $\Game_W$ or even $\Game'_W$.

(2) W is called a strong barrier if player II wins $\Game_W$ and even $\Game'_W$.

(3) W is called disjoint if for distinct $(\mathcal{N}^1, f^1), (\mathcal{N}^2, f^2)$ in W, f^1 and f^2 have no common branch.

We are now ready to state the version of Shelah's black box theorem that is needed in this paper.

Lemma 4.47 (The Black Box Theorem)**.** *Suppose $\lambda > \kappa$ are infinite cardinals, $\mathrm{cf}(\lambda) > \aleph_0$, $\lambda^{\aleph_0} = \lambda^\kappa$ and $S \subseteq S^\lambda_{\aleph_0}$ is stationary. Then there is a sequence*

$$W = \{(\mathcal{N}^\alpha, f^\alpha) : \alpha < \alpha_*\} \subseteq \mathcal{F}_\omega$$

and a non-decreasing function

$$\zeta : \alpha_* \to S$$

such that the following properties are satisfied:

(1) *W is a disjoint strong barrier,*

(2) *every branch of f^α is an increasing sequence converging to $\zeta(\alpha) \in S$,*

(3) *each $\mathcal{N}^\alpha$ is transitive,*

(4) *if $\zeta(\beta) = \zeta(\alpha), \beta + \kappa^{\aleph_0} \leq \alpha < \alpha_*$ and η is a branch of f^α, then for some $k < \omega, \eta \restriction k \notin \mathcal{N}^\beta$,*

(5) *if $\lambda = \lambda^\kappa$, we can also demand that if η is a branch of f^α and $\eta \restriction k \in \mathcal{N}^\beta$ for all $k < \omega$, then $\mathcal{N}^\alpha \subseteq \mathcal{N}^\beta$, and even for all $n < \omega, \mathcal{N}^\alpha_n \in \mathcal{N}^\beta$.*

Proof. This is in [44, 2.8]. □

We now state and prove the main result of this section.

Theorem 4.48. *Assume $\mathfrak{m}$ is a non-trivial context. Suppose $\lambda > ||\mathfrak{m}||$ is such that $\mathrm{cf}(\lambda) \geq \aleph_1 + \kappa(\mathfrak{m})$, for $\mu < \lambda, \mu^{\aleph_0} < \lambda$, $\kappa = \mathrm{cf}(\lambda) > \kappa(\mathfrak{m})$ and $S \subseteq S^\kappa_{\aleph_0}$ is such that S and $S^\kappa_{\aleph_0} \setminus S$ are stationary in κ. Then*

(i) *There is a semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$.*

(ii) *If in addition λ is regular, then there is a nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$.*

(iii) *In part (i) if we omit the assumption $\mathrm{cf}(\lambda) = \kappa$, and let $\bar{\gamma}^* = \langle \gamma^*_\alpha : \alpha \leq \kappa \rangle$ be such that $\gamma^*_0 = ||\mathbb{M}_*||$ and for all $\alpha < \kappa$ $\gamma^*_{1+\alpha} = \gamma^*_0 + \lambda \cdot \alpha$. Then there is a semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$.*

(iv) *In part (iii) if $\kappa = \lambda$, then there is a nice construction with respect to $(\lambda, \mathfrak{m}, S, \bar{\gamma}^*)$.*

Proof. (i) We start by fixing some notation and facts:

(i_1) Without loss of generality for $\mathfrak{e} \in \mathfrak{E}$ and $n < \omega$, the universe of $\mathbb{N}^{\mathfrak{e}}_n$ is a cardinal.

(i_2) As $\mathrm{cf}(\lambda) = \kappa$, let $\bar{\gamma} = \langle \gamma^0_\alpha : \alpha < \kappa \rangle$ be an increasing and continuous sequence cofinal in λ.

(i_3) Let $\bar{\gamma}^* = \langle \gamma^*_\alpha : \alpha \leq \kappa \rangle$ be an increasing and continuous sequence of ordinals with limit λ such that $\gamma^*_0 > ||\mathfrak{m}||$ and for each $\alpha < \kappa, \gamma^*_{\alpha+1} - \gamma^*_\alpha$ is > 0 and divisible by $||\mathfrak{m}||^{\aleph_0} + \gamma^*_\alpha$. We further assume that for each $\alpha < \kappa, \gamma^*_\alpha \geq \gamma_\alpha$.

(i_4) Let $\langle \mathbb{N}^*_\alpha : \alpha < |\mathcal{K}| \rangle$ be an enumeration of elements of $\mathcal{K}$.

Set

$$\bar{S} := \{\gamma^*_\alpha : \alpha \in S\}.$$

Then $\bar{S}$ is a stationary subset of λ. For each $n < \omega$, let τ_n be the following countable vocabulary

$$\tau_n := \{\in, F, G, P_0, \cdots, P_{n-1}, c_0, c_1, \cdots, c_i, \cdots\},$$

where F, G are unary function symbols, $P_0, \cdots, P_{n-1}$ are unary predicate symbols and c_i's, for $i < \omega$ are constant symbols. We now apply the black box theorem (see Lemma 4.47) to get a sequence

$$W = \{(\mathcal{N}^\alpha, f^\alpha) : \alpha < \alpha_*\} \subseteq \mathcal{F}_\omega$$

and a non-decreasing function

$$\zeta : \alpha_* \to \bar{S}.$$

By induction on $\epsilon \leq \kappa$ we will construct $\mathcal{A}_\epsilon$ which is going to be a semi-nice construction up to ϵ in the following sense:

(a) $\mathcal{A}_\epsilon$ consists of

(a-1) $\langle \mathbb{M}_\xi : \xi \leq \epsilon \rangle$,

(a-2) $\langle \mathbb{N}_t, h_t : t \in J_\xi, \xi \in \epsilon \setminus S_\epsilon \rangle$,

(a-3) $S_\epsilon \subseteq S \cap \epsilon$,

(a-4) $\mathcal{T}_\epsilon$,

(a-5) $\langle (\mathfrak{e}_s, \mathbb{P}_s, \bar{\alpha}_s, \bar{t}_s, \bar{g}_s, h_s, q_s) : s \in J_\delta \text{ and } \delta \in S_\epsilon \rangle$,

(a-6) $\langle \gamma^*_\xi : \xi \leq \epsilon \rangle$.

(b) $\mathcal{A}$ satisfies all the relevant parts of Definition 4.35.

(c) we have

(c-1) if $\xi < \epsilon$, then $S_\xi = S_\epsilon \cap \xi$,

(c-2) for $\delta \in S \cap \epsilon$,

$$J_\delta \subseteq \{\beta < \alpha_* : \zeta(\beta) = \gamma_\delta\},$$

(c-3) if $\epsilon \notin S$, then $\epsilon \notin S_{\epsilon+1}$ and

$$J_\epsilon = \{\beta : \exists \eta \in \mathcal{T}_\epsilon,\ \eta^\frown\langle\beta\rangle \in \mathcal{T}_{\epsilon+1}\}.$$

(d) Let $\langle \gamma'_\epsilon : \epsilon < \kappa\rangle$ be an increasing and continuous sequence of ordinals cofinal in λ with $\gamma'_0 = \|\mathbb{M}^{\mathfrak{m}}_*\|$. The sequence is such that

(d-1) $\langle (n_\gamma, \mathfrak{e}_\gamma, f_\gamma) : \gamma < \gamma'_\epsilon\rangle$ is a collection of triples $(n, \mathfrak{e}, f)$ where $n < \omega$, $\mathfrak{e} \in \mathfrak{E}$ and f is a bimodule homomorphism from $\mathbb{N}^{\mathfrak{e}}_n$ into $\mathbb{M}_\epsilon$,

(d-2) every triple $(n, \mathfrak{e}, f)$ as above appears as some $(n_\gamma, \mathfrak{e}_\gamma, f_\gamma)$, for some $\gamma < \gamma'_\epsilon$, for some $\epsilon < \kappa$ large enough.

(e) we have

(e-1) $\langle \mathcal{T}_\epsilon : \epsilon < \kappa\rangle$ is increasing and continuous,

(e-2) $\mathcal{T}_0 = \{<>\}$,

(e-3) if $\mathcal{T}_{\epsilon+1} \setminus \mathcal{T}_\epsilon \neq \emptyset$, then

$$\mathcal{T}_{\epsilon+1} = \{\eta^\frown\langle\beta\rangle : \eta \in \mathcal{T}_\epsilon \text{ and } \beta \in [\gamma'_\epsilon, \gamma'_{\epsilon+1})\}.$$

We now start defining $\mathcal{A}_\epsilon$ for $\epsilon \leq \kappa$.

We distinguish five possibilities for ϵ:

ϵ_1) $\epsilon = 0$,

ϵ_2) ϵ is a limit ordinal,

ϵ_3) $\epsilon = \xi + 1$ and $\xi \notin S$,

ϵ_4) $\epsilon = \delta + 1$, $\delta \in S$ and $\gamma^*_\delta \neq \gamma^0_\delta$,

ϵ_5) $\epsilon = \delta + 1$, $\delta \in S$ and $\gamma^*_\delta = \gamma^0_\delta$.

In the case ϵ_1), we set $\mathbb{M}_0 := \mathbb{M}_*$ and let γ_0^* be the universe of $\mathbb{M}_*$. In the case ϵ_2), we set $\mathbb{M}_\epsilon := \bigcup_{\xi<\epsilon} \mathbb{M}_\xi$, and for everything else, just take the union of the previous ones. Suppose we are in the situation of ϵ_3). In the same vein as of Definition 4.35(E), we define $\mathbb{M}_\epsilon$, so

$$\mathbb{M}_\epsilon := \mathbb{M}_\xi \oplus \bigoplus_{t\in J_\xi} \mathbb{N}_t,$$

where J_ξ and $\mathbb{N}_t$, for $t \in J_\xi$ are defined similar to Definition 4.35(E). Everything else in the definition of $\mathcal{A}_\epsilon$ is clear how to define.

In the case of ϵ_4) we define $\mathbb{M}_\epsilon$ as in the previous case (i.e., as in Definition 4.35(E)). Note that in all of the above cases, if $\delta < \epsilon$ is in S, $s \in J_\delta$ and in stage δ the type $q_s = q_s^\delta$ is defined, then in view of Lemma 3.14 (and its natural generalization to arbitrary direct sums), q_s continues to be omitted at ϵ.

Finally, we deal with ϵ_5) which is the most important part of the induction which corresponds to the case (F) of Definition 4.35. Set

$$\alpha_\delta^* = \sup\{\alpha + 1 : \zeta(\alpha) \leq \gamma_\delta^0\}.$$

By induction on $\alpha \leq \alpha_\delta^*$ we define

- $\mathbb{M}_{\delta,\alpha}$,
- $w_{\delta,\alpha}$,
- η_β for $\beta \in w_{\delta,\alpha}$,
- $\mathfrak{x}_\beta = (\mathfrak{e}_\beta, \mathbb{P}_\beta, \bar{\alpha}_\beta, \bar{t}_\beta, \bar{g}_\beta, h_\beta, q_\beta)$ and η_β for $\beta \in w_{\delta,\alpha}$,

such that:

(α) $w_{\delta,\alpha} \subseteq \{\beta < \alpha : \zeta(\beta) = \gamma_\delta^0\}$ and for $\beta < \alpha$, $w_{\delta,\alpha} \cap \beta = w_{\delta,\beta}$.

(β) $\eta_\beta \in \lim(f^\beta)$ for $\beta \in w_{\delta,\alpha}$.

(γ) clause (F) of Definition 4.35 is satisfied when we consider

$$(M_{\delta,\alpha}, w_{\delta,\alpha}, \langle \mathfrak{x}_\beta : \beta \in w_{\delta,\alpha}\rangle)$$

instead of

$$(\mathbb{M}_{\delta+1}, J_\delta, \langle(\mathfrak{e}_s, \mathbb{P}_s, \bar{\alpha}_s, \bar{t}_s, \bar{g}_s, h_s, q_s) : s \in J_\delta\rangle).$$

We distinguish four possibilities for α:

α_1) $\alpha = 0$;

α_2) α is a limit ordinal;

α_3) $\alpha = \beta + 1$ and $\zeta(\beta) \neq \gamma^0_\delta$;

α_4) $\alpha = \beta + 1$ and $\zeta(\beta) = \gamma^0_\delta$.

In the case α_1), we set $\mathbb{M}_{\delta,0} = \mathbb{M}_\delta$ and $w_{\delta,0} = \emptyset$. Recall that there is nothing else to define. In the case α_1), we set $M_{\delta,\alpha} := \bigcup\limits_{\xi<\alpha} M_{\delta,\xi}$ and $w_{\delta,\alpha} := \bigcup\limits_{\xi<\alpha} w_{\delta,\xi}$. Also, for $\beta \in w_{\delta,\alpha}$ pick some $\xi < \alpha$ such that $\beta \in w_{\delta,\xi}$ and then define η_β and $\mathfrak{x}_\beta$ as defined at step ξ of the induction.

In the case α_3), we set $\alpha = \beta + 1$ and $\zeta(\beta) \neq \gamma^0_\delta$, here we set $M_{\delta,\alpha} = M_{\delta,\beta}$ and $w_{\delta,\alpha} = w_{\delta,\beta}$.

Finally, we deal with the forth possibility: In this case we decide if we add β to our final J_δ or not, where J_δ is going to be $w_{\delta,\alpha^*_\delta}$. So, we consider two possibilities: case α_4.1) and case α_4.2), see below:

Case α_4.1): There are $i < 2$ and $\eta \in \lim(f^\beta)$ such that

$$(\mathbb{M}^{\eta,i}_{\delta,\beta}, w_{\delta,\beta+1}, \langle \mathfrak{x}_\nu : \nu \in w_{\delta,\beta+1} \rangle) \quad (+)$$

satisfies the conclusion of clause (F) of Definition 4.35, where (+) is defined as follows:

(δ) $w_{\delta,\alpha} := w_{\delta,\beta} \cup \{\beta\}$,

(ϵ) $\eta_\beta := \eta$,

(ε) $\mathfrak{x}_\beta := (\mathfrak{e}_\beta, \mathbb{P}_\beta, \bar{\alpha}_\beta, \bar{t}_\beta, \bar{g}_\beta, h_\beta, q_\beta)$, where this sequence is defined as follows:

(ε-1) $\bar{t}_\beta := \langle t_n : n < \omega \rangle$, where for each $n < \omega, t_n = G^{\mathcal{N}^\beta}(\eta(n))$,

(ε-2) for some $\alpha_n < \delta, t_n \in J_{\alpha_n}$ (note that such an α_n is unique if it exists),

(ε-3) $\bar{\alpha}_\beta := \langle \alpha_n : n < \omega \rangle$ is an increasing sequence of ordinals in $\delta \setminus S$ cofinal in δ,

(ε-4) for each $n, m < \omega, \mathfrak{e}_{t_n} = \mathfrak{e}_{t_m}$. Set $\mathfrak{e} := \mathfrak{e}_{t_0}$,

(ε-5) $\{x^{\mathfrak{e}}_n, h_{t_n}, t_n, \alpha_n\} \in N^\beta_{n+1}$,

(ε-6) $\bar{g}_\beta := \bar{g}^{\eta,i}_\beta := \langle g^{\eta,i}_{\beta,n} : n < \omega\rangle$ where $g_n = g^{\eta,i}_{\beta,n} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_{\gamma^{**}_\delta}$, for some γ^{**}_δ, is defined by $g_n = i \cdot f_{c^{\mathcal{N}^\beta}_n}$,

(ε-7) $z^{\eta,i}_{\beta,n} := F^{\mathcal{N}^\beta}\big(\sum\limits_{\ell<n}(g_\ell + h_{t_\ell})(x^{\mathfrak{e}}_\ell)\big)$,

(ε-8) $q_\beta := q^{\eta,i}_{\delta,\beta} := \{\varphi^{\mathfrak{e}}_n(y - z^{\eta,i}_{\beta,n}) : n < \omega\}$,

(ε-9) $h_\beta := h^{\eta,i}_{\delta,\beta} : \mathbb{P}^{\mathfrak{e}} \to \mathbb{P}_\beta$ is onto,

(ε-10) $\mathbb{M}^{\eta,i}_{\delta,\beta}$ is generated by $\mathbb{M}_{\delta,\beta} \cup \mathbb{P}_\beta$.

In this case we set

- $\mathbb{M}_{\delta,\beta+1} = \mathbb{M}^{\eta,i}_{\delta,\beta}$,
- $w_{\delta,\beta+1} = w_{\delta,\beta} \cup \{\beta\}$,
- $\eta_\beta = \eta$,
- for $\nu \in w_{\delta,\beta+1}$ we let $\mathfrak{x}_\nu$ be as above.

Case $\alpha_4.2$): The above situation does not hold. Then we set $\mathbb{M}_{\delta,\beta+1} := \mathbb{M}_{\delta,\beta}$ and $w_{\delta,\beta+1} := w_{\delta,\beta}$.

Having considered the above cases, we set

- $\mathbb{M}_{\delta+1} := \mathbb{M}_{\delta,\alpha^*_\delta}$,
- $J_\delta := w_{\delta,\alpha^*_\delta}$,
- for $s \in J_\delta$, $\langle(\mathfrak{e}_s, \mathbb{P}_s, \bar{\alpha}_s, \bar{t}_s, \bar{g}_s, h_s, q_s) : s \in J_\delta\rangle$ is defined to be $\mathfrak{x}_s$.

This completes our inductive construction of $\mathcal{A}_\epsilon$'s for $\epsilon \leq \kappa$. Finally, we set $\mathcal{A} := \mathcal{A}_\kappa$ and $\mathbb{M} := \mathbb{M}_\kappa$. We are going to show that $\mathcal{A}$ is as required.

Clearly, we have gotten a weakly semi-nice construction. We now show that it is indeed a semi-nice construction. Let us prove the property presented in Definition 4.36(3)$(G)_1$. Thus suppose that $\mathfrak{e} \in \mathfrak{E}$, $\mathbf{f}$ is an endomorphism of $\mathbb{M}$ as an $\mathbf{R}$-module, $\gamma < \kappa$ and for each $n < \omega$ suppose $\mathbf{g_n} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\gamma$ is a bimodule homomorphism. Suppose by contradiction that the property presented in Definition 4.36(3)$(G)_1$ fails. Let

$$\mathbf{g} : \mathcal{T}_\kappa \to \mathcal{T}_\kappa,$$

where $\mathcal{T}_\kappa = \lambda^{<\omega}$, be a one-to-one order preserving (i.e, $\eta \lhd \nu$ iff $\mathbf{g}(\eta) \lhd \mathbf{g}(\nu)$) embedding such that for each $\eta \in \mathcal{T}_\kappa \setminus \mathcal{T}_0, \mathfrak{e}_{\mathbf{g}(\eta)} = \mathfrak{e}$. Let

$$\mathcal{B} = \langle B, \in^{\mathcal{B}}, F^{\mathcal{B}}, G^{\mathcal{B}}, (c_n^{\mathcal{B}})_{n<\omega}, (P_n^{\mathcal{B}})_{n<\omega}\rangle$$

be a $\tau = \bigcup_{n<\omega} \tau_n$-structure satisfying the following properties:

$\tau_1)$: $(\mathcal{B}, \in^{\mathcal{B}})$ expands $(\mathcal{H}_{<\aleph_1}(\mathbb{M}), \in)$,

$\tau_2)$: $F^{\mathcal{B}} \restriction \mathbb{M} = \mathbf{f}$,

$\tau_3)$: $G^{\mathcal{B}} = \mathbf{g}$,

$\tau_4)$: for $n < \omega$, $c_n^{\mathcal{B}} = \gamma_n$, where γ_n is such that $(n, \mathfrak{e}, \mathbf{g}_n) = (n_{\gamma_n}, \mathfrak{e}_{\gamma_n}, f_{\gamma_n})$.

Pick $\gamma^{**} \in \kappa \setminus S$ such that $\gamma^{**} > \gamma$ and for each $n < \omega$, $\mathbf{f}(\mathbf{g}_n(x_n^{\mathfrak{e}})) \in \mathbb{M}_{\gamma^{**}}$. Recall that $\mathrm{Rang}(\mathbf{g}_n) \subseteq \mathbb{M}_\gamma \subseteq \mathbb{M}_{\gamma^{**}}$. Note that the set

$$\{\delta \in S : \gamma_\delta^* = \gamma_\delta\}$$

is a stationary set. We are going to use the black box theorem. In the light of Lemma 4.47 we observe that there are $\delta \in S$ and $\beta \in J_\delta$ such that

(1) $\zeta(\beta) = \gamma_\delta$,

(2) $\gamma_\delta^* = \gamma_\delta$,

(3) $\mathcal{N}^\beta \prec \mathcal{B}$, in particular,

(a) $f^\beta = \mathbf{f} \restriction \mathcal{N}^\beta$,

(b) for all $n < \omega, c_n^{\mathcal{N}^\beta} = c_n^{\mathcal{B}}$,

(c) for $n < \omega, \mathbf{g_n} = f_{c_n^{\mathcal{N}^\beta}}$.

Let $\epsilon = \delta + 1$. Note that as $\delta \in S$ and $\gamma_\delta^* = \gamma_\delta$, at step ϵ of the construction we are at one of the cases (1) or (2). The nontrivial part is to check the property presented in Definition 4.35(n). If case (1) occurs we are immediately done. So suppose that we are in case (2), and hence, by the construction, we are in one of the following situations:

$(\star)_1$ for all $\eta \in \lim(f^\beta)$ and all $i < 2$ the type $q_{\delta,\beta}^{\eta,i}$ is realized in $\mathbb{M}_{\delta,\beta}^{\eta,i}$.

$(\star)_2$ there is some $\delta' \in S_\delta$ and some $\beta' \in J_{\delta'}$ such that for some $\eta \in \lim(f^{\beta'})$ and some $i < 2, q^{\eta,i}_{\delta',\beta'}$ is well-defined and is omitted in $\mathbb{M}^{\eta,i}_{\delta',\beta'}$, but it is realized in $\mathbb{M}^{\eta,i}_{\delta,\beta}$.

$(\star)_3$ there is some $\beta' \in w_{\delta,\beta}$, some $\eta \in \lim(f^{\beta'})$ and some $i < 2$ such that $\beta' \leq 2^{\aleph_0} + \beta$ and $q^{\eta,i}_{\delta,\beta'}$ is well-defined and is omitted in $\mathbb{M}^{\eta,i}_{\delta,\beta'}$, but it is realized in $\mathbb{M}^{\eta,i}_{\delta,\beta}$.

We only consider the case $(\star)_1$ and leave the other cases to the reader, as they are easier to prove. Pick some $\eta \in \lim(f^\beta)$. For $i < 2$ the following type

$$q^{\eta,i}_{\delta,\beta} = \{\varphi^{\mathfrak{c}}_n(y - z^{\eta,i}_{\beta,n}) : n < \omega\}$$

is realized in $\mathbb{M}^{\eta,i}_{\delta,\beta}$. By our construction, $\mathbb{M}^{\eta,i}_{\delta,\beta}$ is generated by $\mathbb{M}_{\delta,\beta} \cup \mathbb{P}_\beta$ and $h^{\eta,i}_{\delta,\beta} : \mathbb{P}^{\mathfrak{c}} \to \mathbb{P}_\beta$ is onto, thus we can find $y_i \in \mathbb{M}_{\delta,\beta}$ and $z_i \in \mathbb{P}^{\mathfrak{c}}$ such that $q^{\eta,i}_{\delta,\beta}$ is realized by $y_i + h^{\eta,i}_{\delta,\beta}(z_i)$. Hence for all $n < \omega$,

$$\mathbb{M}^{\eta,i}_{\delta,\beta} \models \varphi^{\mathfrak{c}}_n(y_i + h^{\eta,i}_{\delta,\beta}(z_i) - z^{\eta,i}_{\beta,n}).$$

Recall from (ε-6) that:

$$g^{\eta,i}_{\beta,n} = i \cdot f_{c^{\mathcal{N}^\beta}_n} = i \cdot \mathbf{g_n} \qquad (*)$$

Hence

$$z^{\eta,i}_{\beta,n} = F^{\mathcal{N}^\beta}\big(\sum_{\ell<n}(h_{t_\ell} + i \cdot \mathbf{g}_\ell)(x^{\mathfrak{c}}_\ell)\big) = \sum_{\ell<n} \mathbf{f}(h_{t_\ell}(x^{\mathfrak{c}}_\ell)) + i \cdot \sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{c}}_\ell)).$$

It follows for each $i < 2$ that

$$\begin{aligned} y_i + h^{\eta,i}_{\delta,\beta}(z_i) - z^{\eta,i}_{\beta,n} \quad &= y_i + h^{\eta,i}_{\delta,\beta}(z_i) - \sum_{\ell<n} \mathbf{f}(h_{t_\ell}(x^{\mathfrak{c}}_\ell)) - i \cdot \sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{c}}_\ell)) \\ &\in \varphi^{\mathfrak{c}}_n(\mathbb{M}^{\eta,i}_{\delta,\beta}). \end{aligned}$$

By applying Lemma 4.37(3) to $z_0 \in \mathbb{P}^{\mathfrak{c}}$ and $y_0 - z^{\eta,0}_{\beta,n} \in \mathbb{M}_{\delta,\beta}$ we can find some $z'_{n,0} \in \sum_{\ell<\omega} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell)$ equipped with the following two properties:

(4) $z_0 - z'_{n,0} \in \varphi^{\mathfrak{c}}_n(\mathbb{P}^{\mathfrak{c}})$,

(5) $y_0 - z^{\eta,0}_{\beta,n} + h^{\eta,0}_{\delta,\beta}(z'_{n,0}) \in \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta})$.

In the same vein, there is $z'_{n,1} \in \sum\limits_{\ell<\omega} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell)$ such that

(6) $z_1 - z'_{n,1} \in \varphi^{\mathfrak{e}}_n(\mathbb{P}^{\mathfrak{e}})$,

(7) $y_1 - z^{\eta,1}_{\beta,n} + h^{\eta,1}_{\delta,\beta}(z'_{n,1}) \in \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta})$.

Given $n < \omega$, pick $k_n < \omega$ such that

$$z'_{n,0}, z'_{n,1} \in \sum_{\ell<k_n} \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell).$$

So we can find $\{z'_{n,0,\ell} : \ell < k_n\}$ and $\{z'_{n,1,\ell} : \ell < k_n\}$ such that for $i < 2$, $z'_{n,i,\ell} \in \mathrm{Rang}(\mathbf{h}^{\mathfrak{c}}_\ell)$ and $z'_{n,i} = \sum\limits_{\ell<k_n} z'_{n,i,\ell}$. For $\ell < k_n$ pick some $z_{n,i,\ell} \in \mathbb{N}^{\mathfrak{c}}_n$ with $\mathbf{h}^{\mathfrak{c}}_\ell(z_{n,i,\ell}) = z'_{n,i,\ell}$.

Set $z = z_1 - z_0$. In view of clauses (4) and (6) above, we have

$$z \in (z'_{n,1} - z'_{n,0}) + \varphi^{\mathfrak{e}}_n(\mathbb{P}^{\mathfrak{e}}) = \sum_{\ell<k_n} \mathbf{h}^{\mathfrak{c}}_\ell(z_{n,1,\ell}) - \sum_{\ell<k_n} \mathbf{h}^{\mathfrak{c}}_\ell(z_{n,0,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{P}^{\mathfrak{e}})$$

for all n. Thus part (a) of Definition 4.36(3)$(G)_1$ is satisfied.

For part (b) of Definition 4.36(3)$(G)_1$, by items (5) and (7) we have

$$\sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{c}}_\ell)) \in (y_1 - y_0) + \Big(h^{\eta,1}_{\delta,\beta}(z'_{n,1}) - h^{\eta,0}_{\delta,\beta}(z'_{n,0})\Big) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta}) \quad (\dagger)$$

Now let $\mathbb{K}$ be such that $\mathbb{M}_{\delta,\beta} = \mathbb{M}_{\gamma^{**}} \oplus \mathbb{K}$ and let $\pi : \mathbb{M}_{\delta,\beta} \to \mathbb{K}$ be the natural projection. We apply $(*)$ for $i = 1$ along with the property presented in Definition 4.35(F)(k) to see:

(8) $h^{\eta,1}_{\delta,\beta}(z'_{n,1}) = \sum\limits_{\ell<k_n} h^{\eta,1}_{\delta,\beta}(z'_{n,1,\ell}) = \sum\limits_{\ell<k_n} h^{\eta,1}_{\delta,\beta}(\mathbf{h}^{\mathfrak{c}}_\ell(z_{n,1,\ell})) = \sum\limits_{\ell<k_n} (h_{t_\ell} + \mathbf{g}_\ell)(z_{n,1,\ell})$.

Also, we apply $(*)$ for $i = 0$ along with the property presented in Definition 4.35(F)(k) to see:

(9) $h^{\eta,0}_{\delta,\beta}(z'_{n,0}) = \sum\limits_{\ell<k_n} h^{\eta,0}_{\delta,\beta}(z'_{n,0,\ell}) = \sum\limits_{\ell<k_n} h^{\eta,1}_{\delta,\beta}(\mathbf{h}^{\mathfrak{c}}_\ell(z_{n,0,\ell})) = \sum\limits_{\ell<k_n} h_{t_\ell}(z_{n,0,\ell})$.

By the choice of γ^{**}, $\mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{c}}_\ell)) \in \mathbb{M}_{\gamma^{**}}$ and $\mathrm{Rang}(\mathbf{g}_\ell) \subseteq \mathbb{M}_\gamma \subseteq \mathbb{M}_{\gamma^{**}}$ and hence the following assertions are valid as well:

(10) for all $\ell < n, \pi(\mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{c}}_\ell))) = 0$,

(11) for all $\ell < k_n$ and $i < 2$, $\pi(\mathbf{g}_\ell(z_{n,i,\ell})) = 0$.

We may assume without loss of the generality that $\alpha_\ell > \gamma^{**}$ for all $\ell < \omega$, where α_ℓ is such that $t_\ell \in J_{\alpha_\ell}$. It then follows from Definition 4.35(E) that for each $\ell < \omega$, $\mathrm{Rang}(h_{t_\ell}) \subseteq \mathbb{K}$ and hence:

(12) for all $\ell < k_n, \pi(h_{t_\ell}(z'_{n,1,\ell})) = h_{t_\ell}(z'_{n,1,\ell})$.

(13) By setting $y' := y_1 - y_0$ and $y := y' - \pi(y')$ in the previous formulas we observe the following equalities and the resulting containment:

$$\begin{aligned}
\sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{e}}_\ell)) &\overset{(10)}{=} \sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{e}}_\ell)) - \pi(\sum_{\ell<n} \mathbf{f}(\mathbf{g}_\ell(x^{\mathfrak{e}}_\ell)))\\
&\overset{(\dagger)}{\in} (y_1 - y_0) + \left(h^{\eta,1}_{\delta,\beta}(z'_{n,1}) - h^{\eta,0}_{\delta,\beta}(z'_{n,0})\right) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta}))\\
&\quad -\pi(y_1 - y_0) - \pi\left(h^{\eta,1}_{\delta,\beta}(z'_{n,1}) - h^{\eta,0}_{\delta,\beta}(z'_{n,0})\right) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta}))\\
&\overset{(13)}{=} y + \left(h^{\eta,1}_{\delta,\beta}(z'_{n,1}) - h^{\eta,0}_{\delta,\beta}(z'_{n,0})\right) - \pi\left(h^{\eta,1}_{\delta,\beta}(z'_{n,1}) - h^{\eta,0}_{\delta,\beta}(z'_{n,0})\right) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta})\\
&\overset{(8+9)}{=} y + \sum_{\ell<k_n} (h_{t_\ell} + \mathbf{g}_\ell)(z_{n,1,\ell}) - \sum_{\ell<k_n} h_{t_\ell}(z_{n,0,\ell})\\
&\quad -\pi \sum_{\ell<k_n} (h_{t_\ell} + \mathbf{g}_\ell)(z_{n,1,\ell}) + \pi \sum_{\ell<k_n} h_{t_\ell}(z_{n,0,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta})\\
&\overset{(11+12)}{=} y + \sum_{\ell<k_n} (h_{t_\ell} + \mathbf{g}_\ell)(z_{n,1,\ell}) - \sum_{\ell<k_n} h_{t_\ell}(z_{n,0,\ell})\\
&\quad - \sum_{\ell<k_n} (h_{t_\ell})(z_{n,1,\ell}) + \sum_{\ell<k_n} h_{t_\ell}(z_{n,0,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta})\\
&= y + \sum_{\ell<k_n} \mathbf{g}_\ell(z_{n,1,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_{\delta,\beta}).
\end{aligned}$$

This completes the proof of Definition 4.36(3)$(G)_1$(b).

We now sketch the proof of property $(G)_2$ of Definition 4.36(3), as its details are similar to the above. Let $\mathfrak{e} \in \mathfrak{E}^{\mathbf{m}}$ and for $n < \omega$ and $\alpha \in \kappa \setminus S$. By $\mathbf{h}_{\alpha,n}$ we mean an embedding of $\mathbb{N}^{\mathfrak{e}}_n$ into $\mathbb{M}_\kappa$ such that

$$\mathbb{M}_\alpha + \mathrm{Rang}(\mathbf{h}_{\alpha,n}) = \mathbb{M}_\alpha \oplus \mathrm{Rang}(\mathbf{h}_{\alpha,n}) \leq_{\aleph_0} \mathbb{M}_\kappa.$$

We look at the following set

$$C := \{\delta < \kappa : \forall\alpha < \delta \ \forall n < \omega, \mathrm{Rang}(\mathbf{h}_{\alpha,n}) \subseteq \mathbb{M}_\delta\},$$

which C is a club subset of κ.[11] Let

$$\mathbf{g} : \mathcal{T}_\kappa \to \mathcal{T}_\kappa$$

[11] Club means closed and unbounded.

be such that for all $\eta, \mu \in \lim(\mathbf{g})$ and all $n < \omega, \mathbf{g}(\eta(n)) = \mathbf{g}(\mu(n))$. Let

$$\mathcal{B} = \langle B, \in^{\mathcal{B}}, F^{\mathcal{B}}, G^{\mathcal{B}}, (c_n^{\mathcal{B}})_{n<\omega}, (P_n^{\mathcal{B}})_{n<\omega} \rangle$$

be a $\tau = \bigcup_{n<\omega} \tau_n$-structure satisfying the following properties:

$\tau_i)$: $(\mathcal{B}, \in^{\mathcal{B}})$ expands $(\mathcal{H}_{<\aleph_1}(\mathbb{M}), \in)$,

$\tau_j)$: $G^{\mathcal{B}} = \mathbf{g}$,

$\tau_k)$: for $n < \omega,\ c_n^{\mathcal{B}} = \gamma_n$, where γ_n is such that $(n, \mathfrak{e}, \mathbf{h}_{\alpha_n,n} - h_{t_n}) = (n_{\gamma_n}, \mathfrak{e}_{\gamma_n}, f_{\gamma_n})$, where $t_n = \mathbf{g}(\eta(n))$ and α_n is such that $t_n \in J_{\alpha_n}$.

In the same vein, we can find $\delta \in S \cap C$, $\beta \in J_\delta$ such that:

($\dagger_1$) $\zeta(\beta) = \gamma_\delta$.

($\dagger_2$) $\gamma_\delta^* = \gamma_\delta$.

($\dagger_3$) $\mathcal{N}^\beta \prec \mathcal{B}$, in particular,

(a) $f^\beta = \mathbf{f} \restriction \mathcal{N}^\beta$,

(b) for all $n < \omega, c_n^{\mathcal{N}^\beta} = c_n^{\mathcal{B}}$,

(c) for $n < \omega, \mathbf{h}_{\alpha_n,n} - h_{t_n} = f_{c_n^{\mathcal{N}^\beta}}$, where t_n and α_n are defined as above.

($\dagger_4$) The properties presented in Clause (F) of Definition 4.35 are hold.

According to Definition 4.35(F)(k), we have

$$\mathbf{h}_\beta^\delta \circ \mathbf{h}_n = h_{t_n} + (\mathbf{h}_{\alpha_n,n} - h_{t_n}) = \mathbf{h}_{\alpha_n,n},$$

as claimed by $(G)_2$ from Definition 4.36(3). The proof of clause (i) is now complete.

(ii): In order to prove the desired claim, suppose further that λ is a regular cardinal. We show that by shrinking S, if necessary, we can assume that the above constructed structure is indeed a nice construction, i.e., we can omit "semi" from it. The map $\delta \mapsto \gamma_\delta^{**}$ is a regressive function on S. We are going to use the Fodor's lemma. This says that $\delta \mapsto \gamma_\delta^{**}$ is constant on some stationary subset S_1 of S, thus for some γ^{**} and for all $\delta \in S_1, \gamma_\delta^{**} = \gamma^{**}$. Now note that for each $\delta \in S_1$ and each $s \in J_\delta, \mathfrak{e}_s^\delta$ is of the form $\mathfrak{e}_\beta$ for some $\beta < \gamma^{**}$, so again by Fodor's lemma and for each $s \in J_\delta, \bar{g}_s^\delta$ is an ω-sequence $\langle g_{s,n}^\delta : n < \omega \rangle$ where for each $n, g_{s,n}^\delta : \mathbb{N}_n^{\mathfrak{e}_s} \to \mathbb{M}_{\gamma^{**}}$

is a bimodule homomorphism. But, the set

$$\{g : g : \mathbb{N}_n^{\mathfrak{e}_s} \to \mathbb{M}_{\gamma^{**}} \text{ is a bimodule homomorphism}\}$$

has size less than λ. Now, let $\delta \in S_1$ be as such that the property presented in clause (F) of Definition 4.35 holds. The sets

$$\{\mathfrak{e}_s : s \in J_\delta\} \;\&\; \{\bar{g}_s^\delta : s \in J_\delta\}$$

have size less than λ. Again, we revisit Fodor's lemma, to find a stationary subset S_2 of S_1 such that the sets $\{\mathfrak{e}_s : s \in J_\delta\}$ and $\{\bar{g}_s^\delta : s \in J_\delta\}$ are the same for all $\delta \in S_2$. This concludes the required result.

(iii) and (iv): These are similar to the previous parts, and we leave the modification to the reader. $\square$

5. Any endomorphism is somewhat definable

In this section we shall investigate what "$\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa\rangle$ is a semi-nice construction" gives us and look somewhat at various implications. The main results of this section are to verify the following property (see Lemma 5.2 and its modifications, see e.g. Lemma 5.8). These enable us to define the concept of strongly semi-nice construction. For more details, see Definition 5.16.

Definition 5.1. *Let $\alpha < \kappa$ and $n < \omega$ be given. Suppose $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$ is a bimodule homomorphism. We say the property $(\mathrm{Pr})_\alpha^n[\mathbf{f}, \mathfrak{e}]$ is satisfied provided*

$$\mathbf{f}(\mathbf{h}(x_n^{\mathfrak{e}})) \in \mathbb{M}_\alpha + \varphi_\ell^{\mathfrak{e}}(\mathbb{M}_\kappa) + \mathrm{Rang}(\mathbf{h}),$$

for all $\ell < \omega$, where $\mathbf{h} : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}_\kappa$ is a bimodule homomorphism.

As an application of the results of Section 4, we present the following statement. This plays an essential role in proving some stronger versions of $(\mathrm{Pr})_\alpha^n[\mathbf{f}, \mathfrak{e}]$, see Lemma 5.8 blow.

Lemma 5.2. *Let $\mathcal{A}$ be a semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$ and suppose $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$ is an endomorphism of $\mathbf{R}$-modules. Also, suppose $\kappa \geq \kappa(\mathfrak{E})$. Then for every $\mathfrak{e} \in \mathfrak{E}$ there are $\alpha \in \kappa \setminus S$ and $n(*) < \omega$ such that the property $(\mathrm{Pr})^{n(*)}_\alpha[\mathbf{f}, \mathfrak{e}]$ holds.*

Proof. Assume not. Then for every $\epsilon \in \kappa \setminus S$ and $n < \omega$ we can find a bimodule homomorphism $\mathbf{h}_{\epsilon,n} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa$ such that for some $\ell(\epsilon, n) < \omega$,

$$\mathbf{f}(\mathbf{h}_{\epsilon,n}(x^{\mathfrak{e}}_n)) \notin \mathbb{M}_\epsilon + \varphi^{\mathfrak{e}}_{\ell(\epsilon,n)}(\mathbb{M}_\kappa) + \mathrm{Rang}(\mathbf{h}_{\epsilon,n}).$$

Without loss of generality, we may assume that $n < \ell(\epsilon, n)$. The following set

$$E := \{\zeta < \kappa : \mathrm{Rang}(\mathbf{f} \restriction \mathbb{M}_\zeta) \subseteq \mathbb{M}_\zeta \text{ and } \forall \xi < \zeta \ \forall n < \omega, \mathrm{Rang}(\mathbf{h}_{\xi,n}) \subseteq \mathbb{M}_\zeta\}$$

is a club subset of κ. Since S is stationary, we have

$$S \cap \lim \big(E \cap (S^\kappa_{\aleph_0} \setminus S)\big) \neq \emptyset,$$

where lim denote the set of limit points. Let ζ be in this intersection. Then $\mathrm{cf}(\zeta) = \aleph_0$ and there exists an increasing sequence $\langle \zeta_n : 0 < n < \omega \rangle$ of elements of $E \cap (S^\kappa_{\aleph_0} \setminus S)$ cofinal in ζ.

For each $0 < \ell < \omega$, as $\zeta_\ell \notin S$, we can find a bimodule $\mathbb{K}_\ell$ such that $\mathbb{M}_{\zeta_{\ell+1}} = \mathbb{M}_{\zeta_\ell} \oplus \mathbb{K}_\ell$. Set also $\mathbb{K}_0 = \mathbb{M}_{\zeta_0}$. Thus, $\mathbb{M}_\zeta = \bigcup\limits_{\ell<\omega} \mathbb{M}_{\zeta_\ell} = \bigoplus\limits_{\ell<\omega} \mathbb{K}_\ell$. Let $\mathbf{g}^*_\ell : \mathbb{M}_{\zeta_{\ell+1}} \to \mathbb{K}_\ell$ be the natural projection, this is well-defined, because $\mathbb{K}_\ell$ is a direct summand of $\mathbb{M}_{\zeta_{\ell+1}}$. Pick also an infinite subset $\mathcal{U} \subseteq \omega$ such that

$$\forall n, m \in \mathcal{U}\, (n < m \implies \ell(\zeta_n, n) < m)\,.$$

For each $n < \omega$, we set $t_n := \mathbf{h}_{\zeta_n,n}(x^{\mathfrak{e}}_n) \in \mathbb{M}_\zeta$. By Definition 4.36(3)$(G)_1$, we can find $y \in \mathbb{M}_\kappa$, $z \in \mathbb{P}^{\mathfrak{e}}$ and a sequence $\{z_{n,\ell}, z'_{n,\ell} : n \in \mathcal{U}, \ell < \omega\}$ with $z_{n,\ell} \in \mathrm{Dom}(\mathbf{h}_{\zeta_\ell,\ell})$ such that for each $n \in \mathcal{U}$, $z_{n,\ell} = 0$ for all large enough ℓ, say for all $\ell \geq k_n$ and for all large enough $n \in \mathcal{U}$ we have

(a) $z \in \sum\limits_{\ell \in \mathcal{U} \cap k_n} \mathbf{h}^{\mathfrak{e}}_\ell(z'_{n,\ell}) + \varphi^{\mathfrak{e}}_n(\mathbb{P}^{\mathfrak{e}})$.

(b) $\sum_{\ell\in\mathcal{U}\cap n}\mathbf{f}(\mathbf{h}_{\zeta_\ell,\ell}(x^{\mathfrak{c}}_\ell))\in y+\sum_{\ell\in\mathcal{U}\cap k_n}\mathbf{h}_{\zeta_\ell,\ell}(z_{n,\ell})+\varphi^{\mathfrak{c}}_n(\mathbb{M}_\kappa)$.

As $\zeta\notin S$ we know $\mathbb{M}_\zeta\leq_{\aleph_0}\mathbb{M}_\kappa$. Hence, for some $\mathbb{K}$ we have $y\in\mathbb{M}_\zeta\oplus\mathbb{K}$ and $\mathbb{M}_\zeta\oplus\mathbb{K}\leq_{\aleph_0}\mathbb{M}_\kappa$. So, in clause (b) all elements are in $\mathbb{M}_{\zeta_n}\oplus\mathbb{K}$. This allows us to replace $\varphi^{\mathfrak{c}}_n(\mathbb{M}_\kappa)$ with $\varphi^{\mathfrak{c}}_n(\mathbb{M}_{\zeta_n}\oplus\mathbb{K})$, and leads us to get the following new clause:

$(b')_n$: $\sum_{\ell\in\mathcal{U}\cap n}\mathbf{f}(\mathbf{h}_{\zeta_\ell,\ell}(x^{\mathfrak{c}}_\ell))\in y+\sum_{\ell\in\mathcal{U}\cap k_n}\mathbf{h}_{\zeta_\ell,\ell}(z_{n,\ell})+\varphi^{\mathfrak{c}}_n(\mathbb{M}_{\zeta_n}\oplus\mathbb{K})$.

Let $\pi:\mathbb{M}_\zeta\oplus\mathbb{K}\to\mathbb{M}_\zeta$ be the natural projection and let $y':=\pi(y)$. Recall that $\pi(\varphi^{\mathfrak{c}}_n(\mathbb{M}_{\zeta_n}\oplus\mathbb{K}))=\varphi^{\mathfrak{c}}_n(\mathbb{M}_{\zeta_n})$. By applying π along with $(b')_n$ we lead to the following equation:

$(b'')_n$: $\sum_{\ell\in\mathcal{U}\cap n}\mathbf{f}(\mathbf{h}_{\zeta_\ell,\ell}(x^{\mathfrak{c}}_\ell))\in y'+\sum_{\ell\in\mathcal{U}\cap k_n}\mathbf{h}_{\zeta_\ell,\ell}(z_{n,\ell})+\varphi^{\mathfrak{c}}_n(\mathbb{M}_{\zeta_n})$.

Pick $n(*)<\omega$ such that $y'\in\mathbb{M}_{\zeta_{n(*)}}$ and choose successive elements $n<m$ in $\mathcal{U}$ bigger than $n(*)$ which are large enough. Also, pick $w\in\varphi^{\mathfrak{c}}_m(\mathbb{M}_{\zeta_m})$ such that

(1) $\sum_{\ell\in\mathcal{U}\cap m}\mathbf{f}(\mathbf{h}_{\zeta_\ell,\ell}(x^{\mathfrak{c}}_\ell))=y'+\sum_{\ell\in\mathcal{U}\cap k_m}\mathbf{h}_{\zeta_\ell,\ell}(z_{m,\ell})+w$.

Then we have:

(2) As $\sum_{\ell\in\mathcal{U}\cap n}\mathbf{f}(\mathbf{h}_{\zeta_\ell,\ell}(x^{\mathfrak{c}}_\ell)),y'\in\mathbb{M}_{\zeta_n}$, by (1) we have

$$\begin{aligned}\mathbf{g}^*_n(\mathbf{h}_{\zeta_n,n}(x^{\mathfrak{c}}_n)) &= \mathbf{g}^*_n\left(\sum_{\ell\in\mathcal{U}\cap m}\mathbf{f}(\mathbf{h}_{\zeta_\ell,\ell}(x^{\mathfrak{c}}_\ell))\right)\\ &= \sum_{\ell\in\mathcal{U}\cap k_m}\mathbf{g}^*_n\left(\mathbf{h}_{\zeta_\ell,\ell}(z_{m,\ell})\right)+\mathbf{g}^*_n(w).\end{aligned}$$

In particular, we drive the following:

(3) $\sum_{\ell\in\mathcal{U}\cap k_m}\mathbf{g}^*_n\left(\mathbf{h}_{\zeta_\ell,\ell}(z_{m,\ell})\right)\in\mathrm{Rang}(\mathbf{g}^*_n\circ\mathbf{h}_{\zeta_n,n})+\varphi^{\mathfrak{c}}_m(\mathbb{M}_{\zeta_m})$.

Due to the definition of $\mathbf{g}^*_n$ we know:

(4) $\mathbf{f}(\mathbf{h}_{\zeta_n,n}(x^{\mathfrak{c}}_n))-\mathbf{g}^*_n\left(\mathbf{f}(\mathbf{h}_{\zeta_n,n}(x^{\mathfrak{c}}_n))\right)\in\mathbb{M}_{\zeta_n}$.

Putting all things together, we have

$$\begin{aligned}\mathbf{f}(\mathbf{h}_{\zeta_n,n}(x^{\mathfrak{c}}_n)) &\in \mathbf{g}^*_n\left(\mathbf{f}(\mathbf{h}_{\zeta_n,n}(x^{\mathfrak{c}}_n))\right)+\mathbb{M}_{\zeta_n}\\ &\subseteq \sum_{\ell\in\mathcal{U}\cap k_m}\mathbf{g}^*_n\left(\mathbf{h}_{\zeta_\ell,\ell}(z_{m,\ell})\right)+\mathbb{M}_{\zeta_n}+\varphi^{\mathfrak{c}}_m(\mathbb{M}_{\zeta_m})\\ &\subseteq \mathbb{M}_{\zeta_n}+\mathrm{Rang}(\mathbf{g}^*_n\circ\mathbf{h}_{\zeta_n,n})+\varphi^{\mathfrak{c}}_m(\mathbb{M}_{\zeta_m})\\ &= \mathbb{M}_{\zeta_n}+\mathrm{Rang}(\mathbf{h}_{\zeta_n,n})+\varphi^{\mathfrak{c}}_m(\mathbb{M}_{\zeta_m}).\end{aligned}$$

This is a contradiction and the lemma follows. $\square$

Definition 5.3. *By* $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_1}(h, \mathbb{N})$ *we mean the following data:*

(1) $\mathbb{M}_1, \mathbb{M}_2, \mathbb{N}$ *are bimodules,*

(2) $\mathbb{M}_1 \subseteq \mathbb{M}_2$,

(3) h *is a bimodule homomorphism from* $\mathbb{N}$ *into* $\mathbb{M}_2$,

(4) $\mathbb{N} \in c\ell_{is}(\mathcal{K})$,

(5) $\mathbb{M}_1 + \mathrm{Rang}(h) = \mathbb{M}_1 \oplus \mathrm{Rang}(h)$,

(6) $\mathbb{M}_1 + \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_2$.

The next lemma summarizes the main properties of **hds**.

Lemma 5.4. (1) *Suppose* $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_1}(h_1, \mathbb{N})$ *holds and let* h_0 *be a bimodule homomorphism from* $\mathbb{N}$ *into* $\mathbb{M}_1$. *If* $h := h_0 + h_1$, *then* $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_1}(h, \mathbb{N})$ *holds as well.*

(2) *Suppose* $\mathbb{M}_0 \subseteq \mathbb{M}_1 \subseteq \mathbb{M}_2$ *are bimodules,* $\mathbb{M}_0 \leq_{\aleph_0} \mathbb{M}_1$ *and suppose* $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_1}(h, \mathbb{N})$ *holds. Then so does* $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_0}(h, \mathbb{N})$.

(3) *Let* $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ *be a weakly semi-nice construction with respect to* $(\lambda, \mathfrak{m}, S, \kappa)$ *and assume that* $\mathbf{hds}^{\mathbb{M}_\lambda}_{\mathbb{M}_\alpha}(h, \mathbb{N})$ *holds where* $\alpha < \kappa$. *Then* $\mathbf{hds}^{\mathbb{M}_\beta}_{\mathbb{M}_\alpha}(h, \mathbb{N})$ *holds for a club of* $\beta \in (\alpha, \kappa)$.

(4) *If* $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_1}(h, \mathbb{N})$ *holds, then so does* $\mathbf{hds}^{\mathbb{M}_1 + \mathrm{Rang}(h)}_{\mathbb{M}_1}(h, \mathbb{N})$.

Proof. (1) We only need to show that $\mathbb{M}_1 + \mathrm{Rang}(h) = \mathbb{M}_1 \oplus \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_2$. But, this is clear as $\mathbb{M}_1 + \mathrm{Rang}(h) = \mathbb{M}_1 + \mathrm{Rang}(h_1)$ and the property $\mathbf{hds}^{\mathbb{M}_2}_{\mathbb{M}_1}(h_1, \mathbb{N})$ holds.

(2) We have

$$\mathbb{M}_0 + \mathrm{Rang}(h) = \mathbb{M}_0 \oplus \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_1 \oplus \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_2,$$

from which the result follows.

(3) By our assumption, $h : \mathbb{N} \to \mathbb{M}_\kappa$ is a bimodule homomorphism and

$$\mathbb{M}_\alpha + \mathrm{Rang}(h) = \mathbb{M}_\alpha \oplus \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_\kappa.$$

But, then for a club $C \subseteq (\alpha, \kappa)$ and for all $\beta \in C$ we have

$$\mathbb{M}_\alpha \oplus \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_\beta.$$

Following definition, $\mathbf{hds}^{\mathbb{M}_\beta}_{\mathbb{M}_\alpha}(h, \mathbb{N})$ holds for all $\beta \in C \subseteq (\alpha, \kappa)$.

(4) Clear from the definition.

$\square$

Definition 5.5. *Let $\alpha < \kappa$ and $n < \omega$ be given. Suppose $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$ is an $\mathbf{R}$-module homomorphism. We say the property $(\mathrm{Pr}^-)^n_\alpha[\mathbf{f}, \mathfrak{e}]$ is valid, provided:*

(1) there is some $\alpha < \beta \in \kappa \setminus S$ and some suitable bimodule homomorphism $h : \mathbb{N} \to \mathbb{M}_2$ so that $\mathbf{hds}^{\mathbb{M}_\beta}_{\mathbb{M}_\alpha}(h, \mathbb{N}^{\mathfrak{e}}_n)$ holds,

(2) for all $\ell < \omega$, we have

$$\mathbf{f}(h(x^{\mathfrak{e}}_n)) \in \mathbb{M}_\alpha + \mathrm{Rang}(h) + \varphi^{\mathfrak{e}}_\ell(\mathbb{M}_\kappa).$$

Lemma 5.6. *Let $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$ be an $\mathbf{R}$-module homomorphism.*

(1) Assume $\alpha \leq \beta < \kappa$, $\alpha \notin S$ and $\beta \notin S$. Then $(\mathrm{Pr}^-)^{n(*)}_\alpha[\mathbf{f}, \mathfrak{e}]$ implies $(\mathrm{Pr}^-)^{n(*)}_\beta[\mathbf{f}, \mathfrak{e}]$.

(2) If $(\mathrm{Pr})^{n(*)}_\alpha[\mathbf{f}, \mathfrak{e}]$ holds, then $(\mathrm{Pr}^-)^{n(*)}_\alpha[\mathbf{f}, \mathfrak{e}]$ holds too.

Proof. (1). This is a combination of Lemma 4.37(1) and Lemma 5.4 (2).

(2). This is clear by definition. $\square$

We need the following membership property:

Definition 5.7. *Let $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$ be a bimodule homomorphism, $\alpha < \kappa$, $n < \omega$ and let $z \in \mathbb{N}^{\mathfrak{e}}_n$. We say the property $(\mathrm{Pr}\,1)^n_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ is valid if*

$$\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa),$$

where h is a bimodule homomorphism from $\mathbb{N}^{\mathfrak{e}}_n$ into $\mathbb{M}_\kappa$.

Lemma 5.8. *Let $\langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ be a weakly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$ and $\mathbf{f} : \mathbb{M}_\kappa \to \mathbb{M}_\kappa$ be an $\mathbf{R}$-endomorphism and let $\alpha \notin S$ be such that the property $(\mathrm{Pr}^-)^{n(*)}_\alpha[\mathbf{f}, \mathfrak{e}]$ holds. Then there is some $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}[\mathcal{K}^{\mathfrak{m}}]$ such that the property $(\mathrm{Pr}\,1)^{n(*)}_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ is satisfied.*

Proof. Let $x := x^{\mathfrak{e}}_{n(*)}$. We shall prove the lemma in a sequence of claims, see Claims 5.9–5.12.

Claim 5.9. *Let $\beta \in (\alpha, \lambda) \setminus S$ be such that $\mathbf{hds}^{\mathbb{M}_\beta}_{\mathbb{M}_\alpha}(h, \mathbb{N}^{\mathfrak{e}}_{n(*)})$ holds. Then*

$$\mathbf{f}(h(x)) \in \mathbb{M}_\alpha + \mathrm{Rang}(h) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

Proof. Let $\mathbb{N} := \mathrm{Rang}(h)$. We know that $\mathbb{M}_\alpha + \mathbb{N} = \mathbb{M}_\alpha \oplus \mathbb{N} \leq_{\aleph_0} \mathbb{M}_\beta$ and $\mathbb{M}_\beta \leq_{\aleph_0} \mathbb{M}_\kappa$. Combining these yields that $\mathbb{M}_\alpha \oplus \mathbb{N} \leq_{\aleph_0} \mathbb{M}_\kappa$. In view of Definition 4.16 and Lemma 4.22(6), there is $\mathbb{K} \subseteq \mathbb{M}_\kappa$ such that

$$(\mathbb{M}_\alpha \oplus \mathbb{N}) + \mathbb{K} = \mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K} \leq_{\aleph_0} \mathbb{M}_\kappa.$$

Then, we may assume that $\mathbf{f}(h(x)) \in \mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K}$. Since $\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K} \leq_{\aleph_0} \mathbb{M}_\kappa$, and in the light of Lemma 4.20 we observe that $\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K} \leq^{pr}_{\varphi^{\mathfrak{e}}_n} \mathbb{M}_\kappa$ holds for all $n < \omega$. According to the property $(\mathrm{Pr}^-)^{n(*)}_\alpha[\mathbf{f}, \mathfrak{e}]$, we know

$$\mathbf{f}(h(x)) \in \mathbb{M}_\alpha + \mathbb{N} + \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa),$$

for all $n < \omega$. We use these to find elements $x_{1,n} \in \mathbb{M}_\alpha$ and $x_{2,n} \in \mathbb{N}$ such that

$$\mathbf{f}(h(x)) - x_{1,n} - x_{2,n} \in \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa).$$

It follows from $\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K} \leq^{pr}_{\varphi^{\mathfrak{e}}_n} \mathbb{M}_\kappa$ that

$$\begin{aligned} \mathbf{f}(h(x)) &= x_{1,n} + x_{2,n} + (\mathbf{f}(h(x)) - x_{1,n} - x_{2,n}) \\ &\in \mathbb{M}_\alpha + \mathbb{N} + \varphi^{\mathfrak{e}}_n(\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K}). \end{aligned}$$

Let g be the projection from $\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K}$ onto $\mathbb{K}$. Recall that its kernel is $\mathbb{M}_\alpha \oplus \mathbb{N}$. Let $z^1_n \in \mathbb{M}_\alpha$, $z^2_n \in \mathbb{N}$ and $z^3_n \in \varphi_n(\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K})$ be such that $\mathbf{f}(h(x)) = z^1_n + z^2_n + z^3_n$.

Let us evaluate g on both sides of this formula to obtain:

$$g(\mathbf{f}(h(x))) = g(z^1_n) + g(z^2_n) + g(z^3_n) = z^3_n \in \varphi^{\mathfrak{e}}_n(\mathbb{M}_\alpha \oplus \mathbb{N} \oplus \mathbb{K}) \subseteq \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa).$$

As this holds for each n, we have

$$g(\mathbf{f}(h(x))) \in \bigcap_{n<\omega} \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa).$$

So,

$$\begin{aligned}\mathbf{f}(h(x)) &= (\mathbf{f}(h(x)) - g(\mathbf{f}(h(x)))) + (g(\mathbf{f}(h(x))) \\ &\in (\mathbb{M}_\alpha \oplus \mathbb{N}) + \textstyle\bigcap_{n<\omega} \varphi_n(\mathbb{M}_\kappa) \\ &= \quad \mathbb{M}_\alpha + \mathrm{Rang}(h) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).\end{aligned}$$

The claim follows. □

Claim 5.10. *For $i = 1, 2$, let $\mathbb{N}^*_i$ and β_i be such that $\mathbb{M}_\alpha \oplus \mathbb{N}^*_i \leq_{\aleph_0} \mathbb{M}_{\beta_i}$ and $\alpha < \beta_i < \kappa$ where β_i is not in S. Let $h_i : \mathbb{N}^{\mathfrak{e}}_{n(*)} \to \mathbb{N}^*_i$ be an isomorphism and $z_i \in \mathbb{N}^{\mathfrak{e}}_{n(*)}$ be such that*

$$\mathbf{f}(h_i(x)) - h_i(z_i) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

Then $z_1 \equiv z_2 \mod \varphi^{\mathfrak{e}}_\omega(\mathbb{N}^{\mathfrak{e}}_{n(*)})$.

Proof. Let $\beta \in \kappa \setminus S$ be large enough such that $\beta > \beta_1$ and $\beta > \beta_2$. Let $\mathbb{N}^*_3$ be isomorphic to $\mathbb{N}^{\mathfrak{e}}_{n(*)}$ and such that for any large enough $\gamma < \kappa$,

$$\mathbb{M}_\beta + \mathbb{N}^*_3 = \mathbb{M}_\beta \oplus \mathbb{N}^*_3 \leq_{\aleph_0} \mathbb{M}_\gamma.$$

Let h_3 be an isomorphism from $\mathbb{N}^{\mathfrak{e}}_{n(*)}$ onto $\mathbb{N}^*_3$. In the light of Claim 5.9 we deduce that

$$\mathbf{f}(h_3(x)) - h_3(z_3) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa),$$

where $z_3 \in \mathbb{N}^{\mathfrak{e}}_{n(*)}$. By the transitivity of the equivalence relation, it is enough to show that $z_3 \equiv z_1$ and $z_3 \equiv z_2 \mod \varphi^{\mathfrak{e}}_\omega(\mathbb{N}^{\mathfrak{e}}_{n(*)})$ in $\mathbb{N}^{\mathfrak{e}}_{n(*)}$; and by the symmetry it is enough to prove $z_3 \equiv z_1 \mod \varphi^{\mathfrak{e}}_\omega(\mathbb{N}^{\mathfrak{e}}_{n(*)})$.

Pick $\gamma > \beta$ large enough. Then $\mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^* \leq_{\aleph_0} \mathbb{M}_\gamma$. Let

$$\mathbb{N}_4^* := \{h_1(z) - h_3(z) : z \in \mathbb{N}_{n(*)}^{\mathfrak{c}}\},$$

and define $h_4 : \mathbb{N}_{n(*)}^{\mathfrak{c}} \longrightarrow \mathbb{M}_{\beta+1}$ by

$$h_4(z) = h_1(z) - h_3(z).$$

Due to its definition, we know $\mathbb{N}_4^*$ is a sub-bimodule of $\mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^*$ and hence of $\mathbb{M}_\kappa$. Also, h_4 is an isomorphism from $\mathbb{N}_{n(*)}^{\mathfrak{c}}$ onto $\mathbb{N}_4^*$, and $\mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^* = \mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_4^*$. Thus, we have

$(*)_1$ $\quad \mathbb{M}_\alpha \oplus \mathbb{N}_4^* \leq_{\aleph_0} \mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^* \leq_{\aleph_0} \mathbb{M}_\gamma$.

Now modulo $\mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa)$, we have

$(*)_2$ $\quad \mathbf{f}(h_4(x)) = \mathbf{f}(h_1(x) - h_3(x)) = \mathbf{f}(h_1(x)) - \mathbf{f}(h_3(x)) \equiv h_1(z_1) - h_3(z_3)$.

Next, by Claim 5.9 and $(*)_1$, we have

$(*)_3$ $\quad \mathbf{f}(h_4(x)) \in \mathrm{Rang}(h_4) + (\mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa))$.

So

$(*)_4$ $\quad h_1(z_1) - h_3(z_3) \in \mathrm{Rang}(h_4) + (\mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa))$.

We combine $(*)_4$ along with the definition of h_4 to deduce that

$$(h_1(z_1) - h_3(z_3) - h_4(z) \in \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa),$$

for some $z \in \mathbb{N}_{n(*)}^{\mathfrak{c}}$. Recall that $h_4(z) = h_1(z) - h_3(z)$. It yields that

$$(h_1(z_1) - h_3(z_3)) - (h_1(z) - h_3(z)) \in \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa).$$

In other words,

$$h_1(z_1 - z) - h_3(z_3 - z) \in \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa).$$

Therefor, there is $y \in \mathbb{M}_\alpha$ such that $h_1(z_1 - z) - h_3(z_3 - z) - y \in \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\kappa)$. As $\mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^* \leq_{\aleph_0} \mathbb{M}_\kappa$, we deduce that

$$h_1(z_1 - z) - h_3(z_3 - z) - y \in \varphi_\omega^{\mathfrak{c}}(\mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^*).$$

Recall that $h_1(z_1 - z) \in \mathbb{N}_1, h_3(z_3 - z) \in \mathbb{N}_3^*$ and $y \in \mathbb{M}_\alpha$. We conclude from Claim 4.21(5) that

$$\varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\alpha) \oplus \varphi_\omega^{\mathfrak{e}}(\mathbb{N}_1^*) \oplus \varphi_\omega^{\mathfrak{e}}(\mathbb{N}_3^*) \cong \varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\alpha \oplus \mathbb{N}_1^* \oplus \mathbb{N}_3^*).$$

Hence

$$h_i(z_i - z) \in h_i'' \left(\varphi_\omega^{\mathfrak{e}}(\mathbb{N}_{n(*)}^{\mathfrak{e}})\right) \quad \text{ for } i = 1, 3,$$

i.e., $z_i - z \in \varphi_\omega^{\mathfrak{e}}(\mathbb{N}_{n(*)}^{\mathfrak{e}})$. It then follows that

$$z_1 - z_3 = (z_1 - z) - (z_3 - z) \in \varphi_\omega^{\mathfrak{e}}(\mathbb{N}_{n(*)}^{\mathfrak{e}}).$$

So $z_1 - z_3 \in \varphi_\omega^{\mathfrak{e}}(\mathbb{N}_{n(*)}^{\mathfrak{e}})$, which finishes the proof of the claim. □

Claim 5.11. *There is $z \in \mathbb{N}_{n(*)}^{\mathfrak{e}}$ such that if $h : \mathbb{N}_{n(*)}^{\mathfrak{e}} \to \mathbb{M}_\kappa$ is a bimodule homomorphism, then $\mathbf{f}(h(z)) - h(z) \in \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\kappa^{\mathfrak{e}})$.*

Proof. Let $\mathbb{N}_0^*$, β_0 and $h : \mathbb{N}_{n(*)}^{\mathfrak{e}} \to \mathbb{N}_0^*$ be as Claim 5.10. In the light of Claim 5.9, there is $z \in \mathbb{N}_{n(*)}^{\mathfrak{e}}$ which satisfies the above requirement for this h. We show that z is as required. Suppose not and let h_0 be a counterexample, i.e.,

$$\mathbf{f}(h_0(z)) - h_0(z) \notin \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\kappa^{\mathfrak{e}}).$$

Choose β such that $\beta_0 < \beta \in \kappa \setminus S$ and $\mathrm{Rang}(h_0) \subseteq \mathbb{M}_\beta$. Such a β exists as $\kappa = \mathrm{cf}(\kappa) \geq \kappa(\mathfrak{E})$. Let h_1 be an isomorphism from $\mathbb{N}_{n(*)}^{\mathfrak{e}}$ onto some $\mathbb{N}_1^*$ such that $\mathbb{M}_\beta \oplus \mathbb{N}_1^* \leq_{\aleph_0} \mathbb{M}_\gamma$ for some $\gamma \in (\beta, \kappa) \setminus S$. So

$$\mathbf{f}(h_1(z)) - h_1(z) \in \mathbb{M}_\alpha + \varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\kappa).$$

Let $h_2 : \mathbb{N}_{n(*)}^{\mathfrak{e}} \longrightarrow \mathbb{M}_\kappa$ be defined by

$$h_2(z) = h_1(z) - h_0(z).$$

Easily h_2 is a bimodule homomorphism. Set $\mathbb{N}_2^* =: \mathrm{Rang}(h_2)$. By the assumptions on $\mathbb{N}_1^*$ and h_1, we deduce that

$$\mathbb{M}_\beta \oplus \mathbb{N}_1^* = \mathbb{M}_\beta \oplus \mathbb{N}_2^* \leq_{\aleph_0} \mathbb{M}_\kappa.$$

In the light of Claim 5.10, we see that

$$\mathbf{f}(h_2(x)) - h_2(z) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\kappa).$$

We apply this to conclude that

$$\begin{aligned} \mathbf{f}(h_0(z)) &= \mathbf{f}(h_1(z) - h_2(z)) \\ &= \mathbf{f}(h_1(z)) - \mathbf{f}(h_2(z)) \\ &\in h_1(z) - h_2(z) + \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\kappa) \\ &= h_0(z) + \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\kappa), \end{aligned}$$

i.e.,

$$\mathbf{f}(h_0(z)) - h_0(z) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}^{\mathfrak{c}}_\kappa),$$

which contradicts our initial assumption on h_0. The claim follows. $\square$

Claim 5.12. *Let z be as Claim 5.11. Then $z \in \mathbb{L}^{\mathfrak{c}}_{n(*)}[\mathcal{K}]$.*

Proof. Let $h_1, h_2 : \mathbb{N}^{\mathfrak{c}}_{n(*)} \to \mathbb{N} \in \mathcal{K}$ be such that $h_1(x^{\mathfrak{c}}_{n(*)}) - h_2(x^{\mathfrak{c}}_{n(*)}) \in \varphi^{\mathfrak{c}}_\omega(\mathbb{N})$ and $\mathbf{f}(h_i(x^{\mathfrak{c}}_{n(*)})) = h_i(z) \mod \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\lambda)$. By the definition of $\mathbb{L}^{\mathfrak{c}}_{n(*)}[\mathcal{K}]$, it is sufficient to show $h_1(z) - h_2(z) \in \varphi^{\mathfrak{c}}_\omega(\mathbb{N})$. To this end, choose $\gamma \in \kappa \setminus S$ large enough and an embedding h of $\mathbb{N}$ into $\mathbb{M}_\kappa$ such that $\mathbb{M}_\gamma + \mathrm{Rang}(h) = \mathbb{M}_\gamma \oplus \mathrm{Rang}(h) \leq_{\aleph_0} \mathbb{M}_\kappa$. Let $i = 1, 2$ and note that $h \circ h_i$ is a bimodule homomorphism from $\mathbb{N}^{\mathfrak{c}}_{n(*)}$ into $\mathbb{M}_\kappa$. Thanks to Claim 5.11 we observe that

$$\mathbf{f}(h(h_i(x^{\mathfrak{c}}_{n(*)}))) - \mathbf{f}(h(h_i(z))) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\kappa).$$

So, by subtracting the above relations for $i = 1, 2$, we have

$$\mathbf{f}(h(h_1 - h_2(x^{\mathfrak{c}}_{n(*)}))) - \mathbf{f}(h(h_1 - h_2(z))) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\kappa).$$

Also, $h_1(x^{\mathfrak{c}}_{n(*)}) = h_2(x^{\mathfrak{c}}_{n(*)}) \mod \varphi^{\mathfrak{c}}_\omega(\mathbb{N})$. It follows that $h(h_1 - h_2(z)) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{c}}_\omega(\mathbb{M}_\kappa)$. We conclude from $\mathbb{M}_\alpha + \mathrm{Rang}(h) = \mathbb{M}_\alpha \oplus \mathrm{Rang}(h)$ that

$$h(h_1 - h_2(z)) \in \varphi^{\mathfrak{c}}_\omega(\mathrm{Rang}(h)).$$

Since h is an embedding, $(h_1 - h_2)(z) \in \varphi^{\mathfrak{e}}_{\omega}(\mathbb{N})$. Consequently,

$$h_1(z) - h_2(z) = (h_1 - h_2)(z) \in \varphi^{\mathfrak{e}}_{\omega}(\mathbb{N}).$$

By definition, $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}[\mathcal{K}]$ as required. $\square$

In sum, Lemma 5.8 follows. $\square$

Lemma 5.13. *Let $\bar{\mathbb{M}}$ be a semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$ and $\mathbf{f}$ be an $\mathbf{R}$-endomorphism of $\mathbb{M}_\kappa$. Then for some $\alpha < \kappa$, $n(*) < \omega$ and $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}$ the statement $(\Pr 1)^{n(*)}_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds.*

Proof. In the light of Lemma 5.2, we can find $\alpha \in \kappa \setminus S$ and $n(*) < \omega$ such that $(\Pr)^{n(*)}_{\alpha}[\mathbf{f}, \mathfrak{e}]$ holds. According to Lemma 5.8, there exists $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}$ such that the desired property $(\Pr 1)^{n(*)}_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds, as claimed. $\square$

Remark 5.14. *The property $(\Pr 1)^{n(*)}_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ is almost what is required, only the "error term" $\mathbb{M}_\alpha$ is too large. However, before we do this, we note that for the solution of Kaplansky test problems, as done later in Section 8, this improvement is immaterial as we just divide by a stronger ideal, i.e., we allow to divide by a submodule of bigger cardinality.*

Definition 5.15. *Assume $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ is a weakly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$, $\mathbf{f}$ is an $\mathbf{R}$-endomorphism of $\mathbb{M}_\kappa$, $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ and $n(*) < \omega$. Let also $\bar{\mathbb{G}} := \langle \mathbb{G}_n : n \geq n(*) \rangle$ be a sequence of additive subgroups of $\varphi^{\mathfrak{e}}_{n(*)}(\mathbb{M}_\kappa)$.*

1. *We say $\bar{\mathbb{G}}$ is $\bar{\varphi}$-appropriate for $\bar{\mathbb{M}}$ if $\mathbb{G}_n \subseteq \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$ for all $n \geq n(*)$.*
2. *We say $\bar{\mathbb{G}}$ is compact with respect to $(\bar{\varphi}^{\mathfrak{e}}, n(*))$ in $\mathbb{M}_\kappa$, if it is $\bar{\varphi}$-appropriate and for any $z^*_\ell \in \mathbb{G}_\ell$ with $\ell \geq n(*)$, there is $z^* \in \mathbb{G}_{n(*)}$ such that*

$$z^* - \sum_{\ell=n(*)}^{n} z^*_\ell \in \varphi_{n+1}(\mathbb{M}_\kappa).^{12}$$

[12] So in appropriate sense, $\sum\limits_{\ell \geq n(*)} z_\ell$ exists; of course we can increase $n(*)$.

(3) *We say* $\mathbb{G}_m$ *is* $(\mathcal{K}, \bar{\varphi})$*-finitary with respect to* $\bar{\mathbb{M}}$*, if* $\mathbb{G}_m \subseteq \sum\limits_{\ell<n} \mathbb{K}_\ell + \varphi_\omega(\mathbb{M}_\kappa)$ *for some finite* $n < \omega$ *and* $\mathbb{K}_\ell \in c\ell_{is}(\mathcal{K})$ *such that* $\sum\limits_{\ell<n} \mathbb{K}_\ell \leq_{\aleph_0} \mathbb{M}_\alpha$ *for* α *large enough in* $\kappa \setminus S$.

(4) *We say* $\bar{\mathbb{G}}$ *is* $(\mathcal{K}, \bar{\varphi})$*-finitary with respect to* $\bar{\mathbb{M}}$ *if* $\mathbb{G}_m$ *is* $(\mathcal{K}, \bar{\varphi})$*-finitary with respect to* $\bar{\mathbb{M}}$ *for some* $m \geq n(*)$.

(5) *We say* $\bar{\mathbb{G}}$ *is non-trivial if* $\mathbb{G}_m \nsubseteq \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa)$ *for all* $m \geq n(*)$.

Let us recall "strongly semi-nice construction":

Definition 5.16. *Assume* $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ *is a weakly semi-nice construction with respect to* $(\lambda, \mathfrak{m}, S, \kappa)$*. Then* $\bar{\mathbb{M}}$ *is strongly semi-nice, if for any* $\mathbf{f} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M}_\kappa)$ *and* $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ *the following properties are valid:*

$(G)^+$: (a) *there is some* $\alpha < \kappa$ *such that* $|\mathbb{M}_\alpha| < \lambda$*;*

(b) *there is some* $n(*) < \omega$ *and* $z \in \mathbb{N}^{\mathfrak{e}}_{n(*)}$ *such that for every bimodule homomorphism* $h : \mathbb{N}^{\mathfrak{e}}_{n(*)} \to \mathbb{M}_\kappa$ *we have*

$$\mathbf{f}h(x^{\mathfrak{e}}_{n(*)}) - h(z) \in \mathbb{M}_\alpha + \varphi_\omega(\mathbb{M}_\kappa).$$

Definition 5.17. *In the previous definition, we replace "strongly" by* strong$^+$ *if* $\mathbb{M}_\alpha$ *is replaced by*

$$\mathbb{M}_* \oplus \mathbb{K} \leq_{\aleph_0} \mathbb{M}_\kappa$$

for some $\mathbb{K} \in c\ell_{is}(\mathcal{K})$.

Lemma 5.18. *Assume* $\kappa = \lambda$ *and* $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ *is a semi-nice construction with respect to* $(\lambda, \mathfrak{m}, S, \kappa)$ *such that* $||\mathbb{M}_\alpha|| < \lambda$ *for* $\alpha < \kappa$*. Suppose for given* $\mathfrak{e} \in \mathfrak{E}$ *and* $\mathbf{f} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M}_\kappa)$ *there are*

- $n(*) < \omega$,
- $\alpha(*) \in \kappa \setminus S$ *and*
- $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}[\mathcal{K}]$

such that the property $(\Pr 1)^{n(*)}_{\alpha(*),z}[\mathbf{f},\mathfrak{e}]$ *holds. Then there exists a decreasing sequence* $\bar{\mathbb{G}}^* = \langle \mathbb{G}^*_n : n \geq n(*)\rangle$ *of additive subgroups of* $\varphi^{\mathfrak{e}}_{n(*)}(\mathbb{M}_\kappa)$ *satisfying* $\mathbb{G}^*_n \subseteq \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$ *and equipped with the following properties:*

(1) *for every* $n \geq n(*)$ *and every bimodule-homomorphism* $h : \mathbb{N}^{\mathfrak{e}}_n \longrightarrow \mathbb{M}_\kappa$*,*
   (a) $\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z_n) \in \mathbb{G}^*_n$ *where* $z_n := g^{\mathfrak{e}}_{n(*),n}(z)$*,*
   (b) $\mathbb{G}_n \subseteq \mathbb{M}_{\alpha(*)} + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa)$.

(2) $\bar{\mathbb{G}}^*$ *is compact with respect to* $\bar{\varphi}^{\mathfrak{e}}$*.*

Proof. For every $n \geq n(*)$ we define $\mathbb{G}^*_n$ by

$$\mathbb{G}^*_n := \left\{\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z_n) : h : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa \text{ is a bimodule homomorphism}\right\} \quad (+)$$

Clearly, $\mathbb{G}^*_n$ is an additive subgroup of $\varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$ and the sequence $\bar{\mathbb{G}}$ is decreasing.

(1). The desired claim $\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z_n) \in \mathbb{G}^*_n$ is in (+). It is easily seen that $\mathbb{G}^*_n \subseteq \mathbb{M}_{\alpha(*)} + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa)$. Indeed, let $h : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa$ be a bimodule homomorphism. Then $h' = h \circ g^{\mathfrak{e}}_{n(*),n} : \mathbb{N}^{\mathfrak{e}}_{n(*)} \to \mathbb{M}_\kappa$ is a bimodule homomorphism. In view of $(\Pr 1)^{n(*)}_{\alpha(*),z}[\mathbf{f},\mathfrak{e}]$ we observe that

$$\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z_n) = \mathbf{f}(h'(x^{\mathfrak{e}}_{n(*)})) - h'(z) \in \mathbb{M}_{\alpha(*)} + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

Also as $x^{\mathfrak{e}}_n, z \in \varphi^{\mathfrak{e}}(\mathbb{N}^{\mathfrak{e}}_n)$, we have $\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(z_n) \in \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$.

(2). Suppose $z^*_\ell \in \mathbb{G}^*_\ell$ for $n(*) \leq \ell < \omega$, so for some bimodule homomorphism $h_\ell : \mathbb{N}^{\mathfrak{e}}_\ell \longrightarrow \mathbb{M}_\kappa$ we have

$$z^*_\ell = \mathbf{f}(h_\ell(x^{\mathfrak{e}}_n)) - h_\ell(z_\ell).$$

Let $\alpha(0)$ with $\alpha(*) < \alpha(0) < \kappa$ be such that $\alpha(0) \notin S$, and for $n(*) \leq \ell$, $\mathrm{Rang}(h_\ell) \subseteq \mathbb{M}_{\alpha(0)}$ and $\mathbf{f}(h_\ell(x^{\mathfrak{e}}_\ell)) \in \mathbb{M}_{\alpha(0)}$. Note that such $\alpha(0)$ necessarily exists as $\kappa = \mathrm{cf}(\kappa) \geq \kappa(\mathfrak{E}) + \aleph_0$. We need the following claim:

Claim 5.19. *For each* $n \geq n(*)$ *and* $\beta > \alpha(1)$ *with* $\beta \in \kappa \setminus S$*, there are* γ *satisfying* $\beta < \gamma \in \kappa \setminus S$*, some embedding* $h_{\beta,n} : \mathbb{N}^{\mathfrak{e}}_n \longrightarrow \mathbb{M}_\gamma$ *and some* $\mathbb{K}_{\beta,n} \in c\ell^{\aleph_0}_{is}(\mathcal{K})$ *such that the following two items hold:*

(1) $\mathbb{M}_\beta \oplus \mathrm{Rang}(h_{\beta,n}) \oplus \mathbb{K}_{\beta,n} \leq_{\aleph_0} \mathbb{M}_\gamma$,

(2) $\mathbf{f}(h_{\beta,n}(x^{\mathfrak{e}}_n)) \in \mathrm{Rang}(h_{\beta,n}) \oplus \mathbb{K}_{\beta,n}$.

In particular, $\mathbf{f}(h_{\beta,n}(x^{\mathfrak{e}}_n)) - h_{\beta,n}(z_n) \in \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa)$.

Proof. Fix n and β as in the claim. For every γ satisfying $\beta < \gamma \in \kappa \setminus S$, let $h_\gamma : \mathbb{N}^{\mathfrak{e}}_n \longrightarrow \mathbb{M}_{\gamma+1}$ and $\mathbb{K}^0_\gamma \in c\ell_{is}(\mathcal{K})$ be such that h_γ is a bimodule embedding and

$$\mathbf{f}(h_\gamma(x^{\mathfrak{e}}_n)) \in \mathbb{M}_\gamma \oplus \mathrm{Rang}(h_\gamma) \oplus \mathbb{K}^0_\gamma \leq_{\aleph_0} \mathbb{M}_\kappa.$$

Let $\epsilon_\gamma > \gamma$ be in $\kappa \setminus S$ such that $\mathbf{f}$ maps $\mathbb{M}_{\epsilon_\gamma}$ into $\mathbb{M}_{\epsilon_\gamma}$ and

$$\mathbb{M}_\gamma \oplus \mathrm{Rang}(h_\gamma) \oplus \mathbb{K}^0_\gamma \leq_{\aleph_0} \mathbb{M}_{\epsilon_\gamma}.$$

Let $\mathbf{f}(h_\gamma(x^{\mathfrak{e}}_n)) = z^1_\gamma + z^2_\gamma + z^3_\gamma$, where $z^1_\gamma \in \mathbb{M}_\gamma$, $z^2_\gamma \in \mathrm{Rang}(h_\gamma)$ and $z^3_\gamma \in \mathbb{K}^0_\gamma$. By Fodor's lemma for some z and for a stationary set $T \subseteq \kappa \setminus S$ we have

- $\min(T) > \beta$,
- $z^1_\gamma = z$, for all $\gamma \in T$.

Let $\gamma(1), \gamma(2) \in T$ be such that $\epsilon_{\gamma(1)} < \gamma(2)$. Now, the following

(i) $\gamma := \epsilon_{\gamma(2)}$,

(ii) $h_{\beta,n} := h_{\gamma(2)} - h_{\gamma(1)}$ and

(iii) $\mathbb{K}_{\beta,n} := \mathbb{K}^0_{\gamma(1)} \oplus \mathbb{K}^0_{\gamma(2)} \oplus \mathrm{Rang}(h_{\gamma(1)})$

are as required.

The particular case follows by the choose of $\alpha(*)$. $\square$

Let us complete the proof of Lemma 5.18. To this end, we look at the following club of κ:

$$A := \{\beta < \kappa : \beta > \alpha(0) \text{ and } \mathbf{f}''(\mathbb{M}_\beta) \subseteq \mathbb{M}_\beta\}.$$

Let $n \geq n(*)$ and $\beta \in A \cap (\kappa \setminus S)$. According to Claim 5.19, we can find some $\gamma_\beta > \beta$ a bimodule embedding $h_{\beta,n} : \mathbb{N}^{\mathfrak{e}}_n \longrightarrow \mathbb{M}_{\gamma_\beta}$, and $\mathbb{K}_{\beta,n} \in c\ell^{\aleph_0}_{is}(\mathcal{K})$ such that

- $\mathbb{M}_\beta \oplus \mathrm{Rang}(h_{\beta,n}) \oplus \mathbb{K}_{\beta,n} \leq_{\aleph_0} \mathbb{M}_{\gamma_\beta}$,
- $\mathbf{f}(h_{\beta,n}(x^{\mathfrak{e}}_n)) \in \mathrm{Rang}(h_{\beta,n}) \oplus \mathbb{K}_{\beta,n}$.

Set $\mathcal{U} =: \{\ell : n(*) \leq \ell < \omega\}$ and $\mathfrak{e}^* = \mathfrak{e} \restriction \mathcal{U}$, so $\mathfrak{e}^* \in \mathfrak{E}^{\mathfrak{m}}$ as well. For $\ell \in \mathcal{U}$ and $\beta \in A \cap (\kappa \setminus S)$, set

$$h'_{\beta,\ell} = h_{\beta,\ell} + h_\ell.$$

We are going to use the property presented in part $(G)_2$ of Definition 4.36(3). In this regard, we can find an embedding $\mathbf{h} : \mathbb{P}^{\mathfrak{e}^*} \to \mathbb{M}_\kappa$ and an increasing sequence $\langle \epsilon_n : n < \omega \rangle$ of elements of $A \cap (\kappa \setminus S)$ such that $\epsilon = \sup\limits_{n<\omega} \epsilon_n \in A \cap (\kappa \setminus S)$ and for each $n \in \mathcal{U}, h'_{\epsilon_n,n} = \mathbf{h} \circ h_n^{\mathfrak{e}^*}$. Then

$$\mathbf{f}(h'_{\epsilon_\ell,\ell}(x_\ell^{\mathfrak{e}})) - h'_{\epsilon_\ell,\ell}(z_\ell) = \mathbf{f}(h_{\epsilon_\ell,\ell}(x_\ell^{\mathfrak{e}})) - h_{\epsilon_\ell,\ell}(z_\ell) - z_\ell^* \in z_\ell^* + \varphi_\omega^{\mathfrak{e}}(\mathbb{M}_\kappa).$$

Let $x := \mathbf{f}^{\mathfrak{e}}_{n(*)}(x_1^{\mathfrak{e}^*})$ and $z' := \mathbf{f}^{\mathfrak{e}}_{n(*)}(z)$ where $\mathbf{f}^{\mathfrak{e}}_{n(*)}$ is in Lemma 4.30. It then follows that

- $x - \sum\limits_{\ell=n(*)}^{\ell=n-1} h_\ell^{\mathfrak{e}}(x_\ell^{\mathfrak{e}}) \in \varphi_n^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$, and
- $z' - \sum\limits_{\ell=n(*)}^{\ell=n-1} h_\ell^{\mathfrak{e}}(z_\ell) \in \varphi_n^{\mathfrak{e}}(\mathbb{P}^{\mathfrak{e}})$.

We deduce from these memberships that

$$\mathbf{h}(x) - \sum_{\ell=n(*)}^{\ell=n-1} \mathbf{h}(h_\ell^{\mathfrak{e}}(x_\ell^{\mathfrak{e}})) \in \varphi_n^{\mathfrak{e}}(\mathbb{M}_\kappa)$$

and

$$\mathbf{h}(z') - \sum_{\ell=n(*)}^{\ell=n-1} \mathbf{h}(h_\ell^{\mathfrak{e}}(z_\ell)) \in \varphi_n^{\mathfrak{e}}(\mathbb{M}_\kappa).$$

As $\mathbf{f}$ is an endomorphism, we have

$$\mathbf{f}(\mathbf{h}(x)) - \sum_{\ell=n(*)}^{\ell=n-1} \mathbf{f}(\mathbf{h}(h_\ell^{\mathfrak{e}}(x_\ell^{\mathfrak{e}}))) \in \varphi_n^{\mathfrak{e}}(\mathbb{M}_\kappa).$$

We plug this in the previous formula and deduce that

$$\mathbf{f}(\mathbf{h}(x)) - \mathbf{h}(z') - \sum_{\ell=n(*)}^{\ell=n-1} (\mathbf{f}(\mathbf{h}(h_\ell^{\mathfrak{e}}(x_\ell^{\mathfrak{e}}))) - \mathbf{h}(h_\ell^{\mathfrak{e}}(z_\ell))) \in \varphi_n^{\mathfrak{e}}(\mathbb{M}_\kappa).$$

Note that for some $\mathbb{K} \in c\ell(\mathcal{K}), \mathbb{M}_\epsilon = \mathbb{M}_{\alpha(0)} \oplus \mathbb{P}^{\mathfrak{e}} \oplus \mathbb{K}$. Let $\pi : \mathbb{M}_\epsilon \to \mathbb{M}_{\alpha(0)}$ be the projection map and set

$$(*) : \quad z^* := \pi(\mathbf{f}(\mathbf{h}(x))) - \pi(\mathbf{h}(z')).$$

We apply $h'_{\epsilon_\ell,\ell} = \mathbf{h} \circ h^{\mathfrak{e}^*}_\ell$ along with the previous observation to see:

$$\begin{aligned}\mathbf{f}(\mathbf{h}(h^{\mathfrak{e}}_\ell(x^{\mathfrak{e}}_\ell))) - \mathbf{h}(h^{\mathfrak{e}}_\ell(z_\ell)) &= \mathbf{f}(h'_{\epsilon_\ell,\ell}(x^{\mathfrak{e}}_\ell)) - h'_{\epsilon_\ell,\ell}(z_\ell)\\ &= \mathbf{f}(h_{\epsilon_\ell,\ell}(x^{\mathfrak{e}}_\ell)) - h_{\epsilon_\ell,\ell}(z_\ell) - z^*_\ell.\end{aligned}$$

This yields that

$$\begin{aligned}(*,*): \quad \pi(\mathbf{f}(\mathbf{h}(h^{\mathfrak{e}}_\ell(x^{\mathfrak{e}}_\ell)))) - \pi(\mathbf{h}(h^{\mathfrak{e}}_\ell(z_\ell))) &= \pi(\mathbf{f}(h_{\epsilon_\ell,\ell}(x^{\mathfrak{e}}_\ell))) - \pi(h_{\epsilon_\ell,\ell}(z_\ell)) - \pi(z^*_\ell)\\ &= z^*_\ell.\end{aligned}$$

Therefore,

$$\begin{aligned}z^* - \sum_{\ell=n(*)}^{n-1} z^*_\ell &\overset{(*)}{=} \pi(\mathbf{f}(\mathbf{h}(x))) - \pi(\mathbf{h}(z')) - \sum_{\ell=n(*)}^{n-1} z^*_\ell\\ &\overset{(*,*)}{=} \pi(\mathbf{f}(\mathbf{h}(x))) - \pi(\mathbf{h}(z')) - \sum_{\ell=n(*)}^{\ell=n-1} \pi(\mathbf{f}(\mathbf{h}(h^{\mathfrak{e}}_\ell(x^{\mathfrak{e}}_\ell)))) - \pi(\mathbf{h}(h^{\mathfrak{e}}_\ell(z_\ell)))\\ &\in \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa).\end{aligned}$$

So, z^* is as required. □

Recall that $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$. For simplicity, we bring the following notation:

Notation 5.20. *For each $\ell = 1, 2$, we take finite subsets J_ℓ of I, and let $y_\ell \in \sum_{t\in J_\ell} \mathbb{K}_t$. Suppose $y_1 - y_2 \in \varphi^{\mathfrak{e}}_n(\mathbb{K})$ where $n < \omega$. We say the property $(*)$ is valid, provided for some $y_3 \in \sum_t \{\mathbb{K}_t : t \in J_1 \cap J_2\}$ we have*

(1) $y_1 - y_3 \in \varphi^{\mathfrak{e}}_n(\mathbb{K})$ and

(2) $y_3 - y_2 \in \varphi^{\mathfrak{e}}_n(\mathbb{K})$.

Let $z \in \mathbb{N}^{\mathfrak{e}}_{n(*)}$. Following the above lemma, let us define, $\mathbb{G}^{\mathfrak{m}}_{n,z}[\bar{\mathbb{M}}]$ as $\langle \mathbb{G}_n : n \geq n(*)\rangle$ where

$$\mathbb{G}_n := \{\mathbf{f}(h(x^{\mathfrak{e}}_n)) - h(g^{\mathfrak{e}}_{n(*),n}(z)) : h \in \mathbf{H}om(\mathbb{N}^{\mathfrak{e}}_n, \mathbb{M}_\kappa)\}.$$

We now show that under some extra assumptions we can get a better $\langle \mathbb{G}_n \rangle$:

Lemma 5.21. *Let $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa\rangle$ be a weakly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$, $\mathbf{f}$ be an $\mathbf{R}$-endomorphism, $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ and $n(*) < \omega$.*

(1) *Let* $\mathbb{K} = \bigoplus_{t\in I} \mathbb{K}_t$ *when we view things as* $\mathbf{R}$*-modules, and let* $\bar{\mathbb{G}} = \mathbb{G}^{\mathfrak{m}}_{n,z}[\bar{\mathbb{M}}]$ *be decreasing and compact for* $(\bar{\varphi}^{\mathfrak{c}}, n(*))$ *in* $\mathbb{K}$ *over* $\mathbb{K}_0$*. Then for some finite* $J \subseteq I$ *and* $m < \omega$*:*

$$\mathbb{G}_m \subseteq \bigoplus_{t\in J} \mathbb{K}_t + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{K}).$$

(2) *Let* $\mathbb{K} = \sum_{t\in I} \mathbb{K}_t$ *when we view things as* $\mathbf{R}$*–modules, and suppose the property* $(*)$ *from Notation 5.20 is valid. Then for some finite subset* $J \subseteq I$ *and* $m < \omega$*:*

$$\mathbb{G}_m \subseteq \bigoplus_{t\in J} \mathbb{K}_t + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{K}).$$

(3) *Let* $\bar{\mathbb{G}}$ *be compact with respect to* $(\bar{\varphi}^{\mathfrak{c}}, n(*))$ *in* $\mathbb{K}$*, as* $\mathbf{R}$*-modules, and let* $h : \mathbb{K} \longrightarrow \mathbb{K}'$ *be an* $\mathbf{R}$*-homomorphism. Suppose for any* $h(x) \in \varphi^{\mathfrak{c}}_{\ell}(\mathbb{K}') \setminus \varphi^{\mathfrak{c}}_{\ell+1}(\mathbb{K}')$ *there is some* $y \in \varphi^{\mathfrak{c}}_{\ell}(\mathbb{K}) \setminus \varphi^{\mathfrak{c}}_{\ell+1}(\mathbb{K})$ *so that* $h(y) = h(x)$*. Then*

$$h''(\bar{\mathbb{G}}) := \big\langle h''(\mathbb{G}_n) : n \geq n(*)\big\rangle$$

is compact with respect to $(\bar{\varphi}^{\mathfrak{c}}, n(*))$ *in* $\mathbb{K}'$*.*

(4) *If* $\mathbb{G} \subseteq \mathbb{K} := \bigoplus_{t=1}^{n} \mathbb{K}_t$ *and the projections from* $\mathbb{G}$ *to each* $\mathbb{K}_t$ *is* $(\mathcal{K}, \bar{\varphi})$*-finitary, then* $\mathbb{G}$ *is* $(\mathcal{K}, \bar{\varphi})$*-finitary.*

(5) *If* $\mathbb{K}_0 \subseteq \mathbb{K}_1 \leq^{ads}_{\mathcal{K},\aleph_0} \mathbb{K}_2$ *and* $\bar{\mathbb{G}}$ *is* $(\bar{\varphi}^{\mathfrak{c}}, n(*))$*-compact in* $\mathbb{K}_2$ *over* $\mathbb{K}_0$*, then* $\langle \mathbb{G}_n \cap \mathbb{K}_1 : n \geq n(*)\rangle$ *is* $(\bar{\varphi}^{\mathfrak{c}}, n(*))$*-compact in* $\mathbb{K}_1$ *over* $\mathbb{K}_0$*.*

Proof. (1) Let us first suppose that the index set I is countable. Suppose by contradiction that for all finite $J \subseteq I$ and $m < \omega$,

$$\mathbb{G}_m \nsubseteq \bigoplus_{t\in J} \mathbb{K}_t + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{K}).$$

We argue by induction on $\ell \geq n(*)$ to find z_ℓ, J_ℓ and n_ℓ such that the following properties hold:

i) J_ℓ is a finite subset of I,

ii) $J_\ell \subseteq J_{\ell+1}$,

iii) $I = \bigcup_{\ell<\omega} J_\ell$,

iv) $z_\ell \in \mathbb{G}_\ell \setminus \left(\bigoplus_{t \in J_\ell} \mathbb{K}_t + \varphi^{\mathfrak{c}}_\omega(\mathbb{K}) \right)$,

v) $z_\ell \notin \bigoplus_{t \in J_\ell} \mathbb{K}_t + \varphi^{\mathfrak{c}}_{n_{\ell+1}}(\mathbb{K})$,

vi) $z_\ell \in \bigoplus_{t \in J_{\ell+1}} \mathbb{K}_t$.

Let $\mathcal{U}$ be an infinite subset of ω such that $0 \in \mathcal{U}$ and

$$\ell < m \text{ and } m \in \mathcal{U} \implies n_{\ell+1} < m.$$

Define $\langle z^*_\ell : \ell < \omega \rangle$ by $z^*_\ell = z_\ell$ if $\ell \in \mathcal{U}$ and $z^*_\ell = 0$ otherwise. By compactness, we can find $z^* \in \mathbb{G}_{n(*)}$ such that for each $\ell \geq n(*)$,

$$z^* - \sum_{i=n(*)}^{\ell} z^*_i \in \varphi^{\mathfrak{c}}_{\ell+1}(\mathbb{K}).$$

Let $J \subseteq I$ be a finite set such that $z^* \in \bigoplus_{t \in J} \mathbb{K}_t$. Pick $m \in \mathcal{U}$ such that $J \subseteq J_m$ and set n be the least element of $\mathcal{U}$ above m. Then $z_m = z^*_m$ and

$$z^* - \sum_{i=n(*)}^{m} z^*_i = z^* - \sum_{i=n(*)}^{n-1} z^*_i \in \varphi^{\mathfrak{c}}_n(\mathbb{K}),$$

an easy contradiction.

Let us now show that we can get the result for arbitrary I. Thus suppose I is uncountable and suppose that the conclusion of the lemma fails for it. Construct z_ℓ, J_ℓ, n_ℓ as before and set $\bar{I} = \bigcup_{\ell=n(*)}^{\omega} J_\ell$. Then $\bar{I}$ is countable. Set $\bar{\mathbb{K}} = \bigoplus_{t \in \bar{I}} \mathbb{K}_t$ and $\bar{\mathbb{G}}_n = \mathbb{G}_n \cap \bar{\mathbb{K}}$. It then follows that the conclusion fails for this case, contradicting the above argument.

(2). This is similar to (1).

(3). Suppose $h(x_\ell) \in h''(\mathbb{G}_\ell)$, for $\ell \geq n(*)$. By our assumption, we can find a sequence $\langle y_\ell : \ell \geq n(*) \rangle$ such that

- $y_\ell \in \mathbb{G}_\ell$,
- $h(y_\ell) = h(x_\ell)$,
- $h(x_\ell) \in \varphi^{\mathfrak{c}}_\ell(\mathbb{K}') \setminus \varphi^{\mathfrak{c}}_{\ell+1}(\mathbb{K}') \implies y_\ell \in \varphi^{\mathfrak{c}}_\ell(\mathbb{K}) \setminus \varphi^{\mathfrak{c}}_{\ell+1}(\mathbb{K})$.

Let $z \in \mathbb{G}_{n(*)}$ be such that for each $n \geq n(*)$, $z - \sum\limits_{\ell=n(*)}^{n} y_\ell \in \varphi^{\mathfrak{c}}_{n+1}(\mathbb{K})$. This implies that

$$h(z) - \sum_{\ell=n(*)}^{n} h(y_\ell) \in \varphi^{\mathfrak{c}}_{n+1}(\mathbb{K}).$$

Thus $h(z) \in h''(\mathbb{G}_{n(*)})$ is as required.

(4). For $t = 1, \cdots, t$ let $\pi_t : \mathbb{K} \to \mathbb{K}_t$ be the projection map to $\mathbb{K}_t$. We have

$$\pi_t''(\mathbb{G}) \subseteq \sum_{i<m_t} \mathbb{K}^t_i + \varphi^{\mathfrak{c}}_\omega(\mathbb{K}),$$

where each $\mathbb{K}^t_i$ is in $c\ell_{\rm is}(\mathbb{K})$. Then

$$\mathbb{G} = \sum_{t=1}^{n} \pi_t''(\mathbb{G}) \subseteq \sum_{t=1}^{n} \left(\sum_{i<m_t} \mathbb{K}^t_i + \varphi^{\mathfrak{c}}_\omega(\mathbb{K}) \right) \subseteq \sum_{t=1}^{n} \sum_{i<m_t} \mathbb{K}^t_i + \varphi^{\mathfrak{c}}_\omega(\mathbb{K}).$$

We are done.

(5). For each $\ell \geq n(*)$, we peak $z_\ell \in \mathbb{G}_\ell \cap \mathbb{K}_1$. Due to our assumption, there exists $z^* \in \mathbb{K}_2$ such that $z^* - \sum\limits_{\ell=n(*)}^{n} z_\ell \in \varphi^{\mathfrak{c}}_{n+1}(\mathbb{K}_2)$ for all $n \geq n(*)$. Consequently,

$$\mathbb{K}_2 \models \exists z^* \bigwedge_{n=n(*)}^{\omega} \varphi^{\mathfrak{c}}_{n+1}(z^* - \sum_{\ell=n(*)}^{n} z_\ell).$$

As $\mathbb{K}_1 \leq^{ads}_{\mathcal{K},\aleph_0} \mathbb{K}_2$ and in the light of Lemma 4.20(1) we have

$$\mathbb{K}_1 \models \exists z^* \bigwedge_{n=n(*)}^{\omega} \varphi^{\mathfrak{c}}_{n+1}(z^* - \sum_{\ell=n(*)}^{n} z_\ell).$$

Let $z^* \in \mathbb{K}_1$ be witness this. Then, $z^* - \sum\limits_{\ell=n(*)}^{n} z_\ell \in \varphi^{\mathfrak{c}}_{n+1}(\mathbb{K}_1)$ for all $n \geq n(*)$, as required. $\square$

Remark 5.22. *Adopt the notation of Lemma 5.21(3).*

i) *We can weaken the assumption on h to the following property: for some $\eta \in {}^\omega\omega$ diverging to infinity if $\ell \geq n(*)$ and $h(x) \in \varphi^{\mathfrak{c}}_\ell(\mathbb{K}') \setminus \varphi^{\mathfrak{c}}_{\ell+1}(\mathbb{K}')$, then $h(x) = h(y)$ for some $y \in \varphi^{\mathfrak{c}}_{n(*)}(\mathbb{K}) \setminus \varphi^{\mathfrak{c}}_{\eta(\ell)}(\mathbb{K})$.*

ii) *Note that if h is a projection, then it satisfies in the presented condition from the first item.*

Lemma 5.23. *Let* $\mathbf{R}$, $\mathbf{S}$ *and every* $\mathbb{N} \in \mathcal{K}$ *be of cardinality* $< 2^{\aleph_0}$ *and let* $\mathfrak{e} \in \mathfrak{E}$. *Then there is no non-trivial compact* $\bar{\mathbb{G}}$ *for* $\bar{\varphi}^{\mathfrak{e}}$ *in any* $\mathcal{K}$*-bimodule* $\mathbb{M}$.

Proof. Let $\langle \mathbb{G}_n : n \geq n_0 \rangle$ be $(\bar{\varphi}^{\mathfrak{e}}, n_0)$-compact in $\mathbb{M}$. According to Definition 5.15 (4) we need to show that $\mathbb{G}_m \subseteq \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$ for some m. Suppose on the contradiction that $\mathbb{G}_m \nsubseteq \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$ for all $m \geq n_0$. Hence, $\mathbb{G}_m \nsubseteq \varphi^{\mathfrak{e}}_{\ell_m}(\mathbb{M})$ for some $\ell_m > n_0$. For each $m \geq n_0$, we pick $z_m \in \mathbb{G}_m \setminus \varphi^{\mathfrak{e}}_{\ell_m}(\mathbb{M})$. For any infinite $\mathcal{U} \subseteq \omega \setminus n_0$, find $z_{\mathcal{U}} \in \mathbb{G}_{n_0}$ such that

$$z_{\mathcal{U}} - \sum \{z_m : m \in \mathcal{U} \cap [n_0, n]\} \in \varphi^{\mathfrak{e}}_{n+1}(\mathbb{M})$$

for all $n \in \mathcal{U}$. Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be two subsets of ω with finite intersection property. Then $z_{\mathcal{U}_1} \neq z_{\mathcal{U}_2}$. It follows that

$$2^{\aleph_0} \leq ||\mathbb{G}_{n_0}|| \leq ||\mathbb{M}|| < 2^{\aleph_0},$$

which is impossible. $\square$

Lemma 5.24. *Assume* $\kappa = \lambda$ *and* $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ *is a semi-nice construction with respect to* $(\lambda, \mathfrak{m}, S, \kappa)$ *such that* $\alpha < \kappa$, $||\mathbb{M}_\alpha|| < \lambda$.

i) *Suppose* $\mathfrak{e} \in \mathfrak{E}$ *and* $\mathbf{f}$ *is an* $\mathbf{R}$*-endomorphism of* $\mathbb{M}_\kappa$, *for some* $n(*) < \omega$, $\alpha(*) \in \kappa \setminus S$ *and* $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}[\mathcal{K}]$ *such that the property* $(\Pr 1)^{n(*)}_{\alpha(*),z}[\mathbf{f}, \mathfrak{e}]$ *holds. Then*

$$\mathbb{G}^{\mathbf{m}}_{n,z}[\bar{\mathbb{M}}] := \langle \mathbb{G}_n : n \geq n(*) \rangle$$

is $(\mathcal{K}, \bar{\varphi}^{\mathfrak{e}})$*-finitary in* $\mathbb{M}_\kappa$, *for some additive subgroups of* $\varphi^{\mathfrak{e}}_{n(*)}(\mathbb{M}_\kappa)$ *such as* $\bar{\mathbb{G}}^* = \langle \mathbb{G}^*_n : n \geq n(*) \rangle$.

ii) *Assume in addition that for* $\mathbb{N} \in \mathcal{K}$, *there is no non-trivial* $\bar{\mathbb{L}} = \langle \mathbb{L}_n : n \geq n(*) \rangle$ *compact with respect to* $(\bar{\varphi}^{\mathfrak{e}}, n(*))$ *in* $\mathbb{M}_* \oplus \mathbb{N}$, *then, by increasing* $n(*)$, *we can take* $\mathbb{G}^{\mathbf{m}}_{n,z}[\bar{\mathbb{M}}] = \bar{0}$, *i.e.*, $\mathbb{G}_n = 0$ *for all* $n \geq n(*)$.

Proof. Let $\bar{\mathbb{G}}^* = \langle \mathbb{G}^*_n : n \geq n(*) \rangle$ be as Lemma 5.18, and pick $\alpha(*) \in \kappa \setminus S$ be such that $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}[\mathcal{K}] \subseteq \mathbb{M}_{\alpha(*)}$. We use the assumption $\mathbb{M}_{\alpha(*)} \in c\ell(\mathcal{K})$ along with

Lemma 5.21(1) to find $m < \omega$ and a finite subset $\{\mathbb{K}_0, \cdots, \mathbb{K}_{n-1}\}$ of $c\ell_{\rm is}(\mathcal{K})$ such that each $\mathbb{K}_i$ is a direct summand of $\mathbb{M}_{\alpha(*)}$ and

$$\mathbb{G}_m \subseteq \sum_{\ell<n} \mathbb{K}_\ell + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

Recall from Lemma 4.37(1) that $\sum\limits_{\ell<n} \mathbb{K}_\ell \leq_{\aleph_0} \mathbb{M}_{\alpha(*)} \leq_{\aleph_0} \mathbb{M}_\alpha$, where $\alpha > \alpha(*)$. Thus $\mathbb{G}^{\mathbf{m}}_{n,z}[\bar{\mathbb{M}}] = \langle \mathbb{G}_n : n \geq n(*)\rangle$ is $(\mathcal{K}, \bar{\varphi}^{\mathfrak{e}})$-finitary in $\mathbb{M}_\kappa$.

Now suppose that for each $\mathbb{N} \in \mathcal{K}$, there is no non-trivial $\bar{\mathbb{L}} = \langle \mathbb{L}_n : n \geq n(*)\rangle$ compact for $(\bar{\varphi}^{\mathfrak{e}}, n(*))$ in $\mathbb{M}_* \oplus \mathbb{N}$. Suppose by contradiction that $\mathbb{G}^{\mathbf{m}}_{n,z}[\bar{\mathbb{M}}] \neq \bar{0}$. Let $\alpha(*)$, $\{\mathbb{K}_0, \cdots, \mathbb{K}_{n-1}\} \subset c\ell_{\rm is}(\mathcal{K})$ and m be as above. By increasing $n(*)$ we may assume that $m = n(*)$. Then $\bigoplus_{\ell<n} \mathbb{K}_\ell$ is a direct summand of $\mathbb{M}_{\alpha(*)}$. Suppose $\mathbb{M}_{\alpha(*)} = \bigoplus_{\ell<n} \mathbb{K}_\ell \oplus \mathbb{M}$ and we look at the natural projection map $\pi : \mathbb{M}_{\alpha(*)} \to \mathbb{M}$. In view of Lemma 5.21(3) (and Remark 5.22) we observe that $\pi''(\mathbb{G}^{\mathbf{m}}_{n,z}[\bar{\mathbb{M}}])$ is non-trivial and compact for $(\bar{\varphi}^{\mathfrak{e}}, n(*))$ in $\mathbb{M}_* \oplus \mathbb{N}$, which is a contradiction. $\square$

Remark 5.25. (1) Suppose for every $\mathcal{K}$-bimodule $\mathbb{M}$ and $\mathfrak{e} \in \mathfrak{E}$ and $\mathbb{G}_n \subseteq \mathbb{M}$ for $n \geq n_0$, if $\langle \mathbb{G}_n : n \geq n_0\rangle$ is $(\bar{\varphi}^{\mathfrak{e}}, n_0)$-compact in $\mathbb{M}$, then for some m, $\mathbb{G}_m \subseteq \varphi^{\mathfrak{e}}_\omega(\mathbb{M})$. Recall that Lemma 5.23 presents a situation for which this property holds. In view of Lemma 5.18 we can choose $\mathbb{G}_{n(*)} = 0$. In particular, the "error term" disappears, i.e., for every endomorphism $\mathbf{f}$ of $\mathbb{M}_\lambda$ as an $\mathbf{R}$–module, for some m we have $\mathbf{f} \restriction \varphi^{\mathfrak{e}}_m(\mathbb{M}_\lambda)/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\lambda)$ is equal to $\mathbf{h}^{\mathfrak{e},m}_{\mathbb{M}_\lambda,z}$ (for its definition, see Definition 6.12 below).

(2) If $\mathbf{R}$, $\mathbf{S}$ have cardinality $< 2^{\aleph_0}$, we have some interesting candidates to define $\mathcal{K}$. For example, let $\mathcal{K}$ be the family of finitely generated finitely presented bimodules.

6. More specific rings and families $\mathfrak{E}$

In this section we introduce some special rings that play an important role in our solution of Kaplansky test problems. In fact, we are going to present the proof of Corollary 2.1. We also specify some specific elements of $\mathfrak{E}$ that we work with them *later. Let us start by extending the notion of pure semisimple from rings to a pair of rings.*

Definition 6.1. *Given a bimodule $\mathbb{M}$ and a sequence $\bar{\varphi}$ of formulas, the notation $(*)^{\bar{\varphi},\mathbb{M}}_{\aleph_0,\aleph_0}$ stands for the following two assumptions:*

(a) $\varphi_n = \varphi_n(x)$ is in $\mathcal{L}^{cpe}_{\aleph_0,\aleph_0}(\tau_{\mathbf{R}})$, and

(b) the sequence $\langle \varphi_n(\mathbb{M}) : n < \omega \rangle$ is strictly decreasing.

Note that if $\bar{\varphi}$ is as above, then it is $(\aleph_0, \aleph_0)$-adequate. Also for simplicity, we can assume that $\varphi_{n+1}(x) \vdash \varphi_n(x)$ holds for all $n < \omega$.

Definition 6.2. *The pair $(\mathbf{R}, \mathbf{S})$ of rings is called purely semisimple if for some bimodule $\mathbb{M}^*$ and a sequence $\bar{\varphi} = \langle \varphi_n(x) : n < \omega \rangle$, the property $(*)^{\bar{\varphi},\mathbb{M}^*}_{\aleph_0,\aleph_0}$ holds.*

Thanks to Theorem 3.22 a ring $\mathbf{R}$ is not purely semisimple if and only if the pair $(\mathbf{R}, \mathbf{R})$ is purely semisimple.

Definition 6.3. *(1) The sequence $\bar{\varphi} := \langle \varphi_n(x) : n < \omega \rangle$ is called very nice if it is as in Definition 6.1 and for some $m_n, k_\ell < \omega$ and $a_\ell, b_{\ell,i} \in \mathbf{R}$ we have*

(a) $\varphi_n(x) = (\exists y_0, \ldots, y_{k_n-1})[\bigwedge\limits_{\ell=0}^{m_n-1} a_\ell x_\ell = \sum\limits_{i<k_\ell} b_{\ell,i} y_i]$,

(b) $m_n < m_{n+1}$, $k_\ell \leq k_{\ell+1}$.

(2) Let $\bar{\varphi}^1 := \langle \varphi^1_n(x) : n < \omega \rangle$ and $\bar{\varphi}^2 := \langle \varphi^2_n(x) : n < \omega \rangle$ be two sequences of formulas. By $\bar{\varphi}^1 \leq \bar{\varphi}^2$ we mean

$$(\forall n < \omega)(\exists m < \omega)[\varphi^2_m(x) \vdash \varphi^1_n(x)].$$

(3) Let $\bar{\varphi}^1$ and $\bar{\varphi}^2$ be two sequences of formulas. We say $\bar{\varphi}^1$ and $\bar{\varphi}^2$ are equivalent if $\bar{\varphi}^1 \leq \bar{\varphi}^2$ and $\bar{\varphi}^2 \leq \bar{\varphi}^1$.

(4) $\mathfrak{e} \in \mathfrak{E}$ is called κ-simple if for each n there are a set $X \subseteq \mathbb{N}^{\mathfrak{e}}_n$ of cardinality $< \kappa$ and a set Σ of $< \kappa$ equations from $\mathcal{L}_{\aleph_0,\aleph_0}(\tau_{\mathbf{R}})$ with parameters from X such that $\mathbb{N}^{\mathfrak{e}}_n$ is generated by X freely except the equations in Σ. We call X a witness.

Lemma 6.4. *Adopt the above notation. The following assertions are true:*

(1) *Suppose $(*)^{\bar{\varphi},\mathbb{M}^*}_{\aleph_0,\aleph_0}$ holds. Then there is a very nice sequence $\bar{\varphi}'$ equivalent to $\bar{\varphi}$.*

(2) *If the φ_n's are from infinitary logic, the same thing holds, only m_n, k_ℓ may be infinite but for each ℓ the set $\{i : b_{\ell,i} \neq 0\}$ is finite.*

Proof. We only prove (1), as clause (2) can be proved in a similar way. Thus assume $(*)^{\bar{\varphi},\mathbb{M}^*}_{\aleph_0,\aleph_0}$ holds. According to Lemma 3.12, we can assume that each φ_n is a simple formula, so it is of the form

$$\varphi_n(x) = (\exists y_0, \ldots, y_{k_n-1}) \left[\bigwedge_{\ell=0}^{m_n-1} a^n_\ell x = \sum_{i<k_n} b^n_{\ell,i} y_i \right],$$

where $a^n_\ell, b^n_{\ell,i}$ are members of $\mathbf{R}$, k_n, m_n are natural numbers. After replacing φ_n with $\bigwedge\limits_{\ell\leq n} \varphi_\ell$, if necessary, we may assume that the sequence is decreasing in the sense that $\varphi_{n+1}(x) \vdash \varphi_n(x)$, for each n. Also without loss of generality and by taking $b^n_{\ell,i} = 0_{\mathbf{R}}$, we can assume that $k_\ell < k_{\ell+1}$ and $m_n < m_{n+1}$. Finally, note that we can even get a better sequence by taking

$$\varphi_0(x) = \exists y_0 (x = y_0).$$

So, $m_0 = 1$, $a_0 = 1_{\mathbf{R}}$, $k_0 = 1$ and $b_{0,0} = 1$. This completes the proof. $\square$

Remark 6.5. *Let $\bar{\varphi}$ be a very nice sequence. According to Lemma 6.4 and its proof, we assume from now on that φ_0 is of the form $\varphi_0(x) = \exists y_0(x = y_0)$.*

We now assign to each very nice sequence $\bar{\varphi}$, an element $\mathfrak{e}(\bar{\varphi}) \in \mathfrak{E}_{\aleph_0,\aleph_0}$ as follows.

Definition 6.6. *Suppose $\bar{\varphi}$ is a very nice sequence. Then*

$$\mathfrak{e}(\bar{\varphi}) = \langle \mathbb{N}_n, x_n, g_n : n < \omega \rangle$$

is defined as follows:

i) $\mathbb{N}_n$ *is the* $(\mathbf{R}, \mathbf{S})$ *bimodule which is generated by*

$$\{x_n\} \cup \{y_{n,i} : i < k_{m_n - 1}\}$$

freely except to the following equations

$$\big\{a_\ell x_n = \sum_{i<k_\ell} b_{\ell,i} y_{n,i} : \ell < m_n\big\}.$$

In other words,

$$\mathbb{N}_n = \frac{\mathbf{R}x\mathbf{S} \oplus (\bigoplus_{i<k_{m_n-1}} \mathbf{R}y_{n,i}\mathbf{S})}{\mathbb{K}},$$

where $\mathbb{K}$ *is the bimodule generated by* $\langle a_\ell x_n - \sum\limits_{i<k_\ell} b_{\ell,i} y_{n,i} : \ell < m_n \rangle$.

ii) $g_n : \mathbb{N}_n \to \mathbb{N}_{n+1}$ *is defined so that* $g_n(x_n) = x_{n+1}$ *and* $g_n(y_{n,i}) = y_{n+1,i}$ *for* $i < k_{m_n - 1}$.

We call $\mathfrak{e}$ *simple if it is of the form* $\mathfrak{e}(\bar{\varphi})$ *for some very nice* $\bar{\varphi}$.

Note that by Remark 6.5, for each n, $x_n = y_{n,0}$. *Then next lemma shows that* $\mathfrak{e}(\bar{\varphi}) \in \mathfrak{E}_{\aleph_0,\aleph_0}$

Lemma 6.7. *Let* $\bar{\varphi}$ *be very nice and set* $\mathfrak{e} := \mathfrak{e}(\bar{\varphi})$. *The following assertions are valid:*

(1) $x_n \in \varphi_n(\mathbb{N}_n)$.

(2) Let $\mathbb{M}$ *be a bimodule. Then* $x^* \in \varphi_n(\mathbb{M})$ *if and only if for some bimodule homomorphism* $h : \mathbb{N}_n \to \mathbb{M}$ *we have* $h(x_n) = x^*$.

(3) $x_n \notin \varphi_{n+1}(\mathbb{N}_n)$.

(4) Each g_n *is a bimodule homomorphism.*

Proof. Clauses (1), (3) and (4) are trivial. Clause (2) follows from Lemma 4.8. $\square$

We will frequently use the following simple observation without any mention of it.

Lemma 6.8. *Let* $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ *be simple. Then for every* $n < \omega$, $\psi_n^{\mathfrak{e}}$ *and* $\varphi_n^{\mathfrak{e}}$ *are equivalent. In other words, if* $\mathbb{M}_*^{\mathfrak{m}} \leq_{\aleph_0} \mathbb{M}$ *and* $x \in \mathbb{M}$ *then* $\mathbb{M} \models$ *"*$\psi_n^{\mathfrak{e}}(x) \leftrightarrow \varphi_n^{\mathfrak{e}}(x)$*".*

We now define some very special bimodules.

Definition 6.9. *Let $n < \omega$.*

(1) *Let $\mathbb{N}'_n$ be the bimodule generated by x_n, $y'_{n,i}$ and $y''_{n,i}$ for $i < k_{m_n-1}$ freely subject to the following relations for $\ell < m_n$:*
   (a) $a_\ell x = \sum_{i<k_\ell} b_{\ell,i} y'_{n,i}$
   (b) $a_\ell x = \sum_{i<k_n} b_{\ell,i} y''_{n,i}$.

(2) *Let $\mathbb{N}^\ell_n$ for $\ell = 1, 2$, be the sub-bimodule of $\mathbb{N}'_n$ generated by:*

$$\{x_n\} \cup \{y'_{n,i} : i < k_{m_n-1}\} \qquad \text{for} \quad \ell = 1,$$
$$\{x_n\} \cup \{y''_{n,i} : i < k_{m_n-1}\} \qquad \text{for} \quad \ell = 2.$$

(3) *Let $f^\ell_n : \mathbb{N}_n \longrightarrow \mathbb{N}^\ell_n$ be the bimodule homomorphisms defined by the following assignments:*
   (a) $f^\ell_n(x_n) = x_n$,
   (b) $f^1_n(y_{n,i}) = y'_{n,i}$ *and*
   (c) $f^2_n(y_{n,i}) = y''_{n,i}$.

(4) $\mathbb{L}^{\mathrm{tr},\bar\varphi}_n := \{z \in \varphi_n(\mathbb{N}_n) : f^1_n(z) - f^2_n(z) \in \varphi_\omega(\mathbb{N}'_n)\}$.

Clearly, it is an abelian subgroup of $\mathbb{N}_n$.

Let $\mathfrak{e} \in \mathfrak{E}$ and suppose $\bar\varphi$ is an adequate sequence for $\mathfrak{e}$. Also, let $(\mathbf{h_1}, \mathbf{h_2})$ be a pair of bimodule homomorphisms from $\mathbb{N}^{\mathfrak{e}}_n$ to $\mathbb{M}$. Recall from Definition 4.29 that

$$\mathbb{L}^{\mathfrak{e},\bar\varphi,\mathbf{h_1},\mathbf{h_2}}_n = \big\{z \in \varphi_n(\mathbb{N}^{\mathfrak{e}}_n) : \mathbf{h_1}(z) = \mathbf{h_2}(z) \mod \varphi_\omega(\mathbb{M})\big\}.$$

Suppose $\mathfrak{m}$ is a context and recall that

$$\mathbb{L}^{\mathfrak{e}}_n[\mathfrak{m}] = \bigcap_{\mathbb{M}\in\mathcal{K}\cup\{\mathbb{M}_*\}} \Big\{\mathbb{L}^{\mathfrak{e},\bar\varphi^{\mathfrak{e}},\mathbf{h_1},\mathbf{h_2}}_n : \mathbf{h_1}, \mathbf{h_2} \in \mathbf{H}om(\mathbb{N}^{\mathfrak{e}}_n, \mathbb{M}) \text{ and } \mathbf{h_1}(x^{\mathfrak{e}}_n) = \mathbf{h_2}(x^{\mathfrak{e}}_n)\Big\}.$$

In the next lemma we show that the bimodule $\mathbb{N}'$ and the homomorphisms f^1_n, f^2_n are sufficient to determine $\mathbb{L}^{\mathfrak{e}(\bar\varphi)}_n[\mathfrak{m}]$ provided $\bar\varphi$ is very nice.

Lemma 6.10. *Let $\mathfrak{m}$ be a nice context, $\bar\varphi$ be very nice and $\mathfrak{e} := \mathfrak{e}(\bar\varphi) \in \mathfrak{E}^{\mathfrak{m}}$. Then $\mathbb{L}^{\mathfrak{e}(\bar\varphi)}_n[\mathfrak{m}] = \mathbb{L}^{\mathrm{tr},\bar\varphi}_n$.*

Proof. Let $z \in \mathbb{L}_n^{\mathfrak{e}(\bar\varphi)}[\mathfrak{m}]$. Since $f_n^1, f_n^2 : \mathbb{N}_n \longrightarrow \mathbb{N}_n^\ell$ satisfy $f_n^1(x_n) = f_n^2(x_n)$, thus it follows from Definition 6.9 that $z \in \mathbb{L}_n^{\mathrm{tr},\bar\varphi}$.

In order to prove $\mathbb{L}_n^{\mathrm{tr},\bar\varphi} \subseteq \mathbb{L}_n^{\mathfrak{e}(\bar\varphi)}[\mathfrak{m}]$, let $\mathbf{h_1}, \mathbf{h_2} : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M} \in \mathcal{K} \cup \{\mathbb{M}_*\}$ be such that $\mathbf{h_1}(x_n) = \mathbf{h_2}(x_n)$. Note that

$$\mathbb{M} \models \varphi_n^{\mathfrak{e}}(\mathbf{h}_1(x_n))$$

and

$$\mathbb{M} \models \varphi_n^{\mathfrak{e}}(\mathbf{h}_2(x_n)).$$

Thus, we can find $\mathbf{h}_1' : \mathbb{N}_n^1 \to \mathbb{M}$ and $\mathbf{h}_2' : \mathbb{N}_n^2 \to \mathbb{M}$ such that $\mathbf{h}_1 = f_n^1 \circ \mathbf{h}_1'$ and $\mathbf{h}_2 = f_n^2 \circ \mathbf{h}_2'$. Take some $z \in \mathbb{L}_n^{\mathrm{tr},\bar\varphi}$. By definition, $z \in \varphi_n^{\mathfrak{e}}(\mathbb{N}_n)$ and $f_n^1(z) - f_n^2(z) \in \varphi_\omega(\mathbb{N}_n')$. But then $\mathbf{h}_1(z) - \mathbf{h}_2(z) \in \varphi_\omega^{\mathfrak{e}}(\mathbb{M})$. From this, $z \in \mathbb{L}_n^{\mathfrak{e},\bar\varphi,\mathbf{h_1},\mathbf{h_2}}$. Since $\mathbf{h_1}, \mathbf{h_2}$ were be arbitrary, $z \in \mathbb{L}_n^{\mathfrak{e}(\bar\varphi)}[\mathfrak{m}]$. $\square$

So, if $\mathfrak{m}$ is a context whose $\mathfrak{E}^{\mathfrak{m}}$ consists of simple $\mathfrak{e}$'s and if $\mathbb{M} \in \mathcal{K}^{\mathfrak{m}}$, then every $\mathbf{R}$-endomorphism is in some sense definable, i.e., by fixing $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ and restricting ourselves to $\varphi_n^{\mathfrak{e}}(\mathbb{M})$ for large enough n, modulo $\varphi_n^{\mathfrak{e}}(\mathbb{M})$, it is determined by some $z \in \mathbb{L}_n^{\mathfrak{e}}[\mathcal{K}^{\mathfrak{m}}]$. However, not every such z may really occur. We try to formalize this in Lemma 6.22.

Notation 6.11. *From now on we fix a context $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$.*

Definition 6.12. *Suppose $\mathfrak{e} \in \mathfrak{E}$, $n < \omega$ and $z \in \mathbb{L}_n^{\mathfrak{e}}$ are given. Let $\mathbb{M}$ be a bimodule which $\leq_{\mathcal{K},\aleph_0}^{ads}$-extends $\mathbb{M}_*$. We define $\mathbf{h}_{\mathbb{M},z}^{\mathfrak{e},n} \in \mathrm{End}(\psi_n^{\mathfrak{e}}(\mathbb{M})/\varphi_\omega^{\mathfrak{e}}(\mathbb{M}))$ as an endomorphism of the additive group $\psi_n^{\mathfrak{e}}(\mathbb{M})/\varphi_\omega^{\mathfrak{e}}(\mathbb{M})$[13] so that for every bimodule homomorphism $h : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}$*

$$\mathbf{h}_{\mathbb{M},z}^{\mathfrak{e},n}\left(h(x_n^{\mathfrak{e}}) + \varphi_\omega^{\mathfrak{e}}(\mathbb{M})\right) := h(z) + \varphi_\omega(\mathbb{M}).$$

Remark 6.13. *According to Lemma 4.8, every $w \in \psi_n^{\mathfrak{e}}(\mathbb{M})$ is of the form $h(x_n^{\mathfrak{e}})$, for some $h : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}$ as above. Also note that as $z \in \mathbb{L}_n^{\mathfrak{e}}$, if $h' : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}$ is another*

[13] Pedantically we should write $\psi_n^{\mathfrak{e}}(\mathbb{M})/\varphi_\omega^{\mathfrak{e}}(\mathbb{M}) \cap \psi_n^{\mathfrak{e}}(\mathbb{M})$.

bimodule homomorphism such that $h'(x^{\mathfrak{e}}_n) = h(x^{\mathfrak{e}}_n)$, then $h'(z) = h(z)$ mod $\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$. This shows that $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ is well-defined, and does not depend on the choice of h.

Definition 6.14. *Let $\mathfrak{e} \in \mathfrak{E}$, $n < \omega$, $z \in \mathbb{L}^{\mathfrak{e}}_n$ and let $\mathbb{M}$ be a bimodule such that $\mathbb{M}_* \leq_{\aleph_0} \mathbb{M}$. Then*

(1) *$z \in \mathbb{L}^{\mathfrak{e}}_n$ is called $(\mathfrak{m}, \mathfrak{e}, n)$-nice if when $h : \mathbb{N}^{\mathfrak{e}}_n \longrightarrow \mathbb{M}$ is a bimodule homomorphism, $m \geq n$ and $\mathbb{M} \models \psi^{\mathfrak{e}}_m(h(x^{\mathfrak{e}}_n))$, then $\mathbb{M} \models \psi^{\mathfrak{e}}_m(h(z))$.*
(2) *$z \in \mathbb{L}^{\mathfrak{e}}_n$ is called $(\mathfrak{m}, \mathfrak{e})$-nice if it is $(\mathfrak{m}, \mathfrak{e}, n)$-nice for every n.*
(3) *$z \in \mathbb{L}^{\mathfrak{e}}_n$ is called weakly $(\mathfrak{m}, \mathfrak{e})$-nice if there is an infinite subset $\mathcal{U} \subseteq \omega$ such that z is $(\mathfrak{m}, \mathfrak{e}, n)$-nice for every $n \in \mathcal{U}$.*

We remove $(\mathfrak{m}, \mathfrak{e})$, when it is clear from the context.

We now define a subgroup of $\mathbb{L}^{\mathfrak{e},n}_n$.

Definition 6.15. *For each $\mathfrak{e} \in \mathfrak{E}$ and $n < \omega$, we define*

$$\mathbb{L}^{\mathfrak{e},*}_n := \{z \in \mathbb{L}^{\mathfrak{e}}_n : z \text{ is } (\mathfrak{m}, \mathfrak{e})\text{-nice}\}.$$

Lemma 6.16. *Let $\mathfrak{e}$ be simple and $\mathbb{M}_\kappa$ be strongly nice with respect to $(\lambda, \mathfrak{m}, S, \kappa)$ and let $\mathbf{f}$ be an $\mathbf{R}$-endomorphism of $\mathbb{M}_\kappa$. The following assertions hold:*

(1) *Let z be as Lemma 5.8. Then $z \in \mathbb{L}^{\mathfrak{e},*}_n$.*
(2) *For some $n < \omega$ and $z \in \mathbb{L}^{\mathfrak{e},*}_n$ there is $\mathbb{K}' \in \mathrm{cl}_{\mathrm{is}}(\mathcal{K})$ such that*
    - *$\mathbb{M}' := \mathbb{M}_* \oplus \mathbb{K}' \leq_{\aleph_0} \mathbb{M}_\kappa$ and*
    - *$x \in \psi^{\mathfrak{e}}_n(\mathbb{M}_\kappa) \Rightarrow \mathbf{f}(x) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa) + \mathbb{M}' = \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}_\kappa,z}(x) + \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa) + \mathbb{M}'$.*

Proof. (1): Since $\mathfrak{e}$ is simple, there is a very nice sequence $\overline{\varphi}$ such that $\mathfrak{e} = \mathfrak{e}(\overline{\varphi})$. Since $z \in \mathbb{L}^{\mathfrak{e}}_n$ we know that $z \in \varphi_n(\mathbb{N}^{\mathfrak{e}}_n)$ and $f^1_n(z) - f^2_n(z) \in \varphi_\omega(\mathbb{N}'_n)$ where $f^\ell_n : \mathbb{N}_n \longrightarrow \mathbb{N}^\ell_n$ are bimodule homomorphisms defined in Definition 6.9(3). Now suppose $\mathbf{h} : \mathbb{N}_n \to \mathbb{M}_\kappa$ is a bimodule homomorphism, $m \geq n$ and suppose

$$\mathbb{M}_\kappa \models \psi^{\mathfrak{e}}_m(\mathbf{h}(x^{\mathfrak{e}}_n)).$$

We are going to show that

$$\mathbb{M}_\kappa \models \psi^{\mathfrak{e}}_m(\mathbf{h}(z)).$$

Recall that $\mathbf{f}(\mathbf{h}(x_n)) - \mathbf{h}(z) \in \mathbb{M}_\alpha + \psi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa)$. This gives an element $y \in \mathbb{M}_\alpha$ such that

$$\mathbf{f}(\mathbf{h}(x_n)) - \mathbf{h}(z) + y \in \psi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

In particular, $\mathbb{M}_\kappa \models \psi^{\mathfrak{e}}_m(\mathbf{f}(\mathbf{h}(x^{\mathfrak{e}}_n)) - \mathbf{h}(z) + y)$. As $\overline{\varphi}$ is very simple,

$$\mathbb{M}_\kappa \models \varphi^{\mathfrak{e}}_m(\mathbf{f}(\mathbf{h}(x^{\mathfrak{e}}_n)) - \mathbf{h}(z) + y).$$

We conclude from this that there is a bimodule homomorphism $H_1 : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa$ such that $H_1(x_m) = \mathbf{f}(\mathbf{h}(x_n)) - \mathbf{h}(z) + y$. Since $\mathbb{M}_\kappa \models \psi^{\mathfrak{e}}_m(\mathbf{h}(x^{\mathfrak{e}}_n))$, we have

$$\mathbb{M}_\kappa \models \psi^{\mathfrak{e}}_m(\mathbf{f}(\mathbf{h}(x^{\mathfrak{e}}_n))).$$

This gives a bimodule homomorphism $H_2 : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa$ such that $H_2(x^{\mathfrak{e}}_m) = \mathbf{f}(\mathbf{h}(x^{\mathfrak{e}}_n))$. Define $K : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa$ by $K := H_2 - H_1$. Clearly, $K(x^{\mathfrak{e}}_m) = \mathbf{h}(z) + y$. So,

$$\mathbb{M}_\kappa \models \varphi^{\mathfrak{e}}_m(\mathbf{h}(z) + y).$$

Take $\mathbb{M} \leq_{\aleph_0} \mathbb{M}_*$ be such that $\mathbf{h}(z) \in \mathbb{M}$ and $\mathbb{M} \oplus \mathbb{M}_\alpha \leq_{\aleph_0} \mathbb{M}_\kappa$. Then

$$\mathbb{M} \oplus \mathbb{M}_\alpha \models \psi^{\mathfrak{e}}_m(\mathbf{h}(z) + y).$$

So, $\mathbb{M} \models \psi^{\mathfrak{e}}_m(\mathbf{h}(z))$ and then $\mathbb{M}_\kappa \models \psi^{\mathfrak{e}}_m(\mathbf{h}(z))$, as required.

(2): In view of Lemmas 5.2 and 5.8, there are $n < \omega$ and $\alpha \in \kappa \setminus S$ such that the property $(\Pr 1)^n_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds. Since $\mathfrak{e}$ is simple, by Lemma 6.8 we have $\psi^{\mathfrak{e}}_n(\mathbb{M}) \equiv \psi^{\mathfrak{e}}_n(\mathbb{M})$.

Now let $x \in \psi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$. Then $x \in \varphi^{\mathfrak{e}}_n(\mathbb{M}_\kappa)$. This gives us a bimodule homomorphism $\mathbf{h} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_\kappa$ such that $\mathbf{h}(x^{\mathfrak{e}}_n) = x$. Since $(\Pr 1)^n_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds, thus we have

$$\mathbf{f}(x) - \mathbf{h}(z) = \mathbf{f}(\mathbf{h}(x^{\mathfrak{e}}_n)) - \mathbf{h}(z) \in \mathbb{M}_\alpha + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\kappa).$$

Also, recall that

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x+\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}_\kappa)) = \mathbf{h}(z)+\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}_\kappa).$$

This completes the argument. □

Remark 6.17. *Adopt the notation of Lemma 6.16, and suppose in addition that* $||\mathbb{M}_*|| < \lambda$ *and* $||\mathbb{K}|| < \lambda$ *for every* $\mathbb{K} \in \mathrm{cl}_{\mathrm{is}}(\mathcal{K})$. *Then* $||\mathbb{M}'|| < \lambda = ||\mathbb{M}_\kappa||$.

Definition 6.18. *Let* $\mathbb{M}$ *be an* $\mathbf{R}$*-module,* $\mathfrak{e} \in \mathfrak{E}$ *and* $n < \omega$.

(1) (a) *For* $\mathbf{f} \in \mathrm{End}(\mathbb{M})$, *we set* $\hat{\mathbf{f}}_n := \mathbf{f} \restriction \varphi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$.

(b) *Let* $\mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) := \{\hat{\mathbf{f}}_n : \mathbf{f} \in \mathrm{End}(\mathbb{M})\}$[14].

(c) *Let* $\Upsilon_{n,\lambda}(\mathbb{M})$ *be the family of all* $\mathbf{f} \in \mathrm{End}(\mathbb{M})$ *such that for some* $A \subseteq \mathbb{M}$ *of cardinality* $< \lambda$ *we have*

$$\mathrm{Rang}(\mathbf{f}\restriction\varphi^{\mathfrak{e}}_n(\mathbb{M})) \subseteq \{x+\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}) : x \in \varphi^{\mathfrak{e}}_n(\langle A\rangle_{\mathbb{M}})\}.$$

(d) *The notation* $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$ *stands for the following two-sided ideal:*

$$\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M}) := \{\hat{\mathbf{f}}_n \in \mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) : \mathbf{f} \in \Upsilon_{n,\lambda}(\mathbb{M})\} \lhd \mathrm{End}^{\mathfrak{e},n}(\mathbb{M}).$$

(e) *Let* $n \leq m$. *The notation* $\mathbf{h}^{\mathfrak{e},n,m}_{<\lambda}[\mathbb{M}]$ *stands for the natural map from* $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$ *to* $\mathrm{End}^{\mathfrak{e},m}_{<\lambda}(\mathbb{M})$. *In particular,* $\{\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M}); \mathbf{h}^{\mathfrak{e},n,m}_{<\lambda}[\mathbb{M}]\}$ *is a directed system.*

(f) $\mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M}) := \varinjlim(\cdots \longrightarrow \mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M}) \longrightarrow \mathrm{End}^{\mathfrak{e},n+1}_{<\lambda}(\mathbb{M}) \longrightarrow \cdots)$.

(g) *We denote the natural maps:*

$$\mathbf{h}^{\mathfrak{e},n,\omega}_{<\lambda}[\mathbb{M}] : \mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M}) \to \mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M}).$$

(2) *Suppose in addition that* $\mathbb{M}$ *is an* $(\mathbf{R},\mathbf{S})$*-bimodule.*

(a) *For any* $\mathbf{f} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M})$, *we assign* $\hat{\mathbf{f}}_n := \mathbf{f} \restriction \varphi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$.[15]

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[14]Recall that $\mathrm{End}(\mathbb{M}) := \mathrm{End}_{\mathbf{R}}(\mathbb{M})$ is the ring of $\mathbf{R}$-endomorphisms of $\mathbb{M}$.

[15]There should be no confusion with clause (1)(a) above, as here we are talking about a bimodule $\mathbb{M}$.

(b) The ring of $\mathbf{R}$*-endomorphisms of* $\mathbb{M}$ *induces the following ring*

$$\mathrm{End}^{\mathfrak{e},n,*}(\mathbb{M}) := \{\hat{\mathbf{f}}_n : \mathbf{f} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M})\}.$$

(c) Let $\Upsilon^*_{n,\lambda}(\mathbb{M})$ *be the family of all* $\mathbf{f} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M})$ *such that for some* $A \subseteq \mathbb{M}$ *of cardinality* $< \lambda$ *we have*

$$\mathrm{Rang}(\mathbf{f}{\restriction}\psi^{\mathfrak{e}}_n(\mathbb{M})) \subseteq \{x + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}) : x \in \varphi^{\mathfrak{e}}_n(\langle A\rangle_{\mathbb{M}})\}.$$

(d) $\mathrm{End}^{\mathfrak{e},n,*}_{<\lambda} := \{\hat{\mathbf{f}}_n : \mathbf{f} \in \Upsilon^*_{n,\lambda}(\mathbb{M})\}.$

It is easily seen that all the above defined notions are rings. We now define some expansions of $\varphi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$.

Definition 6.19. *Let* $\mathbb{M}, \mathfrak{e}$ *and* n *be as above.*

(1) The notation $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$ *stands for* $\varphi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$ *expanded by the finitary relations definable by formulas in* $\mathcal{L}^{pe}_{\infty,\omega}(\tau_{\mathbf{R}})$ *(so actually even if we use this notation for a bimodule* $\mathbb{M}$*, it counts only as an* $\mathbf{R}$*-module).*

(2) Similarly, we define ${}^+\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$*, where p.e. formulas are replaced by "formulas preserved by direct sums".*

(3) For a bimodule $\mathbb{M}$ *we define* $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$ *and* ${}^+\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$ *similarly let restricting ourselves to* $\psi^{\mathfrak{e}}_n(\mathbb{M})$.

So,

$$\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}) = (\varphi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}), \langle R\rangle_{R\in\mathcal{R}}),$$

where $\mathcal{R}$ *consists of all finitary relations defined by a formula from* $\mathcal{L}^{pe}_{\infty,\omega}(\tau_{\mathbf{R}})$*. Similarly*

$${}^+\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}) = (\varphi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}), \langle R\rangle_{R\in{}^+\mathcal{R}}),$$

where ${}^+\mathcal{R}$ *consists of all finitary relations which are defined by a formula from* $\mathcal{L}_{\infty,\omega}(\tau_{\mathbf{R}})$ *which is preserved under direct sums. Since, by Lemma 3.14, pe-formulas are preserved by direct limits,* ${}^+\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$ *expands* $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$.

Lemma 6.20. *The following assertions are hold:*

(1) *Adopt the notation of Definition 6.18(1). Then* $\mathrm{End}^{\mathfrak{e},n}(\mathbb{M})$ *is a ring with* 1 *and* $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$ *is a two-sided ideal of* $\mathrm{End}^{\mathfrak{e},n}(\mathbb{M})$.[16] *Suppose in addition* $\mathbb{M}$ *is a bimodule, then* $\mathbf{S}$ *is naturally maps to* $\mathrm{End}^{\mathfrak{e},n}(\mathbb{M})$.

(2) $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$ *is a two-sided subideal of* $\mathrm{End}^{\mathfrak{e},n}_{<\mu}(\mathbb{M})$ *when* $\lambda < \mu$.

(3) $\mathrm{End}^{\mathfrak{e},n}_{<\|\mathbb{M}\|^+}(\mathbb{M}) = \mathrm{End}^{\mathfrak{e},n}(\mathbb{M})$.

(4) *If* $\mathbb{M}_1, \mathbb{M}_2$ *are* $\mathbf{R}$*-modules and* $\mathbf{h}$ *is an* $\mathbf{R}$*-homomorphism from* $\mathbb{M}_1$ *to* $\mathbb{M}_2$*, then* $\mathbf{h}$ *induces a homomorphism from* $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}_1)$ *into* $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}_2)$.

Proof. (1). It is clear that all the defined notions are rings (not necessarily with 1). Now suppose $\mathbb{M}$ is a bimodule. Then one can easily show that for each $s \in \mathbf{S}$, s defines an $\mathbf{R}$-endomorphism of $\mathbb{M}$ by $x \to xs$.

Items (2) and (3) are clear. To prove (4), first note that, as $\mathbf{h}(\varphi^{\mathfrak{e}}_n(\mathbb{M}_1)) \subseteq \varphi^{\mathfrak{e}}_n(\mathbb{M}_2)$,

$$\hat{\mathbf{h}}_n = \mathbf{h} \restriction \varphi^{\mathfrak{e}}_n(\mathbb{M}_1)/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_1) : \varphi^{\mathfrak{e}}_n(\mathbb{M}_1)/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_1) \to \varphi^{\mathfrak{e}}_n(\mathbb{M}_2)/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_2)$$

is well-defined. Now let $\varphi(\nu_1, \cdots, \nu_n) \in \mathcal{L}^{\mathrm{pe}}_{\infty,\omega}(\tau_{\mathbf{R}})$, and for $\ell = 1, 2$ set

$$R^\ell = \{\langle x_1, \cdots, x_n\rangle \in \mathbb{M}_\ell : \mathbb{M}_\ell \models \varphi(x_1, \cdots, x_n)\}.$$

Clearly $\mathbf{h}(R^1) = R^2$, and hence $\hat{\mathbf{h}}_n(R^1 + \varphi_\omega(\mathbb{M}_1)) = R^2 + \varphi_\omega(\mathbb{M}_2)$. The result follows immediately. $\square$

Remark 6.21. *Adopt the notions presented in the proof of Lemma 6.20(4). Let* $\mathbf{h}$ *be an* $\mathbf{R}$*-homomorphism from* $\mathbb{M}_1$ *to* $\mathbb{M}_2$ *and* $n < \omega$*. The notation* $\hat{\mathbf{h}}_n$ *may stand for the homomorphism from* $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}_1)$ *into* $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}_2)$.

We now define some rings derived from the ring of $\mathbf{R}$*-endomorphism of bimodules. Before doing that we need the following lemma which shows that the maps* $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ *from Definition 6.12 are definable.*

[16] Note that this ideal may be proper, i.e., $1 \notin \mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$ or not.

Lemma 6.22. *Let $\mathbb{M}$ be a bimodule, $\mathfrak{e} \in \mathfrak{E}$ be simple and $n < \omega$.*

(1) *Suppose $\mathbf{h} : \mathbb{M}_1 \to \mathbb{M}_2$ is an $\mathbf{R}$-homomorphism, $\mathfrak{e} \in \mathfrak{E}$, $n < \omega$ and $z \in \mathbb{L}^{\mathfrak{e}}_n$. Then $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}_1,z} \circ \hat{\mathbf{h}}_n = \hat{\mathbf{h}}_n \circ \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}_2,z}$.*

(2) *Suppose $z \in \mathbb{L}^{\mathfrak{e}}_n$ is n-nice and $m > n$. Then there is $y \in \mathbb{N}^{\mathfrak{e}}_m$ such that for every bimodule $\mathbb{M}$,*

$$\mathbf{h}^{\mathfrak{e},m}_{\mathbb{M},y} = \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z} \restriction (\psi_n(\mathbb{M})/\varphi_\omega(\mathbb{M})).$$

(3) *Suppose $\psi(x,y) \in \mathcal{L}^{\mathrm{pe}}_{\infty,\omega}(\tau_{(\mathbf{R},\mathbf{S})})$ is such that*

(i) $\psi^{\mathfrak{e}}_n(x) \vdash \exists y \psi(x,y)$,

(ii) $\psi(x,y) \vdash \psi^{\mathfrak{e}}_n(x) \ \wedge \ \psi^{\mathfrak{e}}_n(y)$,

(iii) $\psi(x,y_1) \wedge \psi(x,y_2) \vdash \psi^{\mathfrak{e}}_\ell(y_1 - y_2)$, *for all $\ell < \omega$.*

Then there is some $z \in \mathbb{L}^{\mathfrak{e}}_n$ such that for every bimodule $\mathbb{M}$ and $x, y \in \psi_n(\mathbb{M})$,

$$\mathbb{M} \models \psi(x,y) \iff \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_\omega(\mathbb{M})) = y + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}).$$

This property is denoted by $(\star)^n_{\psi,z}$.

(4) *For every $z \in \mathbb{L}^{\mathfrak{e}}_n$, there exists $\psi(x,y) \in \mathcal{L}^{\mathrm{pe}}_{\infty,\omega}(\tau_{(\mathbf{R},\mathbf{S})})$ satisfying the above conditions, such that $(\star)^n_{\psi,z}$ holds.*

(5) *If $z_1, z_2 \in \mathbb{L}^{\mathfrak{e}}_n$, then for some $z_3 \in \mathbb{L}^{\mathfrak{e}}_n$ and for all bimodule $\mathbb{M}$,*

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z_3} = \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z_1} \circ \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z_2}.$$

Furthermore,

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z_1} \pm \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z_2} = \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z_1 \pm z_2}.$$

(6) *If $z \in \mathbb{L}^{\mathfrak{e}}_n$ and $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ is a bijection, then for some $z' \in \mathbb{L}^{\mathfrak{e}}_n$ and for all bimodule $\mathbb{M}$, $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z'}$ is the inverse of $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$.*

(7) *If $\mathbb{N}^{\mathfrak{e}}_n$ is finitely presented, then the formula $\psi(x,y)$ is first order. Furthermore, if $\mathbb{N}^{\mathfrak{e}}_n$ is generated by $\{y_i : i < m_n\}$ freely except equations involving $r \in \mathbf{R}$ only (no $s \in \mathbf{S}$), and $z \in \sum_{i<m_n} \mathbf{R} y_i$, then $\psi(x,y) \in \mathcal{L}^{\mathrm{pe}}_{\omega,\omega}(\tau_{\mathbf{R}})$.*

Proof. (1). Let $\mathbf{g} : \mathbb{N}_n^{\mathfrak{e}} \to \mathbb{M}_1$ be a bimodule homomorphism and let $x = \mathbf{g}(x_n^{\mathfrak{e}})$. Then

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}_2,z}(\hat{\mathbf{h}}_n(x+\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_1))) = \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}_2,z}(\mathbf{h}(x)+\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_2))) = \mathbf{h}(\mathbf{g}(z))+\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_2).$$

In a similar way, we have

$$\hat{\mathbf{h}}_n(\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}_1,z}(x)) = \hat{\mathbf{h}}_n(\mathbf{g}(z)+\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_1)) = \mathbf{h}(\mathbf{g}(z))+\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_2)).$$

The desired claim follows immediately.

(2). Set $y = g_{n,m}(z)$. It is easily seen that y is as required.

(3). Let z be such that $\psi^{\mathfrak{e}}_n(x^{\mathfrak{e}}_n) \vdash \psi(x^{\mathfrak{e}}_n, z)$. In view of (i) such an element z exists, and it is unique mod $\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$ by (iii). Now define $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ as given. Thanks to items (i) and (iii) we know $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ is a well-defined function and by item (ii) we deduce that

3.1) $\mathrm{Dom}(\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}) = \psi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$,

3.2) $\mathrm{Rang}(\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}) = \psi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$.

So, we are done.

(4). Suppose $z \in \mathbb{L}^{\mathfrak{e}}_n$ is given, and let $\mathbb{N}^{\mathfrak{e}}_n$ be as Definition 6.6. Then for some $r^*, r^*_i \in \mathbf{R}$ and $s^*, s^*_i \in \mathbf{S}$, for $i < k_{m_n-1}$, we have

$$z = r^* x_n s^* + \sum_{i<k_{m_n-1}} r^*_i y_{n,i} s^*_i \quad (+)$$

Let $\psi(x,y)$ be the formula

$$\psi^{\mathfrak{e}}_n(x) \ \wedge \ \psi^{\mathfrak{e}}_n(y) \wedge y = r^* x s^*.$$

We are going to show that $\psi(x,y)$ is as required. Clauses (i)-(iii) are clearly satisfied. To prove $(\star)^n_{\psi,z}$, let $\mathbb{M}$ be a bimodule and $x,y \in \psi^{\mathfrak{e}}_n(\mathbb{M})$. Suppose first that $\psi(x,y)$ holds. Let also $\mathbf{g} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}$ be a bimodule homomorphism, defined on generators $x_n, y_{n,i}$ by the following assignments:

4.1) $\mathbf{g}(x_n) = x$,

4.2) $\mathbf{g}(y_{n,i}) = 0$.

In view of (+) we observe that

$$\mathbf{g}(z) = r^* x s^* = y.$$

According to the definition

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) = y = \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}).$$

Conversely, suppose that $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) = y = \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$. Let $\mathbf{g}$ be defined as above, so that $\mathbf{g}(x_n) = x$. Now it is clear that $\mathbf{g}(z) = r^* x s^* = y$, and thus $\psi(x, y)$ holds.

Clauses (5) and (6) follows from a combination of (4) and (5). Clause (7) is clear as well. The lemma follows. $\square$

The above lemma allows us to define $\mathbf{h}^{\mathfrak{e},n}_z$, *independent of the choice of the bimodule* $\mathbb{M}$.

Definition 6.23. *Suppose* $\mathfrak{e} \in \mathfrak{E}$ *is simple as witness by* X *(see Definition 6.3(4)) and* $n < \omega$.

(1) *Let* $DE^{\mathfrak{e}}_n$ *be the following ring, whose universe is:*

$$\{\mathbf{h}^{\mathfrak{e},n}_z : z \in \mathbb{L}^{\mathfrak{e}}_n\},$$

such that:

(a) $\mathbf{h}^{\mathfrak{e},n}_{z_1} = \mathbf{h}^{\mathfrak{e},n}_{z_2}$ *if and only if* $z_1 - z_2 \in \varphi^{\mathfrak{e}}_{\omega}(\mathbb{N}^{\mathfrak{e}}_n)$,

(b) $\mathbf{h}^{\mathfrak{e},n}_{z_1} \pm \mathbf{h}^{\mathfrak{e},n}_{z_2} = \mathbf{h}^{\mathfrak{e},n}_{z_1 \pm z_2}$,

(c) $\mathbf{h}^{\mathfrak{e},n}_{z_1} \circ \mathbf{h}^{\mathfrak{e},n}_{z_2} = \mathbf{h}^{\mathfrak{e},n}_{z_3}$, *where* z_3 *is as in Lemma 6.22(5), and it is unique modulo* $\varphi^{\mathfrak{e}}_{\omega}(\mathbb{N}^{\mathfrak{e}}_n)$,

(d) *the zero element is* $\mathbf{h}^{\mathfrak{e},n}_0$, *the identity element is* $\mathbf{h}^{\mathfrak{e},n}_{x^{\mathfrak{e}}_n}$. [17]

(2) $De^{\mathfrak{e}}_n := \big\{\mathbf{h}^{\mathfrak{e},n}_z \in DE^{\mathfrak{e}}_n : z \in \sum\{\mathbf{R}x : x \in X\}\big\}$.

(3) $dE^{\mathfrak{e}}_n := \big\{\mathbf{h}^{\mathfrak{e},n}_z \in DE^{\mathfrak{e}}_n : \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z} \in \mathrm{End}(\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}))$ for all bimodule $\mathbb{M} \geq_{\aleph_0} \mathbb{M}_*\big\}$.

[17] Note that DE^n is embedded into the endomorphism ring of $\psi^{\mathfrak{e}}_n(\mathbb{N}^{\mathfrak{e}}_n)/\varphi^{\mathfrak{e}}_{\omega}(\mathbb{N}^{\mathfrak{e}}_n)$ as an abelian group.

(4) $dE^{\mathfrak{e}}_{n,*} := \{\mathbf{h}^{\mathfrak{e},n}_z \in dE^{\mathfrak{e}}_n : z$ *is n-nice*$\}$.

(5) $de^{\mathfrak{e}}_n := De^{\mathfrak{e}}_n \cap dE^{\mathfrak{e}}_n$.

(6) $de^{\mathfrak{e}}_{n,*} := De^{\mathfrak{e}}_n \cap dE^{\mathfrak{e}}_{n,*}$.[18]

(7) $DE^{\mathfrak{e}}_n(\mathbf{R})$ *is* $DE^{\mathfrak{e}}_n$*, when we choose* $\mathbf{S} = \mathbf{T} = \mathrm{Cent}(\mathbf{R})$*; similarly for the others.*

In the following diagram we display these rings. By $A \to B$ we mean A is a subring of B:

$$\begin{tikzcd}
& de^{\mathfrak{e}}_n \arrow[rr] \arrow[dl] \arrow[dd, dotted] & & De^{\mathfrak{e}}_n \arrow[dl] \arrow[dd] \\
dE^{\mathfrak{e}}_n \arrow[rr] \arrow[dd] & & DE^{\mathfrak{e}}_n \arrow[dd] & \\
& de^{\mathfrak{e}}_{n+1} \arrow[rr, dotted] \arrow[dl, dotted] & & De^{\mathfrak{e}}_{n+1} \arrow[dl] \\
dE^{\mathfrak{e}}_{n+1} \arrow[rr] & & DE^{\mathfrak{e}}_{n+1} &
\end{tikzcd}$$

Lemma 6.24. *The following assertions are hold:*

(1) $DE^{\mathfrak{e}}_n$ *is a ring. Furthermore if* $\mathfrak{e}$ *is simple, then* $De^{\mathfrak{e}}_n$ *and* $dE^{\mathfrak{e}}_n$ *and* $de^{\mathfrak{e}}_n$ *are subrings of* $DE^{\mathfrak{e}}_n$*. The unit (resp. zero) element all of them is* $1 = \mathbf{h}^{\mathfrak{e},n}_{x^{\mathfrak{e}}_n}$ *(resp.* $0 = \mathbf{h}^{\mathfrak{e},n}_0$*).*

(2) All rings from part (1) are extensions of the ring $\mathbf{T}$*.*

(3) $De^{\mathfrak{e}}_n$*,* $dE^{\mathfrak{e}}_n$ *commute.*

(4) The ring $de^{\mathfrak{e}}_n$ *is commutative.*

(5) There is a natural homomorphism $DE^{\mathfrak{e}}_n \to DE^{\mathfrak{e}}_{n+1}$ $(n < \omega)$*. Similarly, there are natural homomorphisms* $De^{\mathfrak{e}}_n \to De^{\mathfrak{e}}_{n+1}, dE^{\mathfrak{e}}_n \to dE^{\mathfrak{e}}_{n+1}$ *and* $de^{\mathfrak{e}}_n \to de^{\mathfrak{e}}_{n+1}$*. By taking directed limit from the corresponding directed system, the following rings are well-defined:*

(a) $De^{\mathfrak{e}} := \varinjlim(De^{\mathfrak{e}}_0 \longrightarrow De^{\mathfrak{e}}_1 \longrightarrow \cdots \longrightarrow De^{\mathfrak{e}}_n \longrightarrow \cdots)$,

[18] So, they two depend on X which witnesses that $\mathfrak{e}$ is simple.

(b) $dE^{\mathfrak{e}} := \varinjlim(dE^{\mathfrak{e}}_0 \longrightarrow dE^{\mathfrak{e}}_1 \longrightarrow \cdots \longrightarrow dE^{\mathfrak{e}}_n \longrightarrow \cdots)$,

(c) $de^{\mathfrak{e}} := \varinjlim(de^{\mathfrak{e}}_0 \longrightarrow de^{\mathfrak{e}}_1 \longrightarrow \cdots \longrightarrow de^{\mathfrak{e}}_n \longrightarrow \cdots)$,

(d) $DE^{\mathfrak{e}} := \varinjlim(DE^{\mathfrak{e}}_0 \longrightarrow DE^{\mathfrak{e}}_1 \longrightarrow \cdots \longrightarrow DE^{\mathfrak{e}}_n \longrightarrow \cdots)$.

(6) The ring $\mathbf{S}$ *is naturally mapped into* $dE^{\mathfrak{e}}_n$.

(7) The ring $de^{\mathfrak{e}}_n$ *is naturally embedded into* $de^{\mathfrak{e}}_{n+1}$ *and* $DE^{\mathfrak{e}}_n$ *into* $DE^{\mathfrak{e}}_{n+1}$.

(8) The abelian group $\psi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})$ *is equipped with a module structure over* $DE^{\mathfrak{e}}_n$ *and it is naturally a* $(De^{\mathfrak{e}}_n, dE^{\mathfrak{e}}_n)$*–bimodule, with* $de^{\mathfrak{e}}_n$ *playing the role of* $\mathbf{T}$.

Proof. (1). This is clear.

(2). This is clear.

(3). Suppose $z, w \in \mathbb{L}^{\mathfrak{e}}_n$ are such that $\mathbf{h}^{\mathfrak{e},n}_z \in De^{\mathfrak{e}}_n$ and $\mathbf{h}^{\mathfrak{e},n}_w \in dE^{\mathfrak{e}}_n$. Without loss of generality, $z = rx$ for some $r \in \mathbf{R}$ and $x \in X$. Furthermore, since $\mathbf{h}^{\mathfrak{e},n}_z \in De^{\mathfrak{e}}_n$, it preserves *pe*-definable relations, so we can assume without loss of generality that

$$X = \{x_n\} \cup \{y_{n,i} : i < k_{m_n - 1}\},$$

where the canonical sequence is taken from Definition 6.6. Let also

$$w = r^* x_n s^* + \sum_{i < k_{m_n - 1}} r_i y_{n,i} s_i.$$

We have to show that

$$\mathbf{h}^{\mathfrak{e},n}_z \circ \mathbf{h}^{\mathfrak{e},n}_w = \mathbf{h}^{\mathfrak{e},n}_w \circ \mathbf{h}^{\mathfrak{e},n}_z.$$

We have two possibilities: 1) $z = r y_{n,i}$ for some $i < k_{m_n - 1}$ or 2) $z = r x_n$.

Case 1: $z = r y_{n,i}$ for some $i < k_{m_n - 1}$.

In this case, first we define

i) $\psi_z(x, y) = \psi^{\mathfrak{e}}_n(x) \wedge \psi^{\mathfrak{e}}_n(y) \wedge y = 0$,

ii) $\psi_w(x, y) = \psi^{\mathfrak{e}}_n(x) \wedge \psi^{\mathfrak{e}}_n(y) \wedge y = r^* x s^*$.

Let $\mathbb{M} \geq_{\aleph_0} \mathbb{M}_*$ and $x \in \psi_n^{\mathfrak{e}}(\mathbb{M})$. It follows from the proof of Lemma 6.22(4) that

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) = y + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}) \iff \psi_z(x, y)$$

and

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) = y + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}) \iff \psi_w(x, y).$$

In particular, we have the following implications

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}))) = y + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$$

$$\Updownarrow$$

$$\exists v \left(\psi_z(x, v) \wedge \psi_w(v, y)\right)$$

$$\Updownarrow$$

$$\exists v \left(\psi^{\mathfrak{e}}_n(x) \wedge \psi^{\mathfrak{e}}_n(v) \wedge \psi^{\mathfrak{e}}_n(y) \wedge v = 0 \wedge y = r^* v s^*\right).$$

Similarly, one deduces the following implications

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}))) = y' + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$$

$$\Updownarrow$$

$$\exists v \left(\psi_w(x, v) \wedge \psi_z(v, y')\right)$$

$$\Updownarrow$$

$$\exists v \left(\psi^{\mathfrak{e}}_n(x) \wedge \psi^{\mathfrak{e}}_n(v) \wedge \psi^{\mathfrak{e}}_n(y') \wedge v = r^* x s^* \wedge y' = 0\right).$$

It follows from the above equations that $y = y' = 0$, and thus the equality follows.

Case 2: $z = r x_n$.

In this case, for each $x \in \mathbb{M}$, $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) = rx + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$. Set $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) = y + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$. By plugging this, we observe that

$$\begin{aligned}
\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}))) &= \mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(rx + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) \\
&= r\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},w}(x + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})) \\
&= ry + \varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}),
\end{aligned}$$

and

$$\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z}(\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},w}(x+\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}))) = \mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z}(y+\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})) = ry+\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}).$$

The equality follows in this case as well.

(4). Recall from definition that $de^{\mathfrak{c}}_n = De^{\mathfrak{c}}_n \cap dE^{\mathfrak{c}}_n$. It remains to apply (3).

(5). In view of Lemma 6.22(2) we observe that the assignment $\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z} \mapsto \mathbf{h}^{\mathfrak{c},n+1}_{\mathbb{M},g_n(z)}$ defines an embedding map $F_n : DE^{\mathfrak{c}}_n \to DE^{\mathfrak{c}}_{n+1}$. This yields a directed system $\{DE^{\mathfrak{c}}_n\}_{n\geq 1}$. Now, we define

$$DE^{\mathfrak{c}} := \varinjlim(DE^{\mathfrak{c}}_0 \xrightarrow{F_0} DE^{\mathfrak{c}}_1 \longrightarrow \cdots \longrightarrow DE^{\mathfrak{c}}_n \xrightarrow{F_n} DE^{\mathfrak{c}}_{n+1} \longrightarrow \cdots).$$

Similarly, one may define the rings $De^{\mathfrak{c}}, dE^{\mathfrak{c}}$, $dE^{\mathfrak{c}}_*$ and $de^{\mathfrak{c}}$ by taking them as a direct limit of the corresponding directed system. In all cases they depend on $\mathfrak{m}$ of course.

(6). To each $s \in \mathbf{S}$ we assign $\mathbf{f}_s \in \mathrm{End}(\mathbb{M})$ defined by $\mathbf{f}_s(x) = xs$. The assignment $s \mapsto \mathbf{f}_s$ defines a map $\mathbf{S} \longrightarrow \mathrm{End}(\mathbb{M})$ which is an embedding. For each $n<\omega$, this induces a map

$$(\hat{\mathbf{f}}_s)_n : \frac{\varphi^{\mathfrak{c}}_n(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})} \longrightarrow \frac{\varphi^{\mathfrak{c}}_n(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})}.$$

We may regard this as an endomorphism of $\mathfrak{B}^{\mathfrak{c}}_n(\mathbb{M})$. Now, let $z := x_n s$. Then $(\hat{\mathbf{f}}_s)_n = \mathbf{h}^{\mathfrak{c},n}_z$. We proved that the assignment $s \mapsto (\hat{\mathbf{f}}_s)_n$ induces a map $\rho_n : S \to dE^{\mathfrak{c}}_n$. Denote the natural map $dE^{\mathfrak{c}}_n \to dE^{\mathfrak{c}}_{n+1}$ by H_n. Now we look at the following commutative diagram:

$$\begin{array}{ccccccccccc}
\mathbf{S} & \xrightarrow{=} & \mathbf{S} & \xrightarrow{=} & \ldots & \xrightarrow{=} & \mathbf{S} & \xrightarrow{=} & \mathbf{S} & \xrightarrow{=} & \ldots \\
\downarrow{\scriptstyle\rho_1} & & {\scriptstyle\rho_2}\downarrow & & {\scriptstyle\cdots}\downarrow & & \downarrow{\scriptstyle\rho_{n-1}} & & {\scriptstyle\rho_n}\downarrow & & \\
dE^{\mathfrak{c}}_1 & \xrightarrow{H_1} & dE^{\mathfrak{c}}_2 & \xrightarrow{H_2} & \ldots & \longrightarrow & dE^{\mathfrak{c}}_{n-1} & \xrightarrow{H_{n-1}} & dE^{\mathfrak{c}}_n & \xrightarrow{H_n} & \ldots
\end{array}$$

Taking direct limits of these directed systems, leads us to a natural map

$$\rho : \mathbf{S} = \varinjlim \mathbf{S} \to \varinjlim dE^{\mathfrak{c}}_n = dE^{\mathfrak{c}},$$

as claimed.

(7). This is similar to (6).

(8). The assignment

$$(m + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}), \mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z}) \mapsto \mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z}(m + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}))$$

defines the scaler multiplication $\frac{\psi^{\mathfrak{c}}_{n}(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})} \times DE^{\mathfrak{c}}_{n} \to \frac{\psi^{\mathfrak{c}}_{n}(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})}$. Now, we take the following:

8.1) $m + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}) \in \frac{\psi^{\mathfrak{c}}_{n}(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})}$,

8.2) $\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z} \in De^{\mathfrak{c}}_{n}$ and

8.3) $\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},w} \in dE^{\mathfrak{c}}_{n}$.

The assignment

$$(\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z}, m + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}), \mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},w}) \mapsto \mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},w}(\mathbf{h}^{\mathfrak{c},n}_{\mathbb{M},z}(m + \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})))$$

defines the scaler multiplication

$$De^{\mathfrak{c}}_{n} \times \frac{\psi^{\mathfrak{c}}_{n}(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})} \times dE^{\mathfrak{c}}_{n} \to \frac{\psi^{\mathfrak{c}}_{n}(\mathbb{M})}{\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})}.$$

Since $de^{\mathfrak{c}}_{n} = dE^{\mathfrak{c}}_{n} \cap De^{\mathfrak{c}}_{n}$ is commutative, this induces the desired bimodule structure on $\psi^{\mathfrak{c}}_{n}(\mathbb{M})/\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})$. $\square$

The following diagram summarizes the relation between the above rings, where by $A \to B$ we mean A is a subring of B:

$$\begin{tikzcd}
de_* \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& de \arrow[r] \arrow[d] & De \arrow[d] \\
& dE \arrow[r] & DE
\end{tikzcd}$$

The following lemma says that for example for strongly semi-nice construction $\bar{\mathbb{M}}$ we have some control over $\mathrm{End}_{\mathbf{R}}(\mathbb{M}_{\kappa})$; note that it only says it is not too large, but we have the freedom to choose the ring $\mathbf{S}$ in order to make $\mathrm{End}(\mathbb{M}_{\lambda})$ have some elements with desirable properties.

Lemma 6.25. *Let $\bar{\mathbb{M}} = \langle \mathbb{M}_\alpha : \alpha \le \kappa \rangle$ be a strongly semi-nice construction with respect to $(\lambda, \mathfrak{m}, S, \kappa)$, let $\mathbb{M} := \mathbb{M}_\kappa$, and suppose every $\mathfrak{e} \in \mathfrak{E}$ is simple. The following assertions are hold:*

(i) *If $(\mathrm{Pr}\,1)^n_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds, then $\mathbf{h}^{\mathfrak{e},n}_z$ is an endomorphism of $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$. Furthermore, $\mathbf{h}^{\mathfrak{e},n}_z \in dE^{\mathfrak{e}}_{n,*}$.*

(ii) *If $(\mathrm{Pr}\,1)^n_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds and $\mathbf{f} \in \boldsymbol{Aut}(\mathbb{M})$, then $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ is an automorphism of $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$ and even of ${}^+\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$.*

(iii) *$\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})/\mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M})$ embedded into $dE^{\mathfrak{e}}$. Suppose in addition that $\lambda = \kappa$ and let $\mathbf{S}$ be a subring of $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})/\mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M})$ of cardinality $< \lambda$. There is a club C of κ such that for any $\alpha \in C \setminus S$ large enough, the ring $\mathbf{S}$ is embedded into $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}/\mathbb{M}_\alpha)$.*

(iv) *Let $\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]$ denote the natural map $\mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) \to \mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})$ and let $\mathbf{E}_n$ be the set of all $\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}](\hat{\mathbf{f}}_n)$ [19] where $\mathbf{f} \in \mathrm{End}(\mathbb{M})$ satisfies in the property $(\mathrm{Pr}\,1)^n_{\alpha_n(\mathbf{f}),z_n(\mathbf{f})}[\mathbf{f}, \mathfrak{e}]$ for some $z_n(\mathbf{f}) \in \mathbb{L}^{\mathfrak{e}}_n$ and $\alpha_n(\mathbf{f}) < \kappa$. Then*

(a) $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}) = \bigcup\limits_{n<\omega} \mathbf{E}_n$,

(b) $\mathbf{E}_n \subseteq \mathbf{E}_{n+1}$,

(c) *$z_n(\mathbf{f})$ is unique modulo $\varphi^{\mathfrak{e}}_\omega(\mathbb{N}^{\mathfrak{e}}_n)$.*

(v) *$\mathbf{E}_n$ is a subring of $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})$ and the mapping $\hat{\mathbf{f}}_n \mapsto \mathbf{h}^{\mathfrak{e},n}_{z_n(\mathbf{f})}$ is a homomorphism from*

$$\left\{\hat{\mathbf{f}}_n : \quad \mathbf{f} \in \mathrm{End}(\mathbb{M}) \textit{ and } (\mathrm{Pr}\,1)^n_{\alpha_n(\mathbf{f}),z_n(\mathbf{f})} \textit{ for some } \ \alpha_n(\mathbf{f}) < \kappa \textit{ and } z_n(\mathbf{f}) \in \mathbb{L}^{\mathrm{tr}}_n\right\}$$

into $dE^{\mathfrak{e}}_n$ with kernel $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$, i.e.,

$$\{\mathbf{f} \in \mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) : z_n(\mathbf{f}) \in \varphi^{\mathfrak{e}}_\omega(\mathbb{N}_n)\}.$$

(vi) *The ring $\mathbf{S}$ is naturally mapped into $\mathrm{End}(\mathbb{M})$, for each $\alpha \le \omega$, there is a natural homomorphism from $\mathrm{End}(\mathbb{M})$ to $\mathrm{End}^{\mathfrak{e},\alpha}(\mathbb{M})$, where for $\alpha < \omega$ has a natural mapping to dE. In particular, $\mathbf{S}$ is naturally mapped into $dE^{\mathfrak{e}}$.*

[19] See Definition 6.18 for the definition of $\hat{\mathbf{f}}_n$.

Proof. (i). In view of Lemma 6.20(ii), $\mathbf{f}$ induces a homomorphism from $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}_1) \to \mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M}_2)$, and we conclude from Lemma 6.16 that $z \in \mathbb{L}^{\mathfrak{e},*}_n$. By definition, $\mathbf{h}^{\mathfrak{e},n}_z \in dE^{\mathfrak{e}}_{n,*}$.

(ii). For some formula $\psi(x,y) \in \mathcal{L}^{pe}_{\mu,\kappa}(\tau_{(\mathbf{R},\mathbf{S})})$, for all $\mathbb{M}$, and all $x,y \in \psi_n(\mathbb{M})$ we have

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}(x+\varphi_\omega(\mathbb{M})) = y + \varphi_\omega(\mathbb{M}) \Leftrightarrow \mathbb{M} \models \psi(x,y).$$

Now, we define $\psi'(x,y)$ by the following role

$$\mathbb{M} \models \psi'(x,y) \Leftrightarrow \mathbb{M} \models \psi(y,x).$$

Since $\mathbf{f}$ is an automorphism, $\psi'(x,y)$ satisfies the assumptions (i)-(iii) of Lemma 6.22(3), and hence for some $z' \in \mathbb{L}^{\mathfrak{e}}_n$ and all $x,y \in \psi_n(\mathbb{M})$,

$$\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z'}(x+\varphi_\omega(\mathbb{M})) = y + \varphi_\omega(\mathbb{M}) \Leftrightarrow \mathbb{M} \models \psi'(x,y).$$

It is now clear that $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z'}$ is the inverse of $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$. In particular, $\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M},z}$ is an automorphism of $\mathfrak{B}^{\mathfrak{e}}_n(\mathbb{M})$.

(iii)+(iv)+(v). For each n, set

$$\mathbf{F}_n = \{\hat{\mathbf{f}}_n : \quad \mathbf{f} \in \mathrm{End}(\mathbb{M}) \text{ and } (\mathrm{Pr}\,1)^n_{\alpha_n(\mathbf{f}),z_n(\mathbf{f})} \text{ for some } \ \alpha_n(\mathbf{f}) < \kappa,\ z_n(\mathbf{f}) \in \mathbb{L}^{\mathrm{tr}}_n\}.$$

We will prove the following four claims:

(1) $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}_\kappa) = \bigcup\limits_{n<\omega} \mathbf{E}_n$.

(2) $\mathbf{E}_n$ is a subring of $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})$.

(3) The assignment $\hat{\mathbf{f}}_n \mapsto \mathbf{h}^{\mathfrak{e},n}_{z_n(\mathbf{f})}$ yields a homomorphism $\varrho_n : \mathbf{F}_n \to dE^{\mathfrak{e}}_n$ with kernel

$$\{\hat{\mathbf{f}}_n \in \mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) : z_n(\mathbf{f}) \in \varphi^{\mathfrak{e}}_\omega(\mathbb{N}_n)\}.$$

(4) The assignment $\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}](\hat{\mathbf{f}}_n) \mapsto \mathbf{h}^{\mathfrak{e},n}_{z_n(\mathbf{f})}$ induces a homomorphism $\mathbf{E}_n \to dE^{\mathfrak{e}}_n$.

Let us prove them:

(1): Clearly, we have $\bigcup\limits_{n<\omega} \mathbf{E}_n \subset \mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}_\kappa)$. To see the reverse inclusion take $\mathbf{g} \in \mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}_\kappa)$. There is some $n < \omega$ such that $\mathbf{g} = \mathbf{h}^{\mathfrak{e},n}[\mathbb{M}](\hat{\mathbf{f}}_n)$ where $\mathbf{f} \in$

$\mathrm{End}^{\mathfrak{c},n}(\mathbb{M})$. Here, we are going to use the a strongly semi-nice construction. In view of this assumption, we can find $\alpha_n(\mathbf{f}) < \lambda$ and $z_n(\mathbf{f}) \in \mathbb{L}_n^{\mathrm{tr}}$ such that the property $(\Pr 1)^n_{\alpha_n(\mathbf{f}),z_n(\mathbf{f})}$ holds. By definition, $\mathbf{g} \in \mathbf{E}_n$. This completes the proof of claim (1).

(2): This is clear and we leave it to the reader.

(3): Note that $z_n(\mathbf{f}_1 \pm \mathbf{f}_2) = z_n(\mathbf{f}_1) \pm z_n(\mathbf{f}_2)$ modulo $\varphi_\omega(\mathbb{N}_n)$. This shows that ϱ_n is additive:

$$\varrho_n(\mathbf{f}_1 \pm \mathbf{f}_2) = \mathbf{h}^n_{z_n(\mathbf{f}_1 \pm \mathbf{f}_2)} = \mathbf{h}^n_{z_n(\mathbf{f}_1)} \pm \mathbf{h}^n_{z_n(\mathbf{f}_2)} = \varrho_n(\mathbf{f}_1) + \varrho_n(\mathbf{f}_2).$$

Similarly, for any $r \in R$ we have $\varrho_n(r\mathbf{f}) = \varrho_n(r\mathbf{f})$. Here, we compute $\ker(\varrho_n)$:

$$\varrho_n(\mathbf{f}) = 0 \Leftrightarrow \mathbf{h}^n_{z_n(\mathbf{f})} = \mathbf{h}^n_0 \Leftrightarrow z_n(\mathbf{f}) \in \varphi_\omega(\mathbb{N}_n),$$

i.e., $\ker(\varrho_n) = \{\hat{\mathbf{f}}_n \in \mathbf{F}_n : z_n(\mathbf{f}) \in \varphi_\omega(\mathbb{N}_n)\}$, as claimed.

(4): This is trivial.

Items (iv) and (v) follow immediately. The assignment

$$\mathbf{h}^{\mathfrak{c},n}[\mathbb{M}](\hat{\mathbf{f}}_n) \mapsto \mathbf{h}^{\mathfrak{c},n}_{z_n(\mathbf{f})}$$

defines a homomorphism from $\mathrm{End}^{\mathfrak{c},\omega}(\mathbb{M})$ into $dE^{\mathfrak{c}}$ with kernel

$$\bigcup_{n<\omega} \Big\{ \mathbf{h}^{\mathfrak{c},n}[\mathbb{M}](\hat{\mathbf{f}}_n) \in \mathrm{End}^{\mathfrak{c},\omega}(\mathbb{M}) : z_n(\mathbf{f}) \in \varphi^{\mathfrak{c}}_\omega(\mathbb{N}_n) \Big\},$$

which is included in $\mathrm{End}^{\mathfrak{c},\omega}_{<\lambda}(\mathbb{M})$. In sum, we have a natural embedding from the ring $\mathrm{End}^{\mathfrak{c},\omega}(\mathbb{M})/\mathrm{End}^{\mathfrak{c},\omega}_{<\lambda}(\mathbb{M})$ into $dE^{\mathfrak{c}}$.

It remains to prove the moreover part of (iii). To this end, let $\mathbf{S}$ be any subring of $\mathrm{End}^{\mathfrak{c},\omega}(\mathbb{M})/\mathrm{End}^{\mathfrak{c},\omega}_{<\lambda}(\mathbb{M})$ of cardinality $< \lambda$. For each $s \in \mathbf{S}$ we find $n_s < \omega$ and $\mathbf{f}_s \in \mathrm{End}(\mathbb{M})$ such that $s = \mathbf{h}^{\mathfrak{c},n}[\mathbb{M}]((\hat{\mathbf{f}}_s)_{n_s}) + \mathrm{End}^{\mathfrak{c},\omega}_{<\lambda}(\mathbb{M})$. We look at the following club of κ:

$$C_s := \{\alpha < \kappa : \mathrm{Rang}(\mathbf{f}_s \restriction \mathbb{M}_\alpha) \subseteq \mathbb{M}_\alpha\}.$$

For each $\alpha \in C_s \setminus S$, we define an endomorphism $\theta(s) : \mathbb{M}/\mathbb{M}_\alpha \to \mathbb{M}/\mathbb{M}_\alpha$ by the following role:

$$\theta(s)(x + \mathbb{M}_\alpha) = \mathbf{f}_s(x) + \mathbb{M}_\alpha.$$

It is now natural to take $\bigcap\limits_{s \in \mathbf{S}} C_s$ as our required club, but we need to do a little more. Indeed, it is not clear that if θ is a homomorphism, as it may not preserve addition or multiplication. To handle this, we shrink the above intersection further as follows. Given $s_1, \cdots, s_m \in \mathbf{S}$, we see the following element

$$\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]((\hat{\mathbf{f}}_{s_1+\cdots+s_m})_{n_{s_1+\cdots+s_m}}) - \mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]((\hat{\mathbf{f}}_{s_1})_{n_{s_1}}) - \cdots - \mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]((\hat{\mathbf{f}}_{s_m})_{n_{s_m}})$$

is in $\mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M})$ and the following element

$$\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]((\hat{\mathbf{f}}_{s_1\cdots s_m})_{n_{s_1\cdots s_m}}) - \Big(\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]((\hat{\mathbf{f}}_{s_1})_{n_{s_1}}) \circ \cdots \circ \mathbf{h}^{\mathfrak{e},n}[\mathbb{M}]((\hat{\mathbf{f}}_{s_m})_{n_{s_m}})\Big)$$

belongs to $\mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M})$. Thus, we can find $\alpha(s_1, \cdots, s_m) < \kappa$ such that

$$\mathrm{Rang}((\mathbf{f}_{s_1+\cdots+s_m} - \mathbf{f}_{s_1} - \cdots - \mathbf{f}_{s_m}) \restriction \psi^{\mathfrak{e}}(\mathbb{M})) \subseteq \mathbb{M}_{\alpha(s_1,\cdots,s_m)}$$

and

$$\mathrm{Rang}((\mathbf{f}_{s_1\cdots s_m} - (\mathbf{f}_{s_1} \circ \cdots \circ \mathbf{f}_{s_m})) \restriction \psi^{\mathfrak{e}}(\mathbb{M})) \subseteq \mathbb{M}_{\alpha(s_1,\cdots,s_m)}.$$

Let

$$\alpha(*) := \sup\{\alpha(s_1, \cdots, s_m) : m < \omega,\ s_1, \cdots, s_m \in \mathbf{S}\} < \kappa.$$

We claim that

$$C := \bigcap_{s \in \mathbf{S}} C_s \setminus (\alpha(*) + 1)$$

is as required. The key point is that for $s_1, \cdots, s_m \in S$, and $\alpha \in C$, modulo $\mathbb{M}_\alpha$, we have

$$\mathbf{f}_{s_1+\cdots+s_m} = \mathbf{f}_{s_1} + \cdots + \mathbf{f}_{s_m}$$

and

$$\mathbf{f}_{s_1\cdots s_m} = \mathbf{f}_{s_1} \circ \cdots \circ \mathbf{f}_{s_m}.$$

(vi). Let $s \in S$. The notation $\mathbf{f}_s$ stands for the multiplication map by s, e.g., $\mathbf{f}_s \in \mathrm{End}(\mathbb{M})$ and it is defined by $\mathbf{f}_s(x) = xs$. The assignment $s \mapsto \mathbf{f}_s$ defines an embedding $S \longrightarrow \mathrm{End}(\mathbb{M})$. Let $\alpha < \omega$. The map $\mathbf{f} \mapsto \hat{\mathbf{f}}_n$ yields a natural homomorphism $\mathrm{End}(\mathbb{M}) \to \mathrm{End}^{\mathfrak{e},n}(\mathbb{M})$. For $\alpha := \omega$, as $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})$ is the direct limit of $\langle \mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) : n < \omega \rangle$, and by the above argument, we have a natural homomorphism $\mathrm{End}(\mathbb{M}) \to \mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})$. Finally, the assignment $\mathbf{h}^{\mathfrak{e},n}[\mathbb{M}](\hat{\mathbf{f}}_n) \mapsto \mathbf{h}^n_{z_n(\mathbf{f})}$ defines a homomorphism $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}) \to dE$. $\square$

Definition 6.26. *By* $\mathrm{End}^{\mathfrak{e},n}_{cpt}(\mathbb{M})$ *we mean*

$$\{h \in \mathrm{End}^{\mathfrak{e},n} : \textit{the range of } h \textit{ is compact for } (\mathfrak{e}, n) \textit{ in } \mathbb{M}_\lambda\}.$$

Remark 6.27. *In Lemma 6.25 we can replace* $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$ *with* $\mathrm{End}^{\mathfrak{e},n}_{cpt}(\mathbb{M})$ *and drive the analogue statement. Since this has no role in this paper, we leave the routine modification to the reader.*

Lemma 6.28. *Let* $\mathfrak{m} := (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ *be a* λ*-context, where* λ *is a regular cardinal such that*

$$\lambda = \lambda^{\aleph_0} > |\mathbf{R}| + |\mathbf{S}| + \aleph_0 + ||\mathbb{M}_*||$$

and for all $\alpha < \lambda$, $\alpha^{\aleph_0} < \lambda$. *Then there is a bimodule* $\mathbb{M} \geq_{\aleph_0} \mathbb{M}_*$ *satisfying* $\|\mathbb{M}\| = \lambda = |\psi^{\mathfrak{e}}_n(\mathbb{M})/\varphi^{\mathfrak{e}}_\omega(\mathbb{M})|$ *such that* $\mathbb{M}$ *has few direct decompositions in the following sense:*

(i) *if* $\mathbb{M} = \bigoplus_{t \in J} \mathbb{M}_t$ *and* $\mathfrak{e} \in \mathfrak{E}$, *then for all but finitely many* $t \in J$ *we have*

$$\bigvee_n \left[\psi^{\mathfrak{e}}_n(\mathbb{M}_t) \subseteq \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_t)\right].$$

In particular,

$$\bigvee_n \left[\varphi^{\mathfrak{e}}_n(\mathbb{M}_t) = \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_t)\right],$$

provided $\mathfrak{e}$ *is simple.*

(ii) *Assume in addition that* $|\mathbf{R}| + |\mathbf{S}| < 2^{\aleph_0}$ *and that* $\mathbb{M} = \mathbb{K}_\alpha \oplus \mathbb{P}_\alpha$ *for* $\alpha < (|\mathbf{R}| + |\mathbf{S}| + \aleph_0)^+$. *Then for some* $\alpha_0 < \alpha_1$ *and some* $n < \omega$ *we have*

$$\psi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) \subseteq \varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_1}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}),$$

where $\ell < 2$. In particular, if $\mathfrak{e}$ is simple, then

$$\varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_1}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}) = \varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_2}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}).$$

(iii) $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M})/\mathrm{End}^{\mathfrak{e},\omega}_{<\lambda}(\mathbb{M})$ *has cardinality* $\leq |\mathbf{R}| + |\mathbf{S}| + \aleph_0$.

Proof. Let $\langle \mathbb{M}_\alpha : \alpha \leq \lambda\rangle$ be a strongly semi-nice construction for $\mathfrak{m}$ and assume $S \subset S^\lambda_{\aleph_0}$ is stationary such that $S^\lambda_{\aleph_0} \setminus S$ is stationary as well. Let $\mathbb{M} := \mathbb{M}_\lambda$. We show $\mathbb{M}$ is as required.

(i). Suppose not. Let $\mathbb{M} = \bigoplus\limits_{t\in J} \mathbb{M}_t$ and $\mathfrak{e} \in \mathfrak{E}$ be a counterexample to the claim. Without loss of the generality, and by shrinking we may and do assume that $J = \omega$. Also, for each $n < \omega$, $\psi^{\mathfrak{e}}_n(\mathbb{M}_n) \nsubseteq \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_n)$. Define $\mathbf{f} : \mathbb{M} \to \mathbb{M}$ in such a way that

$$\mathbf{f}(x) = \begin{cases} 0 & x \in \mathbb{M}_{2n}, \\ x & x \in \mathbb{M}_{2n+1} \end{cases}$$

In other words, $\mathbf{f}$ is the natural projection from $\mathbb{M}$ onto $\bigoplus\limits_{n<\omega} \mathbb{M}_{2n+1}$. Recall that there is some $n(*) < \omega$, $\alpha < \lambda$ and $z \in \mathbb{L}^{\mathfrak{e}}_{n(*)}$ so that the property $(\mathrm{Pr}\, 1)^{n(*)}_{\alpha,z}[\mathbf{f}, \mathfrak{e}]$ holds. Let $\bar{\mathbb{G}}^* = \langle \mathbb{G}^*_n : n \geq n(*)\rangle$ be a decreasing sequence of additive subgroups of $\varphi^{\mathfrak{e}}_{n(*)}(\mathbb{M})$ is taken from Lemma 5.18. In particular, the following is satisfied:

(1) If $n \geq n(*)$, $\mathbf{h} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}$ and $z_n := g_{n(*),n}(z)$, then $\mathbf{f}(\mathbf{h}(x^{\mathfrak{e}}_n)) - \mathbf{h}(z_n) \in \mathbb{G}^*_n$.

(2) If $z^*_\ell \in \mathbb{G}^*_\ell$, for $\ell \geq n(*)$, then there exists $z^* \in \mathbb{G}^*_{n(*)}$ such that

$$z^* - \sum_{\ell=n(*)}^{n} z^*_\ell \in \varphi^{\mathfrak{e}}_{n+1}(\mathbb{M}).$$

Due to Lemma 5.21(1) we know that there are $k, m < \omega$ such that

(3) $\mathbb{G}^*_m \subseteq \bigoplus\limits_{\ell<k} \mathbb{M}_\ell + \varphi^{\mathfrak{e}}_\omega(\mathbb{M})$.

For each $n > n(*) + m + k$, we pick some $y_n \in \psi^{\mathfrak{e}}_n(\mathbb{M}_n) \setminus \varphi^{\mathfrak{e}}_\omega(\mathbb{M}_n)$. Let also $\mathbf{h}_n : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{M}_n$ be a bimodule homomorphism such that $y_n = \mathbf{h}_n(x^{\mathfrak{e}}_n)$. We choose n large enough. The assignment

$$x \mapsto (\mathbf{h}_n(x), \mathbf{h}_{n+1}(g_n(x)) \in \mathbb{M}_n \oplus \mathbb{M}_{n+1} \subset \mathbb{M}$$

defines a map $\mathbf{h} : \mathbb{N}_n \to \mathbb{M}$. This unifies $\mathbf{h}_{n+1}$ and $\mathbf{h}_n$. By symmetry, we may assume that n is even. It then follows from clause (3) that

$$\mathbf{h}(z_n) = \big(\mathbf{h}_n(z_n), \mathbf{h}_{n+1}(z_{n+1})\big) \in \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}_n) \subset \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})$$

and

$$y_n - \mathbf{h}(z_{n+1}) = y_n - (\mathbf{h}_{n+1}(z_{n+1}), \mathbf{h}_{n+2}(z_{n+2})) \in \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}_n) \subset \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}).$$

It follows from the first one that $\mathbf{h}_n(z_n)$ and $\mathbf{h}_{n+1}(z_{n+1})$ are in $\varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})$. By repetition, $\mathbf{h}_{n+2}(z_{n+2}) \in \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})$. We plug these in the second containment to see

$$y_n \in \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}).$$

This is a contradiction that we searched for it.

Here, we assume that $\mathfrak{c}$ is simple. According to Lemma 6.8, $\psi^{\mathfrak{c}}_n(\mathbb{M}_t) = \varphi^{\mathfrak{c}}_n(\mathbb{M}_t)$ for all n and $t \in J$. Hence for the n as chosen above, we have

$$\varphi^{\mathfrak{c}}_n(\mathbb{M}_t) = \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M}_t),$$

as required.

(ii). For each $\alpha < (|\mathbf{R}| + |\mathbf{S}| + \aleph_0)^+$, let $\mathbf{f}_\alpha$ be the projection onto $\mathbb{K}_\alpha$, i.e.,

$$\mathbf{f}_\alpha(x) = \begin{cases} x & x \in \mathbb{K}_\alpha, \\ 0 & x \in \mathbb{P}_\alpha \end{cases}$$

Pick $n_*(\alpha) < \omega$, $z_*(\alpha) \in \mathbb{L}^{\mathfrak{c}}_{n_*(\alpha)}$ and $\beta_*(\alpha)$ such that $(\mathrm{Pr}\,1)^{n_*(\alpha)}_{\beta_*(\alpha), z_*(\alpha)}[\mathbf{f}_\alpha, \mathfrak{c}]$ holds. We combine Lemma 5.18 along with Lemma 5.21(1) to find $m(\alpha) < \omega$ and a compact and decreasing sequence $\langle \bar{\mathbb{G}}^*_\ell : \ell \geq m(\alpha) \rangle$ of additive subgroups of $\varphi^{\mathfrak{c}}_{n(*)}(\mathbb{M})$. Furthermore, in the light of Lemma 5.23 we can assume that $\mathbb{G}^*_{m(\alpha)} \subseteq \varphi^{\mathfrak{c}}_{\omega}(\mathbb{M})$. There exists a stationary set S of $\alpha < (|\mathbf{R}| + |\mathbf{S}| + \aleph_0)^+$ such that for some $n < \omega$ and some fixed $z \in \mathbb{L}^{\mathfrak{c}}_n$, and for all $\alpha \in S$, we have $n_*(\alpha) = n$ and $z_*(\alpha) = z$. Let $\alpha_0 < \alpha_1$ be in S and $\ell < 2$. It follows that:

$$\begin{aligned} y \in \psi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) &\Rightarrow \exists \mathbf{h} : \mathbb{N}^{\mathfrak{e}}_n \to \mathbb{K}_{\alpha_\ell} \textit{ such that } \quad \mathbf{h}(x^{\mathfrak{e}}_n) = y \\ &\Rightarrow \mathbf{f}(y) - \mathbf{h}(z_n) \in \mathbb{G}^*_n \subset \varphi^{\mathfrak{e}}_\omega(\mathbb{M}) \\ &\Rightarrow y - \mathbf{h}(z_n) \in \varphi^{\mathfrak{e}}_\omega(\mathbb{M}) \\ &\Rightarrow y \in \varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}). \end{aligned}$$

We proved that $\psi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) \subseteq \varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M})$.

Now suppose that $\mathfrak{e}$ is simple. In view of Lemma 6.8 we have $\psi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) = \varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell})$. It then follows that

$$\varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_\ell}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}) = \varphi^{\mathfrak{e}}_n(\mathbb{K}_{\alpha_{1-\ell}}) + \varphi^{\mathfrak{e}}_\omega(\mathbb{M}).$$

(iii). We apply the notation introduced in Lemma 6.25. We say $\mathbf{f} \in \mathrm{End}(\mathbb{M})$ is nice at level n. if it satisfies in $(\mathrm{Pr}\,1)^n_{\alpha_n(\mathbf{f}),z_n(\mathbf{f})}$ for some $\alpha_n(F) < \lambda$ and $z_n(\mathbf{f}) \in \mathbb{L}^{\mathrm{tr}}_n$. Recall that E_n is defined by the natural image of

$$F_n := \left\{\mathbf{f} \restriction (\varphi^{\mathfrak{e}}_n(\mathbb{M}_\lambda)/\varphi^{\mathfrak{e}}_\omega(\mathbb{M}_\lambda) : \mathbf{f} \in \mathrm{End}(\mathbb{M}) \text{ which is nice at level n}\right\}$$

in $\mathrm{End}^{\bar{\varphi},\omega}(\mathbb{M})$. Also, the mapping $\mathbf{f} \mapsto \mathbf{h}^{\mathbf{n}}_{\mathbf{z_n(f)}}$ induces a homomorphism from F_n into $dE^{\mathfrak{e}}_n$ with kernel $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$. Since $\mathrm{End}^{\mathfrak{e},\omega}(\mathbb{M}) = \bigcup\limits_{n<\omega} \mathbf{E}_n$, it is enough to show that $\{\mathbf{h}^{\mathbf{n}}_{\mathbf{z_n(f)}} : \mathbf{z_n(f)} \in \mathbb{L}^{\mathrm{tr}}_{\mathbf{n}}\}$ is of cardinality at most $\leq |\mathbf{R}| + |\mathbf{S}| + \aleph_0$. Thus it is enough to show that the cardinality of $\mathbb{N}^{\mathfrak{e}}_n$ is at most $\leq |\mathbf{R}| + |\mathbf{S}| + \aleph_0$. This holds, because $\mathbb{N}^{\mathfrak{e}}_n$ is finitely generated as an $(\mathbf{R}, \mathbf{S})$-bimodule. $\square$

Remark 6.29. *Adopt the notation of Lemma 6.28(ii). If we omit "$|\mathbf{R}|+|\mathbf{S}| < 2^{\aleph_0}$", we get by the same proof weaker conclusions: with an "error term" which is included in a finitely generated bimodule.*

The following is supposed to be used together with any of the later lemmas here as its conclusion is in their assumptions.

Lemma 6.30. *Let $\mathbf{R}$ be a ring which is not purely semisimple and let $\mathbf{S} := \mathbf{T} := \langle 1\rangle_{\mathbf{R}}$. Let $\mathcal{K} = \mathcal{K}[\mathbf{R}, \mu]$ be the family of $(\leq \mu_1)$ generated and $(\leq \theta)$ presented bimodules where $\mu_1^\theta \leq \mu$ and let $\mathbb{M}_*$ be a bimodule of cardinality $< \mu$. Finally, let*

$\mathfrak{E} := \mathfrak{E}_{\mathbf{R},\mu}$ be the set of $(<\kappa)$-simple non trivial $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ wherever $(||\mathbf{R}||+\aleph_0)^{<\kappa} \leq \mu$. The following assertions are true.

- *(a) There is $\mathfrak{e} \in \mathfrak{E}$ which is very nice.*
- *(b) If $(|\mathbf{R}|+\aleph_0)^{\aleph_0} \leq \mu$, then $\mathfrak{e}(\bar{\varphi}) \in \mathfrak{E}$ for every very nice $\bar{\varphi}$.*
- *(c) $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ is a non trivial context which is simple.*

Proof. Since $\mathbf{R}$ is not purely semisimple, and in the light of Theorem 3.22, we can find a sequence $\bar{\varphi} = \langle \varphi_n(x) : n < \omega \rangle$ as in Definition 6.2. Then $\mathfrak{e} = \mathfrak{e}(\bar{\varphi}) \in \mathfrak{E}$ is very nice. This confirms (a). Items (b) and (c) are clear. $\square$

Lemma 6.31. *Suppose $\mathbf{R}$ is a ring, $\mathcal{T}$ is a complete first order theory of $\mathbf{R}$-modules which is not superstable, $\mathbf{S} = \mathbf{T} = \langle 1 \rangle_{\mathbf{R}}$ and $\mu \geq |\mathbf{R}| + \kappa$. Then there is a family $\mathcal{K} \cup \{\mathbb{M}_*\}$ of $\mathbf{R}$-modules with μ members such that:*

- *(a) $\mathbb{M}_* \oplus \bigoplus\limits_{t \in I} \mathbb{M}_t$ is model of $\mathcal{T}$ whenever $\mathbb{M}_t \in \mathcal{K}$, moreover*
$$\mathbb{M}_* \prec_{\mathcal{L}(\tau_{\mathbf{R}})} \mathbb{M}_* \oplus \bigoplus_{t \in I} \mathbb{M}_t.$$
- *(b) For any $\mathbb{N} \in \mathcal{K}$ we have $||\mathbb{N}|| \leq \mu$.*
- *(c) If $2^{\mu_1} \leq \mu$ and $\mu_1 \geq ||\mathbf{R}|| + \kappa$, then every model $\mathbb{N}$ of $\mathcal{T}$ satisfying*
$$\mathbb{M}_* \prec_{\mathcal{L}(\tau_{\mathbf{R}})} \mathbb{M}_* \oplus \mathbb{N}$$
belongs to $\mathcal{K}$.
- *(d) Let $2^{\mu_1} \leq \mu$ and $\mu_1 \geq ||\mathbf{R}||+\aleph_0$. Then every appropriate sequence $\langle \mathbb{N}_n, g_n, x_n : n < \omega \rangle$ with $||\mathbb{N}_n|| \leq \mu_1$ belongs to $\mathcal{K}$.*
- *(e) Let $\mathfrak{E}$ be the set of $(<\aleph_0)$-simple non trivial $\mathfrak{e} \in \mathfrak{E}_{(\mathbf{R},\mathbf{S})}$ such that each $\mathbb{N}_n^{\mathfrak{e}}$ is in $\mathcal{K}$. Then $\mathfrak{m} = (\mathcal{K}, \mathbb{M}_*, \mathfrak{E}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ is a μ-context; note that a bimodule for $\mathfrak{m}$ is just an $\mathbf{R}$-module.*

Proof. Let $\mathbb{M}_*$ be any $\aleph_1$-saturated model of $\mathcal{T}$ of size $\leq \mu$ and set

$$\mathcal{K} := \left\{ \mathbb{N} : \mathbb{N} \text{ is an } \mathbf{R}\text{–module such that } \mathbb{M}_* \prec_{\mathcal{L}(\tau_{\mathbf{R}})} \mathbb{M}_* \oplus \mathbb{N} \text{ and } 2^{||\mathbb{N}||} \leq \mu \right\}.$$

It is easily seen that $\mathcal{K}$ is as required. Since this has no role in the paper, we leave the routine details to the reader. $\square$

The following easy lemma plays an essential role in the sequel:

Lemma 6.32. *Let $\mathfrak{m}$ be a λ-context and $\mathbf{S}$ be a free $\mathbf{T}$-module with a base $\{c^*_\beta : \beta < \alpha\}$. Let $\bar\varphi$ be very nice, $\varphi_n = (\exists y_0, \ldots, y_{k_n-1})\varphi'_n$ where*

$$\varphi'_n = \bigwedge_{\ell=0}^{m_n-1}\Big[a^n_\ell x - \sum_{i=0}^{k_n-1} b^n_{\ell,i} y_i\Big].$$

Let $\mathfrak{e} = \mathfrak{e}(\bar\varphi) \in \mathfrak{E}^{\mathfrak{m}}$ and $\mathbb{N}_n = \mathbb{N}^{\mathfrak{e}}_n$ (see Definition 6.6). The following assertions hold:

(1) *Let $\mathbb{N}_{n,0}$ be the $\mathbf{R}$-submodule of $\mathbb{N}_n$ generated by $\{x, y_i : i < k_{m_n-1}\}$. Then $\mathbb{N}_n$ is the direct sum $\sum\limits_{\beta<\alpha} \mathbb{N}_{n,\beta}$ and $h_\beta : \mathbb{N}_{n,0} \xrightarrow{\cong} \mathbb{N}_{n,\beta}$, as $\mathbf{R}$-modules, where $\mathbb{N}_{n,\beta}$ is the $\mathbf{R}$-module generated by $\{xc^*_\beta\} \cup \{y_i c^*_\beta : i < k_{m_n-1}\}$ freely except the equations φ'_n and h_0 is the identity.*

(2) *$\varphi_n(\mathbb{N}_n)/\varphi_\omega(\mathbb{N}_n) \cong \sum\limits_{\beta<\alpha} \varphi_n(\mathbb{N}_{n,\beta})/\varphi_\omega(\mathbb{N}_{n,\beta})$ as a $\mathbf{T}$-module.*

(3) *For any $z \in \mathbb{L}^{\mathfrak{e}}_n$, there are $z_\beta \in \mathbb{N}_{n,0} \cap \mathbb{L}^{\mathfrak{e}}_n \cap \varphi^{\mathfrak{e}}_n(\mathbb{N}_{n,0})$ such that $z = \sum\limits_{\beta<\alpha} h_\beta(z_\beta)$. Also, $\mathbf{h}^{\mathfrak{e},n}_z = \sum\limits_{\beta<\alpha} \mathbf{h}^{\mathfrak{e},n}_{h_\beta(z_\beta)}$. In particular, z is n-nice if and only if each z_β is n-nice.*

(4) *The rings $de^{\mathfrak{e}}_n$ and $\mathbf{S}$ generate $dE^{\mathfrak{e}}_n$. In fact each element of $dE^{\mathfrak{e}}_n$ has the form $\sum\limits_{\beta<\alpha} x_\beta s_\beta$, where $x_\beta \in de^{\mathfrak{e}}_n$ and $s_\beta \in S_\beta$. Also we have $dE^{\mathfrak{e}}_n = de^{\mathfrak{e}}_n \otimes_{\mathbf{T}} \mathbf{S}$.*

(5) *Let $\mathcal{I}_n$ be a maximal ideal of $de^{\mathfrak{e}}_n$. Then $\mathbf{D}_n := de^{\mathfrak{e}}_n/\mathcal{I}_n$ is a field.*

(6) *Let $\mathbf{T}' := \mathbf{T}/(\mathcal{I}_n \cap \mathbf{T})$, $\mathbf{S}' := \mathbf{S}/(\mathcal{I}_n \cap \mathbf{T})$ and let $\mathbb{M}$ be from a strongly semi-nice construction. Then any set of equations on $\mathbf{S}$ which has a solution in $\mathrm{End}_{\mathbf{R}}(\mathbb{M})$ has a solution in $\mathbf{D}_n \otimes_{\mathbf{T}'} \mathbf{S}'$.*

Proof. (1). Since $\mathbf{S}$ is free as a $\mathbf{T}$-module with the base $\{c^*_\beta : \beta < \alpha\}$, we have $\mathbf{S} = \bigoplus_{\beta<\alpha} \mathbf{T}c^*_\beta$. Now, we apply this through the following natural identifications of $\mathbf{R}$-modules:

$$\begin{aligned}\mathbf{R}x\mathbf{S} \oplus \bigoplus_{i<k_m-1} \mathbf{R}y_i\mathbf{S} &= \bigoplus_{\beta<\alpha} \mathbf{R}xc^*_\beta \oplus \bigoplus_{i<k_m-1} \bigoplus_{\beta<\alpha} \mathbf{R}y_i c^*_\beta \\ &= \bigoplus_{\beta<\alpha} \Big(\mathbf{R}xc^*_\beta \oplus \bigoplus_{i<k_m-1} \mathbf{R}y_i c^*_\beta\Big).\end{aligned}$$

It turns out from the previous displayed identification that

$$\mathbb{N}_n \cong \bigoplus_{\beta<\alpha} \mathbb{N}_{n,\beta},$$

as an $\mathbf{R}$-module. Also for any $y \in \mathbb{N}_{n,0}$, we set $h_\beta(y) := yc_\beta$. This yields an isomorphism $h_\beta : \mathbb{N}_{n,0} \xrightarrow{\cong} \mathbb{N}_{n,\beta}$ of $\mathbf{R}$-modules.

(2). We apply Lemma 4.21 along with (1) to conclude that $\varphi_n(\mathbb{N}_n) = \bigoplus_{\beta<\alpha} \varphi_n(\mathbb{N}_{n,\beta})$. Consequently, $\varphi_n(\mathbb{N}_n)/\varphi_\omega(\mathbb{N}_n) = \bigoplus_{\beta<\alpha} \varphi_n(\mathbb{N}_{n,\beta})/\varphi_\omega(\mathbb{N}_{n,\beta})$.

(3). Let $z \in \mathbb{L}^{\mathfrak{c}}_n$. Then,

$$z \in \varphi_n(\mathbb{N}_n) = \bigoplus_{\beta<\alpha} \varphi^{\mathfrak{c}}_n(\mathbb{N}_{n,\beta}) \cong \bigoplus_{\beta<\alpha} h''_\beta(\varphi^{\mathfrak{c}}_n(\mathbb{N}_{n,0})).$$

Let $z_\beta \in \mathbb{N}_{n,0}$ be such that $z = \sum\limits_{\beta<\alpha} h_\beta(z_\beta)$. It is evident that $z_\beta \in \mathbb{N}_{n,0} \cap \mathbb{L}^{\mathfrak{c}}_n \cap \varphi^{\mathfrak{c}}_n(\mathbb{N}_{n,0})$.

Now suppose that $\mathbf{g} : \mathbb{N}_n \to \mathbb{M}$ is a bimodule homomorphism. Then

$$\begin{aligned}\mathbf{h}^{\mathfrak{c},n}_z(\mathbf{g}(x_n) + \varphi_\omega(\mathbb{M})) &= \mathbf{g}(z) + \varphi_\omega(\mathbb{M}) \\ &= \big(\sum_{\beta<\alpha} \mathbf{g}(h_\beta(z_\beta))\big) + \varphi_\omega(\mathbb{M}) \\ &= \sum_{\beta<\alpha} (\mathbf{g}(h_\beta(z_\beta)) + \varphi_\omega(\mathbb{M})) \\ &= \sum_{\beta<\alpha} \mathbf{h}^{\mathfrak{c},n}_{h_\beta(z_\beta)}(\mathbf{g}(x_n) + \varphi_\omega(\mathbb{M})).\end{aligned}$$

It follows that $\mathbf{h}^{\mathfrak{c},n}_z = \sum\limits_{\beta<\alpha} \mathbf{h}^{\mathfrak{c},n}_{h_\beta(z_\beta)}$.

(4). Let $x \in de^{\mathfrak{c}}_n$ and $s \in \mathbf{S}$. Then $xs \in dE^{\mathfrak{c}}_n$. This implies the existence of a $\mathbf{T}$-linear map $de^{\mathfrak{c}}_n \times \mathbf{S} \to dE^{\mathfrak{c}}_n$. By the universal property of tensor products, there is a map $f : de^{\mathfrak{c}}_n \otimes_{\mathbf{T}} \mathbf{S} \to dE^{\mathfrak{c}}_n$. Now, let $z \in dE^{\mathfrak{c}}_n$. By clause (3), there are $z_\beta \in de^{\mathfrak{c}}_n$ and $s_\beta \in \mathbf{S}$ such that $z = \sum\limits_\beta z_\beta s_\beta$. Moreover, by its proof, we know that such a presentation is unique. This shows that f is an isomorphism. Up to this identification, $dE^{\mathfrak{c}}_n = de^{\mathfrak{c}}_n \underset{\mathbf{T}}{\otimes} \mathbf{S}$. Thanks to Lemma 6.24 we know the rings $de^{\mathfrak{c}}_n$ and $\mathbf{S}$ commute with each other.

(5). As $\mathcal{I}_n$ is a maximal ideal, it is clear that $\mathbf{D}_n$ is a division ring. Since the ring $de_n^{\mathfrak{e}}$ is commutative (see Lemma 6.24(4)) we deduce that $\mathbf{D}_n$ is a field.

(6). Let

$$(*)_1 \qquad \textstyle\sum_j s_{ij} X_i^j = 0,$$

be a system of polynomial equations with parameters $s_{ij} \in S$ and indeterminates $\{X_i\}$. Suppose these equations have a solution $\mathbf{f} \in \mathrm{End}(\mathbb{M})$. This means that

$$(*)_2 \qquad \textstyle\sum_j s_{ij} \mathbf{f}^j = 0.$$

There are $z_n(\mathbf{f}) \in \mathbb{L}_n^{\mathfrak{e}}$ and $\alpha_n(\mathbf{f}) < \lambda$ such that the property $(\Pr 1)^n_{\alpha_n(\mathbf{f}), z_n(\mathbf{f})}[\mathbf{f}, \mathfrak{e}]$ holds. Let $f_n := \hat{\mathbf{f}}_n$. Then

$$(*)_3 \qquad \textstyle\sum_j s_{ij} f_n^j = 0.$$

Recall from Lemma 6.25(v) that the natural mapping $\varrho_n : \hat{\mathbf{f}}_n \mapsto \mathbf{h}^n_{z_n(\mathbf{f})}$ is a homomorphism from

$$\left\{\hat{\mathbf{f}}_n : \quad \mathbf{f} \in \mathrm{End}(\mathbb{M}) \text{ and } (\Pr 1)^n_{\alpha_n(\mathbf{f}), z_n(\mathbf{f})} \text{ holds for some } \ \alpha_n(\mathbf{f}) < \lambda \textit{ and } z_n(\mathbf{f}) \in \mathbb{L}_n^{\mathrm{tr}}\right\}$$

into $dE_n^{\mathfrak{e}}$ with kernel included in $\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})$. Let

$$\pi : \mathrm{End}^{\mathfrak{e},n}(\mathbb{M}) \twoheadrightarrow \frac{\mathrm{End}^{\mathfrak{e},n}(\mathbb{M})}{\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})}$$

be the canonical surjection and let $g_n = \pi(\mathbf{h}^n_{z_n(\mathbf{f})})$. Applying $\pi \circ \varrho_n$ to both sides of $(*)_3$, we get

$$(*)_4 \qquad \textstyle\sum_j s_{ij} g_n^j = 0.$$

Since there is an embedding

$$\rho_n : \frac{\mathrm{End}^{\mathfrak{e},n}(\mathbb{M})}{\mathrm{End}^{\mathfrak{e},n}_{<\lambda}(\mathbb{M})} \hookrightarrow dE_n^{\mathfrak{e}},$$

by setting $e_n := \rho_n(g_n)$, we have

$$(*)_5 \qquad \textstyle\sum_j s_{ij} e_n^j = 0.$$

In view of Clause (4) we see that $dE^{\mathfrak{c}}_n = de^{\mathfrak{c}}_n \underset{\mathbf{T}}{\otimes} \mathbf{S}$. Recall that there are natural surjective maps $de^{\mathfrak{c}}_n \twoheadrightarrow \mathbf{D}_n$ and $\mathbf{S} \twoheadrightarrow \mathbf{S}'$. These induce the following natural map

$$\sigma : de^{\mathfrak{c}}_n \underset{\mathbf{T}}{\otimes} \mathbf{S} \longrightarrow \mathbf{D}_n \underset{\mathbf{T}'}{\otimes} \mathbf{S}'.$$

Set $t_n := \sigma(e_n)$. By applying σ to $(*)_5$ we obtain

$(*)_6$ $$\textstyle\sum_j s_{ij} t^j_n = 0.$$

This essentially says that the polynomial equations from $(*)_1$ with parameters in S have a solution in $\mathbf{D}_n \underset{\mathbf{T}'}{\otimes} \mathbf{S}'$ as well. This is what we want to prove. $\square$

In what follows we will use the following two consequences of Lemma 6.32:

Corollary 6.33. *Suppose that the following three items are valid:*

- *(a)* $\mathbf{R}$ *is a ring which is not pure semisimple and let* $\mathbf{T}$ *be the subring of* $\mathbf{R}$ *generated by* 1, *i.e.,* $\mathbf{T} \cong \mathbb{Z}/n\mathbb{Z}$, *where* $n := \mathrm{char}(\mathbf{R})$ *which is not necessarily prime.*
- *(b)* $\mathbf{S}$ *is a ring containing* $\mathbf{T}$ *such that* $(\mathbf{S}, +)$ *is a free* $\mathbf{T}$*-module and suppose that for every* $s \in \mathbf{S} \setminus \{0_{\mathbf{S}}\}$ *for some* $\mathbb{N} \in \mathcal{K} \cup \{\mathbb{M}_*\}$ *we have* $\mathbb{N}s \neq \{0_{\mathbb{N}}\}$.
- *(c)* $\lambda = \mathrm{cf}(\lambda) > ||\mathbf{R}|| + ||\mathbf{S}|| + \aleph_0$ *and* $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$.

Then we can find an $\mathbf{R}$*-module* $\mathbb{M}$ *of cardinality* λ, *and a homomorphism* $\mathbf{h}$ *from* $\mathbf{S}$ *into* $\mathrm{End}(\mathbb{M})$ *such that:*

- *(d)* $\mathrm{Ker}(\mathbf{h}) = \{0\}$.
- *(e) If* Σ *is a set of equations with parameters in* $\mathbf{S}$ *such that* $\mathbf{h}(\Sigma)$ *is solvable in* $\mathrm{End}_{\mathbf{R}}(\mathbb{M})$, *then* Σ *is solvable in* $\mathbf{D} \otimes \mathbf{S}$ *for some field* $\mathbf{D}$.
- *(f) If* $s \in \mathbf{S} \setminus \{0_{\mathbf{S}}\}$ *and* $\mathbb{N} \in \mathcal{K}$ *is such that* $\mathbb{N}s \neq \{0_{\mathbb{N}}\}$, *then the image of* $\mathbb{M}$ *under* $\mathbf{h}(s)$ *has cardinality* λ.

Recall that the notation $|$ *means divides.*

Corollary 6.34. *Suppose* $\mathbf{S}$ *is a ring extending* $\mathbb{Z}$ *such that* $(\mathbf{S}, +)$ *is free, and let* $\mathbf{R}$ *be a ring which is not pure semisimple. Let* $\mathbf{D}$ *be a field such that*

$$p := \operatorname{char}(\mathbf{D}) | \operatorname{char}(\mathbf{R})$$

and set

$$\mathbb{Z}_p := \begin{cases} \mathbb{Z}/p\mathbb{Z} & p > 0, \\ \mathbb{Z} & \textit{otherwise}. \end{cases}$$

Suppose Σ *is the set of equations over* $\mathbf{S}$ *which is not solvable in* $\mathbf{D} \otimes_{\mathbb{Z}_p} (\mathbf{S}/p\mathbf{S})$. *Finally, let* $\mathbb{M}$ *be strongly nicely constructed. Then* Σ *is not solvable in* $\operatorname{End}(\mathbb{M})$.

Recall that a module is $\aleph_0$*-free if each of its finitely generated submodules are free. This yields the following statement:*

Remark 6.35. In Corollary 6.34, if $(\mathbf{S}, +)$ is an $\aleph_0$-free $\mathbf{T}$-module, the similar conclusions are hold.

7. Dropping Gödel's axiom of constructibility

Our aim in this section is to extend the main results of [46] *to the ordinary set theory. From now on we assume that* $\mathbf{R}$ *is a ring which is not pure semisimple.*

Theorem 7.1. *Let* λ *be a regular cardinal of the form* $(\mu^{\aleph_0})^+$ *such that* $\lambda > |R|$. *Then there are* $\mathbf{R}$*-modules* $\mathbb{M}$, $\mathbb{M}_1$ *and* $\mathbb{M}_2$ *of cardinality* λ *such that:*

(1) $\mathbb{M} \oplus \mathbb{M}_1 \cong \mathbb{M} \oplus \mathbb{M}_2$,

(2) $\mathbb{M}_1 \ncong \mathbb{M}_2$,

(3) $\mathbb{M}_1 \equiv_{\mathcal{L}_{\infty,\lambda}} \mathbb{M}_2$.

Proof. $(1) + (2)$: Let $\mathbf{T}$ be the subring of $\mathbf{R}$ which 1 (the unit) generates.

Step A): Here, we introduce the auxiliary ring $\mathbf{S}$:

Let $\mathbf{S} := \frac{\mathbf{T}\langle \mathcal{X}, \mathcal{W}_1, \mathcal{Y}, \mathcal{W}_2\rangle}{I}$ where $\mathbf{T}\langle \mathcal{X}, \mathcal{W}_1, \mathcal{Y}, \mathcal{W}_2\rangle$ is the skew polynomial ring in non commuting variables $\{\mathcal{X}, \mathcal{W}_1, \mathcal{Y}, \mathcal{W}_2\}$ with coefficients in the commutative ring $\mathbf{T}$, and I is its two-sided ideal generated by:

(∗): $\mathcal{X}\mathcal{X} = \mathcal{X}$,

$\mathcal{Y}\mathcal{Y} = \mathcal{Y}$,

$\mathcal{X}\mathcal{W}_1\mathcal{W}_2 = \mathcal{X}$,

$\mathcal{Y}\mathcal{W}_2\mathcal{W}_1 = \mathcal{Y}$,

$\mathcal{X}\mathcal{W}_1\mathcal{Y} = \mathcal{X}\mathcal{W}_1$,

$(1-\mathcal{X})(1-\mathcal{Y}) = 1-\mathcal{X}$,

$\mathcal{Y}\mathcal{X} = \mathcal{Y}$,

$\mathcal{Y}\mathcal{W}_2\mathcal{X} = \mathcal{Y}\mathcal{W}_2$.

We call them "test equations". In other words, $\mathbf{S}$ is the ring generated by $\mathbf{T} \cup \{\mathcal{X}, \mathcal{W}_1, \mathcal{Y}, \mathcal{W}_2\}$ extending $\mathbf{T}$ freely except the test equations (to understand these equations see the definition of $\mathbb{M}^{\otimes}$ as a bimodule below).

Step B): Let $\alpha < \beta < \gamma$ be additively indecomposable ordinals[20] and let $\mathbb{M}$ be an $\mathbf{R}$-module. We define a new bimodule $\mathbb{M}^{\otimes}$ related to $\mathbb{M}$ and ordinals α, β, γ.

For $i < \gamma$, let $h_i : \mathbb{M} \xrightarrow{\cong} \mathbb{M}_i^{\otimes}$ (where $\mathbb{M}_i^{\otimes}$ is an $\mathbf{R}$-module) and set $\mathbb{M}^{\otimes} := \bigoplus\limits_{i<\gamma} \mathbb{M}_i^{\otimes}$. We expand $\mathbb{M}^{\otimes}$ to an $(\mathbf{R}, \mathbf{S})$-bimodule. To this end, we take $x \in \mathbb{M}_i^{\otimes}$. Due to the axioms of bimodules, it is enough to define $\{h_i(x)\mathcal{X}, h_i(x)\mathcal{Y}, h_i(x)\mathcal{W}_1, h_i(x)\mathcal{W}_2\}$. We define these via the following rules:

$$h_i(x)\mathcal{X} := \begin{cases} h_i(x) & i \geq \alpha, \\ 0 & i < \alpha, \end{cases}$$

$$h_i(x)\mathcal{Y} := \begin{cases} h_i(x) & i \geq \beta, \\ 0 & i < \beta, \end{cases}$$

$$h_i(x)\mathcal{W}_1 := \begin{cases} h_j(x) & \text{if for some } \epsilon,\ i = \alpha + \epsilon < \gamma,\ j = \beta + \epsilon < \gamma, \\ 0 & \text{otherwise,} \end{cases}$$

[20] Recall that an ordinal γ is additively indecomposable if for all ordinals $\alpha, \beta < \gamma$ we have $\alpha + \beta < \gamma$.

and

$$h_i(x)\mathcal{W}_2 := \begin{cases} h_j(x) & \text{if for some } \epsilon,\ i=\beta+\epsilon<\gamma,\ j=\alpha+\epsilon<\gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathfrak{m} := (\mathcal{K}, \mathbb{M}_*, \mathfrak{C}, \mathbf{R}, \mathbf{S}, \mathbf{T})$ be a λ-context, where $\mathbb{M}_*$ is an $\aleph_1$-saturated R-module of size λ, see Definition 4.10.

Let $\bar{\mathbb{M}} := \langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ be a strongly semi-nice construction with respect to $\mathfrak{m}$. Recall that semi-nice construction is a consequence of Section 4, and its strong form was constructed in Section 5.

Let $\mathbb{P} := \mathbb{M}_\kappa$ and let ${}_{\mathbf{R}}\mathbb{P}$ be $\mathbb{P}$ as an $\mathbf{R}$-module.

Step C): Here, we define the $\mathbf{R}$-modules $\mathbb{M}$, $\mathbb{M}_1$ and $\mathbb{M}_2$ of cardinality λ such that $\mathbb{M} \oplus \mathbb{M}_1 \cong \mathbb{M} \oplus \mathbb{M}_2$.

To this end, recall from the second step that every element of $\mathbf{S}$ may be considered as an endomorphism of ${}_{\mathbf{R}}\mathbb{P}$. Set ${}_{\mathbf{R}}\mathbb{M}^1 := ({}_{\mathbf{R}}\mathbb{P})\mathcal{X}$ and ${}_{\mathbf{R}}\mathbb{M}_1 := ({}_{\mathbf{R}}\mathbb{P})(1-\mathcal{X})$. We conclude from the formula $\mathcal{X}\mathcal{X} - \mathcal{X} = 0$ that ${}_{\mathbf{R}}\mathbb{M}^1 \cap {}_{\mathbf{R}}\mathbb{M}_1 = 0$. Let us use from the formula $\mathcal{X}(1-\mathcal{X}) = 1$ that

$${}_{\mathbf{R}}\mathbb{P} = {}_{\mathbf{R}}\mathbb{P}(\mathcal{X} + (1-\mathcal{X})) = {}_{\mathbf{R}}\mathbb{P}\mathcal{X} + (1-\mathcal{X}){}_{\mathbf{R}}\mathbb{P} = {}_{\mathbf{R}}\mathbb{M}^1 + {}_{\mathbf{R}}\mathbb{M}_1 = {}_{\mathbf{R}}\mathbb{M}^1 \oplus {}_{\mathbf{R}}\mathbb{M}_1.$$

Let ${}_{\mathbf{R}}\mathbb{M}^2 := ({}_{\mathbf{R}}\mathbb{P})\mathcal{Y}$ and ${}_{\mathbf{R}}\mathbb{M}_2 := ({}_{\mathbf{R}}\mathbb{P})(1-\mathcal{Y})$. In the same vein, the above formula leads us to the following decomposition

$${}_{\mathbf{R}}\mathbb{P} = {}_{\mathbf{R}}\mathbb{M}^2 \oplus {}_{\mathbf{R}}\mathbb{M}_2.$$

In view of the equation $\mathcal{X}\mathcal{W}_1\mathcal{Y} = \mathcal{X}\mathcal{W}_1$, we have

$$\mathbb{M}^1\mathcal{W}_1 = {}_{\mathbf{R}}\mathbb{P}\mathcal{X}\mathcal{W}_1 = {}_{\mathbf{R}}\mathbb{P}\mathcal{X}\mathcal{W}_1\mathcal{Y} \subset {}_{\mathbf{R}}\mathbb{P}\mathcal{Y} = \mathbb{M}^2.$$

This yields a homomorphism from $\mathbb{M}^1$ to $\mathbb{M}^2$, defined by the help of the following assignment

$$a \mapsto a\mathcal{W}_1.$$

Similarly, $\mathcal{W}_2$ provides a homomorphism from $\mathbb{M}^2$ onto $\mathbb{M}^1$, defined via

$$b \mapsto b\mathcal{W}_2.$$

Thanks to the equations $\mathcal{Y}\mathcal{W}_2\mathcal{W}_1 = \mathcal{Y}$ and $\mathcal{Y}\mathcal{X} = \mathcal{Y}$, it is easily seen that the multiplication maps by $\mathcal{W}_1$ and $\mathcal{W}_2$ are inverse to each other. This provides an isomorphism from $\mathbb{M}^1$ onto $\mathbb{M}^2$, so let ${}_{\mathbf{R}}\mathbb{M} := {}_{\mathbf{R}}\mathbb{M}^1 \cong {}_{\mathbf{R}}\mathbb{M}^2$.

Step D): One has ${}_{\mathbf{R}}\mathbb{M}_1 \ncong {}_{\mathbf{R}}\mathbb{M}_2$.

Assume towards contradiction that ${}_{\mathbf{R}}\mathbb{M}_1 \cong {}_{\mathbf{R}}\mathbb{M}_2$. Thus there are $f_1 : {}_{\mathbf{R}}\mathbb{M}_1 \to {}_{\mathbf{R}}\mathbb{M}_2$ and $f_2 : {}_{\mathbf{R}}\mathbb{M}_2 \to {}_{\mathbf{R}}\mathbb{M}_1$ such that $f_1 f_2 = 1$ and $f_2 f_1 = 1$. Recall that ${}_{\mathbf{R}}\mathbb{P} = {}_{\mathbf{R}}\mathbb{M}^1 \oplus {}_{\mathbf{R}}\mathbb{M}_1 = {}_{\mathbf{R}}\mathbb{M}^2 \oplus {}_{\mathbf{R}}\mathbb{M}_2$. Define $\mathcal{Z}_1 \in \mathrm{End}_{\mathbf{R}}({}_{\mathbf{R}}\mathbb{P})$ by applying the following assignment

$$(a, b) \in {}_{\mathbf{R}}\mathbb{M}^1 \oplus {}_{\mathbf{R}}\mathbb{M}_1 \mapsto (a\mathcal{W}_1, f_1(b)) \in {}_{\mathbf{R}}\mathbb{M}^2 \oplus {}_{\mathbf{R}}\mathbb{M}_2.$$

In the same vein, define $\mathcal{Z}_2 \in \mathrm{End}_{\mathbf{R}}({}_{\mathbf{R}}\mathbb{P})$ via

$$(a, b) \in {}_{\mathbf{R}}\mathbb{M}^2 \oplus {}_{\mathbf{R}}\mathbb{M}_2 \mapsto (a\mathcal{W}_2, f_2(b)) \in {}_{\mathbf{R}}\mathbb{M}^1 \oplus {}_{\mathbf{R}}\mathbb{M}_1.$$

Clearly, $\mathcal{Z}_1\mathcal{Z}_2 = \mathcal{Z}_2\mathcal{Z}_1 = 1 = \mathrm{id}_{\mathbb{P}}$. It is also easy to check that:

$$\begin{aligned}
&\mathcal{X}\mathcal{Z}_1 = \mathcal{X}\mathcal{Z}_1\mathcal{Y},\\
&(1-\mathcal{X})\mathcal{Z}_1 = (1-\mathcal{X})\mathcal{Z}_1(1-\mathcal{Y}),\\
&\mathcal{Y}\mathcal{Z}_2 = \mathcal{Y}\mathcal{Z}_2\mathcal{X},\\
&(1-\mathcal{Y})\mathcal{Z}_2 = (1-\mathcal{Y})\mathcal{Z}_2(1-\mathcal{X}).
\end{aligned}$$

We use just one very simple non trivial $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$. In the light of Lemma 5.13 there are

- $n(*) < \omega$,
- $\alpha(*) \in \kappa \setminus S$ and
- $z_i \in \mathbb{L}^{\mathfrak{e}}_{n(*)}[\mathcal{K}]$

such that the property $(\mathrm{Pr}\,1)^{n(*)}_{\alpha(*),z_i}$ holds. This allows us to apply Lemma 5.18 to conclude that the equations above hold in the endomorphism ring of the abelian group $\varphi^{\mathfrak{e}}_{n(*)}(\mathbb{M})/\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M})$ for any bimodule $\mathbb{M}$,[21] when we replace $\mathcal{Z}_1$ (resp. $\mathcal{Z}_2$) by $\mathbf{h}^{\mathfrak{e},n(*)}_{\mathbb{M},z_1}$ (resp. $\mathbf{h}^{\mathfrak{e},n(*)}_{\mathbb{M},z_2}$) and interpret $\mathcal{X},\mathcal{Y},\mathcal{W}_1,\mathcal{W}_2 \in \mathbf{S}$ naturally. This holds in particular for the bimodule $\mathbb{M}^{\otimes}$ we defined in Step B). So, the following equations hold:

$$\begin{aligned}
&\mathcal{X}\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_1} = \mathcal{X}\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_1}\mathcal{Y},\\
&(1-\mathcal{X})\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_1} = (1-\mathcal{X})\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_1}(1-\mathcal{Y}),\\
&\mathcal{Y}\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_2} = \mathcal{Y}\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_2}\mathcal{X},\\
&(1-\mathcal{Y})\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_2} = (1-\mathcal{Y})\mathbf{h}^{\mathfrak{e},n}_{\mathbb{M}^{\otimes},z_2}(1-\mathcal{X}).
\end{aligned}$$

These equations in turn define certain decompositions of $\varphi^{\mathfrak{e}}_{n}(\mathbb{M}^{\otimes})/\varphi^{\mathfrak{e}}_{\omega}(\mathbb{M}^{\otimes})$ which yield to the following isomorphism

$$\frac{\varphi^{\mathfrak{e}}_n\left(\sum_{i<\beta}\mathbb{M}^{\otimes}_i\right)}{\varphi^{\mathfrak{e}}_\omega\left(\sum_{i<\beta}\mathbb{M}^{\otimes}_i\right)} \xrightarrow{\cong} \frac{\varphi^{\mathfrak{e}}_n\left(\sum_{i<\alpha}\mathbb{M}^{\otimes}_i\right)}{\varphi^{\mathfrak{e}}_\omega\left(\sum_{i<\alpha}\mathbb{M}^{\otimes}_i\right)}.$$

The cardinality of left (resp. right) hand side is $|\beta|$ (resp. $|\alpha|$). Thus if we choose $|\beta| > |\alpha|$, we get a contradiction that we searched for it.

(3): This follows from (1)+(2) and Lemma 4.20. $\square$

Remark 7.2. *Adopt the notation of Theorem 7.1.*

(a) Note that (1) becomes trivial if we remove the "of cardinality λ". To see this, take $\mathbb{M}$, $\mathbb{M}_1$ and $\mathbb{M}_2$ to be free $\mathbf{R}$-modules with

$$\|\mathbb{M}\| > \|\mathbb{M}_1\| > \|\mathbb{M}_2\| \geq |\mathbf{R}| + \aleph_0.$$

(b) Recall that $\mathbb{M}_1 \equiv_{\mathcal{L}_{\infty,\lambda}} \mathbb{M}_2$ means for every sentence $\sigma \in \mathcal{L}_{\infty,\lambda}$,

$$\mathbb{M}_1 \models \sigma \iff \mathbb{M}_2 \models \sigma.$$

Notation 7.3. $\forall^{\infty}$ *means for all but finitely many $n \in \omega$.*

In 8.3, see blow, we will reconstruct the following:

[21]Recall from Lemma 6.8 that $\varphi^{\mathfrak{e}}_n(\mathbb{M}) \equiv \psi^{\mathfrak{e}}_n(\mathbb{M})$ holds for all $n < \omega$.

Theorem 7.4. *Let $\lambda = (\mu^{\aleph_0})^+ > |\mathbf{R}|$ be a regular cardinal. Then there are $\mathbf{R}$-modules $\mathbb{M}_1$ and $\mathbb{M}_2$ of cardinality λ such that:*

(1) $\mathbb{M}_1$, $\mathbb{M}_2$ are not isomorphic,

(2) $\mathbb{M}_1$ is isomorphic to a direct summand of $\mathbb{M}_2$,

(3) $\mathbb{M}_2$ is isomorphic to a direct summand of $\mathbb{M}_1$.

Proof. Let $\mathbf{T}$ be the subring of $\mathbf{R}$ which 1 generates. As before, we need to choose a ring $\mathbf{S}$ (essentially the ring of endomorphisms we would like).

Step A): To make things easier, we first introduce a ring $\mathbf{S}_0$ which is easy compared to $\mathbf{S}$.

Let A_1 (resp. A_{-1}) be the set of even (resp. odd) integers and let f be the following function:

$$f(i) := \begin{cases} i+1 & \text{if } i \geq 0, \\ i-1 & \text{if } i < 0. \end{cases}$$

Thus, f maps A_1 (resp. A_{-1}) into A_{-1} (resp. A_1). Also, $A_1 \setminus \mathrm{Rang}(f \restriction A_{-1}) = \{0\}$ and $A_{-1} \setminus \mathrm{Rang}(f \restriction A_1) = \{-1\}$. Let i vary on the integers. Let $\mathbf{S}_0$ be the ring extending $\mathbf{T}$ generated freely by $\{\mathcal{X}_1, \mathcal{X}_{-1}, \mathcal{W}_1, \mathcal{W}_{-1}, \mathcal{Z}_1, \mathcal{Z}_{-1}\}$. Let $\mathbf{D}$ be a field such that if $\|\mathbf{T}\|$ is finite, then the characteristic of $\mathbf{D}$ is finite and divides $\|\mathbf{T}\|$. Then, we set $\mathbf{S}^*_{\mathbf{D}} = \mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}_0$.

We are going to define a right $(\mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}_0)$-module $\mathbb{M}^*_{\mathbf{D}^*}$. We first equip a left $\mathbf{D}$-module structure over $\mathbb{M} := \sum\{\mathbf{D}x_i : i \in \mathbb{Z}\}$ via the following rule

$$b(\sum_i a_i x_i) := \sum_i (ba_i)x_i,$$

where a_i and b are in $\mathbf{D}$. As mentioned earlier, we like to make $\mathbb{M}$ a structure of right $(\mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}_0)$–module over $\mathbb{M}$ which will be called $\mathbb{M}^*_{\mathbf{D}}$. Let $x \in \mathbb{M}$ and $c \in \mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}_0$. In order to define xc, as $\mathbf{D}$ and $\mathbf{S}_0$ commute, it is enough to define it for $x = x_i$ and $c \in \{\mathcal{X}_1, \mathcal{X}_{-1}, \mathcal{W}_1, \mathcal{W}_{-1}, \mathcal{Z}_1, \mathcal{Z}_{-1}\}$. In sum, the desired scaler multiplication, can be completed if we follow the following table of assignments:

$$x_i\mathcal{X}_1 := \begin{cases} x_i & \text{if } i \in A_1 \\ 0 & \text{if } i \in A_{-1}, \end{cases}$$

$$x_i\mathcal{X}_{-1} := \begin{cases} 0 & \text{if } i \in A_1 \\ x_i & \text{if } i \in A_{-1}, \end{cases}$$

$$x_i\mathcal{W}_1 := x_{f(i)},$$

$$x_i\mathcal{W}_{-1} := \begin{cases} x_{f^{-1}(i)} & \text{if } i \in \mathrm{Rang}(f) \\ 0 & \text{otherwise}, \end{cases}$$

$$x_i\mathcal{Z}_1 := \begin{cases} x_i & \text{if } i \in A_1 \cap (\mathbb{Z} \setminus \{0\}) \\ 0 & \text{otherwise}, \end{cases}$$

and

$$x_i\mathcal{Z}_{-1} := \begin{cases} x_i & \text{if } i \in A_{-1} \cap (\mathbb{Z} \setminus \{0\}) \\ 0 & \text{otherwise}. \end{cases}$$

Recall that we equipped $\mathbb{M}^*_{\mathbf{D}}$ with a structure of right $\mathbf{S}^*_{\mathbf{D}}$–module. Let $g^*_{\mathbf{D}}$ be the natural ring homomorphism from $\mathbf{S}_0$ into $\mathbf{S}^*_{\mathbf{D}}$. By using $g^*_{\mathbf{D}}$, the $\mathbf{S}^*_{\mathbf{D}}$–module $\mathbb{M}^*_{\mathbf{D}}$ becomes a right $\mathbf{S}_0$-module.

Step B): In this step we define the auxiliary ring $\mathbf{S}$.

Let $\mathbf{S}$ be the ring with 1, associative but not necessarily commutative, extending $\mathbf{T}$ generated by $\mathcal{X}_1$, $\mathcal{X}_{-1}$, $\mathcal{W}_1$, $\mathcal{W}_{-1}$, $\mathcal{Z}_1$, $\mathcal{Z}_{-1}$ freely except the following list of test equations (to understand them see below):

($*$) $\sigma = 0$ if σ is a term[22], $\mathbb{M}^*_{\mathbf{D}}\sigma = 0$ for $\mathbb{M}^*_{\mathbf{D}}$ as defined in Step A), for every field $\mathbf{D}$ such that if $\mathbf{T}$ is of finite cardinality n, then $\mathrm{char}(\mathbf{D})|n$.

In the course of proof, we need some explicit test equations. Let us drive some of them from ($*$):

[22]i.e., in the language of rings, in the variables $\mathcal{X}_1, \mathcal{X}_{-1}, \mathcal{W}_1, \mathcal{W}_{-1}, \mathcal{Z}_1, \mathcal{Z}_{-1}$.

$(\star)_1$: $\mathcal{X}_1^2 = \mathcal{X}_1$,

$\mathcal{X}_{-1}^2 = \mathcal{X}_{-1}$,

$\mathcal{X}_1 + \mathcal{X}_{-1} = 1$,

$\mathcal{X}_1\mathcal{X}_{-1} = \mathcal{X}_{-1}\mathcal{X}_1 = 0$,

$\mathcal{Z}_1^2 = \mathcal{Z}_1$, $\mathcal{Z}_1\mathcal{X}_1 = \mathcal{Z}_1 = \mathcal{X}_1\mathcal{Z}_1$,

$\mathcal{X}_{-1}\mathcal{W}_1 = (\mathcal{X}_{-1}\mathcal{W}_1)\mathcal{X}_1\mathcal{Z}_1$,

$\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1} = (\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1})\mathcal{X}_{-1}$,

$\mathcal{X}_{-1}\mathcal{W}_1\mathcal{W}_{-1} = \mathcal{X}_{-1}$, and

$\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1}\mathcal{W}_1 = \mathcal{Z}_1 = \mathcal{X}_1\mathcal{Z}_1$.

Let us show, for example, that $\mathcal{X}_1 + \mathcal{X}_{-1} = 1$. We need to show the left hand side is the identity map when we consider it as an endomorphism of $\mathbb{M}^*_{\mathbf{D}}$. To this end, we evaluate $\mathcal{X}_1 + \mathcal{X}_{-1}$ at any generator of $\mathbb{M}^*_{\mathbf{D}}$, say x_i. Recall from

$$x_i\mathcal{X}_1 = \begin{cases} x_i & \text{if } i \in A_1, \\ 0 & \text{if } i \in A_{-1}, \end{cases}$$

and

$$x_i\mathcal{X}_{-1} = \begin{cases} 0 & \text{if } i \in A_1, \\ x_i & \text{if } i \in A_{-1} \end{cases}$$

that $x_i(\mathcal{X}_1 + \mathcal{X}_{-1}) = x_i$. Hence, $\mathcal{X}_1 + \mathcal{X}_{-1} = 1$, as claimed. The other relations will follow in the same way.

Step C): In this step, we introduce the $\mathbf{R}$-modules $\mathbb{M}_1$ and $\mathbb{M}_2$ of cardinality λ such that:

(i) $\mathbb{M}_1$ is isomorphic to a direct summand of $\mathbb{M}_2$ and

(ii) $\mathbb{M}_2$ is isomorphic to a direct summand of $\mathbb{M}_1$.

Let $\langle \mathbb{M}_\alpha : \alpha \leq \lambda \rangle$ be a strongly semi-nice construction for $(\lambda, \mathfrak{m}, S, \lambda)$ and let $\mathbb{M} := \mathbb{M}_\lambda$. Recall that such a thing exists. Let $\mathbb{M}_1 := \mathbb{M}\mathcal{X}_1$ and $\mathbb{M}_{-1} := \mathbb{M}\mathcal{X}_{-1}$. Since $\mathcal{X}_1\mathcal{X}_{-1} = \mathcal{X}_{-1}\mathcal{X}_1 = 0$, we have $\mathbb{M}_1 \cap \mathbb{M}_{-1} = 0$. Thanks to the formula

$\mathcal{X}_1 + \mathcal{X}_{-1} = 1$,

$$\mathbb{M} = \mathbb{M}(\mathcal{X}_1 + \mathcal{X}_{-1}) = \mathbb{M}\mathcal{X}_1 + \mathbb{M}\mathcal{X}_{-1} = \mathbb{M}_1 + \mathbb{M}_{-1} = \mathbb{M}_1 \oplus \mathbb{M}_{-1}.$$

Then $\mathbb{M}_1$ and $\mathbb{M}_{-1}$ are equipped with $\mathbf{R}$-module structure, and there is the identification $\mathbb{M} = \mathbb{M}_1 \oplus \mathbb{M}_{-1}$. We shall show that $\mathbb{M}_1$, $\mathbb{M}_{-1}$ are as required in Theorem 7.4 (with respect to $\mathbb{M}_1$ and $\mathbb{M}_2$).

The relations $\mathcal{Z}_1^2 = \mathcal{Z}_1$ and $\mathcal{Z}_1\mathcal{X}_1 = \mathcal{Z}_1 = \mathcal{X}_1\mathcal{Z}_1$ imply that

$$\mathbb{M}_1 = \mathbb{M}_1(1 - \mathcal{Z}_1) \oplus \mathbb{M}_1\mathcal{Z}_1,$$

i.e., $\mathbb{M}_1\mathcal{Z}_1$ is a direct summand of $\mathbb{M}_1$. Since $\mathcal{X}_{-1}\mathcal{W}_1 = (\mathcal{X}_{-1}\mathcal{W}_1)\mathcal{X}_1\mathcal{Z}_1$, we have

$$\mathbb{M}_{-1}\mathcal{W}_1 = \mathbb{M}\mathcal{X}_{-1}\mathcal{W}_1 = \mathbb{M}(\mathcal{X}_{-1}\mathcal{W}_1)\mathcal{X}_1\mathcal{Z}_1 \subset \mathbb{M}\mathcal{X}_1\mathcal{Z}_1 = \mathbb{M}_1\mathcal{Z}_1.$$

Thus, $\mathcal{W}_1$ maps $\mathbb{M}_{-1}$ into $\mathbb{M}_1\mathcal{Z}_1$. In the light of the formula $\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1} = (\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1})\mathcal{X}_{-1}$ we observe that

$$\mathbb{M}_1\mathcal{Z}_1\mathcal{W}_{-1} = \mathbb{M}\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1} = \mathbb{M}\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1}\mathcal{X}_{-1} \subset \mathbb{M}\mathcal{X}_{-1} = \mathbb{M}_{-1}.$$

In other words, $\mathcal{W}_{-1}$ maps $\mathbb{M}_1\mathcal{Z}_1$ into $\mathbb{M}_{-1}$. We are going to combine the formula $\mathcal{X}_{-1}\mathcal{W}_1\mathcal{W}_{-1} = \mathcal{X}_{-1}$ along with $\mathcal{X}_1\mathcal{Z}_1\mathcal{W}_{-1}\mathcal{W}_1 = \mathcal{Z}_1 = \mathcal{X}_1\mathcal{Z}_1$ to deduce that the multiplication maps by $\mathcal{W}_1$ and $\mathcal{W}_{-1}$ are the inverse of each other. This implies that $\mathbb{M}_{-1}$ is isomorphic to a direct summand of $\mathbb{M}_1$ (as left $\mathbf{R}$-modules). Similarly, we obtain:

$$\mathbb{M}_{-1} = \mathbb{M}_{-1}(1 - \mathcal{Z}_{-1}) \oplus \mathbb{M}_{-1}\mathcal{Z}_{-1}.$$

So, $\mathbb{M}_{-1}\mathcal{Z}_{-1}$ is a direct summand of $\mathbb{M}_{-1}$ and $\mathbb{M}_{-1}\mathcal{Z}_{-1}$ is isomorphic to $\mathbb{M}_1$. By the same argument, $\mathbb{M}_1$ is isomorphic to a direct summand of $\mathbb{M}_2$ as left $\mathbf{R}$-modules. In summary, we showed that

- $\mathbb{M}_1 \cong \mathbb{M}_1(1 - \mathcal{Z}_1) \oplus \mathbb{M}_{-1}$ and
- $\mathbb{M}_{-1} \cong \mathbb{M}_{-1}(1 - \mathcal{Z}_{-1}) \oplus \mathbb{M}_1$.

This completes the proof of Step C).

It remains to show that $\mathbb{M}_1 \ncong \mathbb{M}_{-1}$. Suppose on the contrary that they are isomorphic, and we get a contradiction, which is presented at Step H) below. Let $\mathfrak{e} \in \mathfrak{E}^{\mathfrak{m}}$ be simple.

Step D): There is a solution $\mathcal{Y} \in dE_n^{\mathfrak{e}}$ to the following equations:

$$(*)_2\textbf{:} \qquad \mathcal{X}_1\mathcal{Y}\mathcal{X}_{-1} = \mathcal{X}_1\mathcal{Y}, \quad \mathcal{X}_{-1}\mathcal{Y}\mathcal{X}_1 = \mathcal{X}_{-1}\mathcal{Y}, \quad \mathcal{Y}\mathcal{Y} = 1.$$

To see this, recall that $\mathbb{M}_1 \cong \mathbb{M}_{-1}$ and $\mathbb{M} = \mathbb{M}_1 \oplus \mathbb{M}_{-1}$. Let h be an isomorphism from $\mathbb{M}_1$ onto $\mathbb{M}_{-1}$. Define $\mathbf{f} : \mathbb{M} \to \mathbb{M}$ by

$$(a,b) \in \mathbb{M} = \mathbb{M}_1 \oplus \mathbb{M}_{-1} \implies \mathbf{f}(a,b) = (h^{-1}(b), h(a)).$$

So, $\mathbf{f} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M})$ and it satisfies $\mathbf{f} \restriction \mathbb{M}_1 = h$ and $\mathbf{f} \restriction \mathbb{M}_{-1} = h^{-1}$. The member in $dE_n^{\mathfrak{e}}$ which $\mathbf{f}$ induces solves the equations in $(*)_2$. This completes the proof of Step D).

In the light of Corollary 6.33, it is enough to prove the following two items:

(a) in $\mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}_0$ there is no solution to $(*)_2$, in particular, there is no such $\mathcal{Y}$. Note that $\mathbf{S}^*_{\mathbf{D}}$ have the same characteristic as $\mathbf{D}$.

(b) $\mathbf{S}_0$ is a free $\mathbf{T}$–module.

Clearly $\mathbf{S}$ is a $\mathbf{T}$-module, generated by the set of monomials in

$$\{\mathcal{X}_1, \mathcal{X}_{-1}, \mathcal{W}_1, \mathcal{W}_{-1}, \mathcal{Z}_1, \mathcal{Z}_{-1}\}.$$

Our aim is to show that $\mathbf{S}$ is a free $\mathbf{T}$-module; in fact we shall exhibit explicitly a free basis. For $\ell \in \{1, -1\}$, $k \in \mathbb{Z}$, $n \geq 0$, $n \geq -k$, we define an endomorphism $\mathbf{f}^{\ell}_{k,n}$ of $\mathbb{M}^*_{\mathbf{D}}$ by

$$\mathbf{f}^{\ell}_{k,n}(x_i) := \begin{cases} x_{f^k(i)} & \text{if } f^{-n}(i) \text{ is well defined and } x_i \in A_\ell, \\ 0 & \text{otherwise} \end{cases}$$

It is easy to see that it is an endomorphism of $\mathbb{M}^*_{\mathbf{D}}$ as a left $\mathbf{D}$-module. We will define a monomial $\mathcal{Y}^{\ell}_{k,n}$, in the following way. For every monomial σ let σ^0 be

$1 = \mathrm{id}_{\mathbb{M}^*_{\mathbf{D}}}$ and remember $n \geq -k$ so $n + k \geq 0$. Now set

$$\mathcal{Y}^\ell_{k,n} = \mathcal{X}_\ell (\mathcal{W}_{-1})^n \mathcal{W}_1^{n+k}.$$

It is easy to see that the operation of $\mathcal{Y}^\ell_{k,n}$ on $\mathbb{M}^*_{\mathbf{D}}$ by right multiplication, is equal to $\mathbf{f}^\ell_{k,n}$.

Let

$$\mathcal{G} := \{\mathcal{Y}^\ell_{k,n} : (\ell, k, n) \in w\}$$

where $w = \Big\{(\ell, k, n) : \ \ell \in \{1, -1\}, \ k \in \mathbb{Z}, \ n \geq 0, \ k + n \geq 0\Big\}$.

In the next step, we show that $\mathcal{G}$ generates $\mathbf{S}$ as a $\mathbf{T}$-module.

Step E): The set $\mathcal{G}$ generates $\mathbf{S}$ as a $\mathbf{T}$-module.

Indeed, it is enough to show that for every monomial σ, some equation $\sigma = \sum a^\ell_{n,k} \mathcal{Y}^\ell_{k,n}$ holds in $\mathbf{S}$, where $\{(\ell, n, k) : a^\ell_{n,k} \neq 0\}$ is finite and $a^\ell_{k,n} \in \mathbf{T}$, i.e., it holds in the endomorphism ring of $\mathbb{M}^*_{\mathbf{D}}$. We prove this by induction on the length of the monomial σ.

If the length is zero, σ is 1. Recall from $(\star)_1$ that $1 = \mathcal{X}_1 + \mathcal{X}_{-1}$. By definition, $\mathcal{X}_\ell = \mathcal{Y}^\ell_{0,0}$; so $1 = \mathcal{Y}^1_{0,0} + \mathcal{Y}^{-1}_{0,0}$ as required.

If the length is > 0, by the induction hypothesis it is enough to prove the following:

$(*)$ Let $\tau \in \{\mathcal{X}_1, \mathcal{X}_{-1}, \mathcal{W}_1, \mathcal{W}_{-1}, \mathcal{Z}_1, \mathcal{Z}_{-1}\}$. Then $\mathcal{Y}^{\ell(*)}_{k(*),n(*)} \tau$ is equal to some $\sum\limits_{\ell,k,n} a^\ell_{k,n} \mathcal{Y}^\ell_{k,n}$.

Indeed, it is enough to check equality on the generators of $\mathbb{M}^*_{\mathbf{D}}$, that is the x_i's. The proof of $(*)$ is divided into the following three cases:

Case 1: $\mathcal{Y}^{\ell(*)}_{k(*),n(*)} \mathcal{W}_\ell$ is:

$$\begin{array}{rll} \mathcal{Y}^{\ell(*)}_{k(*)+1,n(*)} & \text{if} & \ell = 1 \\ \mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)} & \text{if} & \ell = -1 \text{ and } \ k(*) + n(*) > 0, \\ \mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)+1} & \text{if} & \ell = -1 \text{ and } \ k(*) + n(*) = 0. \end{array}$$

First assume that $\ell = 1$. Then

$$\begin{aligned}\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{W}_\ell &= \mathcal{X}_1(\mathcal{W}_{-1})^n\mathcal{W}_1^{n+k}\mathcal{W}_1\\ &= \mathcal{X}_1(\mathcal{W}_{-1})^n\mathcal{W}^{n+k+1}\\ &= \mathcal{Y}^{\ell(*)}_{k(*)+1,n(*)}.\end{aligned}$$

Now assume that $\ell = -1$ and $k(*)+n(*) > 0$. If $i \in \mathrm{Rang}(f)$, then $x_i\mathcal{W}_1\mathcal{W}_{-1} = x_i$. From this,

$$x_i\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{W}_\ell = x_i\mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)}.$$

If $i \notin \mathrm{Rang}(f)$, then $i \in \{0,1\}$. It is easy to see that

$$x_0\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{W}_{-1} = x_0\mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)}$$

and

$$x_{-1}\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{W}_{-1} = x{-}1\mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)}.$$

Thus, the functions $\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{W}_{-1}$ and $\mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)}$ are the same. Finally, assume that $\ell = -1$ and $k(*)+n(*) = 0$. Then

$$\begin{aligned}\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{W}_{-1} &= \mathcal{X}_{-1}(\mathcal{W}_{-1})^n\mathcal{W}_1^{n+k}\mathcal{W}_{-1}\\ &= \mathcal{X}_{-1}(\mathcal{W}_{-1})^n\mathcal{W}_{-1}\\ &= \mathcal{X}_{-1}(\mathcal{W}_{-1})^{n+1}\\ &= \mathcal{Y}^{\ell(*)}_{k(*)-1,n(*)+1}.\end{aligned}$$

This completes the argument.

Case 2: $\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{X}_\ell$ is:

$$\text{zero} \quad \text{if} \quad [\ell(*) = \ell \iff k(*) \text{ odd}],$$

$$\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\text{if} \quad [\ell(*) = \ell \iff k(*) \text{ even}].$$

The proof of this is similar to Case 1.

Case 3: $\mathcal{Y}^{\ell(*)}_{k(*),n(*)}\mathcal{Z}_\ell$ is:

$$\begin{array}{rl} \mathcal{Y}^{\ell(*)}_{k(*),n(*)} & \text{if} \quad n(*)+k(*)>0 \text{ and } [\ell(*)=\ell \iff k(*) \text{ even}],\\ Y^{\ell(*)}_{k(*),n(*)+1} & \text{if} \quad n(*)+k(*)=0 \text{ and } [\ell(*)=\ell \iff k(*) \text{ even}],\\ \text{zero} & \text{if} \quad [\ell(*)=\ell \iff k(*) \text{ odd}]. \end{array}$$

It is enough to check equality on the generators of $\mathbb{M}^*_{\mathbf{D}}$, that is the x_i's. The proof is again similar to the proof of Case 1.

Step F): The set $\mathcal{G}=\{\mathcal{Y}^\ell_{k,n}:(\ell,k,n)\in w\}$ generates $\mathbf{S}$ freely as a $\mathbf{T}$-module.

Indeed, in the light of Step E) we see that $\mathcal{G}$ generates $\mathbf{S}$ as a $\mathbf{T}$-module. Toward a contradiction suppose that

$$0=\sum\{a^\ell_{k,n}\mathcal{Y}^\ell_{k,n}:(\ell,k,n)\in w\},$$

when we view both sides in $\mathbf{S}$, where $w\subseteq w^*$ is finite, $a^\ell_{k,n}\in\mathbf{T}$ and not all of them are zero. If $n=|\mathbf{T}|$ is finite, we take the field $\mathbf{D}$ such that $\mathrm{char}(\mathbf{D})|n$, and some $a^\ell_{k,n}$ is not zero in $\mathbf{D}$. Hence,

$$0=\sum\{a^\ell_{k,n}:(\ell,k,n)\in w\}\in\mathrm{End}_{\mathbf{D}}(\mathbb{M}^*_{\mathbf{D}}),$$

where $a^\ell_{k,n}\in\mathbf{D}$ and $w\subseteq w^*$ is finite. We shall prove that $a^\ell_{k,n}=0$ for every $(\ell,k,n)\in w$.

If $i\in A_1$ and $i\geq 0$, then

$$\begin{aligned} 0 &= x_i(\sum_{(\ell,k,n)\in w} a^\ell_{k,n}\mathcal{Y}^\ell_{k,n})\\ &= \sum_{(\ell,k,n)\in w} a^\ell_{k,n}(x_i\mathcal{Y}^\ell_{k,n})\\ &= \sum_{(\ell,k,n)\in w}\{a^\ell_{k,n}x_{i+k}:\ell=1, \quad \text{and } n\leq i\}\\ &= \sum_{j\geq 0}(\sum_{(1,k,n)\in w}\{a^1_{k,n}: \quad i\geq n,\ i+k=j\})x_j\\ &= \sum_{j\geq 0}(\sum_{(1,j-i,n)\in w}\{a^1_{j-i,n}:\ i\geq n,\ \})x_j. \end{aligned}$$

Hence, for every $i\in A_1$, $i\geq 0$ and $j\geq 0$ we have

$(*)^a_{i,j}$: $$0=\sum\{a^1_{j-i,n}:\ n\geq 0,\ n\leq i \text{ and } n+(j-i)\geq 0\}.$$

Similarly, for $i \in A_{-1}$, $i \geq 0$ (equivalently, $i > 0$ as $i \in A_{-1} \Rightarrow i \neq 0$) and $j \geq 0$ we can prove

$$(*)^b_{i,j}\textbf{:} \qquad 0 = \sum\{a^{-1}_{j-i,n} :\ n \geq 0,\ n \leq i \text{ and } n + (j-i) \geq 0\}.$$

Similarly, for $i \in A_1$, $i < 0$,

$$\begin{aligned} 0 &= x_i\Big(\sum_{(\ell,k,n)\in w} a^\ell_{k,n}\mathcal{Y}^\ell_{k,n}\Big) \\ &= \sum_{(\ell,k,n)\in w} a^\ell_{k,n}(x_i\mathcal{Y}^\ell_{k,n}) \\ &= \sum_{(1,k,n)\in w} \{a^1_{k,n}x_{i-k} :\ -i > n\} \\ &= \sum_{j<0}\Big(\textstyle\sum_{(1,i-j,n)\in w}\{a^1_{i-j,n} :\ n < -i\}\Big)x_j. \end{aligned}$$

Thus,

$$(*)^c_{i,j}\textbf{:} \qquad 0 = \sum\{a^1_{i-j,n} : n \geq 0 \text{ and } n + (i-j) \geq 0 \text{ and } n < -i\}$$

for every negative $i \in A_1$ and $j < 0$. Similarly, for every $i \in A_{-1}$, $i < 0$ and $j < 0$

$$(*)^d_{i,j}\textbf{:} \qquad 0 = \sum\{a^{-1}_{i-j,n} :\ n \geq 0 \text{ and } n + (i-j) \geq 0 \text{ and } n < -i\}.$$

Choose, if possible, (k,m) such that:

(1) $(1,k,m)$ belongs to w,

(2) $a^1_{k,m} \neq 0$,

(3) m is minimal under (1)+(2).

Note that $m \geq 0$ by the definition of w. First assume that m is even. Let $i = m$ and $j = i + k$. So $i \in A_1$ (being even), $i \geq 0$ and $j = m + k$ is ≥ 0 as $(1,k,m) \in w$. In the equation $(*)^a_{i,j}$ the term $a^1_{k,m}$ appears in the sum, and for every other term $a^1_{k_1,m_1}$ which appears in the sum, we have $m_1 < m$ (and $k_1 = k$), and hence by (3) above is zero. It follows that $a^1_{k,m}$ is zero, a contradiction.

If m is odd, we get a similar contradiction using $(*)^c_{i,j}$. Let $i = -m-1$ and $j = i - k$. Note that $m \geq 0$, hence $i < 0$ and i is even as m is odd, so $i \in A_1$. Also,

$$j = i - k = -m - 1 - k \leq -1 < 0.$$

Recalling $k+m \geq 0$ as $(1,k,m) \in w^*$. In the equation $(*)^c_{i,j}$, the term $a^1_{i-j,n} = a^1_{k,n}$ appears in the sum if and only if

i) $0 \leq n < -i = m+1$, and

ii) $n + (i-j) = n+k \geq 0$

(but if the later fails, $a^1_{k,m}$ is not defined). So, $a^1_{k,m}$ appears, and if another term $a^1_{k_1,m_1}$ occurs then $m_1 \leq m$ (and $k_1 = k$). Hence, $m_1 < m$, and so $a^1_{k_1,m_1} = 0$. Necessarily, $a^1_{k,m}$ is zero, a contradiction. In sum, $a^1_{k,n} = 0$ whenever it is defined.

In the same vein, $a^{-1}_{k,n} = 0$ whenever it is defined (use $(*)^b_{i,j} + (*)^d_{i,j}$). So, $\mathbf{S}_0$ is a free module over $\mathbf{T}$, as required.

Step G): In this step we get a contradiction that we searched for it. This will show that $\mathbb{M}_1 \ncong \mathbb{M}_{-1}$.

To this end, recall that there are finitely many nonzero $a^\ell_{k,n} \in \mathbf{D}$ such that

$$\boxtimes\text{:} \qquad \mathcal{Y} = \sum \left\{ a^\ell_{k,n} \mathcal{Y}^\ell_{k,n} : \ n \geq 0 \text{ and } k+n \geq 0 \text{ and } \ell \in \{1,-1\} \right\}.$$

Let $n(*) < \omega$ be such that

$$a^\ell_{k,n} \neq 0 \quad \Rightarrow \quad |k|, n < n(*).$$

For $\ell = 1, -1$ let

$$\mathbb{M}^{\text{pos}}_\ell := \Big\{ \sum_{i \geq 0} d_i x_i : d_i \in \mathbf{D}, \quad \forall^\infty d_i = 0 \text{ and } d_i \neq 0 \ \Rightarrow \ i \in A_\ell \Big\},$$
$$\mathbb{M}^{\text{neg}}_\ell := \Big\{ \sum_{i < 0} d_i x_i : d_i \in \mathbf{D}, \quad \forall^\infty d_i = 0 \text{ and } d_i \neq 0 \ \Rightarrow \ i \in A_\ell \Big\}.$$

Clearly as a $\mathbf{D}$-module

$$\mathbb{M}^*_{\mathbf{D}} = \mathbb{M}^{\text{pos}}_1 \oplus \mathbb{M}^{\text{pos}}_{-1} \oplus \mathbb{M}^{\text{neg}}_1 \oplus \mathbb{M}^{\text{neg}}_{-1}.$$

Let $\mathcal{Y}^{\text{pos}}_\ell := \mathcal{Y} \restriction \mathbb{M}^{\text{pos}}_\ell$ and $\mathcal{Y}^{\text{neg}}_\ell := \mathcal{Y} \restriction \mathbb{M}^{\text{neg}}_\ell$ for $\ell \in \{1,-1\}$.

Now each $\mathcal{Y}^\ell_{k,n}$ maps $\mathbb{M}^{\text{pos}} = \mathbb{M}^{\text{pos}}_1 \oplus \mathbb{M}^{\text{pos}}_{-1}$ to itself, and $\mathbb{M}^{\text{neg}} = \mathbb{M}^{\text{neg}}_1 \oplus \mathbb{M}^{\text{neg}}_{-1}$ to itself, and hence by $\boxtimes$ above also $\mathcal{Y}$ does it. According to $(*)_2$ from Step B) we have $\mathcal{X}_1 \mathcal{Y} \mathcal{X}_{-1} = \mathcal{X}_1 \mathcal{Y}$. This implies that

$$\mathbb{M}_1 \mathcal{Y} = \mathbb{M} \mathcal{X}_1 \mathcal{Y} = \mathbb{M} \mathcal{X}_1 \mathcal{Y} \mathcal{X}_{-1} \subset \mathbb{M} \mathcal{X}_{-1} = \mathbb{M}_{-1},$$

i.e., $\mathcal{Y}$ maps $\mathbb{M}_1$ to $\mathbb{M}_{-1}$. Thus, by the previous sentence, $\mathcal{Y}$ maps $\mathbb{M}_1^{\text{pos}}$ into $\mathbb{M}_{-1}^{\text{pos}}$, and $\mathbb{M}_1^{\text{neg}}$ into $\mathbb{M}_{-1}^{\text{neg}}$, i.e., $\mathcal{Y}_1^{\text{pos}}$ (resp. $\mathcal{Y}_1^{\text{neg}}$) is into $\mathbb{M}_{-1}^{\text{pos}}$ (resp. $\mathbb{M}_{-1}^{\text{neg}}$).

By the same reasoning, and in view of $(*)_2$ we deduce that $\mathcal{X}_{-1}\mathcal{Y}\mathcal{X}_1 = \mathcal{X}_{-1}\mathcal{Y}$. Hence, $\mathcal{Y}$ maps $\mathbb{M}_{-1}^{\text{pos}}$ into $\mathbb{M}_1^{\text{pos}}$ and $\mathbb{M}_{-1}^{\text{neg}}$ into $\mathbb{M}_1^{\text{neg}}$. Also, the mappings $\mathcal{Y}_1^{\text{pos}}$, $\mathcal{Y}_{-1}^{\text{pos}}$, $\mathcal{Y}_1^{\text{neg}}$, $\mathcal{Y}_{-1}^{\text{neg}}$ are endomorphisms of $\mathbf{D}$-modules. As $\mathcal{Y}^2 = 1$ (again by $(*)_2$) we conclude that $\mathcal{Y}_1^{\text{pos}}$ and $\mathcal{Y}_{-1}^{\text{pos}}$ are the inverse of each other, so both of them are isomorphisms. Similarly for $\mathcal{Y}_1^{\text{neg}}$ and $\mathcal{Y}_{-1}^{\text{neg}}$.

Let

$$\mathbb{M}_1^{\text{stp}} := \{\sum_{i>0} d_i x_i : d_i \in \mathbf{D}, \forall^\infty d_i = 0 \textit{ and } d_i \neq 0 \ \Rightarrow \ i \in A_1\}.$$

Clearly, $\mathbb{M}_1^{\text{stp}}$ is a sub $\mathbf{D}$-module of $\mathbb{M}_1^{\text{pos}}$. Note that $x_0 \in \mathbb{M}_1^{\text{pos}} \setminus \mathbb{M}_1^{\text{stp}}$. This yields the difference between $\mathbb{M}_1^{\text{stp}}$ and $\mathbb{M}_1^{\text{pos}}$.

Let

$$\mathbb{N} := \{\sum_{i>n(*)} d_i x_i : \quad d_i \in \mathbf{D}, \forall^\infty d_i = 0 \text{ and } d_i \neq 0 \ \Rightarrow \ i \in A_1\}.$$

Let $H^{\text{pos}} : \mathbb{M}_1^{\text{stp}} \longrightarrow \mathbb{M}_1^{\text{neg}}$ (resp. $H^{\text{neg}} : \mathbb{M}_{-1}^{\text{neg}} \longrightarrow \mathbb{M}_{-1}^{\text{stp}}$) be defined by the assignment $x_i H^{\text{pos}} = x_{-i}$ (resp. $x_i H^{\text{neg}} = x_{-i}$). Both of them are isomorphisms of $\mathbf{D}$-modules. Note that $\mathcal{Y}_1^{\text{pos}}$ is an isomorphism from $\mathbb{M}_1^{\text{pos}}$ onto $\mathbb{M}_{-1}^{\text{pos}}$ and $H^{\text{pos}}\mathcal{Y}_1^{\text{neg}}H^{\text{neg}}$ is an isomorphism from $\mathbb{M}_1^{\text{stp}}$ onto $\mathbb{M}_{-1}^{\text{pos}}$. Note that

$$\mathbb{M}_1^{\text{stp}} \xrightarrow{H^{\text{pos}}} \mathbb{M}_1^{\text{neg}} \xrightarrow{\mathcal{Y}_1^{\text{neg}}} \mathbb{M}_{-1}^{\text{neg}} \xrightarrow{H^{\text{neg}}} \mathbb{M}_{-1}^{\text{pos}}.$$

We claim that

$$\mathcal{Y}_1^{\text{pos}} \restriction \mathbb{N} = (H^{\text{pos}}\mathcal{Y}_1^{\text{neg}}H^{\text{neg}}) \restriction \mathbb{N}.$$

To see this, it is enough to check equality on the generators of $\mathbb{N}$, that is over the x_i's where i is even and it is larger than $n(*)$. In particular, by choosing $n(*)$ large enough, we may assume that $i \gg 0$. Recall that

$$\mathcal{Y} = \sum \left\{ a_{k,n}^\ell \mathcal{Y}_{k,n}^\ell : \ n \geq 0 \text{ and } k + n \geq 0 \text{ and } \ell \in \{1, -1\} \right\}.$$

By definition $x_i\mathcal{X}_{-1} = 0$. Then

$$\begin{aligned}
x_i(H^{\mathrm{pos}}\mathcal{Y}_1^{\mathrm{neg}}H^{\mathrm{neg}}) &= x_{-i}(\sum\{a^\ell_{k,n}\mathcal{Y}^\ell_{k,n}\})H^{\mathrm{neg}}\\
&= \sum a^\ell_{k,n}x_{-i}(\mathcal{X}_1\mathcal{W}^n_{-1}\mathcal{W}^{n+k}_1)H^{\mathrm{neg}}\\
&= \sum a^1_{k,n}(x_{-i}\mathcal{W}^n_{-1}\mathcal{W}^{n+k}_1)H^{\mathrm{neg}}\\
&= \sum a^1_{k,n}(x_{f^{-n}(-i)}\mathcal{W}^{n+k}_1)H^{\mathrm{neg}}\\
&= \sum a^1_{k,n}(x_{f^{n+k}(f^{-n}(-i))})H^{\mathrm{neg}}\\
&= \sum a^1_{k,n}(x_{f^k(-i)})H^{\mathrm{neg}}\\
&= \sum a^1_{k,n}(x_{-i-k})H^{\mathrm{neg}}\\
&= \sum a^1_{k,n}x_{i+k}.
\end{aligned}$$

Also, $(x_i)\mathcal{Y}_1^{\mathrm{pos}} \restriction \mathbb{N} = \sum a^1_{k,n}x_{i+k}$. Thus we have $\mathcal{Y}_1^{\mathrm{pos}} \restriction \mathbb{N} = (H^{\mathrm{pos}}\mathcal{Y}_1^{\mathrm{neg}}H^{\mathrm{neg}}) \restriction \mathbb{N}$, as claimed.

Let $\mathbb{N}^* := \mathrm{Rang}(\mathcal{Y}_1^{\mathrm{pos}} \restriction \mathbb{N})$. Then $\mathbb{N}^* = \mathrm{Rang}((H^{\mathrm{pos}}\mathcal{Y}_1^{\mathrm{neg}}H^{\mathrm{neg}}) \restriction \mathbb{N})$. So, as $\mathcal{Y}_1^{\mathrm{pos}}$ is an isomorphism from $\mathbb{M}_1^{\mathrm{pos}}$ onto $\mathbb{M}_{-1}^{\mathrm{pos}}$, and $\mathbb{N} \subseteq \mathbb{M}_1^{\mathrm{pos}}$ we have that $\mathbb{N}^*$ is a $\mathbf{D}$-submodule of $\mathbb{M}_{-1}^{\mathrm{pos}}$. This means that $\mathbb{M}_{-1}^{\mathrm{pos}}/\mathbb{N}^*$ is isomorphic to $\mathbb{M}_1^{\mathrm{pos}}/\mathbb{N}$ (as $\mathbf{D}$-modules).

But $H^{\mathrm{pos}}\mathcal{Y}_1^{\mathrm{neg}}H^{\mathrm{neg}}$ is an isomorphism from $\mathbb{M}_1^{\mathrm{stp}}$ onto $\mathbb{M}_{-1}^{\mathrm{pos}}$ and $\mathbb{N} \subseteq \mathbb{M}_1^{\mathrm{stp}}$, and it maps $\mathbb{N}$ onto $\mathbb{N}^*$ (see above), so $\mathbb{M}_1^{\mathrm{stp}}/\mathbb{N}$ is isomorphic to $\mathbb{M}_{-1}^{\mathrm{pos}}/\mathbb{N}^*$. By the previous paragraph we get

$$\mathbb{M}_1^{\mathrm{stp}}/\mathbb{N} \cong \mathbb{M}_{-1}^{\mathrm{pos}}/\mathbb{N}^* \cong \mathbb{M}_1^{\mathrm{pos}}/\mathbb{N}.$$

On the one hand, $\mathbb{M}_1^{\mathrm{pos}}/\mathbb{N}$ is free as a $\mathbf{D}$-module, because $\{x_{2i}+\mathbb{N} : 0 \leq 2i \leq n(*)\}$ is a free basis for it. Also, $\mathbb{M}_1^{\mathrm{stp}}/\mathbb{N}$ is a free $\mathbf{D}$-module with the base $\{x_{2i} + \mathbb{N} : 0 < 2i \leq n(*)\}$. On the other hand, the number of their generators differ by 1. This is a contradiction that we searched for it.

The theorem follows. $\square$

8. All things together: the test problem

Recall that $\mathbf{R}$ *is a ring which is not pure semisimple. We now prove the existence of an* $\mathbf{R}$*-module equipped with the Corner pathology. We present such a thing by applying Corollary 6.33.*

Theorem 8.1. *Let* $m(*)$ *be an integer bigger that* 1 *and let* $\lambda > |\mathbf{R}|$ *be a cardinal of the form* $\lambda = \left(\mu^{\aleph_0}\right)^+$. *Then there is an* $\mathbf{R}$*-module* $\mathbb{M}$ *of cardinality* λ *such that:*

$$\mathbb{M}^n \cong \mathbb{M} \Longleftrightarrow m(*) \text{ divides } n-1.$$

Proof. We divide the proof into nine steps.

Step A) We first introduce rings $\mathbf{S}_0$ and $\mathbf{S}$. The ring $\mathbf{S}_0$ is incredibly easy compared to $\mathbf{S}$ and $\mathbf{S}$ is essentially the ring of endomorphisms we would like.

To this end, let $\mathbf{T}$ be the subring of $\mathbf{R}$ which 1 generates. Let $\mathbf{S}_0$ be the ring extending $\mathbf{T}$ generated by $\{\mathcal{X}_0, \ldots, \mathcal{X}_{m(*)}, \mathcal{W}, \mathcal{Z}\}$ freely except the following list of test equations:

$(*)_1$**:** $\mathcal{X}_\ell^2 = \mathcal{X}_\ell$,

$\mathcal{X}_\ell \mathcal{X}_m = 0 \ (\ell \neq m)$,

$1 = \mathcal{X}_0 + \ldots + \mathcal{X}_{m(*)}$,

$\mathcal{X}_\ell \mathcal{W} \mathcal{X}_m = 0$ for $\ell + 1 \neq m \mod m(*)+1$,

$\mathcal{W}^{m(*)+1} = 1$,

$\mathcal{Z}^2 = 1$,

$\mathcal{X}_0 \mathcal{Z}(1-\mathcal{X}_0) = \mathcal{X}_0 \mathcal{Z}$,

$(1-\mathcal{X}_0)\mathcal{Z}\mathcal{X}_0 = (1-\mathcal{X}_0)\mathcal{Z}$.

The meaning of these equations will become clear when we use them, see below.

Similarly, we define $\mathbf{S}$, but in addition we require $\sigma = 0$, where σ is a term in the language of rings, when ${}_{\mathbf{D}}\mathbb{M}^*\sigma = 0$ for every field $\mathbf{D}$ and where ${}_{\mathbf{D}}\mathbb{M}^*$ is the $(\mathbf{D}, \mathbf{S}_0)$-bimodule as defined below. Now "$\mathbf{S}$ is a free $\mathbf{T}$-module" will be proved later.

For an integer m, the notation $[-\infty, m)$ stands for $\{n : n \text{ is an integer } < m\}$ and if $\eta \in {}^{[-\infty,m)}\omega$, then we set $\eta \restriction k := \eta \restriction [-\infty, \min\{m, k\})$.

We look at the following set:

$$W_1 := W_0 \times \{0, \ldots, m(*)\},$$

where

$$W_0 := \Big\{\eta : \begin{array}{l} \eta \text{ is a function with domain of the form } [-\infty, n) \\ \text{and range } \subseteq \{1, \ldots, m(*)\}, \text{ and such that} \\ \text{for every small enough } m \in \mathbb{Z},\ \eta(m) = 1 \end{array}\Big\}.$$

Let $\mathbf{D}$ be a field such that if $\mathbf{T}$ is finite and of cardinality n, then $\mathrm{char}(\mathbf{D})$ divides n. So, $\mathbf{D} \otimes \mathbf{S}$ is the ring extending $\mathbf{D}$ by adding

$$\{\mathcal{X}_0, \ldots, \mathcal{X}_{m(*)}, \mathcal{W}, \mathcal{Z}\}$$

as non-commuting variables over $\mathbf{D}$ act freely except satisfying the equation in $(*)_1$, and if $||\mathbf{T}||$ is finite, we divide $\mathbf{S}$ by $p\mathbf{S}$ where $p := \mathrm{char}(\mathbf{D})$. So, there is a homomorphism $g_{\mathbf{D}}$ from $\mathbf{S}$ to $\mathbf{D} \otimes \mathbf{S}$ such that

$$\{0_{\mathbf{S}}\} = \cap\{\mathrm{Ker}(g_{\mathbf{D}}) : \mathbf{D} \textit{ as above}\}.$$

Let $\mathbb{M}^* = {}_{\mathbf{D}}\mathbb{M}^*$ be the left $\mathbf{D}$-module freely generated by

$$\{x_{\eta,\ell} : \eta \in W_0,\ \ell < m(*) + 1\}.$$

We make ${}_{\mathbf{D}}\mathbb{M}^*$ to a right $(\mathbf{D} \otimes \mathbf{S}_0)$-module by defining xz when $x \in {}_{\mathbf{D}}\mathbb{M}^*$ and $z \in \mathbf{S}_0$. It is enough to deal with

$$z \in \{\mathcal{X}_m : m < m(*) + 1\} \cup \{\mathcal{Z}, \mathcal{W}\}.$$

Let $x := \sum\limits_{\eta,\ell} a_{\eta,\ell} x_{\eta,\ell}$ where

(1) (η, ℓ) vary on W_0,

(2) $a_{\eta,\ell} \in \mathbf{D}$ and $\{(\eta, \ell) : a_{\eta,\ell} \neq 0\}$ is finite.

It is natural to extend things linearly, that is

$$(\sum_{\eta,\ell} a_{\eta,\ell} x_{\eta,\ell}) z := \sum_{\eta,\ell} a_{\eta,\ell}(x_{\eta,\ell} z),$$

where our table of undefined actions of $\{\mathcal{X}_m, \mathcal{Z}, \mathcal{W}\}$ on $x_{\eta,\ell}$ is as follows:

$$x_{\eta,\ell}\mathcal{X}_m \quad := \quad \begin{cases} x_{\eta,\ell} & \text{if } \ell = m \\ 0 & \text{if } \ell \neq m, \end{cases}$$

$$x_{\eta,\ell}\mathcal{Z} \quad := \quad \begin{cases} x_{\eta^\frown\langle\ell\rangle,0} & \text{if } \ell > 0 \\ x_{\eta\restriction[-\infty,n-1),\eta(n-1)} & \text{if } \ell = 0, \text{ and } (-\infty, n) = \mathrm{Dom}(\eta), \end{cases}$$

and

$$x_{\eta,\ell}\mathcal{W} \quad := \quad x_{\eta,m} \quad \text{when } m = \ell + 1 \mod m(*) + 1.$$

In sum, we get a $(\mathbf{D}, \mathbf{S})$-bimodule as the identities in the definition of $\mathbf{S}_0$ and $\mathbf{S}$ holds. If $\mathbf{D} = \mathbf{T}$, some by inspection (those of $(*)_1$), the rest by the choice of $\mathbf{S}$. If $\mathbf{D} \neq \mathbf{T}$, by the restriction on $\mathbf{D}$. Let

$${}_{\mathbf{D}}\mathbb{M}^*_\ell := \left\{ \sum_{\eta} d_{\eta,\ell} x_{\eta,\ell} : \ \eta \in W_0 \text{ and } d_{\eta,\ell} \in \mathbf{D} \right\}.$$

So, clearly

$${}_{\mathbf{D}}\mathbb{M}^* = \bigoplus_{\ell=0}^{m(*)} {}_{\mathbf{D}}\mathbb{M}^*_\ell.$$

Step B) In this step we introduce an $\mathbf{R}$-module $\mathbb{M}$ such that $\mathbb{M}^{m(*)} \cong \mathbb{M}$.

Let $\langle \mathbb{M}_\alpha : \alpha \leq \kappa \rangle$ be a strongly semi-nice construction. Set $\mathbb{P} := \mathbb{M}_\kappa$. We look at $\mathbb{P}_\ell := \mathbb{P}\mathcal{X}_\ell$. Let $i \neq j$. We use the formula $\mathcal{X}_i\mathcal{X}_j = 0$ to observe that $\mathbb{P}_i \cap \mathbb{P}_j = 0$. Since $\sum\limits_{\ell=0}^{m(*)} \mathcal{X}_\ell = 1$ we have

$${}_{\mathbf{R}}\mathbb{P} = \sum_{\ell=0}^{m(*)} \mathbb{P}\mathcal{X}_\ell = \bigoplus_{\ell=0}^{m(*)} {}_{\mathbf{R}}\mathbb{P}_\ell.$$

Suppose $\ell+1 \neq m \mod m(*)+1$. We combine the formula $\mathcal{X}_\ell \mathcal{W} \mathcal{X}_m = 0$ with the formula $\sum\limits_{\ell=0}^{m(*)} \mathcal{X}_\ell = 1$ to observe that

$$\mathbb{P}_\ell \mathcal{W} = \mathbb{P}\mathcal{X}_\ell \mathcal{W}(\mathcal{X}_0 + \ldots + \mathcal{X}_{m(*)}) = \mathbb{P}_\ell \mathcal{W} \mathcal{X}_{\ell+1} \subset \mathbb{P}_{\ell+1},$$

i.e., $\mathcal{W} : {}_{\mathbf{R}}\mathbb{P}_\ell \twoheadrightarrow {}_{\mathbf{R}}\mathbb{P}_{\ell+1}$ is surjective. Here, we use the relation $\mathcal{W}^{m(*)+1} = 1$ to consider $\mathcal{W}$ as an embedding from $\mathbb{P}$ onto $\mathbb{P}$. It turns out that $\mathcal{W} \restriction {}_{\mathbf{R}}\mathbb{P}_\ell$ is an isomorphism from ${}_{\mathbf{R}}\mathbb{P}_\ell$ onto ${}_{\mathbf{R}}\mathbb{P}_{\ell+1}$. So

$${}_{\mathbf{R}}\mathbb{P}_{m(*)} \cong \ldots \cong {}_{\mathbf{R}}\mathbb{P}_1 \cong {}_{\mathbf{R}}\mathbb{P}_0.$$

Thanks to the formulas $\mathcal{X}_0 \mathcal{Z}(1-\mathcal{X}_0) = \mathcal{X}_0 \mathcal{Z}$ and $\sum\limits_{\ell=0}^{m(*)} \mathcal{X}_\ell = 1$ we conclude that $\mathcal{Z}$ maps ${}_{\mathbf{R}}\mathbb{P}_0$ into

$$\begin{aligned} \mathbb{P}\mathcal{X}_0\mathcal{Z} &= \mathbb{P}\mathcal{X}_0\mathcal{Z}(1-\mathcal{X}_0) \\ &= \mathbb{P}\mathcal{X}_0\mathcal{Z}(\mathcal{X}_1 + \ldots + \mathcal{X}_{m(*)}) \\ &\subseteq \bigoplus_{\ell=1}^{m(*)} \mathbb{P}\mathcal{X}_\ell \\ &= \bigoplus_{\ell=1}^{m(*)} \mathbb{P}_\ell \\ &\cong \mathbb{P}_0. \end{aligned}$$

By the same vein, $\mathcal{Z}$ maps $\bigoplus\limits_{\ell=1}^{m(*)} \mathbb{P}_\ell$ into ${}_{\mathbf{R}}\mathbb{P}_0$. We are going to use the formula $\mathcal{Z}^2 = 1$ to exemplifies $\mathcal{Z}$ with the following isomorphism

$$\bigoplus_{\ell=1}^{m(*)} {}_{\mathbf{R}}\mathbb{P}_\ell \cong {}_{\mathbf{R}}\mathbb{P}_0.$$

But, we have just shown

$$\bigoplus_{\ell=1}^{m(*)} {}_{\mathbf{R}}\mathbb{P}_\ell \cong ({}_{\mathbf{R}}\mathbb{P}_0)^{m(*)}.$$

This completes the proof of Step B).

So, it is enough to show

$(*)_2$: $$1 < k < m(*) \quad \Rightarrow \quad {}_{\mathbf{R}}\mathbb{P}_0^k \ncong {}_{\mathbf{R}}\mathbb{P}_0.$$

Assume k is a counterexample. The desired contradiction will be presented in Step I), see below. To this end we need some preliminaries steps.

Step C) There exists a field $\mathbf{D}$ and $\mathcal{Y} \in \mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}$ satisfying the following equations:

$$\begin{aligned}
(*)_3 : \ & \mathcal{Y} \restriction {}_{\mathbf{D}}\mathbb{M}^*_0 \text{ is an isomorphism from } {}_{\mathbf{D}}\mathbb{M}^*_0 \text{ onto } \bigoplus_{\ell=1}^{k} {}_{\mathbf{D}}\mathbb{M}^*_\ell, \\
& \mathcal{Y} \restriction \bigoplus_{\ell=1}^{k} {}_{\mathbf{D}}\mathbb{M}^*_\ell \text{ is an isomorphism from } \bigoplus_{\ell=1}^{k} {}_{\mathbf{D}}\mathbb{M}^*_\ell \text{ onto } {}_{\mathbf{D}}\mathbb{M}^*_0, \\
& \mathcal{Y} \restriction \bigoplus_{\ell=k+1}^{m(*)+1} {}_{\mathbf{D}}\mathbb{M}^*_\ell \text{ is the identity, and} \\
& \mathcal{Y}^2 = 1.
\end{aligned}$$

Indeed, recall from ${}_{\mathbf{R}}\mathbb{P}^k_0 \cong {}_{\mathbf{R}}\mathbb{P}_0$ that ${}_{\mathbf{R}}\mathbb{P}$ is equipped with an endomorphism $\mathbf{f}$ such that:

$$\begin{aligned}
(*)_4 : \ & \mathbf{f} \restriction {}_{\mathbf{R}}\mathbb{P}_0 \text{ is an isomorphism from } {}_{\mathbf{R}}\mathbb{P}_0 \text{ onto } {}_{\mathbf{R}}\mathbb{P}_1 \oplus \ldots \oplus {}_{\mathbf{R}}\mathbb{P}_k, \\
& \mathbf{f} \restriction ({}_{\mathbf{R}}\mathbb{P}_1 \oplus \ldots \oplus {}_{\mathbf{R}}\mathbb{P}_k) \text{ is an isomorphism from } {}_{\mathbf{R}}\mathbb{P}_1 \oplus \ldots \oplus {}_{\mathbf{R}}\mathbb{P}_k \text{ onto } {}_{\mathbf{R}}\mathbb{P}_0, \\
& \mathbf{f} \restriction {}_{\mathbf{R}}\mathbb{P}_j = \mathrm{id}, \text{ for } k < j < m(*)+1, \text{ and} \\
& \mathbf{f}^2 = 1.
\end{aligned}$$

Assuming $(\mathbf{S}, +)$ is a free $\mathbf{T}$-module, according to Corollary 6.33, there is a field $\mathbf{D}$ with the property that $p := \mathrm{char}(\mathbf{D})$ divides $n := \mathrm{char}(\mathbf{R})$, when $n > 0$, and there exists $\mathcal{Y} \in \mathbf{D} \underset{\mathbf{T}}{\otimes} \mathbf{S}$ satisfying the the desired equations. This completes the proof of Step C).

In sum, we are reduced things to showing that $(\mathbf{S}, +)$ is a free $\mathbf{T}$-module. Note that each of $\{1_S, \mathcal{X}_0, \ldots, \mathcal{X}_{m(*)}, \mathcal{W}, \mathcal{Z}\}$ map each generator $x_{\eta,\ell}$ to another generator with some vanishing exceptions. So this applies to any composition of them, in fact, following a quite specific way with respect to the quadric (ρ, ν, k, m), where $\{\rho, \nu\} \subset \{1, \ldots, m(*)\}^{<\omega}$ and k, $m < \omega$. To this run, we are going to define $\mathcal{Y}^{k,m}_{\rho,\nu}$ as a "monomial operator" with respect to the generators of $\mathbf{S}$ so that:

$$x_{\eta,\ell}\mathcal{Y}^{k,m}_{\rho,\nu} = \begin{cases} x_{\eta_1 \frown \nu, m} & \text{if } \ell = k, \text{ and } \eta = \eta_1 \frown \rho \text{ for some } \eta_1 \in W_0 \\ 0 & \text{otherwise.} \end{cases}$$

In order to define these operators, first we define $\mathcal{Y}^{k,k}_{\rho,\langle\rangle}$ and $\mathcal{Y}^{k,k}_{\langle\rangle,\nu}$ by induction on $\lg(\rho)$ and $\lg(\nu)$ respectively. Also, negative powers of $\mathcal{W}$ can be defined, because $\mathcal{W}$ is invertible. Now, we let

$$\mathcal{Y}^{k,m}_{\rho,\nu} := \mathcal{Y}^{k,k}_{\rho,\langle\rangle}\mathcal{Y}^{k,k}_{\langle\rangle,\nu}\mathcal{W}^{m-k}.$$

Step D): Let $\rho,\nu \in [1, m(*)+1)^{<\omega}$. We say they have the same root provided

$$\lg(\rho) > 0 \textit{ and } \lg(\nu) > 0, \textit{ then } \rho(0) = \nu(0).$$

In this step we claim that the following set

$$\Omega := \left\{\mathcal{Y}^{k,m}_{\rho,\nu} : \rho,\nu \in [1, m(*)+1)^{<\omega},\ k,m \in [0, m(*)+1) \textit{ and } \rho,\nu \textit{ have same root}\right\}$$

generates $\mathbf{S}$ as a $\mathbf{T}$-module.

To prove this, first note that if $\mathcal{Y}^{k,m}_{\rho_1,\nu_1}$ is not in the family, then for some (ϱ, ρ, ν) we have $\mathcal{Y}^{k,m}_{\rho,\nu}$ belongs to the family and $(\rho,\nu) = (\varrho^\frown\rho_1, \varrho^\frown\nu_1)$. Hence $\mathcal{Y}^{k,m}_{\rho,\nu} = \mathcal{Y}^{k,m}_{\rho_1,\nu_1}$. So, Ω generates all of $\mathcal{Y}^{k,m}_{\rho,\nu}$'s and thus we can use them.

Let $\mathbf{S}'$ be the $\mathbf{T}$-submodule of $\mathbf{S}$ generated by $\mathbf{T}$ and the family above. Now $\mathbf{T} \subseteq \mathbf{S}'$ because

$$(\dagger)_1 \qquad 1 = \sum\left\{\mathcal{Y}^{m,m}_{\langle\rangle,\langle\rangle} : m = 0,\dots,m(*)\right\}.$$

To see this, we evaluate both sides at $x_{\eta,\ell}$. The right hand side of $(\dagger)_1$ is equal to

$$\begin{aligned} x_{\eta,\ell}\sum\left\{\mathcal{Y}^{m,m}_{\langle\rangle,\langle\rangle} : m = 0,\dots,m(*)\right\} &= x_{\eta^\frown\langle\rangle,\ell}\sum\left\{\mathcal{Y}^{m,m}_{\langle\rangle,\langle\rangle} : m = 0,\dots,m(*)\right\} \\ &= x_{\eta^\frown\langle\rangle,\ell}\mathcal{Y}^{\ell,\ell}_{\langle\rangle,\langle\rangle} \\ &= x_{\eta^\frown\langle\rangle,\ell} \\ &= x_{\eta,\ell}. \end{aligned}$$

Since the left hand side of $(\dagger)_1$ is the identity, we get the desired equality.

Next we show $\mathcal{X}_m \in \mathbf{S}'$ for each $m < m(*)+1$. It suffices to show that

$$(\dagger)_2 \qquad \mathcal{X}_m = \mathcal{Y}^{m,m}_{\langle\rangle,\langle\rangle}.$$

In order to see $(\dagger)_2$, we evaluate both sides at $x_{\eta,\ell}$. Let $n = \lg(\eta)+1$. First, assume that $\ell = m$. Then, the right hand side of $(\dagger)_2$ is equal to

$$x_{\eta,\ell}\mathcal{Y}^{m,m}_{\langle\rangle,\langle\rangle} = x_{\eta,m}\mathcal{Y}^{m,m}_{\langle\rangle,\langle\rangle} = x_{\eta,m},$$

which is equal to $x_{\eta,\ell}\mathcal{X}_m$. Now, we show the claim when $\ell \neq m$. In this case both sides of $(\dagger)_2$ are equal to zero, e.g., $(\dagger)_2$ is valid.

In order to show $\mathcal{W} \in \mathbf{S}'$ we bring the following claim:

$$(\dagger)_3 \qquad \mathcal{W} = \sum\big\{\mathcal{Y}^{\ell,m}_{\langle\rangle,\langle\rangle} : \ell = m+1 \mod m(*)+1\big\}.$$

As before, it is enough to evaluate both sides of $(\dagger)_3$ at $x_{\eta,\ell}$. Let m be such that $\ell \equiv_{m(*)+1} m+1$. The right hand side of $(\dagger)_3$ is equal to

$$\begin{aligned} x_{\eta,\ell}\sum\big\{\mathcal{Y}^{\ell,m}_{\langle\rangle,\langle\rangle} : \ell = m+1 \mod m(*)+1\big\} &= x_{\eta,\ell}\mathcal{Y}^{\ell,m} \\ &= x_{\eta,m}. \end{aligned}$$

By definition, this is equal to the left hand side of $(\dagger)_3$.

Finally, we claim that $\mathcal{Z} \in \mathbf{S}'$. Indeed, it suffices to prove that

$$(\dagger)_4 \quad \mathcal{Z} = \big\{\mathcal{Y}^{0,m}_{\langle m\rangle,\langle\rangle} : m = 1,\dots,m(*)\big\} + \sum\big\{\mathcal{Y}^{m,0}_{\langle\rangle,\langle m\rangle} : m = 1,\dots,m(*)\big\}.$$

Again we evaluate both sides of $(\dagger)_4$ at $x_{\eta,\ell}$. Let $n = \lg(\eta)+1$ and first suppose that $\ell = 0$. Then, the right hand side of $(\dagger)_4$ is equal to

$$\begin{aligned} x_{\eta,0}\big\{\mathcal{Y}^{0,m}_{\langle m\rangle,\langle\rangle} : m = 1,\dots,m(*)\big\} \quad &+ x_{\eta,0}\sum\big\{\mathcal{Y}^{m,0}_{\langle\rangle,\langle m\rangle} : m = 1,\dots,m(*)\big\} \\ &= x_{\eta,0}\big\{\mathcal{Y}^{0,m}_{\langle m\rangle,\langle\rangle} : m = 1,\dots,m(*)\big\} \\ &= x_{\eta\restriction n-1\frown\langle\eta(n)\rangle,0}\mathcal{Y}^{0,\eta(n)} \\ &= x_{\eta\restriction n-1\frown\langle\eta(n)\rangle,\eta(n)}. \end{aligned}$$

This is equal to the left hand side of $(\dagger)_4$. Suppose now that $\ell > 0$. Then the right hand side is equal to $x_{\eta\frown\langle\ell\rangle,0}$. By definition, this is equal to the left hand side of $(\dagger)_4$.

So far, we have proved that the subring $\mathbf{S}'$ includes $\mathbf{T}, \mathcal{X}_m (m < m(*)+1), \mathcal{Z}$ and $\mathcal{W}$. Indeed, it is a $\mathbf{T}$-module. To finish this step, it suffices to prove that $\mathbf{S}'$ is closed under product. For this, it is enough to prove that $\mathbf{S}'$ includes the product $\mathcal{Y}^{k^1,m^1}_{\rho_1,\nu_1}\mathcal{Y}^{k^2,m^2}_{\rho_2,\nu_2}$ of any two members of Ω.

Now if $m^1 \neq k^2$ or no one of ν_1, ρ_2 is an end-segment of the other then this is not the case. So easily the product is $\mathcal{Y}^{k^1,m^2}_{\rho_3,\nu_3}$ where:

- if $\nu_1 = \rho_2$, then $(\rho_3, \nu_3) = (\rho_1, \nu_2)$,
- if $\nu_1 = (\varrho^1)^\frown\rho_2$, then $(\rho_3, \nu_3) = (\rho_1, (\varrho^1)^\frown\nu_2)$,
- if $\rho_2 = (\varrho^1)^\frown\nu_1$, then $(\rho_3, \nu_3) = ((\varrho^1)^\frown\nu_1, \nu_2)$.

This completes the proof of Step D).

Step E) The set Ω from Step D) is a free basis of $\mathbf{S}$ as a $\mathbf{T}$-module.

Indeed, assume that

$$\mathcal{X} := \sum\{a^{k,m}_{\nu,\rho}\mathcal{Y}^{k,m}_{\nu,\rho} : \rho, k, \nu, m\} = 0,$$

where $\{(\rho, \nu, k, m) : a^{k,m}_{\nu,\rho} \neq 0\}$ is a finite set. We need to show $a^{k,m}_{\nu,\rho} = 0$, for all such ρ, k, ν, m.

Let $k^* < \omega$ and $\rho^* \in \{0, \ldots, m(*)\}^{<\omega}$ be such that

$$\lg(\rho^*) > \sup\{\lg(\rho) : a^{k,n}_{\rho,\nu} \neq 0\}.$$

We look at $x_{\rho^*,k^*}\mathcal{X}$. Let $A(k^*, \rho^*)$ be the set of all ν in the finite sequence above where $\rho^* = \rho^*_\nu{}^\frown\nu$, for some ρ^*_ν. Then

$$\begin{aligned} x_{\rho^*,k^*}\mathcal{X} &= \sum_{\nu,\rho,k,m} a^{k,m}_{\nu,\rho}(x_{\rho^*,k^*}\mathcal{Y}^{k,m}_{\nu,\rho}) \\ &= \sum_{\nu,\rho,m} \{a^{k^*,m}_{\nu,\rho}(x_{\rho^*,k^*}\mathcal{Y}^{k^*,m}_{\nu,\rho}) : \nu \in A(k^*, \rho^*)\} \\ &= \sum_{\nu,\rho,m} \{a^{k^*,m}_{\nu,\rho}x_{(\rho^*_\nu)^\frown\rho,m} : \nu \in A(k^*, \rho^*)\} \\ &= 0. \end{aligned}$$

Recall that $\mathbb{M}^*$ is the left $\mathbf{D}$-module freely generated by $x_{\eta,\ell}$'s, so as we are free in choosing k^*, ρ^*, we can easily show that for any tuple (ρ, ν, k, n) in the finite set above, for some suitable choice of ρ^*, $a^{k,n}_{\rho,\nu}$ is the only component of $x_{(\rho^*_\nu)^\frown\rho,m}$ in the sum above, and hence $a^{k,n}_{\rho,\nu} = 0$. This completes the proof of Step E).

Before we continue, let us introduce some notations and definitions. Clearly, $\mathcal{Y}$ can be represented as a finite sum of the form:

$$\mathcal{Y} = \sum \big\{ d^{k,m}_{\rho,\nu} \mathcal{Y}^{k,m}_{\rho,\nu} : k, m, \rho, \nu \big\}.$$

Choose $n(*)$ large enough such that

$$n(*) > \max\{\lg(\rho), \lg(\nu) : d^{k,m}_{\rho,\nu} \neq 0\}.$$

For any

$$u \subseteq \big\{(\eta,\ell) : \eta \in W_0,\ \ell < m(*)+1\big\},$$

we set $\mathbb{N}_u$ be the subspace generated by $\{x_{\eta,\ell} : (\eta,\ell) \in u\}$. , Clearly $\mathbb{N}_u \subseteq {}_{\mathbf{D}}\mathbb{M}^*$, when we view them as $\mathbf{D}$-modules. For $\eta \in w_0$ and $\ell = 1, \ldots, m(*)+1$, we define

$${}_{\mathbf{D}}\mathbb{M}^*_{[1,k]} := \bigoplus_{j=1}^{k} \mathbb{M}^*_\ell,$$

and let

$$\begin{aligned}
w_{\eta,\ell} &:= \big\{(\nu,m) | (\nu,m) = (\eta,\ell) \text{ or } \eta^\frown\langle\ell\rangle \trianglelefteq \nu \text{ and } m < m(*)+1\big\},\\
u_\ell &:= \big\{(\eta,m) | m = \ell,\ \eta \in w_0\big\},\\
w^m_{\eta,\ell}, &:= \ w_{\eta,\ell} \cap u_m,\\
w^{[1,n]}_{\eta,\ell} &:= w_{\eta,\ell} \cap \bigcup_{m\in[1,n]} u_m.
\end{aligned}$$

Step F) For any $\eta \in w_0$ and $\ell < m(*)+1$, there is a finite subset $u \subseteq w^0_{\eta,\ell}$ satisfying the following three items:

(α) $\mathcal{Y}$ maps $\mathbb{N}_{w^0_{\eta,\ell}\setminus u}$ into $\mathbb{N}_{w_{\eta,\ell}}$ in fact into $\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}$

(β) if v is finite and $u \subseteq v \subseteq w^0_{\eta,\ell}$ then $\mathbb{N}_{w^0_{\eta,\ell}\setminus v}\mathcal{Y}$ is a $\mathbf{D}$-vector subspace of $\mathbb{N}_{w^0_{\eta,\ell}}\mathcal{Y}$ of cofinite dimension.

(γ) for any finite subsets v_1, v_2 of $w^0_{\eta,\ell}$ extending u the following equalities are true:

$$\begin{aligned}\dim(\mathbb{N}_{w^0_{\eta,\ell}}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}) &- \dim(\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}/(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}\mathcal{Y}))\\ &= \dim(\mathbb{N}_{w^0_{\eta,\ell}}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2})\\ &- \dim(\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}/(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}\mathcal{Y})),\end{aligned}$$

where the dimensions are computed as $\mathbf{D}$-vector spaces.

Indeed, the case $\ell = 0$ is easy. Recalling the representation of $\mathcal{Y}$ and the choice of $n(*) < \omega$, if we set

$$u = \big\{(\nu, m) \in w_{\eta,\ell} : \ |\operatorname{Dom}(\nu) \setminus \operatorname{Dom}(\eta)| < n(*)\big\},$$

then we have

$$(\nu, m) \in w_{\eta,\ell} \setminus u \quad \Rightarrow \quad x_{\nu,m}\mathcal{Y} \in \mathbb{N}_{w_{\eta,\ell}}.$$

Note that u is finite and clause (α) holds. As $\mathcal{Y}\mathcal{Y} = 1$ etc., we can show that (β) holds. For checking the equalities in clause (γ), let $v_3 := v_1 \cup v_2$. Due to the transitivity of equality, it is enough to prove the required equality for pairs (v_1, v_3) and (v_2, v_3). This enables us to assume without loss of generality that $v_1 \subseteq v_2$. Also, we stipulate $v_0 = \emptyset$.

Now

$$\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2} \subseteq \mathbb{N}_{w^0_{\eta,\ell}\setminus v_1} \subseteq \mathbb{N}_{w^0_{\eta,\ell}\setminus w_0} = \mathbb{N}_{w^0_{\eta,\ell}},$$

and

$$\dim(\mathbb{N}_{w^0_{\eta,\ell}}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_i}) = |v_i|.$$

Manipulating the equalities in clause (α), they are equivalent to

$$\begin{aligned}\dim(\mathbb{N}_{w^0_{\eta,\ell}}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}) &- \dim(\mathbb{N}_{w^0_{\eta,\ell}}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2})\\ &= \dim(\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}/(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}\mathcal{Y}))\\ &- \dim(\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}/(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}\mathcal{Y})) \quad (*)\end{aligned}$$

Since $\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2} \subseteq \mathbb{N}_{w^0_{\eta,\ell}\setminus v_1} \subseteq \mathbb{N}_{w^0_{\eta,\ell}}$, the left hand side of $(*)$ is equal to

$$\dim(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}).$$

Also,

$$\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}\mathcal{Y} \subseteq \mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}\mathcal{Y} \subseteq \mathbb{N}_{w^{[1,k]}_{\eta,\ell}}.$$

Indeed, the first inclusion holds as $\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2} \subseteq \mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}$ and the second one holds by clause (β) and $u \subseteq v_1 \cap v_2$. Hence the right hand side of $(*)$ is equal to

$$\dim(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}\mathcal{Y}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}\mathcal{Y}).$$

Thus, $(*)$ is reduced in proving

$(*)'$ $$\dim(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}/\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}) = \dim(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}/(\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}\mathcal{Y})).$$

Since $\mathcal{Y}$ is an isomorphism from $\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}$ onto $\mathbb{N}_{w^0_{\eta,\ell}\setminus v_2}\mathcal{Y}$, it implies the validity of $(*)'$. This completes the proof.

Let $\mathbf{n}_{\eta,\ell}$ be the natural number so that for every large enough finite subset $v \subseteq w_{\eta,\ell}$

$$\mathbf{n}_{\eta,\ell} := \dim\left(\frac{\mathbb{N}_{w^0_{\eta,\ell}}}{\mathbb{N}_{w^0_{\eta,\ell}\setminus v}}\right) - \dim\left(\frac{\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}}{\mathbb{N}_{w^0_{\eta,\ell}\setminus v}\mathcal{Y}}\right).$$

In view of Step F), this is a well-defined notion, which does not depend on the choice of v.

Step G) If η, $\nu \in w_0$ and $\ell \in \{0, 1, \dots, m(*)\}$, then $\mathbf{n}_{\eta,\ell} = \mathbf{n}_{\nu,\ell}$.

To see this, we define a function $f : w_{\eta,\ell} \longrightarrow w_{\nu,\ell}$ by $f(\eta^\frown\rho, m) = (\nu^\frown\rho, m)$ for any $\rho \in \{0, \dots, m(*)\}^{<\omega}$. This function is one-to-one and onto, and it induces an isomorphism from $\mathbb{N}_{w^{[0,m(*)+1)}_{\eta,\ell}}$ onto $\mathbb{N}_{w^{[0,m(*)+1)}_{\nu,\ell}}$. It almost commutes with all of our operations (the problems are "near" η and ν). So choose $v_1 \subseteq w^0_{\eta,\ell}$ large enough, as required in the definition of $\mathbf{n}_{\eta,\ell}$, $\mathbf{n}_{\nu,\ell}$ and to make $\mathbb{N}_{w^0_{\eta,\ell}\setminus v_1}\mathcal{Y} \subseteq \mathbb{N}_{w^{[1,k]}_{\eta,\ell}}$, and let $v_2 = f(v_1)$. Now, it is easy to check the desired claim.

So, we shall write $\mathbf{n}_\ell$ instead of $\mathbf{n}_{\nu,\ell}$. By Step G), this is well-defined.

Step H) The following equations are valid:

$$\mathbf{n}_\ell = \begin{cases} 0 & \text{if } \ell \in [1, k] \\ \mathbf{n}_0 + \mathbf{n}_1 + \ldots + \mathbf{n}_{m(*)} & \text{if } \ell \in [k+1, m(*)+1) \textit{ or } \ell = 0 \end{cases}$$

To prove this, first note that

$$w_{\eta,\ell} = \{(\eta,\ell)\} \cup \bigcup_{m<m(*)+1} w_{\eta^\frown\langle\ell\rangle,m}.$$

In order to apply Step F), we fix a pair (η,ℓ) and we choose a finite subset $v \subseteq w^0_{\eta,\ell}$ large enough as there. Let $(\eta^\frown\langle\ell\rangle, m)$ be such that

$$\{(\eta,\ell)\} \cup \{(\eta^\frown\langle\ell\rangle, m) : \ m < m(*)+1\} \subseteq v.$$

Now, we compute $\mathbf{n}_{\eta,\ell}$ which is equal to $\mathbf{n}_\ell$, and $\mathbf{n}_{\eta^\frown\langle\ell\rangle,m}$ which is equal to $= \mathbf{n}_m$ for $m < m(*)+1$ and we shall get the equality.

Now set

$$\mathbf{n}^1_{v,\eta,\ell} := \dim\left(\frac{\mathbb{N}_{w^0_{\eta,\ell}}}{\mathbb{N}_{w^0_{\eta,\ell}\setminus v}}\right) - \sum_{m<m(*)+1} \dim\left(\frac{\mathbb{N}_{w^0_{\eta^\frown\langle\ell\rangle,m}}}{\mathbb{N}_{w^0_{\eta^\frown\langle\ell\rangle,m}\setminus v}}\right).$$

According to the definition, we lead to the following equality:

$$\mathbb{N}_{w^0_{\eta,\ell}} = \bigoplus_{m<m(*)+1} \mathbb{N}_{w^0_{\eta^\frown\langle\ell\rangle,m}} \oplus \mathbf{D}x_{\eta,\ell}.$$

Using this equality, we can easily see that

($\otimes_1$) $\mathbf{n}^1_{v,\eta,\ell}$ is equal to 1 when $\eta \in w_0, \ell < m(*)+1$ and $v \subseteq w_0$ is finite and sufficiently large enough. Of course, we can replace v by $v \cap w^0_{\eta,\ell}$.

In particular, the following definition makes sense:

($\otimes_2$) $$\mathbf{n}^2_{v,\eta,\ell} := \dim\left(\frac{\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}}{\mathbb{N}_{w^{[1,k]}_{\eta,\ell}\setminus v}}\right) - \sum_{m<m(*)+1} \dim\left(\frac{\mathbb{N}_{w^{[1,k]}_{\eta^\frown\langle\ell\rangle,m}}}{\mathbb{N}_{w^{[1,k]}_{\eta^\frown\langle\ell\rangle,m}\setminus v}}\right).$$

Suppose $\eta \in w_0, \ell < m(*)+1$ and $v \subseteq w_0$ is finite and large enough. In view of definition, and as the argument of ($\otimes_1$) we present the following three implications:

($\otimes_{2.1}$) $\ell = 0 \ \Rightarrow \ \mathbf{n}^2_{v,\eta,\ell} = 0$,

($\otimes_{2.2}$) $\ell \in [1,k] \ \Rightarrow \ \mathbf{n}^2_{v,\eta,\ell} = 1$,

($\otimes_{2.3}$) $\ell \in [k+1, m(*)+1) \Rightarrow \ \mathbf{n}^2_{v,\eta,\ell} = 0$.

Next, we show that

($\otimes_3$) $$\mathbf{n}^1_{v,\eta,\ell} - \mathbf{n}^2_{v,\eta,\ell} = \mathbf{n}_\ell - \sum_{m<m(*)+1} \mathbf{n}_m.$$

Indeed, we have

$$\begin{aligned}
\mathbf{n}^1_{v,\eta,\ell}-\mathbf{n}^2_{\nu,\eta,\ell} &= \dim(\frac{\mathbb{N}_{w^0_{\eta,\ell}}}{\mathbb{N}_{w^0_{\eta,\ell}}\setminus v})-\dim(\frac{\mathbb{N}_{w^{[1,k]}_{\eta,\ell}}}{w^{[1,k]}_{\eta,\ell}\setminus v})\\
&-\sum_{m<n(*)}\dim(\frac{\mathbb{N}_{w^0_{\eta^\frown\langle\ell\rangle,m}}}{\mathbb{N}_{w^0_{\eta^\frown\langle\ell\rangle,m}}\setminus v})\\
&+\sum_{m<n(*)}\dim(\frac{\mathbb{N}_{w^0_{\eta^\frown\langle\ell\rangle,m}}}{\mathbb{N}_{w^{[1,k]}_{\eta^\frown\langle\ell\rangle,m}}\setminus v})\\
&=\mathbf{n}_{\eta,\ell}-\sum_{\ell<m(*)+1}\mathbf{n}_{\eta^\frown\langle\ell\rangle,m}\\
&=\mathbf{n}_\ell-\sum_{m<m(*)+1}\mathbf{n}_m.
\end{aligned}$$

Now we combine $(\otimes_1)$, $(\otimes_{2.1})$ and $(\otimes_{2.2})$ along with $(\otimes_{2.3})$ to deduce the following formula.

$(\otimes_4)$ $\mathbf{n}^1_{v,\eta,\ell}-\mathbf{n}^2_{v,\eta,\ell}=\begin{cases}0 & \text{if } \ell\in[1,k]\\ 1 & \text{otherwise}\end{cases}$

Step H) follows from $(\otimes_3)+(\otimes_4)$.

Step I) In this step, we present our desired contradiction. To see this, recall from Step H) that

$$\begin{aligned}
\sum_{\ell=0}^{m(*)}\mathbf{n}_\ell &= (\mathbf{n}_0+\ldots+\mathbf{n}_{m(*)})\\
&+\sum_{\ell=1}^{k}(\mathbf{n}_0+\ldots+\mathbf{n}_{m(*)})+\sum_{\ell=k+1}^{m(*)}(\mathbf{n}_0+\ldots+\mathbf{n}_{m(*)})\\
&=(m(*)+1)\big(\sum_{\ell=0}^{m(*)}\mathbf{n}_\ell\big)-k.
\end{aligned}$$

In other words,

$$m(*)\left(\sum_{\ell=1}^{m(*)}\mathbf{n}_\ell\right)=k,$$

i.e.,

$$\sum_{\ell=1}^{m(*)}\mathbf{n}_\ell=\frac{k}{m(*)}.$$

Now, the left hand side is an integer, while as $1<k<m(*)$, the right hand side is not an integer. This is the contradiction that we searched for it.

So, we proved $(*)_2$:

$$1<k<m(*)\quad\Rightarrow\quad {}_{\mathbf{R}}\mathbb{P}^k_0\ncong{}_{\mathbf{R}}\mathbb{P}_0.$$

The proof is now complete. $\square$

An additional outcome is a slight improvement of the above pathological property. Now, we are ready to proof Corollary 1.2 from the introduction:

Corollary 8.2. *Assume $\lambda = (\mu^{\aleph_0})^+ > |\mathbf{R}|$. Let $m(*) > 2$ be an integer and assume that $\mathbf{R}$ is not pure semisimple. Then there is an $\mathbf{R}$-module $\mathbb{M}$ of cardinality λ such that:*

$$\mathbb{M}^{n_1} \cong \mathbb{M}^{n_2} \iff m(*)|(n_1 - n_2)$$

Proof. Let $\mathbf{S}_1$ be the ring constructed in the proof of Theorem 8.1. Let also $\mathbb{M}^*$ be the $(\mathbf{R}, \mathbf{S}_1)$-bimodule constructed as there. Choose cardinals

$$\lambda_{m_1} > \lambda_{m_1 - 1} > \ldots > \lambda_0 > \|\mathbb{M}^*\| + \|\mathbf{R}\| + \|\mathbf{S}_1\| + \aleph_0,$$

where m_1 is any integer bigger than n_1. Let $I := \bigcup\limits_{m \leq m_1} \prod\limits_{\ell=0}^{m-1} \lambda_\ell$ and define a function $f : I \longrightarrow I$ by the following rules:

$$f(\eta) := \begin{cases} \eta \restriction k & \text{if } \eta \in I, \text{ and } \lg(\eta) = k+1 \\ \eta & \text{if } \eta = \langle\rangle. \end{cases}$$

The notation $\mathbb{M}^{\otimes}$ stands for the bimodule $\bigoplus\limits_{\eta \in I} \mathbb{M}^{\otimes}_\eta$ where $\mathbb{M}^{\otimes}_\eta \cong \mathbb{M}^*$ for each $\eta \in I$, and we denote $h_\eta : \mathbb{M}^* \longrightarrow \mathbb{M}^{\otimes}_\eta$ for such an isomorphism.

For every endomorphism $\mathcal{Y}$ of $\mathbb{M}^*$ we define an endomorphism $\mathcal{Y}^{\otimes} \in \mathrm{End}_{\mathbf{R}}(\mathbb{M}^{\otimes})$ of $\mathbb{M}^{\otimes}$ as an $\mathbf{R}$-module as follows. We are going to define the action of $\mathcal{Y}^{\otimes}$ on each $\mathbb{M}^{\otimes}_\eta$ $(\eta \in I)$. To this end, we take $x \in \mathbb{M}^{\otimes}_\eta$ and define the action via:

$$x\mathcal{Y}^{\otimes} := h_{f(\eta)}\left((h_\eta^{-1}(x)\mathcal{Y})\right).$$

Let $\mathbf{S}^{\otimes}$ be the subring of the ring of endomorphisms of $\mathbb{M}^{\otimes}$ as an $\mathbf{R}$-module generated by

$$\{1^{\otimes}, \mathcal{X}_0^{\otimes}, \mathcal{X}_1^{\otimes}, \ldots, \mathcal{X}_{m(*)}^{\otimes}, \mathcal{W}^{\otimes}, \mathcal{Z}^{\otimes}\}.^{23}$$

[23]Each one is a member of $\mathbf{S}$ and as is such an endomorphism of $\mathbb{M}^{\otimes}$ as an $\mathbf{R}$-module.

Note also that $1^{\otimes}$ is not a unit in $\mathbf{S}^{\otimes}$. Now we continue as in the proof of Theorem 7.1, with $\mathbb{M}^{\otimes}$ here instead of $\mathbb{M}$ there, and $\mathbf{S}^{\otimes}$ here instead of $\mathbf{S}$ there. We leave the details to the reader. $\square$

We close the paper by reproving Theorem 7.4 from Theorem 8.1:

Corollary 8.3. *Let $\lambda = (\mu^{\aleph_0})^+ > |\mathbf{R}|$ be a regular cardinal. Then there are $\mathbf{R}$-modules $\mathbb{M}_1$ and $\mathbb{M}_2$ of cardinality λ such that:*

(1) $\mathbb{M}_1$, $\mathbb{M}_2$ are not isomorphic,

(2) $\mathbb{M}_1$ is isomorphic to a direct summand of $\mathbb{M}_2$,

(3) $\mathbb{M}_2$ is isomorphic to a direct summand of $\mathbb{M}_1$.

Proof. We apply Theorem 8.1, for $m(*) := 2$ to find an $\mathbf{R}$-module M such that

$$M^n \cong M \Longleftrightarrow 2|n-1 \quad (*)$$

Now, let $\mathbb{M}_1 := M$ and $\mathbb{M}_2 := M^2$. Then

(1) $\mathbb{M}_1$, $\mathbb{M}_2$ are not isomorphic. This follows from $(*)$.

(2) $\mathbb{M}_1$ is isomorphic to a direct summand of $\mathbb{M}_2$. Indeed,

$$\mathbb{M}_1 \oplus M = M \oplus M = M^2 = \mathbb{M}_2.$$

(3) $\mathbb{M}_2$ is isomorphic to a direct summand of $\mathbb{M}_1$. Indeed,

$$\mathbb{M}_2 \oplus M = M^2 \oplus M = M^3 \overset{(*)}{\cong} M = \mathbb{M}_1.$$

$\square$

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