FUNDAMENTA
MATHEMATICAE
236 (2017)

Avoiding equal distances

by

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Abstract. We show that it is consistent that there is a nonmeager set of reals each of whose nonmeager subsets contains equal distances.

1. Introduction. In [1], Erdős and Kakutani showed that the continuum hypothesis (CH) is equivalent to the following statement: There is a partition of the set $\mathbb{R}$ of reals into countably many rationally independent sets. It follows that, under CH, every nonmeager set of reals contains a nonmeager (in fact, everywhere nonmeager) subset avoiding equal distances. The aim of this note is to show that CH is needed here.

THEOREM 1.1. *It is consistent that there is a nonmeager $X \subseteq \mathbb{R}$ such that for every nonmeager $Y \subseteq X$, there are $a < b < c < d \in Y$ such that $b - a = d - c$.*

Note that, by a result of Rado (see [2, Theorem 3.2]), we cannot require Y to avoid arithmetic progressions of length 3. Also by [1], X cannot have size $\aleph_1$. So we start by adding $\aleph_2$ Cohen reals and consider the set X of their pairwise sums. We then make every small subset Y of X meager using a finite support product where Y is small if it avoids equal distances. To capture the new subsets of X that may appear later, we use a σ-ideal $\mathcal{I}$ (see below). The rest of the work is to show that X remains nonmeager in the final model. The dual problem for the null ideal will be dealt with in a forthcoming work.

2. Proof

NOTATION. We sometimes identify $x \in 2^\omega$ with a real whose binary expansion is x. Addition is always the usual addition in $\mathbb{R}$. We also sometimes

2010 *Mathematics Subject Classification*: Primary 03E35; Secondary 03E75.
Key words and phrases: forcing, Baire category.
Received 1 November 2015; revised 6 April 2016.
Published online 5 August 2016.

DOI: 10.4064/fm169-5-2016

interpret $y \in \mathbb{R}$ as a member of 2^ω which is the binary expansion of the fractional part of y. The relevant point here is that these transformations preserve meager sets.

Assume CH. Let $S_\star = [\omega_2]^2$. Let $\mathcal{I} = \{S \subseteq S_\star : (\exists c : S \to \omega)(\forall A \in [\omega_2]^{\aleph_1})([A]^2 \subseteq S \Rightarrow |c[[A]^2]| \geq 2)$. Note that, by the Erdős–Rado theorem, $\mathcal{I}$ is a proper σ-ideal over $S_\star$.

Let $\langle S_\gamma : \gamma < \gamma_\star \rangle$ be a one-one listing of $\mathcal{I}$. Let $\mathbb{P}$ add $\aleph_2$ Cohen reals $\langle x_\alpha : \alpha < \omega_2 \rangle$. In $V^{\mathbb{P}}$, let $\mathbb{Q} = \prod\{\mathbb{Q}_\gamma : \gamma < \gamma_\star\}$ be the finite support product where $\mathbb{Q}_\gamma = \mathbb{Q}_{S_\gamma}$ is a σ-centered forcing making $\{x_\alpha + x_\beta : \{\alpha, \beta\} \in S_\gamma\}$ meager. For $S \subseteq S_\star$, $\mathbb{Q}_S$ is defined as follows: $p \in \mathbb{Q}_S$ iff

- $p = (F_p, \bar{n}_p, \bar{\sigma}_p, N_p) = (F, \bar{n}, \bar{\sigma}, N)$,
- $F \subseteq [S]^2$ is finite,
- $\bar{n} = \langle n_k : k \leq N \rangle$ is an increasing sequence of integers with $n_0 = 0$ and $n_{k+1} - n_k > 2^{n_k - n_{k-1}}$,
- $\bar{\sigma} = \langle \sigma_k : k < N \rangle$ where $\sigma_k \in {}^{[n_k, n_{k+1})}2$ for each k.

We write $p \leq q$ iff $F_p \subseteq F_q$, $\bar{n}_p \preceq \bar{n}_q$, $\bar{\sigma}_p \preceq \bar{\sigma}_q$ and for all $N_p \leq k < N_q$ and $\{\alpha, \beta\} \in F_p$, $(x_\alpha + x_\beta) \restriction [n_{q,k}, n_{q,k+1}) \neq \sigma_{q,k+1}$. It is clear that $\mathbb{Q}_S$ is a σ-centered forcing adding a meager set covering $X_S = \{x_\alpha + x_\beta : \{\alpha, \beta\} \in S\}$. We write X for $X_{S_\star}$.

Note that the set of conditions $p = (p(0), p(1)) \in \mathbb{P} \star \mathbb{Q}$ such that for each $\gamma \in \mathrm{dom}(p(1))$, $p(0)$ forces an actual value $p(1)(\gamma)$, and for every $\{\alpha, \beta\} \in F_{p(1)(\gamma)}$, $\{\alpha, \beta\} \subseteq \mathrm{dom}(p(0))$, is dense in $\mathbb{P} \star \mathbb{Q}$. We will always assume that our conditions have this form.

Claim 2.1. *In $V^{\mathbb{P} \star \mathbb{Q}}$, whenever $Y \subseteq X$ is nonmeager, there are $y_1 < y_2 < y_3 < y_4$ in Y such that $y_2 - y_1 = y_4 - y_3$.*

Proof. Choose $S \subseteq S_\star$ such that $Y = X_S$ is nonmeager and suppose p forces this. Let $S_1 = \{\{\alpha, \beta\} : (\exists p_{\alpha,\beta} \geq p)(p_{\alpha,\beta} \Vdash \{\alpha, \beta\} \in \mathring{S})\}$. Then $S_1 \in \mathcal{I}^+$. Define an equivalence relation E on S_1 as follows: $\{\alpha_0, \beta_0\} E \{\alpha_1, \beta_1\}$ iff

- $|\mathrm{dom}(p_{\alpha_0,\beta_0}(i))| = |\mathrm{dom}(p_{\alpha_1,\beta_1}(i))| = l_i$ for $i \in \{0, 1\}$; let $\{\gamma^i_{\alpha_j,\beta_j,k} : k < l_i\}$ list $\mathrm{dom}(p_{\alpha_j,\beta_j}(i))$ in increasing order for $i, j \in \{0, 1\}$.
- $p_{\alpha_0,\beta_0}(0)(\gamma^0_{\alpha_0,\beta_0,k}) = p_{\alpha_1,\beta_1}(0)(\gamma^0_{\alpha_1,\beta_1,k})$ for each $k < l_0$.
- $p_{\alpha_0,\beta_0}(1)(\gamma^1_{\alpha_0,\beta_0,k})$ and $p_{\alpha_1,\beta_1}(0)(\gamma^1_{\alpha_1,\beta_1,k})$ have the same $\bar{n}, \bar{\sigma}, N$ (but not necessarily F), for each $k < l_1$.

It is clear that E is an equivalence relation on S_1 with countably many equivalence classes. Since $S_1 \in \mathcal{I}^+$, we can choose $A \in [\omega_2]^{\aleph_1}$ such that $[A]^2 \subseteq S_1$ and $\{\alpha_0, \beta_0\} E \{\alpha_1, \beta_1\}$ for all $\{\alpha_0, \beta_0\}, \{\alpha_1, \beta_1\} \in [A]^2$. Let l_0, l_1 be the corresponding domain sizes. By the Ramsey theorem, there is an infinite $A_1 \subseteq A$ such that whenever $\alpha_0 < \beta_0$ and $\alpha_1 < \beta_1$ are from A_1, for all $i \in \{0, 1\}$ and $k_0, k_1 < l_i$ the truth value of $\gamma^i_{\alpha_0,\beta_0,k_0} = \gamma^i_{\alpha_1,\beta_1,k_1}$ depends only on the order type of $\langle\alpha_0,\beta_0,\alpha_1,\beta_1\rangle$. Choose $\alpha_1<\alpha_2<\alpha_3<\alpha_4\in A_1$ such that A_1 has at least two members between any two α_i's. It is easy to check that the conditions $p_{\alpha_1,\alpha_3}, p_{\alpha_1,\alpha_4}, p_{\alpha_2,\alpha_3}, p_{\alpha_2,\alpha_4}$ have a least common extension q. For example, to see that p_{α_1,α_3} and p_{α_2,α_3} have a least common extension, choose some $\beta\in A_1\cap(\alpha_2,\alpha_3)$ and use the fact that $\langle\alpha_1,\alpha_3,\beta,\alpha_3\rangle$, $\langle\alpha_2,\alpha_3,\beta,\alpha_3\rangle$ and $\langle\alpha_1,\alpha_3,\alpha_2,\alpha_3\rangle$ have the same order type. Similarly, for p_{α_2,α_3} and p_{α_1,α_4}, choose $\beta_1<\beta_2$ from $(\alpha_2,\alpha_3)\cap A_1$ and use the fact that $\langle\alpha_1,\alpha_4,\beta_1,\beta_2\rangle$, $\langle\alpha_2,\alpha_3,\beta_1,\beta_2\rangle$ and $\langle\alpha_1,\alpha_4,\alpha_2,\alpha_3\rangle$ have the same order type. Now q forces that $x_{\alpha_1}+x_{\alpha_3}$, $x_{\alpha_1}+x_{\alpha_4}$, $x_{\alpha_2}+x_{\alpha_3}$ and $x_{\alpha_1}+x_{\alpha_4}$ are in Y and $(x_{\alpha_1}+x_{\alpha_3})+(x_{\alpha_2}+x_{\alpha_4})=(x_{\alpha_1}+x_{\alpha_4})+(x_{\alpha_2}+x_{\alpha_3})$. ∎

Claim 2.2. *X is nonmeager in $V^{\mathbb{P}\star\mathbb{Q}}$.*

Proof. Suppose not. Let $p_\star$ and $\langle\mathring{T}_m : m<\omega\rangle$ be such that $p_\star \Vdash (\forall m)(\mathring{T}_m\subseteq {}^{<\omega}2$ is a nowhere dense subtree$)\wedge \mathring{X}\subseteq\bigcup_m[\mathring{T}_m]$. Since $\mathbb{P}\star\mathbb{Q}$ is ccc, we can assume that each $\mathring{T}_m$ is in $V^{\mathbb{P}\star\prod_{k\geq 1}\mathbb{Q}_k}$ where $\mathbb{Q}_k=\mathbb{Q}_{S_k}$ for some $S_k\in\mathcal{I}$. For each $\{\alpha,\beta\}\in S_\star$, choose $p_{\alpha,\beta}$, $m(\alpha,\beta)$, $k(\alpha,\beta)$, $v(\alpha,\beta)$, $l(\alpha,\beta)$, $n(\alpha,\beta)$ etc. such that:

- $p_{\alpha,\beta}\geq p_\star$ and $p_{\alpha,\beta}\Vdash(\mathring{x}_\alpha+\mathring{x}_\beta)\in[\mathring{T}_{m(\alpha,\beta)}]$.
- $p_{\alpha,\beta}=\langle p_{\alpha,\beta}(k) : k\leq k(\alpha,\beta)\rangle$ where $p_{\alpha,\beta}(0)$ is the Cohen part and $p_{\alpha,\beta}(k)\in\mathbb{Q}_k$.
- $\mathrm{dom}(p_{\alpha,\beta}(0))=v(\alpha,\beta)$, $|v(\alpha,\beta)|=l(\alpha,\beta)$ and $\alpha,\beta\in v(\alpha,\beta)$.
- For each $\gamma\in v(\alpha,\beta)$, $p_{\alpha,\beta}(0)(\gamma)\in {}^{n(\alpha,\beta)}2$.
- For each $1\leq k\leq k(\alpha,\beta)$, $p_{\alpha,\beta}(k)=(F_{\alpha,\beta,k},\bar{n}_{\alpha,\beta,k},\bar{\sigma}_{\alpha,\beta,k},N_{\alpha,\beta,k})$ where $F_{\alpha,\beta,k}\subseteq[v_{\alpha,\beta}]^2\cap[S_k]^2$ and $n_{N_{\alpha,\beta,k}}=n(\alpha,\beta)$ does not depend on k.

Let $\{\gamma_{\alpha,\beta,l} : l<l(\alpha,\beta)\}$ list $v(\alpha,\beta)$ in increasing order. Since $S_\star\in\mathcal{I}^+$, as before, we can choose $A\in[\omega_2]^{\aleph_0}$ such that for every $\alpha<\beta$ from A the following hold:

- $\{\alpha,\beta\}\notin\bigcup_{k\geq 1}S_k$.
- $m(\alpha,\beta)=m_\star$, $k(\alpha,\beta)=k_\star$, $l(\alpha,\beta)=l_\star$, $n(\alpha,\beta)=n_\star$.
- For each $l<l_\star$, $p_{\alpha,\beta}(0)(\gamma_{\alpha,\beta,l})=\eta^l_\star\in {}^{n_\star}2$.
- For each $1\leq k\leq k_\star$, $\bar{n}_{\alpha,\beta,k}=\bar{n}^k_\star$, $\bar{\sigma}_{\alpha,\beta,k}=\bar{\sigma}^k_\star$, $N_{\alpha,\beta,k}=N^k_\star$.
- For all $\alpha_1<\alpha_2$ and $\beta_1<\beta_2$ from A and $l_1,l_2<l_\star$, the truth value of $\gamma_{\alpha_1,\beta_1,l_1}=\gamma_{\alpha_2,\beta_2,l_2}$ depends only on the order type of $\langle\alpha_1,\alpha_2,\beta_1,\beta_2\rangle$.

Let $\langle\alpha_i : i<\omega\rangle$ be increasing members of A. Choose $n_{\star\star}>n_\star+(k_\star+l_\star+10)!$. Choose $q_1\geq p_{\alpha_0,\alpha_1}$ such that

- $q_1=\langle q_1(k) : k\leq k_\star\rangle$.
- $\mathrm{dom}(q_1(0))=\mathrm{dom}(p_{\alpha_0,\alpha_1})$ and $q_1(0)(l)=\eta^l_\star{}^\frown 0^{n_{\star\star}-n_\star}$ for each $l<l_\star$.
- $q_1(k)=(F_{\alpha_\star,\beta_\star,k},\bar{n}^k_\star{}^\frown n_{\star\star},\bar{\sigma}^k_\star{}^\frown 01^{n_{\star\star}-n_\star-2}0,N^k_\star+1)$ for each $1\leq k\leq k_\star$.

Since $p_\star$ forces that $[\mathring{T}_{m_\star}]$ is nowhere dense, we can find $n_{\star\star\star} > n_{\star\star}$, $q_2 \geq q_1$ and $\rho \in {}^{[n_{\star\star}, n_{\star\star\star})}2$ such that:

- $q_2 = \langle q_2(k) : k \leq K \rangle$ for some $k_\star \leq K < \omega$.
- For $1 \leq k \leq k_\star$, if $q_2(k) = (F_k, \bar{n}_k, \bar{\sigma}_k, N_k)$, then $n_{\star\star\star} < n_{k,N_k}$.
- $q_2 \Vdash (\forall x \in [\mathring{T}_{m_\star}])(x \restriction [n_{\star\star}, n_{\star\star\star}) \neq \rho)$.

For $j \geq 2$, consider the set $s_j = \{l < l_\star : (\exists l' < l_\star)(\gamma_{\alpha_0,\alpha_1,l} = \gamma_{\alpha_j,\alpha_{j+1},l'})\}$. We claim that $s_j = s_\star$ is constant. To see this, suppose $2 \leq j_1 < j_2$. Choose j_3 much larger than j_2 and use the fact that the order types of $\langle \alpha_0, \alpha_1, \alpha_{j_i}, \alpha_{j_i+1} \rangle$, $\langle \alpha_0, \alpha_1, \alpha_{j_3}, \alpha_{j_3+1} \rangle$ and $\langle \alpha_{j_i}, \alpha_{j_i+1}, \alpha_{j_3}, \alpha_{j_3+1} \rangle$ are the same for $i \in \{1, 2\}$. It also follows that $\{\gamma_{\alpha_{j_1},\alpha_{j_1+1},l} : l \in l_\star \setminus s_\star\} \cap \{\gamma_{\alpha_{j_2},\alpha_{j_2+1},l} : l \in l_\star \setminus s_\star\} = \emptyset$ whenever $2 \leq j_1 < j_2 - 1$. So we can choose j large enough such that $(\mathrm{dom}(q_2(0)) \cup \bigcup_{k \leq K} F_{q_2(k)}) \cap (\{\gamma_{\alpha_j,\alpha_{j+1},l} : l \in l_\star \setminus s_\star\} \cup \{\alpha_j, \alpha_{j+1}\}) = \emptyset$. The next claim gives us the desired contradiction.

Claim 2.3. *For some q_3, $q_3 \geq q_2$, $q_3 \geq p_{\alpha_j,\alpha_{j+1}}$ and $q_3 \Vdash \rho \subseteq \mathring{x}_{\alpha_j} + \mathring{x}_{\alpha_{j+1}}$.*

Proof. Set $q_3 = \langle q_3(k) : k \leq K \rangle$ and $\mathrm{dom}(q_3(0)) = \mathrm{dom}(q_2(0)) \cup \{\gamma_{\alpha_j,\alpha_{j+1},l} : l \in l_\star \setminus s_\star\}$. For each $l \in l_\star \setminus s_\star$, we would like to find $\eta^l_{\star\star} \succeq \eta^l_\star$ such that:

(a) If $l_1 < l_2 \in l_\star \setminus s_\star$, $1 \leq k \leq k_\star$, $\{\gamma_{\alpha_j,\alpha_{j+1},l_1}, \gamma_{\alpha_j,\alpha_{j+1},l_2}\} \in S_k$ and $N^k_\star \leq i < N_{q_2(k)}$, then $(\eta^{l_1}_{\star\star} + \eta^{l_2}_{\star\star}) \restriction [n_{q_2(k),i}, n_{q_2(k),i+1}) \neq \sigma_{q_2(k),i}$.

(b) If $l \in s_\star, l' \in l_\star \setminus s_\star$, $1 \leq k \leq k_\star$ and $N^k_\star \leq i < N_{q_2(k)}$, then $(q_2(0)(\gamma_{\alpha_0,\alpha_1,l}) + \eta^{l'}_{\star\star}) \restriction [n_{q_2(k),i}, n_{q_2(k),i+1}) \neq \sigma_{q_2(k),i}$.

(c) If $\gamma_{\alpha_j,\alpha_{j+1},l_1} = \alpha_j$, $\gamma_{\alpha_j,\alpha_{j+1},l_2} = \alpha_{j+1}$ (so that $l_1, l_2 \in l_\star \setminus s_\star$), then $(\eta^{l_1}_{\star\star} + \eta^{l_2}_{\star\star}) \restriction [n_{\star\star}, n_{\star\star\star}) = \rho$.

Here, for $\sigma, \tau \in 2^{<\omega}$, $m < n < \omega$ and $\rho : [m, n) \to 2$, by $(\sigma + \tau) \restriction [m, n) \neq \rho$ we mean that $(x + y) \restriction [m, n) \neq \rho$ for all $x \in [\sigma]$ and $y \in [\tau]$.

This suffices since then we can let $q_3(0) = q_2(0) \cup \{(\gamma_{\alpha_j,\alpha_{j+1},l}, \eta^l_{\star\star}) : l \in l_\star \setminus s_\star\}$ and $q_3(k) = (F_{q_2(k)} \cup F_{p_{\alpha_j,\alpha_{j+1}}(k)}, \bar{n}_{q_2(k)}, \bar{\sigma}_{q_2(k)}, N_{q_2(k)})$ for $1 \leq k \leq K$.

First set $\eta^l_{\star\star} \restriction [n_\star, n_{\star\star}) = 0^{n_{\star\star} - n_\star}$ for every $l \in l_\star \setminus s_\star$. Next let $W = \{(k, i) : 1 \leq k \leq k_\star,\ N^k_\star + 1 \leq i < N_{q_2(k)}\}$. Note that for $(k, i) \in W$, we have $n_{q_2(k),i+1} - n_{q_2(k),i} > 2^{i - N^k_\star}(n_{\star\star} - n_\star) > 2^{i - N^k_\star}(k_\star + l_\star + 10)!$.

Inductively choose pairwise disjoint intervals $\langle I_{k,i} : (k, i) \in W \rangle$ with $I_{k,i} = [m_{k,i}, m_{k,i} + (l_\star + 5)!) \subseteq [n_{q_2(k),i}, n_{q_2(k),i+1})$. We claim that for each $(k, i) \in W$, we can choose $\langle \eta^l_{\star\star} \restriction I_{k,i} : l \in l_\star \setminus s_\star \rangle$ such that the (k, i)th instance of requirements (a), (b) are met. To see this, note that we have at most $\binom{l_\star - |s_\star|}{2} + |s_\star|(l_\star - |s_\star|)$ inequalities (coming from (a) and (b)) and one equality from (c) to satisfy, and since $\{\alpha_j, \alpha_{j+1}\} \notin \bigcup\{S_k : k \geq 1\}$, there is no conflict between requirements (a) and (c). ∎

Acknowledgments. Research of A. Kumar was supported by a Postdoctoral Fellowship at the Einstein Institute of Mathematics funded by European Research Council grant 338821.

Research of S. Shelah was partially supported by European Research Council grant 338821; publication no. 1079.

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