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BLACK BOXES

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ABSTRACT. We shall deal comprehensively with Black Boxes, the intention being that provably in ZFC we have a sequence of guesses of extra structure on small subsets, where the guesses are pairwise "almost disjoint;" by this we mean they have quite little interaction, and are far apart but together are "dense". We first deal with the simplest case, where the existence comes from winning a game by just writing down the opponent's moves. We show how it helps when instead of orders we have trees with boundedly many levels, having freedom in the last. After this we quite systematically look at existence of black boxes, and make connection to non-saturation of natural ideals and diamonds on them.

1. Introduction

The non-structure theorems we have discussed in [17] usually rest on some freedom on finite sequences and on a kind of order. When our freedom is related to infinite sequences, and to trees, our work is sometimes harder. In particular,

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This is a revised version of [35, Ch.III,§4,§5]; it has existed (and been occasionally revised) for many years. It was mostly ready in the early nineties, and was made public to some extent. This was written as Chapter IV of the book [51], which hopefully will materialize some day, but in the meantime it is [10]. The intention was to have [16] (revising [30]) for Ch.I, [2] for Ch.II, [17] for Ch.III, [10] for Ch.IV, [19] for Ch.V, [8] for Ch.VI, [11] for Ch.VII, [15] (a revision of [29]) for Ch.VIII, [12] for the appendix, and probably [46], [13], [14], and [18]. References like [10, 3.26 = L6.12] means that 6.12 is the label of Lemma 4.26 in [10].

The reader should note that the version in my website is usually more up-to-date than the one in the mathematical archive.

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we may consider, for $\lambda \geq \chi$, χ regular, and $\varphi = \varphi(\bar{x}_0, \ldots, \bar{x}_\alpha, \ldots)_{\alpha<\chi}$ in a vocabulary τ:

(∗) For any $I \subseteq {}^{\chi\geq}\lambda$ we have a τ-model M_I and sequences $\bar{a}_\eta$ (for $\eta \in {}^{\chi>}\lambda$), where

$$[\eta \lhd \nu \Rightarrow \bar{a}_\eta \neq \bar{a}_\nu], \qquad \ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)}),$$

such that for $\eta \in {}^{\chi}\lambda$ we have:

$$M_I \models \varphi(\ldots, \bar{a}_{\eta\restriction\alpha}, \ldots)_{\alpha<\chi} \text{ if and only if } \eta \in I.$$

(Usually, M_I is to some extent "simply defined" from I). Of course, if we do not ask more from M_I, we can get nowhere: we certainly restrict its cardinality and/or usually demand it is φ-representable[1] in (a variant of) $\mathscr{M}_{\mu,\kappa}(I)$ (for suitable μ, κ). Certainly for T un-superstable we have such a formula φ:

$$\varphi(\ldots, \bar{a}_{\eta\restriction n}, \ldots) = (\exists \bar{x}) \bigwedge_n \varphi_n(\bar{x}, \bar{a}_{\eta\restriction n}).$$

There are many natural examples.

Formulated in terms of the existence of I for which our favorite "anti-isomorphism" player has a winning strategy, we proved this in 1969/70 (in proofs of lower bounds of $\dot{\mathbb{I}}(\lambda, T_1, T)$, T un-superstable), but it was shortly superseded. However, eventually the method was used in one of the cases in [24, Ch.VIII,§2] — for strong limit singular [24, Ch.VIII,2.6], which comes from [20]. It was developed in [27], [28] for constructing Abelian groups with prescribed endomorphism groups. See further a representation of one of the results here in Eklof–Mekler [4], [5] a version which was developed for a proof of the existence of an Abelian (torsion-free $\aleph_1$-free) group G with

$$G^{***} = G^* \oplus A \qquad \big(G^* := \mathrm{Hom}(G, \mathbb{Z})\big)$$

in a work by Mekler and Shelah. A preliminary version of this paper appeared in [35, Ch.III,§4,§5], but §3 here was just almost ready and §4 (on partitions of stationary sets and $\diamondsuit_D$) was written up as a letter to Foreman in the late nineties.

The saturation of ideals was continued much later in Gitik–Shelah [48] and more recently in [49] and Asgarzadeh–Golshani–Shelah [1].

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[1] see Definition [17, 2.7 = Lf4] clauses (c),(d).

2. The easy black box and an easy application

In this section we do not try to get the strongest results, but just provide some examples (e.g. we do not present the results when $\lambda = \lambda^\chi$ is replaced by $\lambda = \lambda^{<\chi}$). By the proof of [24, Ch.VIII,2.5] (see later for a complete proof):

THEOREM 2.1. *Suppose that*

$(*)$ (a) $\lambda = \lambda^\chi$

(b) *τ is a vocabulary and $\varphi = \varphi(\bar{x}_0, \bar{x}_1, \ldots, \bar{x}_\alpha \ldots)_{\alpha<\chi}$ is a formula in $\mathscr{L}(\tau)$ for some logic $\mathscr{L}$.*

$(c)_{\tau,\varphi}$ *For any I such that ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$, we have a τ-model M_I and sequences $\bar{a}_\eta$ (for $\eta \in {}^{\chi>}\lambda$), where*

$$[\eta \lhd \nu \Rightarrow \bar{a}_\eta \neq \bar{a}_\nu], \qquad \ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)}),$$

such that for $\eta \in {}^\chi\lambda$ we have:

$$M_I \models \varphi(\ldots, \bar{a}_{\eta\restriction\alpha}, \ldots)_{\alpha<\chi} \text{ if and only if } \eta \in I.$$

(c) *$\|M_I\| = \lambda$ for every I satisfying ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\leq}\lambda$, and $\ell g(\bar{a}_\eta) \leq \chi$ or just $\lambda^{\ell g(\bar{a}_\eta)} = \lambda$.*

Then (using ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$):

(1) There is no model M of cardinality λ into which every M_I can be $(\pm\varphi)$-embedded (i.e., by a function preserving φ and $\neg\varphi$).

(2) For any M_i (for $i < \lambda$), $\|M_i\| \leq \lambda$, for some I satisfying ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$, the model M_I cannot be $(\pm\varphi)$-embedded into any M_i.

EXAMPLE 2.2. Consider the class of Boolean algebras and the formula

$$\varphi(\ldots, x_n, \ldots) := \Big(\bigcup_n x_n\Big) = 1$$

(i.e., there is no $x \neq 0$ such that $x \cap x_n = 0$ for each n).

For ${}^{\omega>}\lambda \subseteq I \subseteq {}^{\omega\geq}\lambda$, let M_I be the Boolean algebra generated freely by x_η (for $\eta \in I$) except the relations: for $\eta \in I$, if $n < \ell g(\eta) = \omega$ then $x_\eta \cap x_{\eta\restriction n} = 0$.

So[2] $\|M_I\| = |I| \in [\lambda, \lambda^{\aleph_0}]$ and in M_I for $\eta \in {}^\omega\lambda$ we have: $M_I \models$ "$\big(\bigcup_n x_{\eta\restriction n}\big) = 1$" if and only if $\eta \notin I$ (work a little in Boolean algebras).

[2] With more work, we can demand that M_I satisfies the c.c.c.

So

CONCLUSION 2.3. *If $\lambda = \lambda^{\aleph_0}$, then there is no Boolean algebra* **B** *of cardinality λ universal under σ-embeddings (i.e., ones preserving countable unions).*

REMARK 2.4. This is from [24, Ch.VIII,Ex.2.5,pg.464].

PROOF OF THE THEOREM 2.1. First we recall the simple black box (and a variant) in 2.5, 2.6 below:

THE SIMPLE B.B. LEMMA 2.5. *There are functions f_η (for $\eta \in {}^\chi\lambda$) such that:*

- (i) $\mathrm{Dom}(f_\eta) = \{\eta \restriction \alpha \colon \alpha < \chi\}$,
- (ii) $\mathrm{Rang}(f_\eta) \subseteq \lambda$,
- (iii) *If $f\colon {}^{\chi>}\lambda \to \lambda$, then for some $\eta \in {}^\chi\lambda$ we have $f_\eta \subseteq f$.*

PROOF. For $\eta \in {}^\chi\lambda$, let f_η be the function (with domain $\{\eta \restriction \alpha \colon \alpha < \chi\}$) such that

$$f_\eta(\eta \restriction \alpha) = \eta(\alpha).$$

So $\langle f_\eta \colon \eta \in {}^\chi\lambda\rangle$ is well defined. Properties (i) and (ii) are straightforward, so let us prove (iii). Let $f\colon {}^{\chi>}\lambda \to \lambda$. We define $\eta_\alpha = \langle \beta_i \colon i < \alpha\rangle$ by induction on α.

For $\alpha = 0$ or α limit — no problem.

For $\alpha + 1$: let β_α be the ordinal such that $\beta_\alpha = f(\eta_\alpha)$.

So $\eta := \langle \beta_i \colon i < \chi\rangle$ is as required. ∎

FACT 2.6. In 2.5:

(1) We can replace the range of f, f_η by any fixed set of power λ.
(2) We can replace the domains of f, f_η by $\{\bar a_\eta \colon \eta \in {}^{\chi>}\lambda\}$, $\{\bar a_{\eta\restriction\alpha} \colon \alpha < \chi\}$, respectively, as long as

$$\alpha < \beta < \chi \wedge \eta \in {}^\chi\lambda \;\Rightarrow\; \bar a_{\eta\restriction\alpha} \neq \bar a_{\eta\restriction\beta}.$$

REMARK 2.7. We can present it as a game. (As in the book [24, Ch.VIII,2.5]).

Continuation of the proof of Theorem 2.1.

It suffices to prove 2.1(2). Without loss of generality $\langle |M_i| \colon i < \lambda\rangle$ are pairwise disjoint. Now we use 2.6; for the domain we use $\langle \bar a_\eta \colon \eta \in {}^{\chi>}\lambda\rangle$ from the assumption of 2.1, and for the range: $\bigcup_{i<\lambda} {}^{\chi\geq}|M_i|$ (it has cardinality $\leq \lambda$ as $\|M_i\| \leq \lambda = \lambda^\chi$).

We define

$$I = ({}^{\chi>}\lambda) \cup \big\{\eta \in {}^{\chi}\lambda : \text{for some } i < \lambda, \operatorname{Rang}(f_\eta) \text{ is a set of sequences from } |M_i| \text{ and } M_i \models \neg\varphi(\dots, f_\eta(\bar a_{\eta\restriction\alpha}), \dots)_{\alpha<\chi}\big\}.$$

Look at M_I. It suffices to show:

⊗ There is no $(\pm\varphi)$-embedding of M_I into M_i for $i < \lambda$.

Why does ⊗ hold?

If $f\colon M_I \to M_i$ is a $(\pm\varphi)$-embedding, then by Fact 2.6, for some $\eta \in {}^{\chi}\lambda$ we have

$$f \restriction \{\bar a_{\eta\restriction\alpha} \colon \alpha < \chi\} = f_\eta.$$

By the choice of f,

$$M_I \models \varphi\big[\dots, \bar a_{\eta\restriction\alpha}, \dots\big]_{\alpha<\chi} \iff M_i \models \varphi\big[\dots, f(\bar a_{\eta\restriction\alpha}), \dots\big]_{\alpha<\chi},$$

but by the choice of I and M_I we have

$$M_I \models \varphi\big[\dots, \bar a_{\eta\restriction\alpha}, \dots\big]_{\alpha<\chi} \iff M_i \models \neg\varphi\big[\dots, f_\eta(\bar a_{\eta\restriction\alpha}), \dots\big]_{\alpha<\chi}.$$

This is a contradiction, as by the choice of η,

$$\bigwedge_{\alpha<\chi} f(\bar a_{\eta\restriction\alpha}) = f_\eta(\bar a_{\eta\restriction\alpha}). \qquad \blacksquare$$

Discussion 2.8. We may be interested whether, in 2.1, when $\lambda^+ < 2^\lambda$ we may

(1) in 2.1(1), allow $\|M\| = \lambda^+$, and/or
(2) get $\geq \lambda^{++}$ non-isomorphic models of the form M_I, assuming $2^\lambda > \lambda^+$.

The following lemma shows that we cannot prove those better statements in ZFC, though (see 2.11) in some universes of set theory we can. So this requires (elementary) knowledge of forcing, but is not used later. It is here just to justify the limitations of what we can prove, and the reader can skip it.

Lemma 2.9. *Suppose that in the universe* $\mathbf{V}$ *we have* $\kappa < \lambda = \operatorname{cf}(\lambda) = \lambda^{<\lambda}$, $(\forall\lambda_1 < \lambda)\big[(\lambda_1)^\kappa < \lambda\big]$, *and* $\lambda < \mu = \mu^\lambda$.

Then, for some notion forcing $\mathbb{P}$*:*

(*a*) $\mathbb{P}$ *is* λ*-complete and satisfies the* λ^+*-c.c., and* $|\mathbb{P}| = \mu$, $\Vdash_{\mathbb{P}}$ "$2^\lambda = \mu$" *(so forcing with* $\mathbb{P}$ *collapses no cardinals, changes no cofinalities, adds no new sequences of ordinals of length* $< \lambda$, *and* $\Vdash_{\mathbb{P}}$ "$\lambda^{<\lambda} = \lambda$"*).*
(*b*) *We can find* φ, M_I *(for* ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\geq}\lambda$*) as in 2.1*$(*)$, *so with* $\|M_I\| = \lambda$, *(*τ*-models with* $|\tau| = \kappa$ *for simplicity) such that:*
   ⊕ *There are, up to isomorphism, exactly* λ^+ *models of the form* M_I *(for* ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\lambda\geq}\lambda$*).*

(c) *In (b), there is a model M such that $\|M\| = \lambda^+$ and every model M_I can be $(\pm\varphi)$-embedded into M.*

REMARK 2.10. (1) M_I is essentially $(I^+, \lhd)$: the addition of level predicates is immaterial, where I^+ extends I "nicely" so that we can let $a_\eta = \eta$ for $\eta \in I$.

(2) Clearly clause (c) also shows that weakening $\|M\| = \lambda$, even when $\lambda^+ < 2^\lambda$, may make 2.1 false.

(3) In the proof of Lemma 2.9, the class of models isomorphic to some N^*_j with $j < \lambda^+$ is not so nice. But the following class of models, which is reasonably well defined, will fail to satisfy the statement in 2.1(2) (in $\mathbf{V}^{\mathbb{P}}$).

⊞ $N \in K$ iff
- (a) N is a τ-model.
- (b) For some ordinal α and $S \subseteq {}^\kappa\alpha$, N is isomorphic to $N_{I[\delta]}$, where $I = \{\eta \restriction \zeta : \eta \in S, \zeta \le \kappa\}$ and $N_{I[\delta]}$ is defined as in the proof below.

PROOF OF LEMMA 2.9. Let $\tau = \{R_\zeta : \zeta \le \kappa\} \cup \{<\}$ with R_ζ being a monadic predicate, and $<$ being a binary predicate. For a set I, ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\ge}\lambda$ let N_I be the τ-model:

$$|N_I| = I, \quad R_\zeta^{N_I} = I \cap {}^\zeta\lambda, \quad <^{N_I} = \{(\eta, \nu) : \eta, \nu \in I, \eta \lhd \nu\},$$

and

$$\varphi(\ldots, x_\zeta, \ldots)_{\zeta<\kappa} = \bigwedge_{\zeta<\xi<\kappa} \big(x_\zeta < x_\xi \wedge R_\zeta(x_\zeta)\big) \wedge (\exists y)\Big[R_\kappa(y) \wedge \bigwedge_{\zeta<\kappa} x_\zeta < y\Big].$$

Now we define the forcing notion $\mathbb{P}$. It is $\mathbb{P}_{\lambda^+}$, where

$$\big\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_j : i \le \lambda^+, j < \lambda^+ \big\rangle$$

is an iteration with support $< \lambda$, of λ-complete forcing notions, where $\underset{\sim}{\mathbb{Q}}_j$ is defined as follows.

For $j = 0$ we add μ many Cohen subsets to λ:

$$\mathbb{Q}_0 = \{f : f \text{ is a partial function from } \mu \text{ to } \{0,1\}, |\mathrm{Dom}(f)| < \lambda\},$$

the order is the inclusion.

For $j > 0$, we define $\underset{\sim}{\mathbb{Q}}_j$ in $\mathbf{V}^{\mathbb{P}_j}$. Let $\big\langle I(j,\alpha) : \alpha < \alpha(j)\big\rangle$ list, without repetition, all sets $I \in \mathbf{V}^{\mathbb{P}_j}$ such that ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\ge}\lambda$. (Note that the interpretation of ${}^{\kappa\ge}\lambda$ does not change from $\mathbf{V}$ to $\mathbf{V}^{\mathbb{P}_j}$ (as $\kappa < \lambda$), but the family of such I-s increases.)

Now

$$\underset{\sim}{\mathbb{Q}}_j = \Big\{\bar f\colon f = \langle f_\alpha\colon \alpha < \alpha(j)\rangle, f_\alpha \text{ is a partial isomorphism from } N_{I(j,\alpha)} \text{ into } N_{({}^{\kappa\geq}\lambda)}, w(\bar f) \coloneqq \{\alpha\colon f_\alpha \neq \emptyset\} \text{ has cardinality } < \lambda, \mathrm{Dom}(f_\alpha) \text{ has the form } \bigcup_{\beta<\gamma} {}^{\kappa\geq}\beta \cap N_{I(j,\alpha)} \text{ for some } \gamma < \lambda; \text{ and if } \alpha_1, \alpha_2 < \alpha(j) \text{ and } \eta_1, \eta_2 \in {}^{\kappa}\lambda, \text{ and for every } \zeta < \kappa\ f_{\alpha_1}(\eta_1 \restriction \zeta), f_{\alpha_2}(\eta_2 \restriction \zeta) \text{ are well-defined and equal, then } \eta_1 \in I(j,\alpha_1) \Leftrightarrow \eta_2 \in I(j,\alpha_2)\Big\}.$$

The order is:

$$\bar f^1 \leq \bar f^2 \quad \text{if and only if } \big(\forall \alpha < \alpha(j)\big)\big[f^1_\alpha \subseteq f^2_\alpha\big] \text{ and for all } \alpha < \beta < \alpha(j), f^1_\alpha \neq \emptyset \wedge f^1_\beta \neq \emptyset \text{ implies } \mathrm{Rang}(f^2_\alpha) \cap \mathrm{Rang}(f^2_\beta) = \mathrm{Rang}(f^1_\alpha) \cap \mathrm{Rang}(f^1_\beta).$$

Then, $\underset{\sim}{\mathbb{Q}}_j$ is λ-complete and it satisfies the '$*^{\omega}_{\lambda}$' version of λ^+-c.c. from [23][3], hence each $\mathbb{P}_j$ satisfies the λ^+-c.c. (by [23]).

Now the $\mathbb{P}_{j+1}$-name $\underset{\sim}{I}_j$ (interpreting it in $\mathbf{V}^{\mathbb{P}_{j+1}}$, we get I^*_j) is:

$$I^*_j = {}^{\kappa>}\lambda \cup \big\{\eta \in {}^{\kappa}\lambda\colon \text{for some } \bar f \in \underset{\sim}{G}_{\mathbb{Q}_j}, \alpha < \alpha(j), \text{ and } \nu \in N_{I(j,\alpha)}, \text{ we have } \ell g(\nu) = \kappa \text{ and } f_\alpha(\nu) = \eta\big\}.$$

This defines also $f^j_\alpha\colon I(j,\alpha) \to I^*_j$, which is forced to be a $(\pm\varphi)$-embedding and also just an embedding.

So now we shall define, for every I such that ${}^{\kappa>}\lambda \subseteq I \subseteq {}^{\kappa\geq}\lambda$, a τ-model M_I: clearly I belongs to some $\mathbf{V}^{\mathbb{P}_j}$. Let $j = j(I)$ be the first such j, and let $\alpha = \alpha(I)$ be such that $I = I(j,\alpha)$. Let $M_{I(j,\alpha)} = N_{I^*_j}$ (and $a_\rho = f^j_\alpha(\rho)$ for $\rho \in I(j,\alpha)$).

We leave the details to the reader. ∎

On the other hand, consistently we may easily have a better result.

Lemma 2.11. *Suppose that, in the universe* $\mathbf{V}$,

$$\lambda = \mathrm{cf}(\lambda) = \lambda^\kappa = \lambda^{<\lambda}, \quad \lambda < \mu = \mu^\lambda.$$

For some forcing notion $\mathbb{P}$:

[3] See more in [45], and much later in [50].

(*a*) $\mathbb{P}$ *is as in 2.9.*

(*b*) *In* $\mathbf{V}^{\mathbb{P}}$, *assume that*

- φ *and the function* $I \mapsto \left(M_I, \langle \bar{a}^I_\eta : \eta \in {}^{\kappa>}\lambda\rangle\right)$ *are as required in clauses (a),(b),(c) of* $(*)$ *of 2.1,*
- $\zeta(*) < \mu$,
- *each* N_ζ *(for* $\zeta < \zeta(*)$*) is a model in the relevant vocabulary,*
- $\sum_{\zeta<\zeta(*)} \|N_\zeta\|^\kappa < \mu$ *(If the vocabulary is of cardinality* $< \lambda$ *and each predicate or relation symbol has finite arity, then requiring just* $\sum\left\{\|N_\zeta\| : \zeta < \zeta(*)\right\} < \mu$ *will suffice.)*

Then for some I, *the model* M_I *cannot be* $(\pm\varphi)$*-embedded into any* N_ζ.

(*c*) *Assume* $\mu_1 = \mathrm{cf}(\mu_1)$, $\lambda < \mu_1 \leq \mu$ *and* $\mathbf{V} \models (\forall\chi < \mu_1)[\chi^\lambda < \mu_1]$. *Then in* $\mathbf{V}^{\mathbb{P}}$, *if* $\langle M_{I_i} : i < \mu_1\rangle$ *are pairwise non-isomorphic,* ${}^{\kappa>}\lambda \subseteq I_i \subseteq {}^{\kappa\geq}\lambda$, *and* $M_{I_i}, \bar{a}^i_\eta$ *(for* $\eta \in I_i$*) are as in 2.1*$(*)$, *then* M_{I_i} *is not embeddable into* M_{I_j} *for some* $i \neq j$.

(*d*) *In* $\mathbf{V}^{\mathbb{P}}$ *we can find a sequence* $\langle I_\zeta : \zeta < \mu\rangle$ *(with* ${}^{\kappa>}\lambda \subseteq I_\zeta \subseteq {}^{\kappa\geq}\lambda$*) such that no* M_{I_ζ} *is* $(\pm\varphi)$*-embeddable into another.*

Proof. $\mathbb{P}$ is $\mathbb{Q}_0$ from the proof of 2.9. Let $\mathbf{F}$ be the generic function that is $\bigcup\{f : f \in \underset{\sim}{G}_{\mathbb{Q}_0}\}$: clearly it is a function from μ to $\{0, 1\}$. Now clause (a) is trivial.

Next, concerning clause (b), we are given $\left\langle N_\zeta : \zeta < \zeta(*)\right\rangle$. Clearly for some $A \in \mathbf{V}$ of size smaller than μ, we have $A \subseteq \mu$. To compute the isomorphism types of N_ζ for $\zeta < \zeta(*)$, it is enough to know $\mathbf{F} \restriction A$. We can force by $\{f \in \mathbb{Q}_0 : \mathrm{Dom}(f) \subseteq A\}$, then $\mathbf{f} \restriction B$ for any $B \subseteq \lambda \setminus A$ of cardinality λ (from $\mathbf{V}$) gives us an I as required.

To prove clause (c) use a Δ-system argument for the names of various M_I-s, and similarly for (d). ∎

3. An application for many models in λ

Discussion 3.1. Next we consider the following:

Assume λ is regular, $(\forall\mu < \lambda)[\mu^{<\chi} < \lambda]$. Let $\mathscr{U}_\alpha$ (for $\alpha < \lambda$) be pairwise disjoint stationary subsets of $\{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$.

For $A \subseteq \lambda$, let

$$\mathscr{U}_A = \bigcup_{i\in A} \mathscr{U}_i.$$

We want to define I_A such that ${}^{\chi>}\lambda \subseteq I_A \subseteq {}^{\chi\geq}\lambda$ and

$$A \not\subseteq B \Rightarrow M_{I_A} \not\cong M_{I_B}.$$

We choose $\left\langle \langle M^i_{I_A} : i < \lambda\rangle : A \subseteq \lambda \right\rangle$ with $M_{I_A} = \bigcup_{i<\lambda} M^i_{I_A}$, $\|M^i_{I_A}\| < \lambda$, $M^i_{I_A}$ increasing continuous.

Of course, we have to strengthen the restrictions on M_I. For $\eta \in I_A \cap {}^{\chi}\lambda$, let $\delta(\eta) := \bigcup\{\eta(i)+1 : i < \chi\}$. We are specially interested in η which are strictly increasing converging to some $\delta(\eta) \in \mathscr{U}_A$; we shall put only such η-s in I_A. The decision whether $\eta \in I_A$ will be done by induction on $\delta(\eta)$ for all sets A. Arriving to η, we assume we know quite a lot on the isomorphism $f : M_{I_A} \to M_{I_B}$: specifically, we know

$$f \restriction \bigcup_{\alpha<\chi} \bar{a}_{\eta\restriction\alpha},$$

which we are trying to "kill". We can assume $\delta(\eta) \notin \mathscr{U}_B$ and δ belongs to a thin enough club of λ, and using all this information we can "compute" what to do.

(Note: though this is the typical case, we do not always follow it.)

Notation 3.2. (1) For an ordinal α and a regular $\theta \geq \aleph_0$, let $\mathcal{H}_{<\theta}(\alpha)$ be the smallest set Y such that:

(i) $i \in Y$ for $i < \alpha$,
(ii) $x \in Y$ for $x \subseteq Y$ of cardinality $< \theta$.

(2) We can agree that $\mathscr{M}_{\lambda,\theta}(\alpha)$ from [17, 2.1=Lf2] is interpretable in $(\mathcal{H}_{<\theta}(\alpha), \in)$ when $\alpha \geq \lambda$, and in particular its universe is a definable subset of $\mathcal{H}_{<\theta}(\alpha)$, and also R is defined to be:

$$R = \Big\{(\sigma^*, \langle t_i : i < \gamma_x\rangle, x) : x \in \mathscr{M}_{\lambda,\theta}({}^{\theta>}\alpha), \sigma^* \text{ is a } \tau_{\lambda,\kappa}\text{-term}, \\ \theta \leq \lambda \leq \alpha, \text{ and } x = \sigma^*(\langle t_i : i < \gamma_x\rangle)\Big\}.$$

Similarly for $\mathscr{M}_{\lambda,\theta}(I)$, where $I \subseteq {}^{\kappa>}\lambda$ is interpretable in $(\mathcal{H}_{<\chi}(\lambda^*), \in)$ if $\lambda \leq \lambda^*$, $\theta \leq \chi$, and $\kappa \leq \chi$.

The main theorem of this section (see [17, 1.4(1)=La11]) is:

Theorem 3.3. *$\dot{I}\dot{E}_{\pm\varphi}(\lambda, K) = 2^\lambda$, provided that:*

(a) $\lambda = \lambda^\chi$
(b) $\varphi = \varphi(\ldots, \bar{x}_\alpha, \ldots)_{\alpha<\chi}$ *is a formula in the vocabulary* τ_K.

(c) *For every I such that ${}^{\chi>}\lambda \subseteq I \subseteq {}^{\chi\geq}\lambda$, we have a model $M_I \in K_\lambda$, a function f_I, and $\bar{a}_\eta \in {}^{\chi\geq}|M_I|$ for $\eta \in {}^{\chi>}\lambda$ with $\ell g(\bar{a}_\eta) = \ell g(\bar{x}_{\ell g(\eta)})$ such that:*

(α) *For $\eta \in {}^{\chi}\lambda$ we have $M_I \models \varphi(\dots, \bar{a}_{\eta\restriction\alpha}, \dots)$ if and only if $\eta \in I$.*

(β) *$f_I\colon M_I \to \mathscr{M}_{\mu,\kappa}(I)$, where $\mu \leq \lambda$ and $\kappa = \chi^+$.*

(γ) *If $\bar{b}_\alpha \in M_I$ is such that $\ell g(\bar{x}_\alpha) = \ell g(\bar{b}_\alpha)$ for $\alpha < \chi$ and $f_I(\bar{b}_\alpha) = \bar{\sigma}_\alpha(\bar{t}_\alpha)$ then:*

- *The truth value of $M_I \models \varphi[\dots, \bar{b}_\alpha, \dots]_{\alpha<\chi}$ can be computed from $\langle \bar{\sigma}_\alpha\colon \alpha < \chi\rangle$ and $\langle \bar{t}_\alpha\colon \alpha < \chi\rangle$ (not just its quantifier-free type in I) and from the truth values of statements of the form*

$$\big(\exists \nu \in I \cap {}^{\chi}\lambda\big)\Big[\bigwedge_{i<\chi} \nu \restriction \epsilon_i = \bar{t}_{\beta_i}(\gamma_i) \restriction \epsilon_i\Big]$$

for $\alpha_i, \beta_i, \gamma_i, \epsilon_i < \chi$ (i.e., in a way not depending on I or f_I). [We can weaken this.]

We shall first prove 3.3 under stronger assumptions.

Fact 3.4. Suppose

$(*)$ $\lambda = \lambda^{2^\chi}$ (so $\mathrm{cf}(\lambda) > \chi$) and $\chi \geq \kappa$.

Then there are $\big\{(M^\alpha, \eta^\alpha)\colon \alpha < \alpha(*)\big\}$ such that:

(i) For every model M with universe $\mathcal{H}_{<\chi^+}(\lambda)$ such that $|\tau(M)| \leq \chi$ (and, e.g., $\tau \subseteq \mathcal{H}_{<\chi^+}(\lambda)$), for some α, we have $M^\alpha \prec M$.

(ii) $\eta^\alpha \in {}^{\chi}\lambda$, $(\forall i < \chi)[\eta^\alpha \restriction i \in M^\alpha]$, $\eta^\alpha \notin M^\alpha$, and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \eta^\beta$.

(iii) For every $\beta < \alpha < \alpha(*)$ we have $\{\eta^\alpha \restriction i\colon i < \chi\} \nsubseteq M^\beta$.

(iv) For $\beta < \alpha$, if $\{\eta^\beta \restriction i\colon i < \chi\} \subseteq M^\alpha$ then $|M^\beta| \subseteq |M^\alpha|$.

(v) $\|M^\alpha\| = \chi$.

Proof. By 4.20 + 4.21 below, with $\lambda, 2^\chi, \chi$ here standing for $\lambda, \chi(*), \theta$ there.

Proof of 3.3 from the Conclusion of 3.4.

Without loss of generality, the universe of M_I is λ in 3.3.

We shall define, for every $A \subseteq \lambda$, a set $I[A]$ satisfying ${}^{\chi>}\lambda \subseteq I[A] \subseteq {}^{\chi\geq}\lambda$; moreover,

$$I[A] \setminus {}^{\chi>}\lambda \subseteq \big\{\eta^\alpha\colon \alpha < \alpha(*)\big\}.$$

For $\alpha < \alpha(*)$, let $\mathscr{U}_\alpha = \big\{\eta \in {}^{\chi}\lambda\colon \{\eta \restriction i\colon i < \chi\} \subseteq M^\alpha\big\}$. We shall define, for every $A \subseteq \lambda$, the set $I[A] \cap \mathscr{U}_\alpha$ by induction on α so that on the one hand, those restrictions are compatible (i.e. in the end we can still define I[A] for each $A \subseteq \lambda$), and on the other hand they guarantee the non-($\pm\varphi$)-embeddability.

For each α, we argue as follows. Essentially, we decide whether $\eta^\alpha \in I[A]$, assuming that M^α correctly "guesses" both a function $g\colon M_{I_1} \longrightarrow M_{I_2}$ (where $I_\ell = I[A_\ell]$) and the set $A_\ell \cap M^\alpha$ for $\ell = 1, 2$, and we make our decision to prevent this.

Case I. There are distinct subsets A_1, A_2 of λ and I_1, I_2 satisfying ${}^{\chi>}\lambda \subseteq I_\ell \subseteq$ $\subseteq {}^{\chi\geq}\lambda$, a ($\pm\varphi$)-embedding g of M_{I_1} into M_{I_2}, and

$$M^\alpha \prec \big(\mathcal{H}_{<\chi^+}(\lambda), \in, R, A_1, A_2, I_1, I_2, M_{I_1}, M_{I_2}, f_{I_1}, f_{I_2}, g\big),$$

where

$$R = \Big\{\big\{(0, \sigma_x, x), (1+i, t_i^x, x)\big\} : i < i_x \text{ and } x \text{ has the form } \sigma_x(\langle t_i^x : i < i_x\rangle)\Big\}$$

(we choose for each x a unique such term σ), $I_2 \cap \mathscr{U}_\alpha \subseteq I_2 \cap (\bigcup_{\beta<\alpha} \mathscr{U}_\beta)$, and I_1, I_2 satisfy the restrictions we already have imposed on $I[A_1]$, $I[A_2]$ respectively, for each $\beta < \alpha$. Computing the truth value of $M_{I_2} \models \varphi[\ldots, f(\bar{a}_{\eta^\alpha\restriction i}), \ldots]_{i<\chi}$ according to clause 3.3(d) (assuming $I_2 \cap \mathscr{U}^\alpha \subseteq \bigcup_{\beta<\alpha} \mathscr{U}^\beta$), we get $\mathbf{t}^\alpha$.

Then we restrict:

(i) If $B \subseteq \lambda$ and $B \cap |M^\alpha| = A_2 \cap |M^\alpha|$ then $I[B] \cap \big(\mathscr{U}^\alpha \setminus \bigcup_{\beta<\alpha} \mathscr{U}^\beta\big) = \emptyset$.

(ii) If $B \subseteq \lambda$, $B \cap |M^\alpha| = A_1 \cap |M^\alpha|$, and $\mathbf{t}^\alpha$ is true, then

$$I[B] \cap \big(\mathscr{U}^\alpha \setminus \bigcup_{\beta<\alpha} \mathscr{U}^\beta\big) = \emptyset$$

or just $\eta^\alpha \notin I[B]$.

(iii) If $B \subseteq \lambda$, $B \cap |M^\alpha| = A_1 \cap |M^\alpha|$, and $\mathbf{t}^\alpha$ is false, then

$$I[B] \cap \big(\mathscr{U}^\alpha \setminus \bigcup_{\beta<\alpha} \mathscr{U}^\beta\big) = \{\eta^\alpha\}$$

or just $\eta^\alpha \in I[B]$.

Case II. Not Case I.

No restriction is imposed. ∎

The point of this is the two facts below, which should be clear.

Fact 3.5. The choice of A_1, A_2, I_1, I_2, g is immaterial (any two candidates lead to the same decision).

Proof. Use clause (d) of 3.3. ∎

Fact 3.6. The $M_{I[A]}$ (for $A \subseteq \lambda$) are pairwise non-isomorphic. Moreover, for $A \neq B \subseteq \lambda$ there is no $(\pm\varphi)$-embedding of $M_{I[A]}$ into $M_{I[B]}$.

Proof. By the choice of the $I[A]$-s and 3.4(i). ∎

* * *

Still, the assumption of 3.4 is too strong: it does not cover all the desirable cases, though it covers many of them. However, a statement weaker than the conclusion of 3.4 holds under weaker cardinality restrictions and the proof of 3.3 above works using it, thus we will finish the proof of 3.3.

Fact 3.7. Suppose $\lambda = \lambda^\chi$.

Then there are $\{(M^\alpha, A_1^\alpha, A_2^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ such that:

$(*)$ (i) For every model M with universe $\mathcal{H}_{<\chi^+}(\lambda)$ such that $|\tau(M)| \leq \chi$ and $\tau(M) \subseteq \mathcal{H}_{<\chi^*}(\lambda)$ (with arity of relations and functions finite) and sets $A_1 \neq A_2 \subseteq \lambda$, for some $\alpha < \alpha(*)$, we have

$$(M^\alpha, A_1^\alpha, A_2^\alpha) \prec (M, A_1, A_2).$$

(ii) $\eta^\alpha \in {}^\chi\lambda$, $\{\eta^\alpha \restriction i : i < \chi\} \subseteq |M^\alpha|$, $\eta^\alpha \notin M^\alpha$, and $\alpha \neq \beta \Rightarrow \eta^\alpha \neq \neq \eta^\beta$.

(iii) For every $\beta < \alpha(*)$, if $\{\eta^\alpha \restriction i : i < \chi\} \subseteq M^\beta$, then $\alpha < \beta + 2^\chi$. Furthermore, $\alpha + 2^\chi = \beta + 2^\chi$ implies $A_1^\alpha \cap |M^\alpha| \neq A_2^\beta \cap |M^\alpha|$.

(iv) For every $\beta < \alpha$, if $\{\eta^\beta \restriction i : i < \chi\} \subseteq M^\alpha$, then $|M^\beta| \subseteq |M^\alpha|$.

(v) $\|M^\alpha\| = \chi$.

Proof. See 4.46. ∎

Proof of Theorem 3.3. Should be clear. We act as in the proof of 3.3 from the conclusion of 3.4 but now we have to use the "or just" version in (ii),(iii) there. ∎

Conclusion 3.8. *(1) If $T \subseteq T_1$ are complete first order theories, T is in the vocabulary τ, $\kappa = \mathrm{cf}(\kappa) < \kappa(T)$ (hence T is un-superstable), and $\lambda = \lambda^{\aleph_0} \geq |T_1|$, then $\dot{\mathbb{I}}_\tau(\lambda, T_1) = 2^\lambda$. (For more on $\dot{\mathbb{I}}_\tau$, see* [17]*.)*

(2) Assume $\kappa = \mathrm{cf}(\kappa)$, Φ is proper and almost nice for K_{tr}^κ (see [17, 1.7]*), $\bar\sigma^i$ $(i \leq \kappa)$ is a finite sequence of terms, $\tau \subseteq \tau_\Phi$, $\varphi_i(\bar x, \bar y)$ is first order in $\mathscr{L}[\tau]$, and for $\nu \in {}^i\lambda$, $\eta \in {}^\kappa\lambda$, $\nu \lhd \eta$ we have that*

$$\mathrm{EM}({}^\kappa\lambda, \Phi) \models \varphi_i\big(\bar\sigma_i^\kappa(x_\eta), \bar\sigma^{i+1}(x_{\eta^\frown\langle\alpha\rangle})\big)$$

holds if and only if $\alpha = \eta(i)$. *Then*

$$\left|\{EM_\tau(S,\Phi)/\cong : {}^{\kappa>}\lambda \subseteq S \subseteq {}^{\kappa\geq}\lambda\}\right| = 2^\lambda.$$

Proof. (1) By [17, 1.10] there is a template Φ which is proper for K^κ_{tr}, as required in part (2).

(2) By 3.3. ∎

Discussion 3.9. What about Theorem 3.3 in the case we assume only $\lambda = \lambda^{<\chi}$? There is some information in [24, Ch.VIII,§2].

Of course, concerning un-superstable T, that is 3.8, more is done there: the assumption is just $\lambda > |T|$.

Claim 3.10. *In 3.3, we can restrict ourselves to* I *such that* $I^0_{\lambda,\chi} \subseteq I \subseteq {}^{\chi\geq}\lambda$, *where*

$$I^0_{\lambda,\chi} = {}^{\chi>}\lambda \cup \{\eta \in {}^\chi\lambda : \eta(i) = 0 \text{ for every } i < \chi \text{ large enough}\}.$$

Proof. By renaming. ∎

4. Black boxes

We try to give comprehensive treatment of black boxes: quite a few few of them are useful in some contexts and some parts are redone here, as explained in §0,§1.

Note that "omitting countable types" is a very useful device for building models of cardinality $\aleph_0$ and $\aleph_1$. The generalization to models of higher cardinality, λ or λ^+, usually requires us to increase the cardinality of the types to λ, and even so we may encounter problems (see [15] and background there). Note that we do not look mainly at the omitting type theorem *per se*, but at its applications.

Jensen defined square and proved existence in **L**: in Facts 4.1–4.8, we deal with slightly weaker related principles which can be proved in ZFC. E.g. for λ regular $> \aleph_1$, $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ is the union of λ sets, each has square (as defined there). You can skip them in first reading — particularly 4.1 (and later take the references on faith).

Then we deal with black boxes. In 4.12 we give the simplest case: λ regular $> \aleph_0$, $\lambda = \lambda^{<\chi(*)}$. (Really, $\lambda^{<\theta} = \lambda^{<\chi(*)}$ is almost the same.) In 4.12 we also assume "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is a good stationary set". In 4.16 we weaken this demand such that enough sets S as required exist (provably in ZFC!). The strength of the cardinality hypothesis ($\lambda = \lambda^{<\chi(*)}$, $\lambda^{<\theta} = \lambda^{<\chi(*)}$, $\lambda^{\theta} = \lambda^{<\chi(*)}$) vary the conclusion. In 4.14–4.17 we prepare the ground for replacing "λ regular" by "$\mathrm{cf}(\lambda) \geq \chi(*)$", which is done in 4.18.

As we noted in §2, it is much nicer to deal with $(\overline{M}^{\beta}, \eta^{\beta})$, this is the first time we deal with η^{β}, i.e., for no $\alpha < \beta$,

$$\{\eta^{\beta} \restriction i : i < \theta\} \subseteq \bigcup_{i<\theta} M_i^{\alpha}.$$

In 4.20, 4.21 (parallel to 4.12, 4.18, respectively) we guarantee this, at the price of strengthening $\lambda^{<\theta} = \lambda^{<\chi(*)}$ to

$$\lambda^{<\theta} = \lambda^{\chi(1)}, \chi(1) = \chi(*) + (<\chi(*))^{\theta}.$$

Later, in 4.46, we draw the conclusion necessary for section 2 (in its proof the function h, which may look redundant, plays the major role). This (as well as 4.20, 4.21) exemplifies how those principles are self propagating — better ones follow from the old variant (possibly with other parameters).

In 4.22–4.27 we deal with the black boxes when θ (the length of the game) is $\aleph_0$. We use a generalization of the Δ-system lemma for trees and partition theorems on trees.[4] We get several versions of the black box — as the cardinality restriction becomes more severe, we get a stronger principle.

It would be better if we can use, for a strong limit $\kappa > \aleph_0 = \mathrm{cf}(\kappa)$,

$$\kappa^{\aleph_0} = \sup\Big\{\lambda : \text{ for some } \kappa_n < \kappa \text{ and uniform ultrafilter } D \text{ on } \omega, \mathrm{cf}\Big(\prod_{n<\omega} \kappa_n / D\Big) = \lambda\Big\}.$$

We know this for the uncountable cofinality case (see [32] or [40]), but then there are other obstacles. Now [39] gives a partial remedy, but lately by [41] there are many such cardinals.

In 4.41, 4.42 we deal with the case $\mathrm{cf}(\lambda) \leq \theta$. Note that $\mathrm{cf}(\lambda^{<\chi(*)}) \geq \chi(*)$ is always true, so you may wonder: why wouldn't we replace λ by $\lambda^{<\chi(*)}$? This is true in many applications: but is not true, for example, when we want to construct structures with density character λ.

[4]See Rubin–Shelah [7, §4], [26, Ch.XI] = [44, Ch.XI], [12, 1.10=L1.7], [12, 1.16=L1.15] and the proof of 4.24 here; see history there, and 4.6.

Several times, we use results quoted from [8, §2], but there are no dependency loops. The pcf results quoted here are gathered in [12, §3], so we will refer to it throughout in addition to quoting the original place.

We end with various remarks and exercises.

4.1. On stationary sets

FACT 4.1. (1) If $\mu^\chi = \mu < \lambda \leq 2^\mu$, χ and λ are regular uncountable cardinals, and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$ is a stationary set, then there are a stationary set $W \subseteq \chi$ and functions $h_a, h_b : \lambda \to \mu$ and $\langle S_\zeta : 0 < \zeta < \lambda \rangle$ such that:

(a) $S_\zeta \subseteq S$ is stationary.
(b) $\xi \neq \zeta \Rightarrow S_\xi \cap S_\zeta = \emptyset$
(c) If $\delta \in S_\xi$, then for some increasing continuous sequence $\langle \alpha_i : i < \chi \rangle$ we have $\delta = \bigcup_{i<\chi} \alpha_i$, $h_b(\alpha_i) = i$, $h_a(\alpha_i) \in \{\xi, 0\}$, and the set $\{i < \chi : h_a(\alpha_i) = \xi\}$ is stationary (in fact, it is W).

(2) If in (1), a sequence $\langle C_\delta : \delta < \lambda, \mathrm{cf}(\delta) \leq \chi \rangle$ satisfying

$$(\forall \alpha \in C_\delta)\big[\alpha \text{ limit } \Rightarrow \ \alpha = \sup(\alpha \cap C_\delta)\big]$$

is given, where C_δ is a closed unbounded subset of δ of order type $\mathrm{cf}(\delta)$, then in the conclusion we can get also S^* and $\langle C^*_\delta : \delta \in S^* \rangle$ such that in addition to (a)–(c), we have:

(c)′ In (c), we add $C_\delta = \{\alpha_i : i < \chi\}$.
(d) $\bigcup_{0<\xi<\lambda} S_\xi \subseteq S^* \subseteq \bigcup_{0<\xi<\lambda} S_\xi \cup \{\delta < \lambda : \mathrm{cf}(\delta) < \chi\}$
(e) $W \subseteq \chi$ is $(> \aleph_0)$-closed and stationary in cofinality $\aleph_0$, which means:
 (*i*) If $i < \chi$ is a limit ordinal such that $i = \sup(i \cap W)$ has cofinality $> \aleph_0$ then $i \in W$.
 (*ii*) $\{i \in W : \mathrm{cf}(i) = \aleph_0\}$ is a stationary[5] subset of χ.
(f) for $\delta \in \bigcup_{0<\xi<\lambda} S_\xi$ we have

$$C^*_\delta = \{\alpha \in C_\delta : \mathrm{otp}(\alpha \cap C_\delta) = \sup(W \cap \mathrm{otp}(\alpha \cap C_\delta))\}$$

(g) C^*_δ is a club of δ included in C_δ for $\delta \in S^*$, and if $\delta(1) \in C^*_\delta$, $\delta \in S^*$, $\delta \in \bigcup_{0<\zeta<\lambda} S_\zeta$, $\delta(1) = \sup(\delta(1) \cap C^*_\delta)$, and $\mathrm{cf}(\delta(1)) > \aleph_0$ then $C^*_{\delta(1)} \subseteq C^*_\delta$,
(h) If C is a closed unbounded subset of λ and $0 < \xi < \lambda$ then the set $\{\delta \in S_\xi : C^*_\delta \subseteq C\}$ is stationary.

[5]We can add '$\notin I$' if I is any normal ideal on $\{i < \chi : \mathrm{cf}(i) = \aleph_0\}$.

Proof. (1) We can find $\{\langle h^1_\xi, h^2_\xi\rangle : \xi < \mu\}$ such that:

(1) For every ξ we have $h^1_\xi : \lambda \to \mu$ and $h^2_\xi : \lambda \to \mu$.
(2) If $A \subseteq \lambda$, $|A| \le \chi$, and $h^1, h^2 : A \to \mu$ then for some ξ, $h^1_\xi \restriction A = h^1$ and $h^2_\xi \restriction A = h^2$.

This holds by Engelking–Karlowicz [3].[6]

(2) For $\alpha < \lambda$, let $C^\bullet_\alpha$ be a closed unbounded subset of α of order type $\mathrm{cf}(\alpha)$. Now for each $\xi < \mu$ and for $a \subseteq \chi$ stationary, we ask whether for every $i < \lambda$, for some $j < \lambda$, we have

$(*)^{\xi,a}_{i,j}$ The following subset of λ is stationary:

$$S^{\xi,a}_{i,j} = \big\{\delta \in S\colon \begin{array}{ll} \text{(i)} & \text{if } \alpha \in C_\delta, \mathrm{otp}(\alpha \cap C_\delta) \notin a \text{ then } h^1_\xi(\alpha) = 0, \\ \text{(ii)} & \text{if } \alpha \in C_\delta, \mathrm{otp}(\alpha \cap C_\delta) \in a \text{ then the } h^1_\xi(\alpha)\text{-th} \\ & \text{member of } C_\alpha \text{ belongs to } [i, j), \\ \text{(iii)} & \text{if } \alpha \in C_\delta \text{ then } h^2_\xi(\alpha) = \mathrm{otp}(\alpha \cap C_\delta)\} \end{array}$$

Subfact 4.2. For some $\xi < \mu$ and a stationary set $a \subseteq \chi$, for every $i < \lambda$, for some $j \in (i, \lambda)$, the statement $(*)^{\xi,a}_{i,j}$ holds.

Proof. If not, then for every $\xi < \mu$ and a stationary $a \subseteq \chi$, for some $i = i(\xi, a) < \lambda$, for every j such that $i(\xi, a) < j < \lambda$, there is a closed unbounded subset $C(\xi, a, i, j)$ of λ disjoint from $S^{\xi,a}_{i,j}$.

Let

$$i(*) = \bigcup\big\{i(\xi, a) + \omega\colon \xi < \mu \text{ and } a \subseteq \chi \text{ is stationary}\big\}.$$

Clearly $i(*) < \lambda$.

For $i(*) \le j < \lambda$, let

$$C(j) = \bigcap\big\{C(\xi, a, i(\xi, a), j)\colon a \subseteq \chi \text{ is stationary and } \xi < \mu\big\} \cap \big(i(*) + \omega, \lambda\big).$$

Clearly it is a closed unbounded subset of λ.

Let

$$C^* = \big\{\delta < \lambda\colon \delta > i(*) \text{ and } (\forall j < \delta)\big[\delta \in C(j)\big]\big\}.$$

So C^* is a closed unbounded subset of λ as well. Let C^+ be the set of accumulation points of C^*. Choose $\delta(*) \in C^+ \cap S$, and we shall define

$$h^1\colon C_{\delta(*)} \to \mu, \quad h^2\colon C_{\delta(*)} \to \mu.$$

[6] See for example [36, AP]; on history see e.g. [43, §5]

For $\alpha \in C_{\delta(*)}$, let $h^0(\alpha)$ be:

$$\min\{\gamma \in (0,\chi)\colon \text{the } \gamma^{\text{th}} \text{ member of } C^\bullet_\alpha \text{ is} > i(*)\}$$

if $\alpha = \sup(C_{\delta(*)} \cap \alpha) > i(*)$, and zero otherwise. Clearly the set

$$\{\alpha \in C_{\delta(*)}\colon \ h^0(\alpha) = 0\}$$

is not stationary. Now we can define $g\colon C_{\delta(*)} \to \delta(*)$ by:

$$g(\alpha) \text{ is the } h^0(\alpha)^{\text{th}} \text{ member of } C_\alpha.$$

Note that g is pressing down and $\{\alpha \in C_{\delta(*)}\colon g(\alpha) \le i(*)\}$ is not stationary. So (by the variant of Fodor's Lemma speaking on an ordinal of uncountable cofinality) for some $j < \sup(C_{\delta(*)}) = \delta(*)$, the set

$$a := \{\alpha \in C_{\delta(*)} \cap C^*\colon i(*) < g(\alpha) < j\}$$

is a stationary subset of $\delta(*)$. Let $h^1\colon C_{\delta(*)} \to \mu$ be

$$h^1(\alpha) = \begin{cases} 0 & \text{if } \operatorname{otp}(\alpha \cap C^\bullet_\delta) \notin a \\ h^0(\alpha) & \text{if } \operatorname{otp}(\alpha \cap C^\bullet_\delta) \in a. \end{cases}$$

Let $h^2\colon C_{\delta(*)} \to \mu$ be $h^2(\alpha) = \operatorname{otp}(\alpha \cap C_{\delta(*)})$. By the choice of $\langle (h^1_\xi, h^2_\xi)\colon \xi < \mu\rangle$, for some ξ we have $h^1_\xi \restriction C_{\delta(*)} = h^1$ and $h^2_\xi \restriction C_{\delta(*)} = h^2$. Easily, $\delta(*) \in S^{\xi,a}_{i,j}$ which is disjoint to $C(\xi, a, i(*), j)$, contradicting $\delta(*) \in C^*$ by the definition of $C(j)$ and C^*.

So we have proved Subfact 4.2. ∎

Continuing the proof of 4.1

Having chosen ξ and a, we define an ordinal $i(\zeta) < \lambda$ by induction on $\zeta < \lambda$ such that $\langle i(\zeta)\colon \zeta < \lambda\rangle$ is increasing continuous, $i(0) = 0$, and $(*)^{\xi,a}_{i(\zeta),i(\zeta+1)}$ holds.

Now, for $\alpha < \lambda$ we define $h_a(\alpha)$ as follows: it is ζ if $h^1_\xi(\alpha) > 0$ and the $h^1_\xi(\alpha)^{\text{th}}$ member of $C^\bullet_\alpha$ belongs to $[i(1+\zeta), i(1+\zeta+1))$, and it is zero otherwise. Lastly, let $h_b(\alpha) := h^2_\xi(\alpha)$ and $W = a$ and

$$\begin{aligned} S_\zeta := \{\delta \in S\colon & \text{(i)} \ \ \text{for } \alpha \in C_\delta, \operatorname{otp}(\alpha \cap C_\delta) = h_b(\alpha), \\ & \text{(ii)} \ \text{for } \alpha \in C_\delta, h_b(i) \in a \ \Rightarrow \ h_a(\alpha) = \zeta, \\ & \text{(iii)} \ \text{for } \alpha \in C_\delta, h_b(i) \notin a \ \Rightarrow h_a(i) = 0\}. \end{aligned}$$

Now, it is easy to check that a, h_a, h_b, and $\langle S_\zeta : 0 < \zeta < \lambda\rangle$ are as required.

(2) In the proof of 4.1(1) we shall now consider only sets $a \subseteq \chi$ which satisfy the demand on W from 4.1(2)(e). (This makes a difference in the definition of $C(j)$ during the proof of Subfact 4.2.) Also, in $(*)^{\xi,a}_{i,j}$ in the definition of $S^{\xi,a}_{i,j}$, we change (iii) to:

(iii)′ If $\alpha \in C_\delta$ then $h^2_\xi(\alpha)$ codes the isomorphism type of (for example)

$$\Big(C^\bullet_\delta \cup \bigcup_{\beta \in C_\delta} C_\beta, <, \alpha, C^\bullet_\delta, \big\{\langle i, \beta\rangle : i \in C_\beta\big\}\Big).$$

In the end, having chosen ξ and a we can define C^*_δ and S^* in the natural way. ∎

FACT 4.3. (1) If λ is regular $> 2^\kappa$, κ regular, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ is stationary, and C^0_δ is a club of δ of order type κ $(= \mathrm{cf}(\delta))$ for $\delta \in S$, then we can find a club c^* of κ (see 4.4(1) below) such that for $\delta \in S$,

$$C_\delta = C^0_\delta[c^*] := \{\alpha \in C^0_\delta : \mathrm{otp}(C^0_\delta \cap \alpha) \in c^*\}.$$

It is a club of δ, and:

(∗) For every club $C \subseteq \lambda$, we have:
- (a) If $\kappa > \aleph_0$ then $\{\delta \in S : C_\delta \subseteq C\}$ is stationary.
- (b) If $\kappa = \aleph_0$, then the set

$$\big\{\delta \in S : (\forall \alpha, \beta \in C_\delta)[\alpha < \beta \Rightarrow (\alpha, \beta) \cap C \neq \emptyset]\big\}$$

is stationary.

(2) If λ is a regular cardinal $> 2^\kappa$, then we can find $\big\langle \langle C^\zeta_\delta : \delta \in S_\zeta\rangle : \zeta < 2^\kappa\big\rangle$ such that:

(1) $\bigcup\{S_\zeta : \zeta < 2^\kappa\} = \{\delta < \lambda : \aleph_0 < \mathrm{cf}(\delta) \leq \kappa\}$
(2) C^ζ_δ is a club of δ of order type $\mathrm{cf}(\delta)$.
(3) If $\alpha \in S_\zeta$, $\mathrm{cf}(\alpha) > \theta > \aleph_0$, then

$$\big\{\beta \in C^\zeta_\alpha : \mathrm{cf}(\beta) = \theta, \beta \in S_\zeta \text{ and } C^\zeta_\beta \subseteq C^\zeta_\alpha\big\}$$

is a stationary subset of α.

(3) If λ is regular and $2^\mu \geq \lambda > \mu^\kappa$ then we can find $\big\langle \langle C^\zeta_\delta : \delta \in S_\zeta\rangle : \zeta < \mu\big\rangle$ such that:

(1) $\bigcup\{S_\zeta : \zeta < 2^\kappa\} = \{\delta < \lambda : \aleph_0 < \mathrm{cf}(\delta) \leq \kappa\}$
(2) C^ζ_δ is a club of δ of order type $\mathrm{cf}(\delta)$.
(3) If $\alpha \in S_\zeta$, $\beta \in C^\zeta_\alpha$, $\mathrm{cf}(\beta) > \aleph_0$, then $\beta \in S_\zeta$ and $C^\zeta_\beta \subseteq C^\zeta_\alpha$.

(4) Moreover, if $\alpha, \beta \in S_\zeta$ and $\beta \in C^\zeta_\alpha$ then

$$\big\{\big(\mathrm{otp}(\gamma \cap C^\zeta_\beta), \mathrm{otp}(\gamma \cap C^\zeta_\alpha)\big) : \gamma \in C_\beta\big\}$$

depends only on $\big(\mathrm{otp}(\beta \cap C_\alpha), \mathrm{otp}(C_\alpha)\big)$.

(4) We can, in clauses (1)(∗)(a)-(b), replace "stationary" by "$\notin I$" for any normal ideal I on λ.

Remark 4.4. (1) Here a club C of δ, where $\mathrm{cf}(\delta) = \aleph_0$, just means an unbounded subset of δ.

(2) In 4.3(1) instead of 2^κ, the cardinal

$$\min\big\{|\mathscr{F}| : \mathscr{F} \subseteq {}^\kappa\kappa \ \wedge \ (\forall g \in {}^\kappa\kappa)(\exists f \in \mathscr{F})(\forall \alpha < \kappa)\big[g(\alpha) < f(\alpha)\big]\big\}$$

suffices.

(3) In 4.3(1)(∗)(b) above, it is equivalent to ask that

$$\big\{\delta \in S : (\forall \alpha, \beta \in C_\delta)[\alpha < \beta \implies \mathrm{otp}((\alpha, \beta) \cap C) > \alpha]\big\}$$

is stationary.

Proof of Fact 4.3. (1) If 4.3(1) fails, then for each club c^* of κ there is a club $C[c^*]$ of λ exemplifying its failure. So $C^+ := \bigcap\{C[c^*] : c^* \subseteq \kappa$ a club$\}$ is a club of λ. Choose a $\delta \in S$ which is an accumulation point of C^+, and get a contradiction easily.

(2) Let $\lambda = \mathrm{cf}(\lambda) > 2^\kappa$, and let C_α be a club of α of order type $\mathrm{cf}(\alpha)$ for each limit $\alpha < \lambda$. Without loss of generality

$$\beta \in C_\alpha \ \wedge \ \beta > \sup(\beta \cap C_\alpha) \implies \beta \text{ is a successor ordinal.}$$

For any sequence $\bar{c} = \langle c_\theta : \aleph_0 < \theta = \mathrm{cf}(\theta) \leq \kappa\rangle$ such that each c_θ is a club of θ, for $\delta \in S^* = \{\alpha < \lambda : \aleph_0 < \mathrm{cf}(\alpha) \leq \kappa\}$ we let:

$$C^{\bar{c}}_\delta = \{\alpha \in C_\delta : \mathrm{otp}(C_\delta \cap \alpha) \in c_{\mathrm{cf}(\delta)}\}.$$

Now to define $S_{\bar{c}}$, we define the set $S_{\bar{c}} \cap \delta$ by induction on $\delta < \lambda$: the only problem is to define whether $\alpha \in S_{\bar{c}}$ knowing $S_{\bar{c}} \cap \delta$. We stipulate

$\alpha \in S_{\bar{c}}$ if and only if (i) $\aleph_0 < \mathrm{cf}(\alpha) \leq \kappa$

(ii) If $\aleph_0 < \theta = \mathrm{cf}(\theta) < \mathrm{cf}(\alpha)$ then the set $\{\beta \in C^{\bar{c}}_\alpha : \mathrm{cf}(\beta) = \theta, \beta \in S_{\bar{c}} \cap \alpha\}$ is stationary in α.

Let $\langle \bar{c}^\zeta : \zeta < 2^\kappa \rangle$ list the possible sequences $\bar{c}$, and let $S_\zeta = S_{\bar{c}^\zeta}$ and $C^\zeta_\delta = C^{\bar{c}^\zeta}_\delta$. To finish, note that for each $\delta < \lambda$ satisfying $\aleph_0 < \mathrm{cf}(\delta) \leq \kappa$, we have $\delta \in S_\zeta$ for some ζ.

(3) Combine the proof of (2) and of 4.1.

(4) Similarly. ∎

We may remark

FACT 4.5. Suppose that λ is a regular cardinal $> 2^\kappa$, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, a set

$$S \subseteq \{\delta < \lambda \colon \mathrm{cf}(\delta) = \kappa\}$$

is stationary, and I is a normal ideal on λ with $S \notin I$. If I is λ^+-saturated (i.e. in the Boolean algebra $\mathcal{P}(\lambda)/I$, there is no family of λ^+ pairwise disjoint elements), then we can find $\langle C_\delta \colon \delta \in S \rangle$, C_δ a club of δ of order type $\mathrm{cf}(\delta)$, such that:

(∗) For every club C of λ we have $\{\delta \in S \colon C_\delta \setminus C$ is unbounded in $\delta\} \in I$.

PROOF. For $\delta \in S$, let C'_δ be a club of δ of order type $\mathrm{cf}(\delta)$. Call $\bar{C} = \langle C_\delta \colon \delta \in S^* \rangle$ (where $S^* \subseteq S \subseteq \lambda$ stationary, $S^* \notin I$, C_δ a club of δ) *I-large* if for every club C of λ, the set

$$\{\delta < \lambda \colon \delta \in S^* \text{ and } C_\delta \setminus C \text{ is bounded in } \delta\}$$

does not belong to I.

We call $\bar{C}$ *I-full* if above $\{\delta \in S^* \colon C_\delta \setminus C$ unbounded in $\delta\} \in I$.

By 4.3(4), for every stationary $S' \subseteq S$ with $S' \notin I$, there is a club c^* of κ such that $\langle C'_\delta[c^*] \colon \delta \in S' \rangle$ is I-large.

Now note:

(∗) If $\langle C_\delta \colon \delta \in S' \rangle$ is I-large and $S' \subseteq S$, then for some $S'' \subseteq S'$ such that $S'' \notin I$, $\langle C_\delta \colon \delta \in S'' \rangle$ is I-full (hence $S'' \notin I$).

PROOF OF (∗). Choose, by induction on $\alpha < \lambda^+$, a club C^α of λ such that:

(1) For $\beta < \alpha$, $C^\alpha \setminus C^\beta$ is bounded in λ.
(2) If $\beta = \alpha + 1$ then $A_\beta \setminus A_\alpha \in I^+$, where

$$A_\gamma := \{\delta \in S' \colon C_\delta \setminus C^\gamma \text{ is unbounded in } \delta\}.$$

As clearly

$$\beta < \alpha \implies A_\beta \setminus A_\alpha \text{ is bounded in } \lambda$$

(by (a) and the definition of A_α, A_β) and as I is λ^+-saturated, clearly for some α we cannot define C^α. This cannot be true for $\alpha = 0$ or a limit α, so necessarily

$\alpha = \beta + 1$. Now $S' \setminus A_\beta$ is not in I as $\bar{C}$ was assumed to be I-large. Check that $S'' := S' \setminus A_\beta$ is as required. ∎

Repeatedly using 4.3(4) and $(*)$, we get the conclusion. ∎

Claim 4.6. *Suppose $\lambda = \mu^+$, $\mu = \mu^\chi$, χ is a regular cardinal and*

$$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$$

is stationary. Then we can find S^*, $\langle C_\delta : \delta \in S^* \rangle$, and $\langle S_\xi : \xi < \lambda \rangle$ such that:

- (a) $\bigcup_{\zeta < \mu} S_\zeta \subseteq S^* \subseteq S \cup \{\delta < \lambda : \mathrm{cf}(\delta) < \chi\}$
- (b) *$S_\zeta \cap S$ is a stationary subset of λ for each $\zeta < \mu$.*
- (c) *For $\alpha \in S^*$, C_α is a closed subset of α of order type $\le \chi$. If $\alpha \in S^*$ is a limit then C_α is unbounded in α (so it is a club of α).*
- (d) *$\langle C_\alpha : \alpha \in S_\zeta \rangle$ is a square on S_ζ; i.e. S_ζ is stationary in $\sup(S_\zeta)$ and:*
  - (i) *C_α is a closed subset of α, unbounded if α is limit.*
  - (ii) *If $\alpha \in S_\zeta$ and $\alpha(1) \in C_\alpha$ then $\alpha(1) \in S_\zeta$ and $C_{\alpha(1)} = C_\alpha \cap \alpha(1)$.*
- (e) *For each club C of λ and $\zeta < \mu$, we have $C_\delta \subseteq C$ for some $\delta \in S_\zeta$.*

Proof. Similar to the proof of 4.1 (or see [33]). Alternatively, see 4.8 below (using 4.10(1) for clause (e)). ∎

We shall use the following in 4.27.

Claim 4.7. *Suppose $\lambda = \mu^+$, γ a limit ordinal of cofinality χ,*

$$h : \gamma \longrightarrow \{\theta : \theta = 1 \text{ or } \theta = \mathrm{cf}(\theta) \le \mu\},$$

$\mu = \mu^{|\gamma|}$, and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \chi\}$ is stationary. Then we can find S^*, $\langle C_\delta : \delta \in S^* \rangle$ and $\langle S_\zeta : \zeta < \lambda \rangle$ such that:

- (a) $\bigcup_{\zeta < \lambda} S_\zeta \subseteq S^* \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) \le \chi\}$
- (b) *$S_\zeta \cap S$ is stationary for each $\zeta < \lambda$.*
- (c) *For $\delta \in S^*$,*
  - (i) *C_δ is a club of δ of order type $\le \gamma$ and*
  - (ii) *$\mathrm{otp}(C_\delta) = \gamma$ iff $\delta \in S \cap S^*$,*
  - (iii) *$\alpha \in C_\delta \wedge \sup(C_\delta \cap \alpha) < \alpha \implies \alpha$ has cofinality $h[\mathrm{otp}(C_\delta \cap \alpha)]$.*
- (d) *If $\delta \in S_\zeta$ and $\delta(1)$ is a limit ordinal $\in C_\delta$ then $\delta(1) \in S_\zeta$ and $C_{\delta(1)} = C_\delta \cap \delta(1)$.*
- (e) *For each club C of λ and $\zeta < \lambda$, for some $\delta \in S_\zeta$, $C_\delta \subseteq C$.*

Proof. Like 4.6. ∎

Claim 4.8. *(1) Suppose λ is regular $> \aleph_1$. Then $\{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ is a good stationary subset of λ^+. (I.e., it is in $\check{I}[\lambda^+]$: see* [12, 3.4=Lcd1.1] *or* [47, 0.6,0.7] *or 4.9(2) below.)*

(2) Suppose λ is regular $> \aleph_1$. Then we can find $\langle S_\zeta : \zeta < \lambda \rangle$ such that:

(a) $\bigcup_{\zeta<\lambda} S_\zeta = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda\}$

(b) *On each S_ζ there is a square (see clause 4.6(d)). Say it is $\langle C^\zeta_\alpha : \alpha \in S_\zeta \rangle$ with $|C^\zeta_\delta| < \lambda$.*

(c) *If $\delta(*) < \lambda$ and $\kappa = \mathrm{cf}(\kappa) < \lambda$ then for some $\zeta < \lambda$, for every club C of λ^+, for some accumulation point δ of C, $\mathrm{cf}(\delta) = \kappa$ and $\mathrm{otp}(C^\zeta_\delta \cap C)$ is divisible by $\delta(*)$.*

(d) *If $\mathrm{cf}(\delta(*)) = \kappa$ as well, then we can add in the conclusion of (c):*

$$C^\zeta_\delta \subseteq C \text{ and } \mathrm{otp}(C^\zeta_\delta) = \delta(*).$$

Remark 4.9. (1) For $\lambda = \aleph_1$ the conclusion of 4.8(1), (2)(a),(b) becomes totally trivial. But for $\delta < \omega_1$, it means something if we add '$\{\alpha \in S_\zeta : \mathrm{otp}(C^\zeta_\alpha) = \delta\}$ is stationary, and for every club C of λ the set $\{\alpha \in S_\delta : \mathrm{otp}(C^\zeta_\alpha) = \delta, C^\zeta_\alpha \subseteq C\}$ is stationary.' So 4.8(2)(c),(d) are not so trivial, but still true. Their proofs are similar so we leave them to the reader (they are used only in [8, 2.7]).

(2) Recall that for a regular uncountable cardinal μ, the family $\check{I}[\mu]$ of good subsets of μ is the family of $S \subseteq \mu$ such that there are a sequence $\bar{a} = \langle a_\alpha : \alpha < \lambda \rangle$ and a club $C \subseteq \mu$ satisfying:

- $a_\alpha \subseteq \alpha$ is of order type $< \alpha$ when λ is a successor cardinal.
- $\beta \in a_\alpha \Rightarrow a_\beta = a_\alpha \cap \beta$
- $(\forall \delta \in S \cap C)\big[\sup(a_\delta) = \delta \wedge \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)\big]$.

We may say that the sequence $\bar{a}$ as above exemplifies that S is good; if $C = \mu$ we say "explicitly exemplifies".

Proof. Appears also in detail in [38] (originally proved for this work but as its appearance was delayed we put it there, too). Of course,

(1) follows from (2).

(2) Let $S = \{\alpha < \lambda^+ : \mathrm{cf}(\alpha) < \lambda\}$. For each $\alpha \in S$, choose $\bar{A}^\alpha$ such that:

(α) $\bar{A}^\alpha = \langle A^\alpha_i : i < \lambda \rangle$ is an increasing continuous sequence of subsets of α of cardinality $< \lambda$ such that $\bigcup_{i<\lambda} A^\alpha_i = \alpha \cap S$.

(β) If $\beta \in A^\alpha_i \cup \{\alpha\}$, β is a limit ordinal and $\mathrm{cf}(\beta) < \lambda$ (this actually follows from the first two conditions), then $\beta = \sup(A^\alpha_i \cap \beta)$.

(γ) If $\beta \in A_i^\alpha \cup \{\alpha\}$ is limit and $\aleph_0 < \mathrm{cf}(\beta) < \lambda$ then A_i^α contains a club of β.
(δ) $0 \in A_i^\alpha$ and $\big(\beta \in S \ \wedge\ \beta+1 \in A_i^\alpha \cup \{\alpha\}\big) \ \Rightarrow\ \beta \in A_i^\alpha$.
(ε) The closure of A_i^α in α (in the order topology) is included in A_{i+1}^α.

There are no problems with choosing $\bar{A}^\alpha$ as required.

We define B_i^α (for $i < \lambda$, $\alpha \in S$) by induction on α as follows:

$$B_i^\alpha = \begin{cases} \mathrm{closure}(A_i^\alpha) \cap \alpha & \text{if } \mathrm{cf}(\alpha) \neq \aleph_1 \\ \bigcap\big\{ \bigcup_{\beta \in C} B_i^\beta : C \text{ a club of } \alpha \big\} & \text{if } \mathrm{cf}(\alpha) = \aleph_1. \end{cases}$$

For $\zeta < \lambda$ we let:

$$S_\zeta = \big\{\alpha \in S: \ \text{(i) } B_\zeta^\alpha \text{ is a closed subset of } \alpha,\ \text{(ii) if } \beta \in B_\zeta^\alpha, \text{ then } B_\zeta^\beta = B_\zeta^\alpha \cap \beta \text{ and (iii) if } \alpha \text{ is limit, then } \alpha = \sup(B_\zeta^\alpha)\big\}$$

and for $\alpha \in S_\zeta$ let $C_\alpha^\zeta = B_\zeta^\alpha$.

Now, demand (b) holds by the choice of S_ζ. To prove clause (a) we shall show that for any $\alpha \in S$, for some $\zeta < \lambda$, $\alpha \in S_\zeta$; moreover we shall prove

$(*)_\alpha^0$ $E_\alpha := \{\zeta < \lambda\colon \text{if } \mathrm{cf}(\zeta) = \aleph_1 \text{ then } \alpha \in S_\zeta\}$ contains a club of λ.

For $\alpha \in S$ define $E_\alpha^0 = \{\zeta < \lambda\colon \text{if } \mathrm{cf}(\zeta) = \aleph_1 \text{ then } B_\zeta^\alpha = \mathrm{closure}(A_\zeta^\alpha) \cap \alpha\}$. We shall prove by induction on $\alpha \in S$ that $E_\alpha \cap E_\alpha^0$ contains a club of λ, and then we will choose such a club E_α^1.

Arriving to α, let

$$E_\alpha^* = \{\zeta < \lambda\colon \text{if } \beta \in A_\zeta^\alpha \text{ then } \zeta \in E_\beta^1 \text{ and } A_\zeta^\beta = A_\zeta^\alpha \cap \beta\}.$$

Clearly E_α^* is a club of λ. Let $\zeta \in E_\alpha^*$ and $\mathrm{cf}(\zeta) = \aleph_1$, and we shall prove that $\alpha \in S_\zeta \cap E_\alpha \cap E_\alpha^0$: clearly this will suffice. By the choice of ζ (and the definition of E) we have: if β belongs to A_ζ^α then $A_\zeta^\beta = A_\zeta^\alpha \cap A$ and $B_\zeta^\beta = \mathrm{closure}(A_\zeta^\beta) \cap \beta$, so

$(*)_1$ $\beta \in A_\zeta^\alpha \ \Rightarrow\ B_\zeta^\beta = \mathrm{closure}(A_\zeta^\alpha) \cap \beta$.

Let us check the three conditions for "$\alpha \in S_\zeta$"; this will suffice for clause (a) of the claim.

Clause (i). B^α_ζ is a closed subset of α. If $\mathrm{cf}(\alpha) \neq \aleph_1$ then $B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \alpha$, hence necessarily it is a closed subset of α.

If $\mathrm{cf}(\alpha) = \aleph_1$ then $B^\alpha_\zeta = \bigcap\{\bigcup_{\beta\in C} B^\beta_\zeta : C \text{ is a club of } \alpha\}$. Now, for any club C of α, $C \cap A^\alpha_\zeta$ is an unbounded subset of α (see clause (γ) above). By $(*)_1$ above,

$$\bigcup_{\beta\in C} B^\beta_\zeta \supseteq \bigcup_{\beta\in C\cap A^\alpha_\zeta} B^\beta_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \beta.$$

To finish proving clause (i), it suffices to note that we have gotten

$(*)_2$ $\alpha \in E^0_\zeta$.

[Why? If $\mathrm{cf}(\alpha) = \aleph_1$ see above, if $\mathrm{cf}(\alpha) \neq \aleph_1$ this is trivial.]

Clause (ii). If $\beta \in B^\alpha_\zeta$ then $B^\beta_\zeta = B^\alpha_\zeta \cap \beta$.

We know that $B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \alpha$ by $(*)_2$ above. If $\beta \in A^\alpha_\zeta$ then (by $(*)_1$) we have $B^\beta_\zeta = \mathrm{closure}(A^\alpha_\zeta) \cap \beta$, so we are done. So assume $\beta \notin A^\alpha_\zeta$. Then by clause (ε), necessarily:

$\odot$ If $\varepsilon < \zeta$ then $\beta > \sup(A^\alpha_\varepsilon \cap \beta)$ and $\sup(A^\alpha_\varepsilon \cap \beta) \in A^\alpha_{\varepsilon+1} \subseteq A^\alpha_\zeta$.

But $\beta \in B^\alpha_\zeta = \mathrm{closure}(A^\alpha_\zeta)$ by $(*)_2$, hence together A^α_ζ contains a club of β and $\mathrm{cf}(\beta) = \mathrm{cf}(\zeta)$, but $\mathrm{cf}(\zeta) = \aleph_1$, so $\mathrm{cf}(\beta) = \aleph_1$. Now, as in the proof of clause (i), we get $B^\beta_\zeta = \bigcup\{B^\gamma_\zeta : \gamma \in A^\alpha_\zeta \cap \beta\}$, so by the induction hypothesis we are done.

Clause (iii). If α is limit then $\alpha = \sup(A^\alpha_i)$.

By clause (β) we know A^α_ζ is unbounded in α, but $A^\alpha_\zeta \subseteq B^\alpha_\zeta$ (by $(*)_2$) and we are done.

So we have finished proving $(*)^0_\alpha$ by induction on α hence clause (a) of the claim.

For proving (c) of 4.8(2), note that above, if α is limit, C is a club of α, $C \subseteq S$, and $|C| < \lambda$, then for every i large enough, $C \subseteq A^\alpha_i$ and even $C \subseteq B^\alpha_i$.

Now assume that the conclusion of (c) fails (for fixed $\delta(*)$ and κ). Then for each $\zeta < \lambda$ we have a club E^0_ζ exemplifying it. Now, $E^0 := \bigcap_{\zeta<\lambda} E^0_\zeta$ is a club of λ^+, hence for some $\delta \in E^0$, $\mathrm{otp}(E^0 \cap \delta)$ is divisible by $\delta(*)$ and $\mathrm{cf}(\delta) = \kappa$. Choose an unbounded in δ set $e \subseteq E^0 \cap \delta$ of cardinality $< \lambda$ and order type divisible by $\delta(*)$. Then, for a final segment of $\zeta < \lambda$ we have $e \cap \delta \subseteq C^\zeta_\delta$.

Note that for any set C_1 of ordinals, $\mathrm{otp}(C_1)$ is divisible by $\delta(*)$ if C_1 has an unbounded subset of order type divisible by $\delta(*)$, so we get a contradiction because by $(*)^0_{\delta(*)}$ for some $\zeta \in E_{\delta(*)}$ (so $\delta(*) \in S_\zeta$) by $E^0_\zeta \cap C^\zeta_\delta \supseteq E^0 \cap \delta \supseteq e$, $\sup(e) = \delta$ and e has order type divisible by $\delta(*)$.

We are left with clause (d) of 4.8(2). Fix κ, $\delta(*)$, and ζ as above, we may add $\leq \lambda$ new sequences of the form $\langle C_\alpha \colon \alpha \in S_\zeta \rangle$ as long as each is a square. First assume that for every $\gamma, \beta < \lambda$, such that $\mathrm{cf}(\beta) = \kappa = \mathrm{cf}(\gamma)$, γ divisible by $\delta(*)$ we have

$(*)^3_{\beta,\gamma}$ There is a club $E_{\beta,\gamma}$ of λ^+ such that for no $\delta \in S_\zeta$ do we have $\mathrm{otp}(C^\zeta_\delta) = \beta$ and $\mathrm{otp}(C^\zeta_\delta \cap E_{\beta,\gamma}) = \gamma$.

Then let

$$E := \bigcap \big\{E_{\beta,\gamma} \colon \gamma < \lambda, \beta < \lambda, \mathrm{cf}(\beta) = \kappa = \mathrm{cf}(\gamma), \gamma \text{ divisible by } \delta(*)\big\}.$$

Applying part (c) we get a contradiction.

So for some γ, $\beta < \lambda$, $\mathrm{cf}(\beta) = \kappa = \mathrm{cf}(\gamma)$, γ divisible by $\delta(*)$ and $(*)^3_{\beta,\gamma}$ fails. Also there is a club E^* of λ^+ such that for every club $E \subseteq E^*$ for some $\delta \in S_\zeta$, $\mathrm{otp}(C^\zeta_\delta) = \beta$, $\mathrm{otp}(C^\zeta_\delta \cap E) = \gamma$ and $C^\zeta_\delta \cap E = C^\zeta_\gamma \cap E^*$ (by 4.10 below). Let $e \subseteq \gamma = \sup(e)$ be closed and such that $\mathrm{otp}(e) = \delta(*)$ and

$$\epsilon \in e \text{ is limit } \Rightarrow \epsilon = \sup(e \cap \epsilon).$$

We define ${}^*C^\zeta_\delta$ (for $\delta \in S_\zeta$) as follows: if $\delta \notin E^*$ then

$${}^*C^\zeta_\delta := C^\zeta_\delta \setminus (\max(\delta \cap E^*) + 1),$$

if $\delta \in E^*$ and $\mathrm{otp}(C^\zeta_\delta \cap E^*) \in e \cup \{\gamma\}$ then

$${}^*C^\zeta_\delta = \{\alpha \in C^\zeta_\delta \cap E^* \colon \mathrm{otp}(\alpha \cap C^\zeta_\delta \cap E^*) \in e\},$$

and if $\delta \in E^*$, $\mathrm{otp}(C^\zeta_\delta \cap E^*) \notin e \cup \{\gamma\}$ let

$${}^*C^\zeta_\delta = C^\zeta_\delta \setminus \big(\max\big\{\alpha \colon \mathrm{otp}(C^\zeta_\delta \cap E^* \cap \alpha) \in e \cup \{\gamma\}\big\} + 1\big).$$

One easily checks that (d) and square hold for $\langle {}^*C^\zeta_\delta \colon \delta \in S_\zeta \rangle$. So, we just have to add $\langle {}^*C^\zeta_\delta \colon \delta \in S_\zeta \rangle$ to $\{\langle C^\zeta_\delta \colon \delta \in S_\zeta \rangle \colon \zeta < \lambda\}$ for any $\zeta, \delta(*), \kappa$ (for which we choose ζ and E^*). ∎

Claim 4.10. *(1) Assume that $\aleph_0 < \kappa = \mathrm{cf}(\kappa)$, $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, $S \subseteq \{\delta < \lambda \colon \mathrm{cf}(\delta) = \kappa\}$ is stationary, C_δ is a club of δ (for $\delta \in S$), and $(\forall \delta \in S)\big[|C_\delta| = \kappa\big]$ (or at least $\sup_{\delta \in S} |C_\delta|^+ < \lambda$). Then for some club $E^* \subseteq \lambda$, for every club $E \subseteq E^*$, the set $\{\delta \in S^* \colon C_\delta \cap E^* \subseteq E\}$ is stationary, where*

$$S^* := \{\delta \in S \colon \delta \in \mathrm{acc}(E^*)\}.$$

(2) Assume that $\kappa = \mathrm{cf}(\kappa)$, $\kappa^+ < \lambda = \mathrm{cf}(\lambda)$, $S \subseteq \{\delta < \lambda \colon \mathrm{cf}(\delta) = \kappa\}$ *is stationary,* C_δ *is a club of* δ *(for* $\delta \in S$*),* $\sup\limits_{\delta \in S} |C_\delta|^+ < \lambda$, I_δ *is an ideal on* C_δ *which includes the bounded subsets, and for every club* E *of* λ*, for stationarily many* $\delta \in S$*, we have* $C_\delta \cap E \notin I_\delta$ *(or* $C_\delta \setminus E \in I_\delta$*).*

Then for some club E^* *of* λ*, for every club* $E \subseteq E^*$ *of* λ *the set* $\{\delta \in\in S^* \colon C_\delta \cap E^* \subseteq E\}$ *is stationary, where*

$$S^* := \big\{\delta \in S \colon \delta \in \mathrm{acc}(E^*), \delta = \sup(C_\delta \cap E^*) \text{ and } C_\delta \cap E^* \notin I_\delta \text{ (or } C_\delta \setminus E^* \in I_\delta)\big\}.$$

REMARK 4.11. This also was written in [42].

PROOF OF CLAIM 4.10. (1) If not, choose by induction on $i < \mu := \sup\limits_{\delta \in S}(|C_\delta|^+)$ a club $E^*_i \subseteq \lambda$, decreasing with i, E^*_{i+1} exemplifies that E^*_i is not as required, i.e.,

$$\{\delta \in S^*(E^*_i) \colon C_\delta \cap E^*_i \subseteq E^*_{i+1}\} = \emptyset.$$

Now, $\mathrm{acc}(\bigcap\limits_{i<\mu} E^*_i)$ is a club of λ, so there is $\delta \in S \cap \mathrm{acc}\big(\bigcap\limits_{i<\mu} E^*_i\big)$. The sequence $\langle C_\delta \cap E^*_i \colon i < \mu\rangle$ is necessarily strictly decreasing, and we get an easy contradiction.

(2) Similarly. ∎

4.2. Black boxes: first round

Now we turn to the main issue: black boxes.

LEMMA 4.12. *Suppose that* λ, θ *and* $\chi(*)$ *are regular cardinals and* $\lambda^\theta = \lambda^{<\chi(*)}$, $\theta < \chi(*) \le \lambda$*, and a set* $S \subseteq \{\delta < \lambda \colon \mathrm{cf}(\lambda) = \theta\}$ *is stationary and in* $\check{I}[\lambda]$*.*[7]

Then we can find

$$\mathbf{W} = \big\{\big(\overline{M}^\alpha, \eta^\alpha\big) \colon \alpha < \alpha(*)\big\}$$

(pedantically, $\mathbf{W}$ *is a sequence) and functions* $\dot\zeta \colon \alpha(*) \to S$ *and* $h \colon \alpha(*) \to \lambda$ *such that (so* $\alpha, \beta < \alpha(*)$*)*

- $(a0)$ $h(\alpha)$ *depends only on* $\dot\zeta(\alpha)$*, and* $\dot\zeta$ *is non-decreasing (but not necessarily strictly increasing).*
- $(a1)$ *We have:*
  - (α) $\overline{M}^\alpha = \langle M^\alpha_i \colon i \le \theta\rangle$ *is an increasing continuous chain. (*$\tau(M^\alpha_i)$*, the vocabulary, may be increasing.)*

[7] If $\theta = \aleph_0$ this holds trivially; see [12, 3.4=Lcd1.1], [47, 0.6,0.7], or just 4.9(2).

(β) *Each* M^α_i *is an expansion of a submodel of* $(\mathcal{H}_{<\chi(*)}(\lambda), \in, <)$ *belonging to* $\mathcal{H}_{<\chi(*)}(\lambda)$ *and* M^α_i *is transitive (i.e. considering the ordinals as atoms,* $x \in M^\alpha_i \Rightarrow x \subseteq M^\alpha_i$*), so* M^α_i *necessarily has cardinality* $< \chi(*)$*. (Of course the order means the order on the ordinals, and for transparency the vocabulary belongs to* $\mathcal{H}_{<\chi(*)}(\chi(*))$*.)*

(γ) $M^\alpha_i \cap \chi(*)$ *is an ordinal,* $\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha_i$*, and* $M^\alpha_i \in \mathcal{H}_{<\chi(*)}(\eta^\alpha(i))$*.*

(δ) $M^\alpha_i \cap \lambda \subseteq \eta^\alpha(i)$

(ε) $\langle M^\alpha_j : j \leq i\rangle \in M^\alpha_{i+1}$

(ζ) $\eta^\alpha \in {}^\theta\lambda$ *is increasing with limit* $\dot\zeta(\alpha) \in S$ *such that for* $i < \theta$*,* $\eta^\alpha \restriction (i+1) \in M^\alpha_{i+1}$*.*

($a2$) *In the following game,* $\Game(\theta, \lambda, \chi(*), \mathbf{W}, h)$*, Player I has no winning strategy. A play lasts* θ *moves. In the* i^{th} *move Player I chooses a model* $M_i \in \mathcal{H}_{<\chi(*)}(\lambda)$*, and then Player II chooses* $\gamma_i < \lambda$*. In the first move, Player I also chooses* $\beta < \lambda$*. In the end Player II wins the play if* $(\alpha) \Rightarrow (\beta)$*, where:*

(α) *The pair* $\big(\langle M_i : i < \theta\rangle, \langle \gamma_i : i < \theta\rangle\big)$ *satisfies the relevant demands on the pair*[8] $(\overline{M}^i \restriction \theta, \eta^\alpha)$ *in clause* $(a1)$*.*

(β) *For some* $\alpha < \alpha(*)$*,* $\eta^\alpha = \langle \gamma_i : i < \theta\rangle$*,* $M_i = M^\alpha_i$ *for* $i < \theta$*, and* $h(\alpha) = \beta$*.*

($b0$) $\eta^\alpha \neq \eta^\beta$ *for* $\alpha \neq \beta$*.*

($b1$) *If* $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta_\theta$ *then*

$\bullet_1$ $\dot\zeta(\alpha) \leq \dot\zeta(\beta)$

$\bullet_2$ $x \in M^\alpha_\theta \Rightarrow x \in M^\beta_\theta$

$\bullet_3$ $\alpha + (<\chi(*))^\theta = \beta + (<\chi(*))^\theta$ *(see 4.13(2) below).*

($b2$) *If in addition* $\lambda^{<\theta} = \lambda^{<\chi(*)}$*, then for every* $\alpha < \alpha(*)$ *and* $i < \theta$*, there is* $j < \theta$ *such that* $\eta^\alpha \restriction j \in M^\beta_\theta$ *implies* $M^\alpha_i \in M^\beta_\theta$ *(hence* $M^\alpha_i \subseteq M^\beta_\theta$*).*

($b3$) *If* $\lambda = \lambda^{<\chi(*)}$ *and* $\eta^\alpha \restriction (i+1) \in M^\beta_j$ *then* $M^\alpha_i \in M^\beta_j$ *(and hence* $M^\alpha_i \subseteq M^\beta_j$*, so* $x \in M^\alpha_i \Rightarrow x \in M^\beta_j$*) and*

$$\eta^\alpha \restriction i \neq \eta^\beta \restriction i \;\Rightarrow\; \eta^\alpha(i) \neq \eta^\beta(i).$$

REMARK 4.13. (1) If $\mathbf{W}$ (with $\dot\zeta, h, \lambda, \theta, \chi(*)$) satisfies (a0), (a1), (a2), (b0), (b1) we call it a *barrier*.

[8] So $\langle M_j : j \leq i\rangle$ is an increasing continuous chain, $M_i \cap \chi(*)$ an ordinal, $\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M_i$, $\langle M_\epsilon : \epsilon \leq j\rangle \in M_{j+1}$ and $\langle \gamma_\epsilon : \epsilon \leq j\rangle \in M_{j+1}$ for $j < i$, $M_i \in \mathcal{H}_{<\chi(*)}(\gamma_i)$, and $\langle \gamma_i : j \leq i\rangle \in M_{i+1}$.

(2) Remember, $(<\chi)^\theta := \sum_{\mu<\chi} \mu^\theta$.

(3) The existence of a good stationary set $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ follows, for example, from $\lambda = \lambda^{<\theta}$ (see [12, 3.4 = Lcd1.1] or [47, 0.6,0.7]) and from "λ is the successor of a regular cardinal and $\lambda > \theta^{+}$". But see 4.16(1),(2),(3).

(4) Compare the proof below with [28, Lemma 1.13,pg.49] and [25].

Proof of Lemma 4.12. First assume $\lambda = \lambda^{<\chi(*)}$.

Let $\langle S_\gamma : \gamma < \lambda \rangle$ be a sequence of pairwise disjoint stationary subsets of S such that $S = \bigcup_{\gamma<\lambda} S_\gamma$, and without loss of generality $\gamma < \min(S_\gamma)$. We define $h^* : S \to \lambda$ by $h^*(\alpha) =$ "the unique γ such that $\alpha \in S_\gamma$", and below we shall let $h(\alpha) := h^*(\dot{\zeta}(\alpha))$.

Let $\mathrm{cd} = \mathrm{cd}_{\lambda,\chi(*)}$ be a one-to-one function from $\mathcal{H}_{<\chi(*)}(\lambda)$ onto λ such that $\mathrm{cd}(\langle\alpha,\beta\rangle)$ is an ordinal

$$\max(\alpha,\beta) < \mathrm{cd}(\langle\alpha,\beta\rangle) < \max(|\alpha+\beta|^+,\omega),$$

and $x \in \mathcal{H}_{<\chi(*)}(\mathrm{cd}(x))$ for every relevant x. For $\xi \in S$ let:

$(*)_1$ (a) $\mathbf{W}^0_\xi :=$

$$\Big\{(\overline{M},\eta) : \text{the pair } (\overline{M},\eta) \text{ satisfies (a1) of 4.12, } \sup\{\eta(i) : i < \theta\} = \xi, \text{ and for every } i < \theta, \text{ for some } y \in \mathcal{H}_{<\chi(*)}(\lambda), \eta(i) = \mathrm{cd}\big(\langle \overline{M} \restriction (i+1), \eta \restriction i, y\rangle\big)\Big\}.$$

(b) $\mathbf{W} = \bigcup\{\mathbf{W}^0_\xi : \xi \in S\}$

Below, we shall choose $\big\langle (\overline{M}^\alpha,\eta^\alpha) : \alpha < \alpha(*)\big\rangle$ listing $\mathbf{W}$.

So (a0), (a1), (b0), (b3) (hence (b2)) should be clear.

We can choose $\big\langle (\overline{M}^\alpha,\eta^\alpha) : \alpha < \alpha(*)\big\rangle$ an enumeration of $\bigcup_{\xi\in S} \mathbf{W}^0_\xi$ to satisfy (b1) (and $\dot{\zeta}(\alpha) = \sup \mathrm{rang}(\eta^\alpha)$, of course) because:

$(*)_2$ If $(\overline{M}^*,\eta^*) \in \bigcup_\xi \mathbf{W}^0_\xi$ then

$$\big|\{\eta \in {}^\theta\lambda : \{\eta \restriction i : i < \theta\} \subseteq M^*_\theta\}\big| \le \|M^*_\theta\|^\theta \le \big(<\chi(*)\big)^\theta.$$

Clearly $(*)_2$ holds, but why does it suffice for choosing our $\big\langle (\overline{M}^\alpha,\eta^\alpha) : \alpha < \alpha(*)\big\rangle$?

$(*)_{2.1}$ We define the partial order $\leq_{\mathbf{W}}$ on $\mathbf{W}$ by

$$(\overline{M}, \eta) \leq_{\mathbf{W}} (\overline{M}', \eta') \text{ iff } M_\theta \subseteq M'_\theta.$$

For each $\xi \in S$, try to choose a sequence $\mathbf{x}_{\xi,\gamma} = \langle (\overline{M}^{\xi,\gamma}, \eta^{\xi,\gamma}) : \alpha < \alpha_\gamma \rangle$ by induction on the order $\gamma < \|\mathbf{W}^0_\xi\|^+$, so it will be $\triangleleft$-increasing with γ such that:

$(*)_{2.2}$ (a) $(\overline{M}^{\xi,\alpha}, \eta^{\xi,\alpha}) \in \mathbf{W}^0_\xi$ for $\alpha < \alpha_\gamma$.

(b) If $(\overline{M}, \eta) \in \mathbf{W}^0_\xi$ and $(\overline{M}, \eta) \leq_\xi (\overline{M}^{\xi,\alpha}, \eta^{\xi,\alpha})$ for some $\alpha < \gamma_\alpha$ then $(\overline{M}, \eta) = (\overline{M}^{\xi,\beta}, \eta^{\xi,\beta})$ for some $\beta < \alpha_\gamma$.

How do we carry the induction? For $\gamma = 0$ let $\alpha_\gamma = 0$; also, for γ a limit ordinal the choice of $\mathbf{x}_{\xi,\gamma}$ is obvious. For $\gamma = \gamma_1 + 1$ $(= \gamma(1) + 1)$, if

$$\{(\overline{M}^{\xi,\alpha}, \eta^{\xi,\alpha}) : \alpha < \alpha_{\gamma_1}\} = \mathbf{W}^0_\xi$$

then we stop. Otherwise, choose

$$(\overline{N}^{\gamma_1}, \eta^{\gamma_1}) \in \mathbf{W}^0_\xi \setminus \{(\overline{M}^{\xi,\alpha}, \eta^{\xi,\alpha}) : \alpha < \alpha_{\gamma_1}\}$$

and let

$$\mathbf{W}_{\xi,\gamma_1} = \Big\{(\overline{M}, \eta) \in \mathbf{W}^0_\xi : (\overline{M}, \eta) \leq_\xi (\overline{N}^{\gamma_1}, \eta^{\gamma_1}), \text{ but } (\overline{M}, \eta) \notin \{(\overline{M}^{\xi,\alpha}, \eta^{\xi,\alpha}) : \alpha < \alpha_{\gamma_1}\}\Big\},$$

so $\mathbf{x}_\gamma$ is defined by letting $\alpha_\gamma := \alpha_{\gamma(1)} + \|\mathbf{W}_{\xi,\gamma(1)}\|$ and

$$\langle (\overline{M}^{\xi,\alpha}, \eta^{\xi,\alpha}) : \alpha \in [\alpha_{\gamma(1)}, \alpha_\gamma) \rangle$$

list the elements of $\mathbf{W}_{\xi,\gamma(1)}$.

So for some $\gamma[\xi] < |\mathbf{W}^0_\xi|^+$, $\mathbf{x}_{\xi,\gamma[\xi]}$ lists $\mathbf{W}^0_\xi$. Lastly, we choose $\alpha(*) = \sum_{\xi \in S} \gamma[\xi]$, and $(\overline{M}^\alpha, \eta^\alpha) = (\overline{M}^{\xi,\beta}, \eta^{\xi,\beta})$ when $\alpha = \sum\{\gamma[\xi'] : \xi' < \xi\} + \beta$ and $\beta < \gamma[\xi]$.

This, in fact, defines the function $\dot{\zeta}$ as follows: we have $\dot{\zeta}(\alpha) = \xi$ if and only if $(\overline{M}^\alpha, \eta^\alpha) \in \mathbf{W}^0_\xi$.

We are left with proving (a2). Let G be a strategy for Player I.

Let $\langle C_\delta : \delta < \lambda \rangle$ exemplify "S is a good stationary subset of λ" (see 4.9(2)) and let $R = \{(i, \alpha) : i \in C_\alpha, \alpha < \lambda\}$.

Let $\langle \mathscr{A}_i : i < \lambda \rangle$ be a representation of the model

$$\mathscr{A} = (\mathcal{H}_{<\chi(*)}(\lambda), \in, G, R, \mathrm{cd}),$$

i.e., it is increasing continuous, $\|\mathscr{A}_i\| < \lambda$, and $\bigcup_i \mathscr{A}_i = \mathscr{A}$. Without loss of generality $\mathscr{A}_i \prec \mathscr{A}$ and $|\mathscr{A}_i| \cap \lambda$ is an ordinal for $i < \lambda$.

Let G "tell" Player I to choose $\beta^* < \lambda$ in his first move. So there is a $\delta \in S_{\beta^*}$ (hence $\delta > \beta^*$: see the beginning of the proof) such that $|\mathscr{A}_\delta| \cap \lambda = \delta$. Now, necessarily $C_\delta \cap \alpha \in \mathscr{A}_\delta$ for $\alpha < \delta$. Let $\{\alpha_i : i < \mathrm{cf}(\delta)\}$ list C_δ in increasing order.

Lastly, by induction on i, we choose $M_i, \eta(i)$ as follows:

$$\eta(i) = \mathrm{cd}\Big(\Big\langle \langle M_j : j \leq i\rangle, \langle \eta(j) : j < i\rangle, \langle \alpha_j : j < i\rangle \Big\rangle\Big),$$

and M_i is what the strategy G "tells" Player I to choose in his i^{th} move if Player II has chosen $\langle \eta(j) : j < i\rangle$ so far.

Now for each $i < \theta$, the sequences $\langle M_j : j \leq i\rangle$, $\langle \eta(j) : j < i\rangle$ are definable in $\mathscr{A}_\delta$ with $\langle \alpha_j : j \leq i\rangle$ as the only parameter, hence they belong to $\mathscr{A}_\delta$. So

$$\sup\{\eta(j) : j < \theta\} \leq \delta.$$

However, by the choice of $\eta(i)$ (and cd), $\eta(i) \geq \sup\{\alpha_j : j < i\}$ and hence $\sup\{\eta(j) : j < \theta\}$ is necessarily δ. Now check.

We have finished the proof, but only by including the assumption $\lambda = \lambda^{<\chi(*)}$. The case $\lambda < \lambda^{<\theta} = \lambda^{<\chi(*)}$ is similar. For a set $A \subseteq \theta$ of cardinality θ we let $\mathrm{cd}^A = \mathrm{cd}^A_{\lambda,\chi(*)}$ be a one-to-one function from $\mathcal{H}_{<\chi(*)}(\lambda)$ onto A_λ, where

$$A_\lambda = \{h : h \text{ is a function from } A \text{ to } \lambda\}.$$

We strengthen (b2) to

$(b2)'$ Let $A_i := \{\mathrm{cd}(i,j) : j < \theta\}$ for $i \in [1,\theta)$ and $A_0 := \theta \setminus \bigcup\{A_{1+i} : i < \theta\}$, so $\langle A_i : i < \theta\rangle$ is a sequence of pairwise disjoint subsets of θ, each of cardinality θ, with $\min(A_i) \geq i$, and we have
$(*)$ $\eta^\alpha \restriction A_i = \mathrm{cd}^{A_i}\big(\overline{M}^\alpha \restriction i, \eta^\alpha \restriction i\big)$. ∎

* * *

What can we do when S is not good? As we say in 4.13(3), in many cases a good S exists (note that for singular λ we will not have one).

The following rectifies the situation in the other cases (but is interesting mainly for λ singular). We shall, for a regular cardinal λ, remove this assumption in 4.16(1)–(3), while 4.17 helps for singular λ. (This is carried in 4.18).

Definition 4.14. Let ∂ be an ordinal greater than 0, and for $\alpha < \partial$ let κ_α be a regular uncountable cardinal and $S_\alpha \subseteq \{\delta < \kappa_\alpha : \mathrm{cf}(\delta) = \theta\}$ be a stationary set.

Assume θ, χ are regular cardinals such that for every $\alpha < \partial$ we have $\theta < \chi \leq \kappa_\alpha$. Let $\bar{S} = \langle S_\alpha : \alpha < \partial \rangle$, $\bar{\kappa} = \langle \kappa_\alpha : \alpha < \partial \rangle$. If $\partial = 1$ we may write S_0, κ_0.

(1) We say that $\bar{S}$ is *good for* $(\bar{\kappa}, \theta, \chi)$ when[9] for every large enough μ and model $\mathscr{A}$ expanding $(\mathcal{H}_{<\chi}(\mu), \in)$ with $|\tau(\mathscr{A})| \leq \aleph_0$, there are M_i (for $i < \theta$) such that:

- $M_i \prec \mathscr{A}$ and $\bar{S} \in M_i$.
- $\langle M_j : j \leq i \rangle \in M_{i+1}$, $\|M_i\| < \chi$, $M_i \cap \chi \in \chi$, and $\chi = \chi^+{}_1 \Rightarrow \chi_1 + 1 \subseteq \subseteq M_i$.
- $\alpha < \partial$, $\alpha \in \bigcup_{j<\theta} M_j$ implies that $\sup\big(\kappa_\alpha \cap (\bigcup_{j<\theta} M_j)\big)$ belongs to S_α.

(2) If $\bar{\kappa}$ is constant (i.e. $i < \partial \Rightarrow \kappa_i = \kappa$) then we may say $\bar{S}$ is good for $(\kappa, \partial, \theta, \chi)$. We may omit ∂ if $\partial = \kappa$.

(3) If $\partial = 1$, we may write S_0, κ_0 instead of $\bar{S}, \bar{\kappa}$. If $\partial < \chi$ then we can demand $\partial \subseteq M_0$.

Definition 4.15. For regular uncountable cardinal λ and regular $\theta < \lambda$, let $\check{J}_\theta[\lambda]$ be the family of subsets S of λ such that $\{\delta \in S : \mathrm{cf}(\delta) = \theta\}$ is not good for $(\lambda, \theta, \lambda)$ (i.e. for $(\lambda, \lambda, \theta, \lambda)$).

Claim 4.16. *Assume $\theta = \mathrm{cf}(\theta) < \chi = \mathrm{cf}(\chi) \leq \kappa = \mathrm{cf}(\kappa)$.*

(1) Then $\{\delta < \kappa : \mathrm{cf}(\delta) = \theta\}$ is good for (κ, θ, χ), i.e. is not in $\check{J}_\theta[\lambda]$.

(2) Any $S \subseteq \kappa$ good for (κ, θ, χ) is the union of κ pairwise disjoint such sets.

(3) In 4.12 it suffices to assume that S is good for (λ, θ, χ).

(4) $\check{J}_\theta[\lambda]$ is a normal ideal on λ and there is no stationary $S \subseteq \{\delta < < \lambda : \mathrm{cf}(\delta) = \theta\}$ which belongs to $\check{J}_\theta[\lambda] \cap \check{I}[\lambda]$.

(5) In Definition 4.14, any $\mu > \lambda^{<\chi}$ is OK; we can pre-assign $x \in \mathcal{H}_{<\chi}(\mu)$ and demand $x \in M_i$.

(6) In 4.12 we can replace the assumption "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary and in $\check{I}[\lambda]$" by "$S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary not in $\check{J}_\theta[\lambda]$" (which holds for $S = \{\delta < \kappa : \mathrm{cf}(\delta) = \theta\}$).

Proof. (1) Straightforward (play the game).

(2) Similar to the proof of 4.1.

(3) Obvious.

(4) Easy.

[9] Note that we can compute ∂ from $\bar{\kappa}$.

(5) Easy.

(6) Follows. ∎

Claim 4.17. *Assume that $\bar\kappa, \theta, \chi$ are as in 4.14 with $|\partial| \le \chi$.*

(1) Then the sequence $\langle \{\delta < \kappa_i : \mathrm{cf}(\delta) = \theta\} : i < \partial\rangle$ is good for $(\bar\kappa, \theta, \chi)$.

(2) If $\partial_1 < \partial$ and $\langle S_i : i < \partial_1\rangle$ is good for $(\bar\kappa \restriction \partial_1, \theta, \chi)$ then

$$\big\langle S_i : i < \partial_1\big\rangle\hat{}\big\langle\{\delta < \kappa_i : \mathrm{cf}(\delta) = \theta\} : \partial_1 \le i < \partial\big\rangle$$

is good for $(\bar\kappa, \theta, \chi)$.

(3) If $\langle S_i : i < \partial_1\rangle$ is good for $(\bar\kappa, \theta, \chi)$ and $i(*) < \partial$, then we can partition $S_{i(*)}$ to pairwise disjoint sets $\langle S_{i(*),\epsilon} : \epsilon < \kappa_i\rangle$ such that for each $\epsilon < \kappa_i$, the sequence

$$\big\langle S_i : i < i(*)\big\rangle\hat{}\big\langle S_{i(*),\epsilon}\big\rangle\hat{}\big\langle\{\delta < \kappa_i : \mathrm{cf}(\delta) = \theta\} : i(*) < i < \partial\big\rangle$$

is good for $(\bar\kappa, \theta, \chi)$.

(4) $\bar S$ good for $(\bar\kappa, \theta, \chi)$ implies that S_i is a stationary subset of κ_i for each $i < \ell g(\bar\kappa)$.

Proof. Like 4.16. [In 4.17(3) we choose, for $\delta \in S_{i(*)}$, a club C_δ of δ of order type $\mathrm{cf}(\delta)$; for $j < \theta$, $\epsilon < \kappa_{i(\alpha)}$, let $S^j_{i(*),\epsilon} = \{\delta \in S_{i(*)} : \epsilon$ is the j^{th} member of $C_\delta\}$; for some j and unbounded $A \subseteq \kappa_{i(*)}$, $\langle S^j_{i(*),\epsilon} : \epsilon \in A\rangle$ are as required.] ∎

Now we remove from 4.12 (and subsequently 4.20) the hypothesis "λ is regular" when $\mathrm{cf}(\lambda) \ge \chi(*)$.

Lemma 4.18. *Suppose $\lambda^\theta = \lambda^{<\chi(*)}$, λ is singular, θ and $\chi(*)$ are regular, $\theta < \chi(*)$ and $\mathrm{cf}(\lambda) \ge \chi(*)$. Suppose further that $\lambda = \sum_{i<\mathrm{cf}(\lambda)} \mu_i$ and each μ_i is regular $> \chi(*)+\theta^+$. Then we can find $\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ and functions $\dot\zeta : \alpha(*) \to \mathrm{cf}(\lambda)$, $\dot\xi : \alpha(*) \to \lambda$, and $h : \alpha(*) \to \lambda$, and $\{\mu'_i : i < \mathrm{cf}(\lambda)\}$ such that ($\{\mu'_i : i < \mathrm{cf}(\lambda)\} = \{\mu_i : i < \mathrm{cf}(\lambda)\}$ and):*

(a0) *$h(\alpha)$ depends only on $\langle \dot\zeta(\alpha), \dot\xi(\alpha)\rangle$, $\alpha < \beta \Rightarrow \dot\zeta(\alpha) \le \dot\zeta(\beta)$,*

$$\alpha < \beta \wedge \dot\zeta(\alpha) = \dot\zeta(\beta) \Rightarrow \dot\xi(\alpha) \le \dot\xi(\beta),$$

and $\dot\xi(\alpha) < \mu'_{\dot\zeta(\alpha)}$.

(a1) *As in 4.12, except that: $\langle \eta^\alpha(3i) : i < \theta\rangle$ is strictly increasing with limit $\dot\zeta(\alpha)$ and $\langle \eta^\alpha(3i+1) : i < \theta\rangle$ is strictly increasing with limit $\dot\xi(\alpha)$ for $i < \theta$,*

$$\sup\big(|M^\alpha_i| \cap \mu'_{\zeta(\alpha)}\big) < \dot\xi(\alpha) = \sup\big(|M^\alpha_\theta| \cap \mu'_{\dot\zeta(\alpha)}\big),$$

and for every $i < \theta$,

$$\sup\big(|M^\alpha_i| \cap \mathrm{cf}(\lambda)\big) < \dot\zeta(\alpha) = \sup\big(|M^\alpha_\theta| \cap \mathrm{cf}(\lambda)\big).$$

$(a2)$ *As in 4.12.*
$(b0), (b1), (b2)$ *As in 4.12, but in clause (b3) we demand* $i = 2 \mod 3$.

Remark 4.19. To make it similar to 4.12, we can fix S^a, S^a_i, S^b_i, $S^b_{i,a}$, μ'_i as in the first paragraph of the proof below.

Proof of Lemma 4.18. First, by 4.16 [(1)+(2)], we can find pairwise disjoint $S^a_i \subseteq \mathrm{cf}(\lambda)$ for $i < \mathrm{cf}(\lambda)$, each good for $(\mathrm{cf}(\lambda), \theta, \chi(*))$ (and $\alpha \in S^a_i \Rightarrow \Rightarrow \alpha > i \wedge \mathrm{cf}(\alpha) = \theta$), and let $S^a = \bigcup_{i<\mathrm{cf}(\lambda)} S^a_i$. We define $\mu'_i \in \{\mu_j : j < i\}$ such that

$$\big(\forall i < \mathrm{cf}(\lambda)\big)[j \in S^a_i \Rightarrow \mu'_j = \mu_i].$$

Then for each i, by 4.17(2),(3) (with $1, 2, S_0, \kappa_0, \kappa_1$ standing for σ_1, σ, S^a_i, $\mathrm{cf}(\lambda)$, μ'_i), we can find pairwise disjoint subsets $\langle S^b_{i,\alpha} : \alpha < \mu'_i\rangle$ of $\{\delta < \mu'_i : \mathrm{cf}(\delta) = \theta\}$ such that for each $\alpha < \mu'_\alpha$, $(S^a_i, S^b_{i,\alpha})$ is good for $(\langle \mathrm{cf}(\lambda), \mu'_i\rangle, \theta, \chi)$. Let $S^b_i = \bigcup\{S^b_{i,\alpha} : \alpha < \mu'_i\}$.

Let cd be as in 4.12's proof coding only for ordinals $i = 2 \mod 3$, and for $\zeta \in S^a_i$, $\xi \in S^a_{i,j}$ let

$$\begin{aligned}\mathbf{W}^0_{\zeta,\xi} = \Big\{\big(\overline{M}, \eta\big) :\ & \overline{M} \text{ satisfies (a1)}, \zeta = \sup\{\eta(3i) : i < \theta\},\\ & \xi = \sup\{\eta(3i+1) : i < \theta\} \text{ and}\\ & \text{for each } i < \theta, \text{ for some } y \in \mathcal{H}_{<\chi(*)}(\lambda),\\ & \eta(3i+2) = \mathrm{cd}\big(\langle M_j : j \le 3i+1\rangle, \eta \restriction (3i+1), y\big)\Big\}.\end{aligned}$$

The rest is as in 4.12's proof. ∎

* * *

The following Lemma improves 4.12 when λ satisfies a stronger requirement, making the distinct $(\overline{M}^\alpha, \eta^\alpha)$ interact less. Lemmas 4.20 + 4.18 were used in the proof of 3.4 (and 3.3).

Lemma 4.20. *(1) In 4.12, if* $\lambda = \lambda^{\chi(*)}$ *and* $\chi(*)^\theta = \chi(*)$ *then we can strengthen clause (b1) to*

$(b1)^+$ *If* $\alpha \neq \beta$ *and* $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta$ *then* $\alpha < \beta$ *and* $x \in M^\alpha_\theta \Rightarrow x \in M^\beta_\theta$.

(2) To clause 4.12(b1), we can add

- *Moreover, if* $\alpha < \chi(*) \Rightarrow |\alpha|^{\aleph_0} < \chi(*)$ *then* $\alpha < \beta + \left(<\chi(*)\right)^{\theta}$.

Proof. (1) Apply 4.12 (actually, its proof) but using λ, $\chi(*)^+$, θ, instead of $\lambda, \chi(*), \theta$; and get $\mathbf{W} = \{(\overline{M}^{\alpha}, \eta^{\alpha}) \colon \alpha < \alpha(*)\}$, and the functions $\dot{\zeta}, h$.

Let cd be as in the proof of 4.12. Let $<^*$ be some well ordering of $\mathcal{H}_{<\chi(*)}(\lambda)$, and let $\mathscr{U}$ be the set of ordinals $\alpha < \alpha(*)$ such that for $i < \theta$, M_i^{α} has the form $(N_i^{\alpha}, \in_i^{\alpha}, <^{\alpha})$ and $(|N_i^{\alpha}|, \in_i^{\alpha}, <^{\alpha}) \prec (\mathcal{H}_{<\chi(*)}(\lambda), \in, <^*)$.

Let $\alpha \in \mathscr{U}$, by induction on $\epsilon < \chi(*)$ we define $M_i^{\epsilon,\alpha}, \eta^{\epsilon,\alpha}$ as follows:

(A) $\eta^{\epsilon,\alpha}(i)$ is $\mathrm{cd}(\langle \eta^{\alpha}(i), \epsilon \rangle)$, (which is an ordinal $< \lambda$ but $> \eta^{\alpha}(i)$ and $> \epsilon$)

(B) $M_i^{\epsilon,\alpha} \prec N_i^{\alpha}$ is the Skolem Hull of $\{\eta^{\epsilon,\alpha} \restriction (j+1) \colon j < i\}$ inside N_i^{α}, using as Skolem functions the choice of the $<^*$-first element and making $M_i^{\epsilon,\alpha} \cap \chi(*)$ an ordinal. [If we want, we can use $\eta^{\epsilon,\alpha}$ such that it fits the definition in the proof of 4.12].

Note that $\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M_i^{\alpha}$ and $M_i^{\epsilon,\alpha}$ is definable in $M_{i+1}^{\epsilon,\alpha}$ as $M_i^{\epsilon,\alpha} \in M_{i+1}^{\epsilon,\alpha}$ (by the definition of $\mathbf{W}_{\xi}^{0}$ in the proof of 4.12). Similarly, $\langle M_j^{\epsilon,\alpha} \colon j \leq i \rangle$ is definable in M_{i+1}^{α}. It is easy to check that the pair $(\overline{M}^{\epsilon,\alpha}, \eta^{\epsilon,\alpha})$ satisfies condition (a1) of 4.12.

Next we choose $\epsilon(\alpha) < \chi(*)$ by induction on $\alpha \in \mathscr{U}$ as follows:

(C) $\epsilon(\alpha)$ is the first $\epsilon < \chi(*)$ such that if $\beta < \alpha$ but $\beta + \chi(*) > \alpha$ then
 $(*)$ $\{\eta^{\alpha,\epsilon} \restriction j \colon j < \theta\} \nsubseteq M_{\theta}^{\beta,\epsilon(\beta)}$.

This is possible and easy, as for $(*)$ it suffices to have for each suitable β, $\epsilon \notin M_{\theta}^{\beta,\epsilon(\beta)}$, so each β "disqualifies" $< \chi(*)$ ordinals as candidates for $\epsilon(\alpha)$, and there are $< \chi(*)$ such β-s, and $\chi(*)$ is by the assumptions (see 4.12) regular.

Now

$$\mathbf{W}' = \left\{(\overline{N}^{\alpha,\epsilon(\alpha)}, \eta^{\alpha,\epsilon(\alpha)}) \colon \alpha \in \mathscr{U}\right\},$$

$\dot{\zeta} \restriction \mathscr{U}, h \restriction \mathscr{U}$ are as required except that we should replace $\mathscr{U}$ by an ordinal (and adjust ζ, h accordingly). In the end replace N_i^{α} by $N_i^{\alpha} \cap \mathcal{H}_{<\chi(*)}(\lambda)$.

(2) We have to prove the version of (b1) with the "Moreover".

Let $\mathcal{S} \subseteq [\mathcal{H}_{<\chi(*)}(\lambda)]^{\aleph_0}$ be MAD (that is, $u \neq v \in \mathcal{S} \Rightarrow |u \cap v| < \aleph_0$ and $\mathcal{S}$ is maximal under $\subseteq$) such that $\mathcal{S} \cap [\mathcal{H}_{<\chi(*)}(\zeta)]^{\aleph_0}$ is MAD for every $\zeta < \lambda$, and demand $|M_{\theta}^{\alpha}| \cap \mathcal{S} \subseteq \left[|M_{\theta}^{\alpha}|\right]^{\aleph_0}$ is MAD. So it is well-known that the order $(\mathbf{W}, \leq_{\mathbf{W}})$ is well founded.[10] ∎

[10]So we use $\zeta < \chi(*) \Rightarrow |\zeta|^{\aleph_0} < \chi(*)$ to ensure that we can demand that M_{θ}^{α} is as required. However, $\lambda \not\to (\omega)_2^{<\omega}$ will suffice.

Claim 4.21. *If in 4.18 we add "$\lambda = \lambda^{\chi(*)^\theta}$" (or the condition from 4.20) then we can replace (b1) by*

$(b1)^+$ *If* $\{\eta^\alpha \restriction i : i < \theta\} \subseteq M^\beta_\theta$ *then* $\alpha \le \beta$.

Proof. The same as the proof of 4.20 combined with the proof of 4.18. ∎

4.3. Black boxes: for θ countable

Next we turn to the case (of black boxes with) $\theta = \aleph_0$. We shall deal with several cases.

Lemma 4.22. *Suppose that*

$(*)$ λ *is a regular cardinal,* $\theta = \aleph_0$, $\mu = \mu^{<\chi(*)} < \lambda \le 2^\mu$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ *is stationary, and* $\aleph_0 < \chi(*) = \mathrm{cf}(\chi(*))$.

Then we can find

$$\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$$

and functions

$$\dot{\zeta} : \alpha(*) \to S \text{ and } h : \alpha(*) \to \lambda$$

such that:

$(a0)$–$(a2)$ *As in 4.12.*

$(b0)$–$(b2)$ *As in 4.12, and even*

$(b1)^*$ $\alpha \ne \beta$, $\{\eta^\alpha \restriction n : n < \omega\} \subseteq M^\beta_\omega$ *implies* $\alpha < \beta$ *and even* $\dot{\zeta}(\alpha) < \dot{\zeta}(\beta)$.

$(c1)$ *If* $\dot{\zeta}(\alpha) = \dot{\zeta}(\beta)$ *then* $|M^\alpha_\omega| \cap \mu = |M^\beta_\omega| \cap \mu$, *there is an isomorphism* $h_{\alpha,\beta}$ *from* M^α_ω *onto* M^β_ω, *mapping* $\eta^\alpha(n)$ *to* $\eta^\beta(n)$ *and* M^α_n *to* M^β_n *for* $n < \omega$, *and* $h_{\alpha,\beta} \restriction (|M^\alpha_\omega| \cap |M^\beta_\omega|)$ *is the identity.*

$(c2)$ *There is* $\overline{C} = \langle C_\delta : \delta \in S\rangle$, C_δ *an* ω*-sequence converging to* δ, $0 \notin C_\delta$, *and letting* $\langle \gamma^\delta_n : n < \omega\rangle$ *enumerate* $\{0\} \cup C_\delta$ *we have, when* $\dot{\zeta}(\alpha) = \delta$:

- (i) $\lambda \cap |M^\alpha_n| \subseteq \gamma^\delta_{n+1}$ *but* $\lambda \cap |M^\alpha_n|$ *is not a subset of* γ^δ_n, *(hence* $M^\alpha_n \cap [\gamma^\delta_n, \gamma^\delta_{n+1}) \ne \emptyset$*).*
- (ii) $C_\delta \cap |M^\alpha_\omega| = \emptyset$
- (iii) *If in addition* $\dot{\zeta}(\beta) = \delta$ *then for each* n, $h_{\alpha,\beta}$ *maps* $|M^\alpha_\omega| \cap \cap [\gamma^\delta_n, \gamma^\delta_{n+1})$ *onto* $|M^\beta_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}]$.
- (iv) *If* $\dot{\zeta}(\beta) = \delta = \dot{\zeta}(\alpha)$ *and* $\lambda = \lambda^{<\chi(*)}$ *then* $|M^\alpha_\omega| \cap \gamma^\delta_1 = |M^\beta_\omega| \cap \gamma^\delta_1$.

Remark 4.23. (1) We only use $\lambda \le 2^\mu$ in order to get "$h_{\alpha,\beta} \restriction (|M^\alpha_\omega| \cap |M^\beta_\omega|) = = \mathrm{id}$" in condition (c1).

(2) Below we quote "guessing of clubs" — that is clause (ii) in the proof; without this we just get a somewhat weaker conclusion.

PROOF OF LEMMA 4.22. Let S be the disjoint union of stationary $S_{\alpha,\beta,\gamma}$, for $\alpha < \mu, \beta < \lambda, \gamma < \lambda$.

For each α, β, γ, let $\langle C_\delta : \delta \in S_{\alpha,\beta,\gamma} \rangle$ satisfy:

⊠ (i) C_δ is an unbounded subset of δ of order type ω.
(ii) For every club C of λ, for stationarily many $\delta \in S_{\alpha,\beta,\gamma}$, we have $C_\delta \subseteq C$.
(iii) $0 \notin C_\delta$

(exists by [8, 2.2] or [42] = [40, Ch.III]).

Let $\mathbf{W}^*$ be the family of quadruples $(\delta, \overline{M}, \eta, C)$ such that:

⊛ (α) $(\overline{M}, \eta)$ satisfies the requirement (a1) (so $\overline{M} = \langle M_n : n < \omega \rangle$).
(β) $0 \notin C$, and letting $\{\gamma_n : n < \omega\}$ enumerate $C \cup \{0\}$ in increasing order, we have $\lambda \cap M_n$ is a subset of γ_{n+1} but not of γ_n, and $\bigcup_{n<\omega} \gamma_n = \delta$ and $C \cap (\bigcup_n M_n) = \emptyset$.
(γ) $\bigcup_n |M_n| \subseteq \mathcal{H}_{<\chi(*)}(\mu + \mu)$
(δ) In $\tau(M_n)$ there is a two-place relation R and a one-place function cd. (We do not necessarily require $\mathrm{cd} \restriction M_n = \mathrm{cd}^{M_n}$; similarly for R — see below. Recall that as usual, $\tau(M_n) \in \mathcal{H}_{<\chi(*)}(\chi(*))$ for transparency.)

As $\mu^{<\chi(*)} = \mu$, clearly $|\mathbf{W}^*| = \mu$, so let

$$\mathbf{W}^* = \big\{(\delta^\alpha, \langle M_{\alpha,n} : n < \omega \rangle, \eta_\alpha, C^\alpha) : \alpha < \mu\big\}.$$

If $\lambda = \lambda^{<\chi(*)}$ let $\{N_\beta : \beta < \lambda\}$ list the models $N \in \mathcal{H}_{<\chi(*)}(\lambda)$ with $\tau(N) \in$ $\in \mathcal{H}_{<\chi(*)}(\chi(*))$.

Also, let $\langle A_\alpha : \alpha < \lambda \rangle$ be a sequence of pairwise distinct subsets of μ, and define the two place relation R on λ by

$$\gamma_1 \; R \; \gamma_2 \Leftrightarrow [\gamma_1 < \mu \;\wedge\; \gamma_1 \in A_{\gamma_2}].$$

Lastly, for $\delta \in S_{\alpha,\beta,\gamma}$ let $\mathbf{W}^0_\delta$ be the set of pairs $(\overline{M}, \eta)$ such that:

⊕ (a) $\overline{M} = \langle M_n : n < \omega \rangle, \eta \in {}^\omega\lambda$
(b) $(\overline{M}, \eta)$ satisfies 4.12(a1). In particular:
(α) η is increasing with limit δ.
(β) there is an isomorphism h from $\bigcup_{n<\omega} M_n$ onto $\bigcup_{n<\omega} M_{\alpha,n}$.
(γ) h maps $\eta(n)$ to $\eta^\alpha(n)$ and M_n onto $M_{\alpha,n}$.

(δ) h preserves $\in$, R, $\mathrm{cd}(x) = y$ and their negations. (For R and cd: in $\bigcup_{n<\omega} M_n$ we mean the standard cd restricted to $\bigcup_{n<\omega} M_{\alpha,n}$ as in clause $\circledast(\delta)$ above.)

(c) $(\forall\epsilon < \lambda)\Big[\epsilon \in \bigcup_n M_n \Rightarrow \mathrm{otp}(C_\delta \cap \epsilon) = \mathrm{otp}\big(C^\alpha \cap h(\epsilon)\big)\Big]$.

(d) If $\lambda = \lambda^{<\chi(*)}$ then $N_\beta = \Big(\bigcup_n M_n\Big) \restriction \big\{x \in \bigcup_n M_n \colon \mathrm{cd}(x) < \min(C_\delta)\big\}$.

We proceed as in the proof of 4.12 after $\mathbf{W}^0_\delta$ was defined (only $\dot\zeta(\alpha) = \delta \in S_{\alpha_1,\beta_1,\gamma_1} \Rightarrow h(\alpha) = \gamma_1$).

Suppose G is a winning strategy for Player I. So suppose that if Player II has chosen $\eta(0), \eta(1), \dots, \eta(n-1)$, Player I will choose M_η. So $|M_\eta|$ is a subset of $\mathcal{H}_{<\chi(*)}(\lambda)$ of cardinality $< \chi(*)$ and $\mathrm{Rang}(\eta) \subseteq M_\eta$. For $\eta \in {}^\omega\lambda$ we define $M_\eta = \bigcup_{\ell<\omega} M_{\eta\restriction\ell}$.

Let $\mathcal{T}_n$ be the set of $\eta \in {}^n\lambda$ such that M_η is well defined, so $\bigcup\{\mathcal{T}_n \colon n < \omega\}$ is a subtree of $({}^{\omega>}\lambda, \lhd)$ with each node having λ immediate successors.

We can find a function $\mathbf{c}_n$ from $\mathcal{T}_n$ into μ such that $\mathbf{c}_n(\eta) = \mathbf{c}_n(\nu)$ iff there is an isomorphism h from M_η onto M_ν mapping $M_{\eta\restriction k}$ onto $M_{\nu\restriction k}$ for every $k < n$. By [12, 1.10=L1.7], or [7], or the proof of 4.24 below, there is $\mathcal{T}$ such that:

$(*)$ (a) $\mathcal{T} \subseteq {}^{\omega>}\lambda$
(b) $\mathcal{T}$ is closed under initial segments.
(c) $\langle\,\rangle \in \mathcal{T}$
(d) $\eta \in \mathcal{T} \Rightarrow (\exists^\lambda\alpha)\big[\eta\hat{\ }\langle\alpha\rangle \in \mathcal{T}\big]$
(e) $\mathbf{c}_n \restriction (\mathcal{T} \cap \mathcal{T}_n)$ is constant.

It follows that for any $\nu_* \in \lim(\mathcal{T})$ we can find $\langle h_\eta \colon \eta \in \mathcal{T}\rangle$ such that h_η is an isomorphism from $M_{\nu_*\restriction \ell g(\eta)}$ onto M_η increasing with η.

Note that above, all those isomorphisms are unique as the interpretation of $\in$ satisfies comprehension. Also, clause (c1) follows from the use of R.

The rest should be clear. ∎

LEMMA 4.24. *Let $S, \lambda, \mu, \theta, \chi(*)$ be as in 4.22$(*)$, and in addition:*

$$\aleph_0 \le \kappa = \mathrm{cf}(\kappa) < \chi(*) = \mathrm{cf}(\chi(*)),$$
$$\big(\forall\chi < \chi(*)\big)\big[\chi^{<\kappa} < \chi(*)\big], \quad (\forall\alpha < \lambda)\big[|\alpha|^{<\kappa} < \lambda\big].$$

Then we can find $\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) \colon \alpha < \alpha(*)\}$ and functions $\dot\zeta \colon \alpha(*) \to S$ and $h \colon \alpha(*) \to \lambda$ such that:

$(a0), (b0), (b2)$ *As in 4.22 (i.e. as in 4.12).*
$(b1)^*, (c1), (c2)$ *As in 4.22.*
$(a1)^*$ *As in in 4.12$(a1)$, except that we omit "$\langle M_j : j \leq i\rangle \in M_{i+1}$" and add: $[a \subseteq |M_i| \wedge |a| < \kappa] \Rightarrow a \in M_i$, and for $i < j$, $M_i \cap \lambda$ is an initial segment of $M_j \cap \lambda$.*
$(a2)^*$ *For every expansion $\mathscr{A}$ of $(\mathcal{H}_{<\chi(*)}(\lambda), \in, <)$ by $\chi < \chi(*)$ relations (with $\tau(\mathscr{A}) \subseteq \mathcal{H}_{<\chi(*)}(\chi(*))$), for some $\alpha < \alpha(*)$, for every n, $M^\alpha_n \prec \mathscr{A}$. In fact, for stationarily many $\zeta \in S$, there is such α satisfying $\dot{\zeta}(\alpha) = \zeta$.*

Remark 4.25. We can retain $(a1)^*$ and add $a \subseteq M_i \wedge |a| < \kappa \Rightarrow a \in M_i$.

Proof of Lemma 4.24. Similar to 4.22, using the proof of [31], but for completeness we give details.

We choose $\langle S_{\alpha,\beta,\gamma} : \alpha < \mu, \beta < \lambda, \gamma < \lambda\rangle$ as there. The main point is that defining $\mathbf{W}^*$ we have one additional demand:

(ε) If $n < \omega$ and $u \subseteq M_n$ has cardinality $< \kappa$, then $u \in M_n$.

We then define $\mathbf{W}^0_\delta$ and $\langle N_\alpha : \alpha < \lambda\rangle$ as there.

This gives the changed demand in (a1)*, but it creates extra work in verifying the demand (a2)*.

So let a model $\mathscr{A}$ and cardinal $\chi = \chi^{<\kappa} < \chi(*)$ be given as there; as usual, $\tau(\mathscr{A}) \in \mathcal{H}_{<\chi(*)}(\chi(*))$ and $\mathscr{A}$ expands $(\mathcal{H}_{<\chi(*)}(\lambda), \in, <)$. For every

$$\mathbf{x} = (\delta_\mathbf{x}, \overline{M}_\mathbf{x}, \eta_\mathbf{x}, C_\mathbf{x}) \in \mathbf{W}^*$$

we define a family $\mathscr{F}_\mathbf{x}$, a function $n : \mathscr{F}_\mathbf{x} \to \omega$ and a function $\mathrm{rank}_\mathbf{x}$ from $\mathscr{F}_\mathbf{x}$ into $\mathrm{Ord} \cup \{\infty\}$ as follows:

(α) $\mathscr{F}_\mathbf{x} = \bigcup\{\mathscr{F}_{\mathbf{x},n} : n < \omega\}$
(β) $\mathscr{F}_{\mathbf{x},n} = \{f : f$ is an elementary embedding of $M_{\mathbf{x},n}$ into $\mathscr{A}\}$
(γ) $n(f) = k$ if and only if $f \in \mathscr{F}_{\mathbf{x},k}$.
(δ) $\mathrm{rank}(f) = \bigcup\big\{\epsilon + 1 :$ for every $\alpha < \lambda$ there is $g \in \mathscr{F}_{\mathbf{x},n(f)}$ extending f such that $\beta = \mathrm{rank}_\mathbf{x}(g)$ and $\mathrm{Rang}(g) \cap \alpha = \mathrm{Rang}(f) \cap \lambda\big\}$.

Now

Case 1. For no $\mathbf{x} \in \mathbf{W}^*$ and $f \in \mathscr{F}_{\mathbf{x},0}$ do we have $\mathrm{rank}_\mathbf{x}(f) = \infty$.

For every $\mathbf{x} \in \mathbf{W}^*$ and $f \in \mathscr{F}_\mathbf{x}$ let $\beta(f, \mathbf{x})$ be the first ordinal $\alpha < \lambda$ such that if $\mathrm{rank}_\mathbf{x}(f) = \epsilon$ then there is no $g \in \mathscr{F}_{\mathbf{x},n(f)+1}$ extending f with $\mathrm{rank}_\mathbf{x}(g) = \epsilon$ and $\mathrm{Rang}(g) \cap \alpha = \mathrm{Rang}(f) \cap \lambda$.

Next, let $\bar{\mathscr{A}} = \langle \mathscr{A}_i : i < \lambda \rangle$ be an increasing continuous sequence of elementary submodels of $\mathscr{A}$, each of cardinality $< \lambda$ such that $\langle \mathscr{A}_j : j \leq i \rangle \in \mathscr{A}_{i+1}$.

Easily the set $E = \{i < \lambda : \mathscr{A}_i \cap \lambda = i > \mu\}$ is a club of λ.

Choose, by induction on $n < \omega$, an ordinal i_n increasing with n such that $i_n \in E$ is of cofinality κ (this is possible as $\kappa = \mathrm{cf}(\kappa) < \lambda$) hence $\mathscr{A}_{i_n}$ is an elementary submodel of $\mathscr{A}$ of cardinality $< \lambda$.

Choose $M \prec \mathscr{A}$ of cardinality χ, including $\{i_n : n < \omega\} \cup \{\bar{\mathscr{A}}, \mathbf{W}^*\} \cup (\chi+1)$ such that every $u \subseteq M$ of cardinality $< \kappa$ belongs to M.

Note that if $u \subseteq \mathscr{A}_{i_n}$ has cardinality $< \kappa$ then $u \in \mathscr{A}_{i_n}$ because $i_n \in E$ and $\mathrm{cf}(i_n) = \kappa$, hence this holds for every $\mathscr{A}_{i_n} \cap M$.

Let M^*_n be $\mathscr{A} \restriction (\mathscr{A}_{i_n} \cap M)$; easily $M^*_n \in \mathscr{A}_{i_n}$, so $[u \subseteq M^*_n \wedge |u| < \kappa] \Rightarrow u \in M^*_n$. We can find $\mathbf{x} \in \mathbf{W}$, and isomorphism f_n from $M_{\mathbf{x},n}$ onto M^*_n increasing with n. Now clearly $\mathbf{x} \in \mathscr{A}_{i_n}$.

[Why? As $\mu = \mu^{<\chi(*)}$ and $\mu + 1 \subseteq \mathscr{A}_{i_n}$. Also, $f_n \in \mathscr{F}_{\mathbf{x},n}$ and these f_n are unique as those models expand a submodel of $(\mathcal{H}_{<\chi(*)}(\lambda), \in, <)$ and are necessarily transitive over the ordinals.]

Similarly by the choice of $\mathbf{x}$, we have $f_n \subseteq f_{n+1}$. So $\langle \mathrm{rank}_{\mathbf{x}}(f_n) : n < \omega \rangle$ is constantly ∞ as otherwise we get an infinite decreasing sequence of ordinals.

But this contradicts our case assumption.

Case 2. Not case 1.

So we choose $\mathbf{x} \in \mathbf{W}^*$ and $f \in \mathscr{F}_{\mathbf{x},0}$ such that $\mathrm{rank}_{\mathbf{x}}(f) = \infty$.

We easily get the desired contradiction and even a Δ-system tree of models. How? Let $\langle \eta_\alpha : \alpha < \lambda \rangle$ list ${}^{\omega>}\lambda$ such that $\eta_\alpha \triangleleft \eta_\beta$ implies $\alpha < \beta$.

Now we choose a pair $(f_{\eta_\alpha}, \gamma_\alpha)$ by induction on $\alpha < \lambda$ such that

- (i) $f_{\eta_\alpha} \in \mathscr{F}_{\mathbf{x},\ell g(\eta_\alpha)}$
- (ii) $\gamma_\alpha = \sup\big(\bigcup\{\lambda \cap \mathrm{Rang}(f_{\eta_\beta}) : \beta < \alpha\}\big)$
- (iii) if $\eta_\beta \triangleleft \eta_\alpha$ and $\ell g(\eta_\alpha) = (\ell g(\eta_\beta) + 1$ then $\gamma_\alpha \cap \mathrm{Rang}(f_{\eta_\alpha}) = \lambda \cap \mathrm{Rang}(f_{\eta_\beta})$.

There is no problem to carry the induction. This finishes the proof. ∎

Lemma 4.26. *(1) In 4.24, if in addition $\lambda = \mu^+$ then we can add:*

- $(c3)$ *If $\dot\zeta(\alpha) = \dot\zeta(\beta)$, then $|M^\alpha_\omega| \cap |M^\beta_\omega| \cap \lambda$ is an initial segment of $|M^\alpha_\omega| \cap \lambda$ and of $|M^\beta_\omega| \cap \lambda$, so when $\alpha \neq \beta$ it is a bounded subset of $\dot\zeta(\alpha)$.*

(2) In 4.24 (and 4.26), when $\kappa > \aleph_0$ then it follows that:

$(c4)^*$ *If* $\alpha \neq \beta$ *and* $\{\eta^\alpha \restriction n : n < \omega\} \subseteq M^\beta_\omega$ *then* $\overline{M}^\alpha, \bar{\eta}^\alpha \in M^\beta_\omega$.

(3) Assume $\lambda = \mu^+$, $\mu = \mu^\kappa$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ *is a stationary subset of* λ, *and* $\langle C_\delta : \delta \in S\rangle$ *guesses clubs (and* C_δ *is an unbounded subset of* δ *of order type* ω, *of course).*

Then we can find $\langle \overline{N}_\eta : \eta \in \Gamma\rangle$ *such that:*

- (a) $\Gamma = \bigcup\{\Gamma_\delta : \delta \in S\}$, *where* $\Gamma_\delta \subseteq \{\eta : \eta$ *an increasing* ω-*sequence of ordinals* $< \delta$ *with limit* $\delta\}$ *and* $\delta(\eta) = \delta$ *when* $\eta \in \Gamma_\delta$ *and* $\delta \in S$.
- (b) $\overline{N}_\eta$ *is* $\langle N_{\eta,n} : n \leq \omega\rangle$, *which is* $\prec$-*increasing continuous, and we let* $N_\eta = N_{\eta,\omega}$.
- (c) *Each* N_η *is a model of cardinality* κ *(with vocabulary* $\subseteq \mathcal{H}(\kappa^+)$ *for notational simplicity), universe* $\subseteq \delta := \delta(\eta)$, $N_{\eta,n} = N_\eta \restriction \gamma^\delta_n$ *(where* γ^δ_n *is the* n^{th} *member of* C_δ*), and* $N_\eta \cap (\gamma^\delta_n, \gamma^\delta_{n+1}) \neq \emptyset$.
- (d) *For every distinct* $\eta, \nu \in \Gamma_\delta$ *with* $\delta \in S$, *for some* $n < \omega$, *we have* $N_\eta \cap N_\nu = N_{\eta,n} = N_{\nu,n}$.
- (e) *For every* $\eta, \nu \in \Gamma_\delta$ *the models* N_η, N_ν *are isomorphic; moreover, there is such an isomorphism* f *which preserves the order of the ordinals and maps* $N_{\eta,n}$ *onto* $N_{\nu,n}$.
- (f) *If* $\mathscr{A}$ *is a model with universe* λ *and vocabulary* $\subseteq \mathcal{H}(\kappa^+)$ *then for stationarily many* $\delta \in S$, *for some* $\eta \in \Gamma_\delta \subseteq \Gamma$, *we have* $N_\eta \prec \mathscr{A}$. *Moreover, if* $\kappa^\partial = \kappa$ *and* h *is a one to one function from* ${}^\partial\lambda$ *into* λ *then we can add: if* $\rho \in {}^\partial(N_{\eta,n})$ *then* $h(\rho) \in N_{\eta,n}$.

Proof. (1) Let g^0, g^1 be two place functions from $\lambda \times \lambda$ to λ such that for $\alpha \in [\mu, \lambda]$, $\langle g^0(\alpha, i) : i < \mu\rangle$ enumerates $\{j : j < \mu\}$ without repetition and $g^1(\alpha, g^0(\alpha, i)) = i$ for $i < \lambda$.

Now we can restrict ourselves to $\overline{M}^\alpha$ such that each M^α_i (for $i \leq \omega$) is closed under g^0, g^1. Then (c3) follows immediately from

$$\dot{\zeta}(\alpha) = \dot{\zeta}(\beta) \Rightarrow |M^\alpha_\omega| \cap \mu = |M^\beta_\omega| \cap \mu$$

(required in (c1)).

(2) Should be clear.

(3) This just rephrases what we have proved above. ∎

Lemma 4.27. *Suppose that* $\lambda = \mu^+$, $\mu = \kappa^{\aleph_0} = 2^\kappa > 2^{\aleph_0}$, $\mathrm{cf}(\kappa) = \aleph_0$ *and* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ *is stationary,* $\theta = \aleph_0$, $\aleph_0 < \chi(*) = \mathrm{cf}(\chi(*)) < \kappa$. *Then we can find* $\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ *and functions*

$$\dot{\zeta} : \alpha(*) \to S, \quad h : \alpha(*) \to \lambda$$

and $\langle C_\delta : \delta \in S\rangle$ *with* $\langle \gamma^\delta_n : n < \omega\rangle$ *listing* C_δ *in increasing order such that:*

$(a0)$–$(a1)$ *As in 4.12.*
$(a2)^*$ *As in 4.24.*
$(b0)$–$(b2)$ *As in 4.12, and even*
$(b1)^*$ *$\alpha \neq \beta$, $\{\eta^\alpha \restriction n \colon n < \omega\} \subseteq M^\beta_\omega$ implies $\alpha < \beta$ and even $\dot\zeta(\alpha) < \dot\zeta(\beta)$.*
$(c1)$–$(c3)$ *As in 4.22 + 4.26(1).*
$(c4)$ *If $\dot\zeta(\alpha) = \dot\zeta(\beta) = \delta$ but $\alpha \neq \beta$ then for some $n_0 \geq 1$, there are no $n > n_0$ and $\alpha_1 \leq \beta_2 \leq \alpha_3$ satisfying:*

$$\alpha_1 \in |M^\alpha_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}),$$
$$\beta_2 \in |M^\beta_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}),$$
$$\alpha_3 \in |M^\alpha_\omega| \cap [\gamma^\delta_n, \gamma^\delta_{n+1}),$$

i.e., either

$$\sup\big([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|\big) < \min\big([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|\big)$$

or $\sup\big([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|\big) < \min\big([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|\big)$.

$(c5)$ *If $\Upsilon < \kappa$ and there is $B \subseteq {}^\omega\kappa$, $|B| = \kappa^{\aleph_0}$ which contains no perfect set with density Υ (this holds trivially if κ is strong limit), then also $\{\eta^\alpha \colon \alpha < \alpha(*)\}$ does not contain such a set. (See 4.28.)*

Proof. We repeat the proof of 4.22 with some changes.

Let $\langle S_{\alpha,\beta,\gamma} \colon \alpha < \mu, \beta < \lambda, \gamma < \lambda\rangle$ be pairwise disjoint stationary subsets of S. Let g^0, g^1 be as in the proof of 4.26. By 4.7 there is a sequence $\langle C_\delta \colon \delta \in S\rangle$ such that:

(i) C_δ is a club of δ of order type κ (not ω!), $0 \notin C_\delta$.
(ii) For $\alpha < \mu$, $\beta < \lambda$, $\gamma < \lambda$, for every club C of λ, the set

$$\{\delta \in S_{\alpha,\beta,\gamma} \colon C_\delta \subseteq C\}$$

is stationary.

We then define $\mathbf{W}^*$, $(\delta^j, \langle M_{j,n} \colon n < \omega\rangle, \eta_j, C^j)$ for $j < \mu$, A_α for $\alpha < \lambda$, and R as in the proof of 4.22.

Now, for $\delta \in S_{\alpha,\beta,\gamma}$ let $\mathbf{W}^1_\delta$ be the collection of all systems $\langle M_\rho, \eta_\rho \colon \rho \in {}^{\omega>}\kappa\rangle$ such that:

(i) η_ρ is an increasing sequence of ordinals of length $\ell g(\rho)$.
(ii) $\text{otp}\big(C_\delta \cap \eta_\rho(\ell)\big) = 1 + \rho(\ell)$ for $\ell < \ell g(\rho)$.
(iii) There are isomorphisms $\langle h_\rho \colon \rho \in {}^{\omega>}\kappa\rangle$ such that h_ρ maps M_ρ onto $M_{\alpha,\ell g(\rho)}$ preserving $\in, R, \text{cd}(x) = y, g^0(x_1, x_2) = y, g^1(x_1, x_2) = y$ (and their negations).

(iv) If $\rho \triangleleft \nu$ then $h_\rho \subseteq h_\nu$, $M_\rho \prec M_\sigma$, and $M_\rho \in M_\nu$.

(v) $M_\rho \cap C_\delta = \emptyset$, and $M_\rho \cap \lambda \subseteq \bigcup_\ell [\gamma_{\rho(\ell)}, \gamma_{\rho(\ell)+1})$, where γ_ζ is the ζ^{th} member of C_δ.

(vi) If $\rho \in {}^{\omega>}\kappa$, $\ell < \ell g(\rho)$, and γ is the $(1+\rho(\ell))^{\text{th}}$ member of C_δ then $M_\ell \cap \gamma$ depends only on $\rho \restriction \ell$ and $M_\rho \restriction \gamma \prec M_\rho$.

(vii) $N_\beta = M_{\langle\ \rangle}$.

Now clearly $|\mathbf{W}^1_\delta| \leq \mu$, so let $\mathbf{W}^1_\delta = \{\langle (M^j_\rho, \eta^j_\rho) \colon \rho \in {}^{\omega>}\kappa \rangle \colon j < \mu\}$. Let $\langle \rho_j \colon j < \mu \rangle$ be a list of distinct members of ${}^\omega\kappa$, for (c5) — choose as there.

Let
$$M^j_\ell = \bigcup_{\ell<\omega} M^j_{\rho_j \restriction \ell}, \quad \eta^j = \langle \eta^j_{\rho_j \restriction (\ell+1)}(\ell+1) \colon \ell \leq \omega \rangle.$$
Now,
$$\{\langle M^j_\ell \colon \ell < \omega \rangle \colon j < \mu\}$$
is as required in (c4). Also, (c5) is straightforward, as taking union for all δ-s changes little. (Of course, we are omitting δ-s where we get unreasonable pairs.)

The rest is as before. ∎

Remark 4.28. The existence of B as in (c5) is proved for some Υ, for all strong limit κ of cofinality $\aleph_0$. By [40, Ch.II,6.9,pg.104], much stronger conclusions hold. If 2^κ is regular and belongs to $\{\mathrm{cf}(\prod \kappa_n/D) \colon D$ an ultrafilter on $\omega, \kappa_n < < \kappa\}$, or 2^κ is singular and is the supremum of this set, then it exists for $\Upsilon = (2^{\aleph_0})^+$. Now, if above we replace D by the filter of co-bounded subsets of ω, then we get it even for $\Upsilon = \aleph_0$; by [9, Part D] the requirement holds, e.g., for $\beth_\delta$ for a club of $\delta < \omega_1$.

Moreover, under this assumption on κ we can demand (essentially, this is expanded in 4.33) We strengthen clause (c4) to:

$(c4)^*$ If $\dot\zeta(\alpha) = \dot\zeta(\beta) = \delta$ but $\alpha \neq \beta$ then for some $n_0 \geq 1$, either for every $n \in [n_1, \omega)$ we have
$$\sup([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|) < \min([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|)$$
or for every $n \in [n_1, \omega)$ we have
$$\sup([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\beta_\omega|) < \min([\gamma^\delta_n, \gamma^\delta_{n+1}) \cap |M^\alpha_\omega|).$$

Lemma 4.29. *We can combine 4.27 with 4.24.*

Proof. Left to the reader. ∎

LEMMA 4.30. *Suppose* $\aleph_0 = \theta < \chi(*) = \mathrm{cf}(\chi(*))$ *and* $\lambda^{\aleph_0} = \lambda^{<\chi(*)}$, $\chi(*) \le \lambda$, $\lambda = \lambda^{+}{}_1$, *and* $(*)_{\lambda_1}$ *(see below) holds.*

Then

$(*)_\lambda$ *We can find* $\mathbf{W} = \{(\overline{M}^{\alpha}, \eta^\alpha) \colon \alpha < \alpha(*)\}$ *and functions* $\dot\zeta \colon \alpha(*) \to S$ *and* $h \colon \alpha(*) \to \lambda$ *such that:*

$(a0)$–$(a2)$ *Are as in 4.12.*

$(b0)$–$(b2)$ *As in 4.12, and even*

$(c3)$ *If* $\dot\zeta(\alpha) = \dot\zeta(\beta)$ *then* $|M_\alpha| \cap |M_\beta|$ *is a bounded subset of* $\dot\zeta(\alpha)$.

PROOF. Left to the reader. ∎

LEMMA 4.31. *Suppose that* λ *is a strongly inaccessible uncountable cardinal,*

$$\mathrm{cf}(\lambda) \ge \chi(*) = \mathrm{cf}(\chi(*)) > \theta = \aleph_0,$$

and let $S \subseteq \lambda$ *consist of strong limit singular cardinals of cofinality* $\aleph_0$ *and be stationary. Then we can find* $\mathbf{W} = \{(\overline{M}^{\alpha}, \eta^\alpha) \colon \alpha < \alpha(*)\}$ *and functions* $\dot\zeta \colon \alpha(*) \to S$ *and* $h \colon \alpha(*) \to \lambda$ *such that:*

$(a0)$–$(a2)$ *As in 4.12 (except that* $h(\alpha)$ *does not only depend on* $\dot\zeta(\alpha)$*).*

$(b0), (b3)$ *As in 4.12.*

$(b1)^{+}$ *As in 4.20.*

$(c3)^{-}$ *If* $\dot\zeta(\alpha) = \delta = \dot\zeta(\beta)$ *then* $|M^\alpha_\omega| \cap |M^\beta_\omega| \cap \delta$ *is a bounded subset of* δ.

REMARK 4.32. (1) See [22] for a use of what is essentially a weaker version.

(2) We can generalize 4.24.

PROOF OF LEMMA 4.31. See the proof of [8, 1.10(3)] (but there $\sup(N_{\langle\ \rangle} \cap \lambda) < \delta$). ∎

LEMMA 4.33. *(1) Suppose that* $\lambda = \mu^+$, $\mu = \kappa^\theta = 2^\kappa$, $\theta < \mathrm{cf}(\chi(*)) = \chi(*) < \kappa$, κ *is strong limit,* $\kappa > \mathrm{cf}(\kappa) = \theta > \aleph_0$, *and* $S \subseteq \{\delta < \lambda \colon \mathrm{cf}(\delta) = \theta\}$ *is stationary.*

Then we can find $\mathbf{W} = \{(\overline{M}^{\alpha}, \eta^\alpha) \colon \alpha < \alpha(*)\}$ *(actually, a sequence), functions* $\dot\zeta \colon \alpha(*) \to S$ *and* $h \colon \alpha(*) \to \lambda$, *and* $\langle C_\delta \colon \delta \in S \rangle$ *such that:*

$(a1), (a2)$ *As in 4.12.*

$(b0)$ $\eta^\alpha \ne \eta^\beta$ *for* $\alpha \ne \beta$.

$(b1)$ *If* $\{\eta^\alpha \restriction i \colon i < \theta\} \subseteq M^\beta_\theta$ *and* $\alpha \ne \beta$ *then* $\alpha < \beta$ *and even* $\dot\zeta(\alpha) < \dot\zeta(\beta)$.

$(b2)$ *If* $\eta^\alpha \restriction (j+1) \in M^\beta_\theta$ *then* $M^\alpha_j \in M^\beta_\theta$.

$(c2)$ $\overline{C} = \langle C_\delta \colon \delta \in S \rangle$, C_δ *is a club of* δ *of order type* θ, *and every club of* λ *contains* C_δ *for stationarily many* $\delta \in S$.

(c3) *If $\delta \in S$, $C_\delta = \{\gamma_{\delta,i} : i < \theta\}$ is the increasing enumeration, and $\alpha < \alpha^*$ satisfies $\dot\zeta(\alpha) = \delta$, then there is $\big\langle \langle \gamma^-{}_{\alpha,i}, \gamma^+{}_{\alpha,i}\rangle : i < \theta \text{ odd}\big\rangle$ such that $\gamma^-{}_{\alpha,i} \in M_i^\alpha$, $M_i^\alpha \cap \lambda \subseteq \gamma^+{}_{\alpha,i}$, $\gamma_{\delta,i} < \gamma^-{}_{\alpha,i} < \gamma^+{}_{\alpha,i} < \gamma_{\delta,i+1}$, and*

(∗) *If $\dot\zeta(\alpha) = \dot\zeta(\beta)$ and $\alpha < \beta$ then for every large enough odd $i < \theta$ we have $\gamma^+{}_{\alpha,i} < \gamma^-{}_{\beta,i}$ (hence $[\gamma^-{}_{\alpha,i}, \gamma^+{}_{\alpha,i}) \cap [\gamma^-{}_{\beta,i}, \gamma^+{}_{\beta,i}) = \emptyset$) and $[\gamma^-{}_{\beta,i}, \gamma^+{}_{\beta,i}) \cap M_\theta^\alpha = \emptyset$.*

(2) In part (1), assume $\theta = \aleph_0$ and $\mathrm{pp}(\kappa) = {}^+2^\kappa$. Then the conclusion holds; moreover, (c3) (from 4.26) does as well.

REMARK 4.34. The assumption $\mathrm{pp}(\kappa) = 2^\kappa$ holds (for example) for $\kappa = \beth_\delta$ for a club of $\delta < \omega_1$ (and for a club of $\delta < \theta$ when $\aleph_0 < \theta = \mathrm{cf}(\theta) < \kappa$: see [41, §5]).

PROOF OF LEMMA 4.33. (1) By 4.6 we can find $\bar C = \langle C_\delta : \delta \in S\rangle$, C_δ a club of δ of order type κ such that for any club C of λ, for stationarily many $\delta \in S$, we have $C_\delta \subseteq C$.

FIRST CASE. Assume μ (= 2^κ) is regular.

By [40, Ch.II,5.9], we can find an increasing sequence $\langle \kappa_i : i < \theta\rangle$ of regular cardinals $> \chi(*)$ such that $\kappa = \sum_{i<\theta} \kappa_i$ and $\prod_{i<\theta} \kappa_i / J_\theta^{\mathrm{bd}}$ has true cofinality μ, and let $\langle f_\epsilon : \epsilon < \mu\rangle$ exemplify this. This means

$$\epsilon < \zeta < \mu \;\Rightarrow\; f_\epsilon < f_\zeta \mod J_\theta^{\mathrm{bd}}$$

and for every $f \in \prod_{i<\theta} \kappa_i$, for some $\epsilon < \mu$, we have $f < f_\epsilon \mod J_\theta^{\mathrm{bd}}$. We may assume that if ϵ is limit and $\bar f \restriction \epsilon$ has a $<_{J_\theta^{\mathrm{bd}}}$-l.u.b. then f_ϵ is a $<_{J_\theta^{\mathrm{bd}}}$-l.u.b., and we know that if $\mathrm{cf}(\epsilon) > 2^\theta$ then this holds, and that without loss of generality $\bigwedge_{i<\theta} \mathrm{cf}(f_\epsilon(i)) = \mathrm{cf}(\epsilon)$. Without loss of generality $\kappa_i > f_\epsilon(i) > \bigcup_{j<i} \kappa_j$.

We shall define $\mathbf{W}$ later. Let St be a strategy for Player I in the game from 4.12(a2). By the choice of $\bar C$, for some $\delta \in S$, for every $\alpha \in C_\delta$ of cofinality $> \theta$, $\mathcal{H}_{<\chi(*)}(\alpha)$ is closed under the strategy St. Let $C_\delta = \{\alpha_i : i < \kappa\}$ be increasing continuous. For each $\epsilon < \mu$ we choose a play of the game with Player I using St. For a play, $\langle M_j^\epsilon, \eta_j^\epsilon : j < \theta\rangle$ satisfies:

$$\langle M_j^\epsilon : j \le j_1\rangle \in \mathcal{H}_{<\chi(*)}(\alpha_{f_\epsilon(j_1)+1}),$$
$$\eta_\gamma^\epsilon = \big\langle \mathrm{cd}\big(\alpha_{f_\epsilon(i)}, \langle M_i^\epsilon : i \le j\rangle\big) : j < \gamma\big\rangle,$$
$$\text{and } \eta_{j+1}^\epsilon \in M_{j+1}^\epsilon.$$

Then let $g_\epsilon \in \prod_{i<\theta} \kappa_i$ be $g_\epsilon(i) = \sup\big(\kappa_i \cap \bigcup_{j<\theta} M_j^\epsilon\big)$, so for some $\beta_\epsilon \in (\epsilon, \mu)$, we have $g_\epsilon < f_{\beta_\epsilon} \mod J_\theta^{\mathrm{bd}}$.

On the other hand, if $\mathrm{cf}(\epsilon) = (2^\theta)^+$ then without loss of generality $\mathrm{cf}\big(f_\epsilon(i)\big) = \mathrm{cf}(\epsilon)$ for every $i < \theta$ (see [40, Ch.II,§1]), so there is $\gamma_\epsilon < \epsilon$ such that

$$h_\epsilon < f_{\gamma_\epsilon} \mod J^{\mathrm{bd}}_\theta, \text{ where } h_\epsilon(i) = \sup(f_\epsilon(i) \cap \bigcup_{j<\theta} M^\epsilon_j).$$

So for some $\gamma(*) < \mu$ we have:

$$S_\delta[\mathsf{St}] = \{\epsilon < \mu \colon \mathrm{cf}(\epsilon) = (2^\theta)^+ \text{ and } \gamma_\epsilon = \gamma(*)\} \text{ is stationary.}$$

Now, for each $\delta \in S$ we can consider the set $\mathbf{C}_\delta$ of all possible such $\langle(\overline{M}^\epsilon, \eta^\epsilon) \colon \epsilon < \mu\rangle$, where $\overline{M}^\epsilon = \langle M^\epsilon_j \colon j < i\rangle$ and η^ϵ_θ are as above (letting St vary on all strategies of Player I for which $[\alpha \in C_\delta \ \wedge\ \mathrm{cf}(\alpha) > \theta] \Rightarrow$ $\Rightarrow [\mathcal{H}_{<\chi(*)}(\alpha)$ is closed under $\mathsf{St}]$).

A better way to write the members of $\mathbf{C}_\delta$ is $\big\langle\langle(\overline{M}^\epsilon_j, \eta^\epsilon_j) \colon j < \theta\rangle \colon \epsilon < \mu\big\rangle$, but for $j < \theta$,

$$f_{\epsilon(1)} \restriction j = f_{\epsilon(2)} \restriction j \ \Rightarrow\ \Big[\overline{M}^{\epsilon(1)}_j = M^{\epsilon(2)}_j \ \wedge\ \eta^{\epsilon(1)}_j = \eta^{\epsilon(2)}_j\Big].$$

Actually, it is a function from $\{f_\epsilon \restriction j \colon \epsilon < \mu, j < \theta\}$ to $\mathcal{H}_{<\chi(*)}(\delta)$. But the domain has power κ, the range has power $|\delta| \le \mu$. So $|\mathbf{C}_\delta| \le \mu^\kappa = (2^\kappa)^\kappa = 2^\kappa =$ $= \mu$.

So we can well order $\mathbf{C}_\delta$ in a sequence of length μ, and choose by induction on $\epsilon < \mu$ a representative of each for $\mathbf{W}$ satisfying the requirements.

Second case. Assume μ is singular.

So let $\mu = \sum_{\xi<\mathrm{cf}(\mu)} \mu_\xi$ with μ_ξ regular. Without loss of generality

$$\mu_\xi > \big(\sum\{\mu_\epsilon \colon \epsilon < \xi\}\big)^+ + \mathrm{cf}(\mu)^+.$$

We know that $\mathrm{cf}(\mu) > \kappa$, and again by [40, Ch.VIII,§1] there are $\langle\kappa_{\xi,i} \colon i < \theta\rangle$, $\langle\kappa_i \colon i < \theta\rangle$ such that

$$\mathrm{tcf}\Big(\prod_{i<\theta} \kappa_{\xi,i}/J^{\mathrm{bd}}_\theta\Big) = \mu_\xi, \qquad \mathrm{tcf}\Big(\prod_{i<\theta} \kappa_i/J^{\mathrm{bd}}_\theta\Big) = \mathrm{cf}(\mu),$$

$$\kappa^a_i < \kappa_{\xi i} < \kappa^b_i, \quad \kappa^a_i < \kappa_i < \kappa^b_i \quad \text{and} \quad i < j \Rightarrow \kappa^b_i < \kappa^a_j$$

(we can even get $\kappa^a_i > \prod_{j<i} \kappa^b_j$ as we can uniformize on ξ).

Let $\langle f^\xi_\epsilon : \epsilon < \mu_\xi \rangle$, $\langle f_\epsilon : \epsilon < \mathrm{cf}(\mu) \rangle$ witness the true cofinalities. Now, for every $f \in \prod_{i<\theta} \kappa_i$ (for simplicity, every f such that $f(i) > \sum_{j<i} \kappa_j$ and $\bigwedge_i \mathrm{cf}(f(i)) = (2^\theta)^+$) and ξ we can repeat the previous argument for $\langle f + f^\xi_\epsilon : \epsilon < \mu_\epsilon \rangle$. After "cleaning inside", replacing by a subset of power μ_ξ, we find a common bound below $\prod_{i<\theta} \kappa_i$ and below $\prod f$, and we can uniformize on ξ.

Thus we apply $\mathrm{cf}(\epsilon) = (2^\theta)^+$ on every f_ϵ, and use the same argument on the bound we have just gotten.

(2) Should be clear. ∎

Similarly to 4.22, with ω^2 for θ (not a cardinal!) we have:

CLAIM 4.35. *Suppose that*

(∗) *λ is a regular cardinal, $\theta = \aleph_0$, $\mu = \mu^{<\chi(*)} < \lambda \le 2^\mu$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \aleph_0\}$ is stationary, and $\aleph_0 < \chi(*) = \mathrm{cf}(\chi(*))$.*

Then we can find

$$\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$$

and functions

$$\dot\zeta : \alpha(*) \to S \text{ and } h : \alpha(*) \to \lambda$$

such that:

$(a0)$ *As in 4.12.*
$(a1)$ *$\overline{M}^\alpha = \langle M^\alpha_i : i \le \omega^2 \rangle$ is an increasing continuous elementary chain,*[11] *each M^α_i is a model belonging to $\mathcal{H}_{<\chi(*)}(\lambda)$ [so necessarily has cardinality $< \chi(*)$], $M^\alpha_i \cap \chi(*)$ is an ordinal, $[\chi(*) = \chi^+ \Rightarrow \chi + 1 \subseteq M^\alpha_i]$, $\eta^\alpha \in {}^{\omega^2}\lambda$ is increasing with limit $\dot\zeta(\alpha) \in S$, $\eta^\alpha \restriction i \in M^\alpha_{i+1}$, M^α_i belongs to $\mathcal{H}_{<\chi(*)}(\eta^\alpha(i))$, and $\langle M^\alpha_i : i \le j \rangle$ belongs to M^α_{j+1}.*
$(a2)$ *Like 4.12 (with ω^2 instead θ).*
$(b0)$–$(b2)$ *As in 4.12.*
$(b1)^*$ *As in 4.22.*
$(c1)$ *If $\dot\zeta(\alpha) = \dot\zeta(\beta)$ then $M^\alpha_{\omega^2} \cap \mu = M^\beta_{\omega^2} \cap \mu$, there is an isomorphism $h_{\alpha,\beta}$ from $M^\alpha_{\omega^2}$ onto $M^\beta_{\omega^2}$ mapping $\eta^\alpha(i)$ to $\eta^\beta(i)$ and M^α_i to M^β_i for $i < \omega^2$, and $h_{\alpha,\beta} \restriction \big(|M^\alpha_{\omega^2}| \cap |M^\beta_{\omega^2}|\big)$ is the identity.*
$(c2)$ *As in 4.22, using $\langle M^\alpha_{\omega n} : n < \omega \rangle$.*
$(c3)$ *As in 4.26, assuming $\lambda = \mu^+$.*
$(c4)$ *$\eta^\alpha(i) > \sup\big(|M^\alpha_i| \cap \lambda\big)$ (so $\sup\big(|M^\alpha_{\omega(n+1)}| \cap \lambda\big) = \bigcup_\ell \eta^\alpha(\omega n + \ell)$).*

[11] $\tau(M^\alpha_i)$, the vocabulary, may be increasing too and belongs to $\mathcal{H}_{<\chi(*)}(\chi(*))$.

Proof. We use $\langle \overline{M}^{\alpha,0} : \alpha < \alpha(*) \rangle$, which we got in 4.22. Now for each α we look at $\bigcup_{n<\omega} M_n^{\alpha,0}$ as an elementary submodel of $(\mathcal{H}_{<\chi(*)}(\lambda), \in)$ with a function St (intended as a strategy for Player I in the play for (a2) above).

Play in $\bigcup_{n<\omega} M_n^{\alpha,0}$ and get

$$\langle M_i^\alpha, \eta^\alpha(i) : i < \omega n \rangle \in M_n^{\alpha,0},$$
$$\sup\{\eta^\alpha(i) : i < \omega n\} \in M_{n+1}^{\alpha,0},$$
$$\eta^\alpha(\omega n) > \sup(M_n^{\alpha,0} \cap \lambda).$$

∎

4.4. Black boxes: third round

Lemma 4.36. *Assume that* $\lambda \geq \chi(*) > \theta$ *are regular cardinals,* $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ *is a stationary set,* $\lambda^{<\chi(*)} = \lambda$, *and the conclusion of 4.33 holds for them. Then it holds for* λ^+ *as well as* λ.

Proof. By [8, 2.10(2)] (or see [42]) we know

- $(*)$ There are $\langle C_\delta : \delta < \lambda^+, \mathrm{cf}(\delta) = \theta \rangle$, $\langle e_\alpha : \alpha < \lambda^+ \rangle$ such that:
  - (i) C_δ is a club of δ of order type θ such that
  $$\alpha \in C_\delta \ \wedge \ \alpha > \sup(C_\delta \cap \alpha) \ \Rightarrow \ \mathrm{cf}(\alpha) = \lambda.$$
  - (ii) e_α is a club of α of order type $\mathrm{cf}(\alpha)$; we let $e_\alpha = \{\beta_i^\alpha : i < \mathrm{cf}(\alpha)\}$ (increasing continuous).
  - (iii) If E is a club of λ^+ then for stationarily many $\delta < \lambda^+$ we have $\mathrm{cf}(\delta) = \theta$, $C_\delta \subseteq E$, and the set
  $$\{i < \lambda : \text{for every } \alpha \in C_\delta, \mathrm{cf}(\alpha) = \lambda \ \Rightarrow \ \beta_{i+1}^\alpha \in E\}$$
  is unbounded in λ.

Now copying the black box of λ on each $\delta < \lambda^+$ with $\mathrm{cf}(\delta) = \theta$, we can finish easily. ∎

Lemma 4.37. *If* λ, μ, κ, θ, $\chi(*)$, S *are as in 4.33, and*

$$\alpha < \chi(*) \ \Rightarrow \ |\alpha|^\theta < \chi(*)$$

then there is a stationary $S^* \subseteq [\lambda]^{<\chi(*)}$ *and a one-to-one function* cd *from* S^* *to* λ *such that*

$$[A \in S^* \ \wedge \ B \in S^* \ \wedge \ A \subsetneq B] \ \Rightarrow \ \mathrm{cd}(A) \in B.$$

Remark 4.38. This gives another positive instance to a problem of Zwicker. (See [31].)

Proof of Lemma 4.37. Similar to the proof of 4.33, only choose cd: $[\lambda]^{<\chi(*)} \to \lambda$ one-to-one, and then define $S^* \cap [\alpha]^{<\chi(*)}$ by induction on α. ∎

Problem 4.39. (1) Can we prove in ZFC that for some regular $\lambda > \theta$:

$(*)_{\lambda,\theta,\chi(*)}$ We can define, for $\alpha \in S^\lambda_\theta = \{\delta < \lambda : \aleph_0 \le \mathrm{cf}(\delta) = \theta\}$, a model M_α with a countable vocabulary and universe an unbounded subset of α of power $< \chi(*)$, such that $M_\delta \cap \chi(*)$ is an ordinal such that for every model M with countable vocabulary and universe λ, for some[12] $\delta \in S^\lambda_\kappa$, we have $M_\delta \subseteq M$.

(2) The same. dealing with relational vocabularies only. (We call it $(*)^{\mathrm{rel}}_{\lambda,\theta,\kappa}$.)

Remark 4.40. Note that by 4.8, if $(*)_{\lambda,\theta,\kappa}$ and $\mu = \mathrm{cf}(\mu) > \lambda$ then $(*)_{\mu^+,\theta,\kappa}$.

* * *

Now (in 4.41–4.45) we return to black boxes for singular λ: i.e. we deal with the case $\mathrm{cf}(\lambda) \le \theta$.

Lemma 4.41. *Suppose that $\lambda^\theta = \lambda^{<\chi(*)}$, λ is a singular cardinal, θ is regular, and $\chi(*)$ is regular $> \theta$.*

Assume further

(α) $\mathrm{cf}(\lambda) \le \theta$

(β) $\lambda = \sum_{i \in w} \mu_i$, $|w| \le \theta$, $w \subseteq \theta^+$ *(usually* $w = \mathrm{cf}(\lambda)$*)*, $[i < j \Rightarrow \mu_i < \mu_j]$, *each* μ_i *is regular* $< \lambda$, *and*

$$\mathrm{cf}(\lambda) > \aleph_0 \wedge \mathrm{cf}(\lambda) = \theta \Rightarrow w = \mathrm{cf}(\lambda).$$

(γ) $\mu > \lambda$, μ *is a regular cardinal,* D *is a uniform filter on* w *(so* $\{\alpha \in w : \alpha > \beta\} \in D$ *for each* $\beta \in w$*),* μ *is the true cofinality of* $\prod_{i \in w}(\mu_i, <)/D$ *(see* [12, 3.7(2) = Lc18] *or* [40]*).*

(δ) $\bar f = \langle f_i/D : i < \mu\rangle$ *exemplifies "the true cofinality of* $\prod_i(\mu_i, <)/D$ *is* μ*": i.e.,*

$$\alpha < \beta < \lambda \Rightarrow f_\alpha/D < f_\beta/D,$$

$$f \in \prod_i \mu_i \Rightarrow \bigvee_\alpha f/D < f_\alpha/D.$$

(ε) $S \subseteq \{\delta < \mu : \mathrm{cf}(\delta) = \theta\}$ *is good for* $(\mu, \theta, \chi(*))$.

[12] Equivalently, stationarily many.

(ζ) *If $\theta > \mathrm{cf}(\lambda)$, $\delta \in S$, then for some $A_\delta \in D$ and unbounded $B_\delta \subseteq \delta$ we have*

$$\alpha, \beta \in B_\delta \ \wedge\ \alpha < \beta \ \wedge\ i \in A_\delta \ \Rightarrow\ f_\alpha(i) < f_\beta(i)$$

i.e. $\langle f_\alpha \restriction A_\delta : \alpha \in B_\delta \rangle$ is $<$-increasing.

Then we can find $\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$ (pedantically, a sequence) and functions $\dot{\zeta} : \alpha(*) \to S$ and $h : \alpha(*) \to \mu$ such that:

$(a0)$–$(a2)$ *As in 4.12, except that we replace $(a1)(*)$ by*

$(*)'$ (i) $\eta^\alpha \in {}^\theta\lambda$

(ii) *If $i < \mathrm{cf}(\lambda)$ then* $\sup(\mu_i \cap \mathrm{Rang}(\eta^\alpha)) = \sup(\mu_i \cap M^\alpha_\theta)$.

(iii) *If $\xi < \dot{\zeta}(\alpha)$ then*

$$f_\xi / E < \left\langle \sup(\mu_i \cap M^\alpha_\theta) : i < \mathrm{cf}(\lambda) \right\rangle / E \le f_{\dot{\zeta}(\alpha)} / E.$$

$(b0)$–$(b3)$ *As in 4.12.*

Proof. For $A \subseteq \theta$ of cardinality θ, let $\mathrm{cd}^A_{\lambda,\chi(*)} : \mathcal{H}_{<\chi(*)}(\lambda) \to {}^A\lambda$ be one-to-one and $G : \lambda \to \lambda$ be such that for γ divisible by $|\gamma|$ and $\alpha < \gamma \le \lambda$ (and $\mu \ge \aleph_0$), the set $\{\beta < \gamma : G(\beta) = \alpha\}$ is unbounded in γ and of order type γ. Let $\bar{A} = \langle A_i : i < \theta \rangle$ be a sequence of pairwise disjoint subsets of θ each of cardinality θ.

For $\delta \in S$, let

$$\mathbf{W}^0_\delta = \Big\{ (\overline{M}, \eta) : \overline{M}, \eta \text{ satisfy (a1), and for some } y \in \mathcal{H}_{<\chi(*)}(\lambda), \text{ for every } i < \theta, \text{ we have } \left\langle G(\eta(i)) : i \in A_j \right\rangle = \mathrm{cd}^A_{\lambda,\chi(*)}\big(\langle \overline{M} \restriction j, \eta \restriction j, y \rangle\big) \Big\}.$$

The rest is as before. ∎

Claim 4.42. *Suppose that $\lambda^\theta = \lambda^{<\chi(*)}$, λ is singular, θ and $\chi(*)$ are regular, and $\chi(*) > \theta$.*

(1) If $(\forall \alpha < \lambda)\big[|\alpha|^{<\chi(*)} < \lambda\big]$ then by $\lambda^\theta = \lambda^{<\chi(*)}$ we know that either $\mathrm{cf}(\lambda) \ge \chi(*)$ (and so lemma 4.18 applies) or $\mathrm{cf}(\lambda) \le \theta$.

(2) We can find regular μ_i (for $i < \mathrm{cf}(\lambda)$) increasing with i such that $\lambda = \sum_{i < \mathrm{cf}(\lambda)} \mu_i$.

(3) For $\langle \mu_i : i \in w \rangle$ as in 4.41(β), we can find $D, \mu, \bar{f}$ as in 4.41(γ),(δ) with D the co-bounded filter plus one unbounded subset of ω.

(4) For $\langle \mu_i : i \in w \rangle$, $D, \mu, \bar{f}$ *as in* (β),(γ),(δ) *of 4.41, we can find* μ *and pairwise disjoint* $S \subseteq \mu$ *as required in 4.41*$(\delta)(\varepsilon)$ *provided that* $\theta > \mathrm{cf}(\lambda) \Rightarrow \Rightarrow 2^\theta < \mu$ *[equivalently,* $< \lambda$*].*

(5) If $\mathrm{cf}(\lambda) > \aleph_0$, $(\forall \alpha < \lambda)\big[|\alpha|^{\mathrm{cf}(\lambda)} < \lambda\big]$, *and* $\lambda < \mu = \mathrm{cf}(\mu) \leq \lambda^{\mathrm{cf}(\lambda)}$ *then we can find* $\langle \mu_i : i < \mathrm{cf}(\lambda) \rangle$ *and the co-bounded filter* D *on* $\mathrm{cf}(\lambda)$ *as required in 4.31*(β), (γ).

Proof. Now (1)–(3) are trivial; for (5) see [37, §9]. As for (4), we should recall that [37, §5] actually says:

Fact 4.43. If $\langle \mu_i : i \in w \rangle, \bar{f}, D$ are as in 4.41, then

$$S = \big\{\delta < \mu : \mathrm{cf}(\delta) = \theta \text{ and there are } A_\delta \in D \text{ and unbounded } B_\delta \subseteq \delta \text{ such that } [\alpha, \beta \in B_\delta \wedge \alpha < \beta \wedge i \in A_\delta \Rightarrow f_\alpha(i) < f_\beta(i)]\big\}.$$

is good for $(\mu, \theta, \chi(*))$. ∎

Lemma 4.44. *Let* $\chi(1) = \chi(*) + (< \chi(*))^\theta$.

In 4.41, if $\lambda^\theta = \lambda^{\chi(1)}$, *we can strengthen (b1) to (b1)*$^+$ *(of 4.20).*

Proof. Combine proofs of 4.41, 4.20. ∎

Lemma 4.45. $\frac{3.17}{3.11} \times 3.29$ *and* $\frac{3.19}{3.11} \times 3.37$ *hold (we need also the parallel to 4.33).*

Proof. Left to the reader. ∎

4.5. Conclusion

Now we draw some conclusions.

The first, 4.46, gives what we need in 3.7 (so 3.3).

Conclusion 4.46. *Suppose* $\lambda^\theta = \lambda^{<\chi(*)}$, $\mathrm{cf}(\lambda) \geq \chi(*) + \theta^+$, $\theta = \mathrm{cf}(\theta) < < \chi(*) = \mathrm{cf}(\chi(*))$. *Then we can find*

$$\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\},$$

where

$$M_i^\alpha = (N_i^\alpha, A_i^\alpha, B_i^\alpha), A_i^\alpha \subseteq \lambda \cap |N_i^\alpha|, B_i^\alpha \subseteq \lambda \cap |N_i^\alpha|, A_i^\alpha \neq B_i^\alpha,$$

and functions $\dot{\zeta}, h$ *such that:*

$(a0), (a1)$ *As in 4.12.*
$(a2)$ *As in 4.12, except that in the game, Player I can choose* M_i *only as above.*
$(b0), (b1), (b2)$ *As in 4.12.*

$(b1)''$ *If* $\{\eta^\alpha \restriction i \colon i < \theta\} \subseteq M^\beta$ *but* $\alpha < \beta$ *(so* $\beta < \alpha + (<\chi(*))^\theta$*) then:*

$$A^\alpha_\theta \cap \big(|M^\alpha_\theta| \cap |M^\beta_\theta|\big) \neq B^\beta_\theta \cap \big(|M^\alpha_\theta| \cap |M^\beta_\theta|\big),$$
$$B^\alpha_\theta \cap \big(|M^\alpha_\theta| \cap |M^\beta_\theta|\big) \neq A^\beta_\theta \cap \big(|M^\alpha_\theta| \cap |M^\beta_\theta|\big).$$

Proof. First assume λ is regular, and $\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) \colon \alpha < \alpha(*)\}, \dot{\zeta}, h^*$ be as in the conclusion of 4.12 (with h^* here standing in for h there). Let $w = \{\mathrm{cd}(\alpha,\beta) \colon \alpha,\beta < \lambda\}$, and $G_1, G_2 \colon w \to \lambda$ be such that for $\alpha \in E$, $\alpha = \mathrm{cd}(G_1(\alpha), G_2(\alpha))$.

Let

$$Y = \big\{\alpha < \alpha(*) \colon \overline{M}^\alpha_i \text{ has the form } (N^\alpha_i, A^\alpha_i, B^\alpha_i),\ A^\alpha_i, B^\alpha_i \text{ distinct subsets of } \lambda \cap |N^\alpha_i| \text{ (equivalently, monadic relations), and } G_2(h(\alpha)) = \min(A^\alpha_i \setminus B^\alpha_i \cup B^\alpha_i \setminus A^\alpha_i)\big\}.$$

Now we let

$$\mathbf{W}^* = \big\{(\overline{M}^\alpha, \eta^\alpha) \colon \alpha \in Y\big\}, \dot{\zeta}^* = \dot{\zeta} \restriction Y, \text{ and } h = G_1 \circ h^*.$$

They exemplify that 4.46 holds.

What if λ is singular? Still, $\mathrm{cf}(\lambda) \geq \chi(*) + \theta^*$, and we can just use 4.18 instead 4.12. ∎

Claim 4.47. *(1) In 4.12, if* $\lambda = \lambda^{<\chi(*)}$ *we can let* $h \colon S \to \mathcal{H}_{<\chi(*)}(\lambda)$ *be onto. Generally, we can still make* $\mathrm{Rang}(h)$ *be* $\subseteq A$ *whenever* $|A| = \lambda$*.*

(2) In 4.12, by its proof, whenever $S' \subseteq S$ *is stationary, and*

$$\bigwedge_\zeta \big[h^{-1}(\zeta) \cap S' \textit{ stationary}\big]$$

then $\big\{(\overline{M}^\alpha, \eta^\alpha) \colon \alpha < \alpha(*), \dot{\zeta}(\alpha) \in S'\big\}$ *satisfies the same conclusion.*

(3) For any unbounded $a \subseteq \theta$*, we can let Player I also choose* $\eta(i)$ *for* $i \in \theta \setminus a$ *without changing our conclusions.*

(4) Similar statements hold for the parallel claims.

(5) It is natural to have $\chi(*) = \chi^+$*.*

Proof. Straightforward. ∎

Fact 4.48. We can make the following changes in (a1), (a2) of 4.12 (and in all similar lemmas here) getting equivalent statements:

(∗) $M_i^\alpha \in \mathcal{H}_{<\chi(*)}(\lambda+\lambda)$: in the game, for some arbitrary $\lambda^* \geq \lambda$ (but fixed during the game) Player I chooses the $M_i^\alpha \in \mathcal{H}(\lambda^*)$ of cardinality $< \chi(*)$, and in the end instead of "$\bigwedge_{i<\theta} [M_i = M_i^\alpha]$" we have

- There is an isomorphism from M_θ onto M_θ^α taking M_i onto M_i^α, is the identity on $M_\theta \cap \mathcal{H}_{<\chi(*)}(\lambda)$, maps $|M_\theta| \setminus \mathcal{H}(\lambda)$ into $\mathcal{H}_{<\chi(*)}(\lambda+\lambda) \setminus \mathcal{H}_{<\chi(*)}(\lambda)$, and preserves $\in$, $\notin$, and '[is/is not] an ordinal'.

Exercise 4.49. If D is a normal fine filter on $\mathcal{P}(\mu)$, λ is regular, $\lambda \leq \mu$, $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$ is stationary, and furthermore

$(*)_{D,S}$ $\{a \subseteq \mu : \ \sup(a \cap \lambda) \in S\} \neq \emptyset \mod D$.

then we can partition S to λ stationary disjoint subsets $\langle S_i : i < \lambda\rangle$ such that $i < \lambda \Rightarrow (*)_{D,S_i}$.

[Hint: like the proof of 4.3.]

Notation 4.50. (1) Let κ be an uncountable regular cardinal. We let $\mathrm{seq}^\alpha_{<\kappa}(\mathscr{A})$ (where $\mathscr{A}$ is an expansion of a submodel of some $\mathcal{H}_{\leq\mu}(\lambda)$ with $|\tau(\mathscr{A})| \leq \chi$) be the set of sequences $\langle M_i : i < \alpha\rangle$ which are increasing continuous with $M_i \prec \mathscr{A}$, $\|M_i\| < \kappa$, $M_i \cap \kappa \in \kappa$, $\kappa = \kappa_1^+ \Rightarrow \kappa_1 + 1 \subseteq M_i$, and $\langle M_j : j \leq i\rangle \in M_{i+1}$. (If $\alpha = \delta$ is limit, $M_\delta := \bigcup_{i<\delta} M_i$).

(2) If $\kappa = \kappa^+{}_1$, we may write $\leq \kappa_1$ instead $< \kappa$.

We repeat the definition of filters introduced in [21, Definition 3.2].

Definition 4.51. (1) $\mathscr{E}^\theta_{<\kappa}(A)$ is a filter on $[A]^{<\kappa}$ defined as follows: $Y \in \mathscr{E}^\theta_{<\kappa}(A)$ iff for (every) χ large enough, for some $x \in \mathcal{H}(\chi)$, the set

$$\Big\{\big(\bigcup_{i<\theta} M_i\big) \cap A : \ \langle M_i : i < \theta\rangle \in \mathrm{seq}^\theta_{<\kappa}\big(\mathcal{H}(\chi), \in, x\big)\Big\}$$

is included in Y.

Exercise 4.52. Let λ, κ, θ, and $Y \subseteq [\lambda]^{<\kappa}$ be given. Then

$$\text{(a)} \ \Rightarrow \ \text{(b)} \ \Rightarrow \ \text{(c)},$$

where

(a) For some $\mathbf{W} = \{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\}$, $\dot{\zeta}$, and h satisfying 4.12, we have
$$Y = \{M_\theta^\alpha \cap \lambda : \ \alpha < \alpha(*)\}$$
and

($*$) $\alpha \neq \beta \wedge \bigwedge_{i<\theta} [\eta_i^\alpha \in M_\theta^\beta] \Rightarrow \alpha < \beta$.

(b) $\diamondsuit_{E^\theta_{<\kappa}(\lambda)}$ holds.

(c) Like (a), but without ($*$).

EXERCISE 4.53. If $\lambda^{2^\kappa} = \lambda$ and $\theta \leq \kappa$ then $\diamondsuit_{E^\theta_{<\kappa}}$. (Main case: $\kappa = \theta$.)

EXERCISE 4.54. If $\lambda = \mu^+$, $\lambda^\kappa = \lambda$, $\theta = \aleph_0$, $\kappa = \kappa^\theta$, then there is a coding set with diamond (see [31]).

EXERCISE 4.55. Suppose that $\mathrm{cf}(\lambda) > \aleph_0$, $2^\lambda = \lambda^{\mathrm{cf}(\lambda)}$, $\chi(*) \geq \theta > \mathrm{cf}(\lambda)$, $(\forall \alpha < \lambda)\big[|\alpha|^{\chi(*)} < \lambda\big]$, and $\mathfrak{C}$ is a model expanding $(\mathcal{H}_{<\chi(*)}(\lambda), \in)$, $|\tau(\mathfrak{C})| \leq \aleph_0$. Then we can find $\{\overline{M}^\alpha : \alpha < \alpha(*)\}$ such that:

(i) $\overline{M}^\alpha = \langle M_i^\alpha : i < \sigma\rangle$, $M_i^\alpha \in \mathcal{H}_{<\chi(*)}(\lambda)$, $M_i^\alpha \cap \chi(*)$ is an ordinal, $M_i^\alpha \restriction \tau(\mathfrak{C}) \prec \mathfrak{C}$, $[i < j \Rightarrow M_i^\alpha \prec M_j^\alpha]$, and $\langle M_j^\alpha : j \leq i\rangle \in M_{i+1}^\alpha$.

(ii) If f_n is a k_n-place function from λ to $\mathcal{H}_{<\chi(*)}(\lambda)$ then for some α, $M_\sigma^\alpha \prec (\mathfrak{C}, f_n)_{n<\omega}$.

EXERCISE 4.56. Suppose $\theta = \mathrm{cf}(\mu) < \mu$, $(\forall \alpha < \mu)\big[|\alpha|^\theta < \mu\big]$, $2^\mu = \mu^\theta$ and $\lambda = (2^\mu)^+$, and $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \theta\}$. Let $\mu = \sum_{i<\theta} \mu_i$, μ_i regular strictly increasing, and $\mathrm{cf}(\prod \mu_i / E) = 2^\mu$. Then we can find

$$\mathbf{W} = \big\{(\overline{M}^\alpha, \eta^\alpha) : \alpha < \alpha(*)\big\}, \quad \dot{\zeta} : \alpha(*) \to S, \quad h : \alpha(*) \to \lambda$$

such that:

($*$) For $\delta \in S$ there is a club C_δ of δ of order type θ such that

$$\alpha \in C_\delta \wedge \mathrm{otp}(\alpha \cap C_\delta) = \gamma + 1 \ \Rightarrow\ \mathrm{cf}(\alpha) = \mu_\gamma.$$

REMARK 4.57. We do not know if the existence of a Black Box for λ^+ with h one-to-one follows from ZFC (of course it is a consequence of $\diamondsuit$). On the other hand, it is difficult to get rid of such a Black Box (i.e., prove the consistency of non-existence).

If $\lambda = \lambda^{<\lambda}$ then we have $h : S \to \lambda$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) < \lambda\}$ such that C_δ is a club of δ, $\mathrm{otp}(C_\delta) = \mathrm{cf}(\delta)$ and

$$(\forall \alpha \in C_\delta)(\forall \text{clubs } C \subseteq \alpha)\big[\mathrm{cf}(\alpha) > \aleph_0 \wedge \min_{C' \text{ club of } C_\alpha} \sup(h \restriction C') = \mathrm{otp}(C \cap \alpha)\big].$$

This is hard to get rid of (i.e. it is hard to find a forcing notion making it no longer a black box without collapsing too many cardinals); compare with Mekler–Shelah [6].

Recall

Definition 4.58. For $\lambda > \theta = \mathrm{cf}(\theta) > \aleph_0$ and stationary $S \subseteq [\lambda]^{<\theta}$, let $\diamondsuit_S$ be defined as follows:

If τ is a countable vocabulary, then there is a diamond sequence $\overline{N} = \langle N_a : a \in S\rangle$ witnessing it, which means

- If N is a τ-model with universe λ then for stationarily many $a \in S$ we have $N_a \prec N$.

(Pedantically, we only consider $a \in S \setminus \emptyset$.)

5. On partitions to stationary sets

We present some results on the club filter on $[\kappa]^{\aleph_0}$ and $[\kappa]^{\theta}$ and some relatives, and on $\diamondsuit$ (see Definition [12, 4.6=Ld12] or 5.4(2) here). There are overlaps of the claims, hence redundant parts, but we believe they are still of some interest.

Claim 5.1. *Assume κ is a cardinal $> \aleph_1$. Then $[\kappa]^{\aleph_0}$ can be partitioned to $\kappa^{\aleph_0}$ (pairwise disjoint) stationary sets.*

Proof. Follows by 5.2 below. In detail, let τ be the vocabulary $\{c_n : n < \omega\}$ where each c_n is an individual constant. By 5.2 below there is a sequence $\overline{M} = \langle M_u : u \in [\kappa]^{\aleph_0}\rangle$ of τ-models, with M_u having universe u such that $\overline{M}$ is a diamond sequence.

For each $\eta \in {}^{\omega}\lambda$, let $\mathcal{S}_\eta$ be the set $u \in [\kappa]^{\aleph_0}$ such that for every $n < \omega$ we have $c_n^{M_u} = \eta(n)$.

By the choice of $\overline{M}$, each set $\mathcal{S}_\eta$ is necessarily a stationary subset of $[\kappa]^{\aleph_0}$, and trivially those sets are pairwise disjoint. ∎

Claim 5.2. *Let $\kappa > \aleph_1$. Then we have diamond on $[\kappa]^{\aleph_0}$ (modulo the filter of clubs on it: see 4.58 or* [12, 4.6=Ld12]*), and we can find $A_\alpha \subseteq [\kappa]^{\aleph_0}$ for $\alpha < \lambda := 2^{\kappa^{\aleph_0}}$ such that each is stationary but the intersection of any two is not.*

Proof. The existence of the A_α-s for $\alpha < \lambda$ follows from the first result. Let τ be a countable vocabulary and $\tau_1 = \tau \cup \{<\}$. First we prove it when $\kappa = \aleph_2$. Without loss of generality $\kappa \le 2^{\aleph_0}$, as otherwise the claim follows by 4.26(3), with $(\aleph_2, \aleph_1, \aleph_0)$ here standing in for (λ, μ, κ) there. Let $\omega \setminus \{0\}$ be the disjoint union of s_n for $n < \omega$, each s_n is infinite with the first element $> n{+}3$ when $n > 0$.

By [8, 2.2] or [42] = [40, Ch.III] we can choose a sequence $\langle C_\delta : \delta \in S^2_0\rangle$ which guesses clubs (where $S^2_0 = \{\delta < \omega_2 : \mathrm{cf}(\delta) = \aleph_0\}$) such that $C_\delta \subseteq \delta = \sup(C_\delta)$ has order type ω.

Let $\big\langle(\mathfrak{A}^\zeta, \bar{a}^\zeta) : \zeta < 2^{\aleph_0}\big\rangle$ list the pairs $(\mathfrak{A}, \bar{\alpha})$ without repetitions, with $\mathfrak{A}$ a model with vocabulary τ_1 and universe a limit countable ordinal, and $\bar{\alpha} = \langle \alpha_n : n < \omega\rangle$ an increasing sequence of ordinals with limit $\sup(\mathfrak{A})$ and $\mathfrak{A} \restriction \alpha_n \prec \mathfrak{A}$. Let E_n be the following equivalence relation relation on $2^{\aleph_0}$: $\varepsilon\, E_n\, \zeta$ iff $(\mathfrak{A}^\varepsilon \restriction \alpha^\varepsilon_n, \bar{\alpha}^\varepsilon \restriction n)$ is isomorphic to $(\mathfrak{A}^\zeta \restriction \alpha^\zeta_n, \bar{\alpha}^\zeta \restriction n)$. By this we mean there is an isomorphism f from $\mathfrak{A} \restriction \alpha^\varepsilon_n$ onto $\mathfrak{A}^\zeta \restriction \alpha^\zeta_n$ which maps $\mathfrak{A}^\varepsilon \restriction \alpha^\varepsilon_k$ onto $\mathfrak{A}^\zeta \restriction \alpha^\zeta_k$ for $k < n$ and is an order preserving function (for the ordinals, alternatively we restrict ourselves to the case where $<$ is interpreted as a well ordering).

We can find subsets t^ζ of ω (for $\zeta < 2^{\aleph_0}$) such that:

(∗) (a) For $\zeta, \varepsilon < 2^{\aleph_0}$ and $n < \omega$ we have $t^\zeta \cap s_n = t^\varepsilon \cap s_n$ iff $\mathfrak{A}^\zeta \restriction \alpha^\zeta_n = \mathfrak{A}^\varepsilon \restriction \alpha^\varepsilon_n$ and $\alpha^\zeta_k = \alpha^\varepsilon_k$ for $k \le n$.
(b) If $\zeta < 2^{\aleph_0}$ and $n < \omega$ then $t^\zeta \cap s_n$ is infinite.
(c) $t^\zeta \cap s_n$ depends only on ζ / E_n.

For $\zeta < 2^{\aleph_0}$ let

$\mathcal{S}_\zeta := \big\{a \in [\kappa]^{\aleph_0} : \mathrm{otp}(a) \text{ is a limit ordinal and } t_\zeta = \{|C_{\sup}(a) \cap \beta| : \beta \in a\}\big\}$

and

$$\mathcal{S}'_\zeta = \big\{a \in \mathcal{S}_t : \mathrm{otp}(a) = \mathrm{otp}(\mathfrak{A}^\zeta)\big\},$$

and for $a \in \mathcal{S}'_\zeta$ let N_a be the model isomorphic to $\mathfrak{A}^\zeta$ by the function f_a, where $\mathrm{Dom}(f_a) = a$, $f_a(\gamma) = \mathrm{otp}(\gamma \cap a)$.

Let $\mathcal{S}$ be the union of $\mathcal{S}'_\zeta$ for $\zeta < 2^{\aleph_0}$. Clearly $\zeta \neq \xi \Rightarrow \mathcal{S}_\zeta \cap \mathcal{S}_\xi = \emptyset$, and so $\mathcal{S}'_\zeta \cap \mathcal{S}'_\xi = \emptyset$. Hence N_a is well defined for $a \in \mathcal{S}$.

Let K_n be the set of pairs $(\mathfrak{A}, \bar{\alpha})$ such that $\mathfrak{A}$ is a τ_1-model with universe a countable subset of κ with no last member, and $\bar{\alpha}$ is an increasing sequence of ordinals $< \kappa$ of length n such that for all $k < n$ we have $\alpha_k < \sup(\mathfrak{A})$, $[\alpha_k, \alpha_{k+1}) \cap \mathfrak{A} \neq \emptyset$, and $\mathfrak{A} \restriction \alpha_k \prec \mathfrak{A}$. So clearly there is a function $\mathrm{cd}_n : K_n \to \mathcal{P}(s_n)$ such that for $\zeta < 2^{\aleph_0}$, $\mathrm{cd}_n(\mathfrak{A}, \bar{\alpha}) = t^\zeta \cap s_n$ iff the pairs $(\mathfrak{A}, \bar{\alpha})$, $(\mathfrak{A}^\zeta, \bar{\alpha}^\zeta \restriction n)$ are isomorphic.

Let M be a τ_1-model with universe κ. Now[13] we can find a full subtree $\mathcal{T}$ of ${}^{\omega>}(\aleph_2)$ (i.e. it is non-empty, closed under initial segments, and each member

[13] See [12, 1.16=L1.15] or history in the introduction of §3, and the proof of 4.24.

has $\aleph_2$ immediate successors) and elementary submodels N_η of M for $\eta \in \mathcal{T}$ such that:

(1) $\mathrm{rang}(\eta) \subseteq N_\eta$
(2) If η is an initial segment of ρ then N_η is a submodel N_ρ. Moreover, $N_\eta \cap \aleph_2$ is an initial segment of N_ρ.

Now let E be the set of $\delta < \kappa = \aleph_2$ satisfying the following condition: if $\rho \in \mathcal{T} \cap {}^{\omega>}\delta$ then $N_\rho \cap \kappa$ is a bounded subset of δ, and δ is a limit ordinal. Let E_1 be the set of $\delta \in E$ such that if $\rho \in \mathcal{T} \cap {}^{\omega>}\delta$ then for every $\beta < \delta$, there is γ such that $\beta < \gamma < \delta$ and $\rho\hat{\ }\langle\gamma\rangle \in \mathcal{T}$. So by the choice of $\langle C_\delta : \delta \in S\rangle$, for some $\delta \in S$ we have $C_\delta \subset E_1$.

Let $\langle \alpha_{\delta,k} : k < \omega\rangle$ list C_δ in increasing order.

Now we choose, by induction on n, a quadruple $(\eta_n, s^*_n, \alpha_n, k_n)$ such that:

$(*)$ (a) $\eta_n \in \mathcal{T}$ has length n (so η_0 is necessarily $\langle\ \rangle$).
(b) If $n = m+1$ then η_n is a successor of η_m.
(c) s^*_n is $\mathrm{cd}_n\big((N_{\eta_n}, \langle\alpha_\ell : \ell < n\rangle)\big)$ if the pair $(N_{\eta_n}, \langle\alpha_\ell : \ell < n\rangle)$ belongs to K_n and is s_n otherwise (so $s^*_n \subseteq s_n$ is infinite).
(d) $\alpha_n = \sup(N_{\eta_n}) + 1$
(e) $k_n = \min\{k : N_{\eta_n} \subseteq \alpha_{\delta,k}\}$ and $k_0 = 0$.
(f) if $n = m+1$ then
  (α) $\min(N_{\eta_n} \setminus N_{\eta_m}) > \alpha_{\delta,k_n-1}$
  (β) $k_m < k_n$
  (γ) $k_n \in \bigcup\{s^*_\ell : \ell < n\}$
  (δ) If $n = (n_1+n_2)^2 + n_2 < (n_1+n_2+1)^2$ (so n_1, n_2 are uniquely determined by n and $n_2 < n$) then $k_n \in s^*_{n_2}$.
  (ε) k_n is minimal under those restrictions.

There is no problem to carry the induction. In the end, let $\eta = \bigcup_n \eta_n \in \lim(\mathcal{T})$, so we get a τ_1-model $N_\eta := \bigcup\{N_{\eta_n} : n < \omega\}$ and an increasing sequence $\langle\alpha_n : n < < \omega\rangle$ of ordinals with limit $\sup(\mathfrak{A})$. Now by the choice of $\langle(\mathfrak{A}^\zeta, \bar{\alpha}^\zeta) : \zeta < 2^{\aleph_0}\rangle$, clearly for some ζ we have $(N_\eta, \bar{\alpha})$ isomorphic to $(\mathfrak{A}^\zeta, \bar{\alpha}^\zeta)$, so necessarily $(N_\eta \restriction \alpha_n, \bar{\alpha} \restriction n)$ belongs to K_n and $\mathrm{cd}_n(N_{\eta\restriction n}, \langle\alpha_\ell : \ell < n\rangle) = s^*_n$.

Also, clearly $\sup(N_\eta) = \delta$ and $\{k_n : n < \omega\} = \{|C_\delta \cap \beta| : \beta \in N_\eta\}$.

Letting a be the universe of N_η, it follows that $a \in \mathcal{S}_\zeta$, so N_a is well defined and isomorphic to $\mathfrak{A}^\zeta$ (hence to N_η). Using $<^M$ we get $N_a = N_\eta$. But $N_\eta \prec M$, so $\langle N_a : a \in \mathcal{S}\rangle$ is really a diamond sequence. (Well, for τ_1-models rather then τ-models, but this does no harm and will even help us for $\kappa > \aleph_2$.)

Second, we consider the case $\kappa > \aleph_2$. Given a countable vocabulary τ, let $\tau_1 = \tau \cup \{<\}$ (pedantically, assuming $<\notin \tau$) and let $\langle N_c : c \in [\aleph_2]^{\aleph_0}\rangle$ be as was proved above with $\kappa = \aleph_2$. For each $c \in [\kappa]^{\aleph_0}$, if $\mathrm{otp}(c) = \mathrm{otp}(c \cap \omega_2, <^{N_{c\cap\omega_2}})$, let g_c be the unique isomorphism from $(c \cap \omega_2, <^{N_{c\cap\omega_2}})$ onto $(c, <)$, $<$ the usual order, and let M_c be the τ-model with universe c such that g is an isomorphism from $N_{c\cap\omega_2} \restriction \tau$ onto M_c. Clearly it is an isomorphism and the M_c-s form a diamond sequence.

[Why? For notational simplicity τ has predicates only (and, of course, $<\notin \tau$). Let $M_0 = M$ be a τ-model with universe κ, let M_1 be an elementary submodel of M of cardinality $\aleph_2$ such that $\omega_2 \subseteq M_1$, let h be a one-to-one function from M_1 onto ω_2, M_2 be a τ-model with universe ω_2 such that h is an isomorphism from M_1 onto M_2, and let M_3 be the τ_1-model expanding M_2 such that

$$<^{M_3} = \big\{\big(h(\alpha), h(\beta)\big) : \alpha < \beta \text{ are from } M_1\big\}.$$

So for some $a \in \mathcal{S} \subseteq [\kappa]^{\aleph_0}$ we have $N_a \prec M_3$ and

$$h(\alpha) = \beta \in N_a \wedge \alpha < \omega_2 \Rightarrow \alpha \in a.$$

(Note that the set of a-s satisfying this contains a club of $[\aleph_2]^{\aleph_0}$.)

Let $c = \{\alpha : h(\alpha) \in a\}$, so clearly $c \cap \omega_2 = a$ and $M_c \prec M_1$ hence $M_c \prec M$, so we are done.] ∎

Discussion 5.3. Some concluding remarks:

(1) We can use other cardinals, but it is natural if we deal with $D_{\kappa,<\theta,\aleph_0}$ (see below).

(2) The context is very near to §3, but the stress is different.

Definition 5.4. Let $\kappa \geq \theta \geq \sigma$ and θ be uncountable regular. If $\theta = \mu^+$ we may write μ instead of $< \theta$.

(1) Let $D = D_1 = D^1_{\kappa,<\theta,\aleph_0}$ be the filter on $[\kappa]^{<\theta}$ generated by $\{A^1_x : x \in \mathcal{H}(\chi)\}$, where

$$A^1_x = \Big\{N \cap \kappa : N = \bigcup_{n<\omega} N_n \text{ is an elementary submodel of } (\mathcal{H}(\chi), \in), \; N_n \text{ is increasing, } N_n \in N_{n+1}, \|N_n\| < \theta, \text{ and } N_n \cap \theta \in \theta\Big\}.$$

(2) Let $D = D_2 = D^2_{\kappa,<\theta,\sigma}$ be the filter on $[\kappa]^{<\theta}$ generated by $\{A^2_x : x \in \mathcal{H}(\chi)\}$, where

$$A^2_x = \Big\{N \cap \kappa \colon\ N = \bigcup_{\zeta<\sigma} N_\zeta \text{ is an elementary submodel of } (\mathcal{H}(\chi), \in),$$
$$N_\zeta \text{ increasing, } \langle N_\varepsilon \colon \varepsilon \leq \zeta\rangle \in N_{\zeta+1}, \text{ and } N_\varepsilon \cap \theta \in \theta\Big\}.$$

(3) For a filter D on $[\kappa]^{<\theta}$, let $\diamondsuit_D$ mean the following: fixing any countable vocabulary τ there are $S \in D$ and $N = \langle N_a \colon a \in S\rangle$, each N_a a τ-model with universe a, such that for every τ-model M with universe λ we have $\{a \in S \colon N_a \subseteq M\} \neq \emptyset \mod D$.

(4) If D is a filter on $[\kappa]^{<\theta}$ and $S \in D^+$, then

$$D \restriction S := \big\{X \subseteq [\kappa]^{<\theta} \colon X \cup \big([\kappa]^{<\theta} \setminus S\big) \in D\big\}.$$

Claim 5.5. *Assume $\theta \leq \sigma$ and $\kappa > \sigma^+$, and let $D = D_{\kappa,\theta,\aleph_0}$.*

(1) $[\kappa]^\theta$ can be partitioned to $\sigma^{\aleph_0}$ (pairwise disjoint) D-positive sets.

(2) Assume in addition that $\sigma^{\aleph_0} \geq 2^\theta$. Then

- (α) *We can find $A_\alpha \subseteq [\kappa]^\theta$ for $\alpha < \lambda := 2^{\kappa^\theta}$ such that each is D-positive but they are pairwise disjoint $\mod D$.*
- (β) *If $\lambda = \kappa^\theta$ and τ is a countable vocabulary then $\diamondsuit_{\lambda,\theta,\aleph_0}$. Moreover, there exist $S^* \subseteq [\lambda]^\theta$ and a function N^* with domain S^* such that*
  - (a) *For distinct a, b from S^* we have $a \cap \kappa \neq b \cap \kappa$.*
  - (b) *For $a \in S^*$ we have that $N^*(a) = N^*_a$ is a τ-model with universe a.*
  - (c) *For a τ-model M with universe λ, the set $\{a \colon N^*_a = M \restriction a\}$ is stationary.*

Proof. Similar to earlier ones: part (1) like Claim 5.1 case (a), part (2) like the proof of Claim 5.2. ∎

Claim 5.6. *(1) If $\theta \leq \kappa_0 \leq \kappa_1$ and $\diamondsuit_{S_0}$ (i.e. $\diamondsuit_{D_{\kappa_0,\theta,\sigma}\restriction S_0}$), where S_0 is a subset of $[\kappa_0]^\theta$ which is $D_{\kappa_0,\theta,\sigma}$-positive and $S_1 := \{a \in [\kappa_1]^\theta \colon a \cap \kappa_0 \in S_0\}$, then $\diamondsuit_{S_1}$ (i.e. $\diamondsuit_{D_{\kappa_1,\theta,\sigma}\restriction S_1}$).*

(2) In part (1), if in addition $\kappa_0 = (\kappa_0)^\theta$ and $\kappa_2 = (\kappa_1)^\theta$ then we can find $S_2 \subseteq [\kappa_2]^\theta$ such that:

- (a) *$a \in S_2 \Rightarrow a \cap \kappa_0 \in S_0$*
- (b) *If $b \neq c \in S_2$ then $b \cap \kappa_1 \neq c \cap \kappa_1$.*
- (c) *$\diamondsuit_{S_2}$*

(3) If $\kappa = \kappa^\theta$ then $\diamondsuit_{D_{\kappa,\theta,\sigma}}$.

REMARK 5.7. This works for other uniform definitions of normal filters.

Above, $\kappa^{\theta^{\sigma}} = \kappa$ can be replaced by "every tree with $\leq \theta$ nodes has at most θ^* branches, and $\kappa^{\theta^*} = \kappa$".

PROOF OF CLAIM 5.6. (1) Easy.

(2) Implicit in earlier proof, 5.2.

(3) See [34], [31] ∎

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