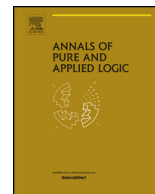




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Abstract elementary classes stable in \aleph_0 [☆]Saharon Shelah ^{a,b}, Sebastien Vasey ^{c,*}

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ABSTRACT

We study abstract elementary classes (AECs) that, in \aleph_0 , have amalgamation, joint embedding, no maximal models and are stable (in terms of the number of orbital types). Assuming a locality property for types, we prove that such classes exhibit superstable-like behavior at \aleph_0 . More precisely, there is a superlimit model of cardinality \aleph_0 and the class generated by this superlimit has a type-full good \aleph_0 -frame (a local notion of nonforking independence) and a superlimit model of cardinality \aleph_1 . We also give a supersimplicity condition under which the locality hypothesis follows from the rest.

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1. Introduction

1.1. Motivation

In [27] (a revised version of which appears as [29, Chapter I], from which we cite), the first author introduced *abstract elementary classes* (AECs): a semantic framework generalizing first-order model theory and also encompassing logics such as $L_{\omega_1, \omega}$. He studied PC_{\aleph_0} -representable AECs (roughly, AECs which

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are reducts of a class of models of a first-order theory omitting a countable set of types) and generalized and improved some of his earlier results on $\mathbb{L}_{\omega_1, \omega}$ [25,26] and $\mathbb{L}_{\omega_1, \omega}(Q)$ [24].

For example, fix a PC_{\aleph_0} -representable AEC \mathfrak{K} and assume that it is categorical in \aleph_0 . Assuming $2^{\aleph_0} < 2^{\aleph_1}$ and $1 \leq \mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$, the first author shows (without even assuming PC_{\aleph_0} -representability) [29, I.3.8] that \mathfrak{K} has amalgamation in \aleph_0 . Further, [29, §I.4, §I.5], it has a lot of structure in \aleph_0 and assuming more set-theoretic assumptions as well as few models in \aleph_2 , \mathfrak{K} has a superlimit model in \aleph_1 [29, I.5.34, I.5.40]. This means roughly (see Section 2) that there is a saturated model in \aleph_1 and that the union of an increasing chain of type ω consisting of saturated models of cardinality \aleph_1 is saturated.

The reader can think of the existence of a superlimit in \aleph_1 as a step toward showing that the models of cardinality \aleph_1 behave in a “superstable-like” way. Indeed several recent works [35,36,6,15] have connected superlimits with other definitions of superstability in AECs, including uniqueness of limit models and local character of orbital splitting.

Another notable consequence of the existence of a superlimit in \aleph_1 is that it implies that there is a model of cardinality \aleph_2 . This ties back to a result of the first author: [29, I.3.11]: for a PC_{\aleph_0} AEC, categoricity in \aleph_0 and \aleph_1 implies the existence of a model in \aleph_2 . The argument first establishes, using only categoricity in \aleph_0 and few models in \aleph_1 , that there is a pair M, N of models in \mathfrak{K}_{\aleph_1} such that $M <_{\mathfrak{K}} N$ and then uses, in essence, that (by \aleph_1 -categoricity) these models are superlimits. In this context, the very strong hypotheses make it possible to avoid referring to any stability-theoretic notions. Still, in more complicated frameworks the existence of a superlimit model in \aleph_1 can be thought of as a key conceptual step toward proving existence of models in higher cardinality and more generally developing a stability theory cardinal by cardinal.

The arguments for the results from [29, §I.5] discussed in the second paragraph of this introduction are complicated by the lack of \aleph_0 -stability: one can only get that there are \aleph_1 -many orbital types over countable models. The workaround there is to redefine the ordering (but not the class of models) to obtain a stable class, see [29, I.5.29]. If the AEC is “nicely-presented”, e.g. a class of models of an $\mathbb{L}_{\omega_1, \omega}$ -sentence or more generally a finitary AEC [18], then this difficulty does not occur (see [2]): \aleph_0 -stability follows from few models in \aleph_1 and $2^{\aleph_0} < 2^{\aleph_1}$. One can also obtain \aleph_0 -stability by starting with only countably-many models in \aleph_1 [4, 3.18]. Finally, it is worth noting that (assuming amalgamation and joint embedding in \aleph_0), \aleph_0 -stability is upward absolute for PC_{\aleph_0} -AECs [22].

1.2. Main results

The bottom line is that \aleph_0 -stability holds in several cases of interest. In fact, there are no known examples which (under $2^{\aleph_0} < 2^{\aleph_1}$) are categorical in \aleph_0 , have few models in \aleph_1 , and are *not* \aleph_0 -stable (see [4, Question 3.15]). Thus in the present paper, we *start* with stability in \aleph_0 (and often amalgamation and categoricity in \aleph_0). Our goal is to say as much as we can on the structure of the class, in particular to get superstable-like behavior in \aleph_0 and \aleph_1 , *without* assuming a non-ZFC hypothesis or $\mathbb{I}(\mathfrak{K}, \aleph_1) < 2^{\aleph_1}$.

One of our first results (Theorem 4.2) is that \aleph_0 -stability (together with amalgamation and \aleph_0 -categoricity) imply that the class \mathfrak{K} is already PC_{\aleph_0} -representable. We also show that the assumption of categoricity in \aleph_0 is not really needed: without assuming it, one can find a superlimit in \aleph_0 and change to the class generated by that superlimit, which will be categorical in \aleph_0 . In fact, we prove (Theorem 4.4) that one can characterize brimmed models (also called limit models in the literature) as those that are homogeneous for orbital types. This has as immediate consequence that the brimmed model of cardinality \aleph_0 is superlimit (Corollary 4.6). This last result sheds light on an argument of Lessmann [23] and answers a question of Fred Drueck (see footnote 3 on [12, p. 25]), who asked when this equality held. The argument works more generally assuming only density of amalgamation bases, as in [32].

For the main result of this paper, we assume that orbital types over countable models are determined by their finite restrictions. The study of statements of the form “orbital types are determined by their small

restrictions” was pioneered by Grossberg and VanDieren [14], who called this condition tameness. Hyttinen and Kesälä [18, §3] were the first to specifically study orbital types over finite sets and the condition that they determine orbital types over countable models.

Following the first author’s terminology [31, 0.1(2)], we call this last condition $(< \aleph_0, \aleph_0)$ -locality (not to be confused with sequence locality [31, 0.1(1)], which is called locality in [3, 1.8] or [1, 11.4]). This is known to hold for several classes of interest:

Example 1.1.

- (1) Let \mathfrak{K} be a finitary AEC (see [18]; this includes classes of models of $L_{\omega_1, \omega}$ -sentences) and assume that \mathfrak{K} is stable in \aleph_0 (finitary AECs have amalgamation and no maximal models by definition). By [18, 4.11], \mathfrak{K} is $(< \aleph_0, \aleph_0)$ -local.
- (2) Finitary AECs are not the only setup where $(< \aleph_0, \aleph_0)$ -locality holds. For example, it is known for quasiminimal pregeometry classes (that may not be finitary [21, Theorem 2]), see [5, 5.2], and more generally in the finite U-rank (FUR) classes of Hyttinen and Kangas [17, 2.17] (we thank Will Boney for pointing us to that reference).

We prove the following:

Theorem 1.2. *Let \mathfrak{K} be an AEC with $LS(\mathfrak{K}) = \aleph_0$ and countable vocabulary. Assume that \mathfrak{K} is categorical in \aleph_0 , \mathfrak{K} is $(< \aleph_0, \aleph_0)$ -local, \mathfrak{K} has amalgamation and no maximal models in \aleph_0 and \mathfrak{K} is stable in \aleph_0 . Then:*

- (1) (Theorem 5.8) *There is a good \aleph_0 -frame on \mathfrak{K}_{\aleph_0} .*
- (2) (Corollary 5.9) *There is a superlimit model of cardinality \aleph_1 .*

The good \aleph_0 -frame (or the superlimit in \aleph_1) imply the nontrivial corollary that \mathfrak{K} has a model of cardinality \aleph_2 [29, II.4.13]. This consequence also follows from a theorem of the second author [41, 12.1] (which however does *not* give a good \aleph_0 -frame or a superlimit in \aleph_1). The conclusion that there is a superlimit model in \aleph_1 seems new, even for finitary AECs or FUR classes.

It is natural to ask whether the locality hypothesis in Theorem 1.2 is really needed.² In fact we do not even know whether the existence of a good \aleph_0 -frame implies $(< \aleph_0, \aleph_0)$ -locality:

Question 1.3. Let \mathfrak{K} be an AEC with $LS(\mathfrak{K}) = \aleph_0$. If there is a good \aleph_0 -frame on \mathfrak{K} , is \mathfrak{K} $(< \aleph_0, \aleph_0)$ -local?

In Section 6, we give a partial answer: any AEC that is \aleph_0 -stable, \aleph_0 -categorical, and *supersimple* (in a sense generalizing that of homogeneous model theory [11]) is $(< \aleph_0, \aleph_0)$ -local. This generalizes the proof of [5, 5.2], which shows that quasiminimal pregeometry classes are $(< \aleph_0, \aleph_0)$ -local (see also [40]). Supersimple \aleph_0 -stable AECs are also much more general than FUR classes.

1.3. Notes

This paper is organized as follows. Section 2 gives some background definitions and fixes the notation. Section 3 is a technical section on good frames (possibly on uncountable models) which sets up the machinery to prove the main theorem (more precisely, to prove a strong symmetry property for nonforking in good frames). Section 4 works with countable models and shows that \aleph_0 -stability implies the existence of a

² In fact, an earlier version of the present paper asserted that it could be derived from the other hypotheses but the argument contained a mistake.

superlimit in \aleph_0 . Section 5 builds the good \aleph_0 -frame and proves the main theorem. Finally, Section 6 studies a sufficient condition to get $(< \aleph_0, \aleph_0)$ -locality.

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Note that at the beginning of several sections, we make global hypotheses assumed throughout the section.

2. Preliminaries

We assume familiarity with the basics of AECs, as presented for example in [13,1], or the first three sections of Chapter I together with the first section of Chapter II in [29]. We also assume familiarity with good frames (see [29, Chapter II] or [7]; it would help the reader to have a copy of both available during the reading of Section 3). This section mostly fixes the notation that we will use.

Given a τ -structure M , we write $|M|$ for its universe and $\|M\|$ for its cardinality. We may abuse notation and write e.g. $a \in M$ instead of $a \in |M|$. We may even write $\bar{a} \in M$ instead of $\bar{a} \in {}^{<\omega}M$.

We write $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ for an AEC. We may abuse notation and write $M \in \mathfrak{K}$ instead of $M \in K$. For a cardinal λ , we write \mathfrak{K}_λ for the AEC restricted to its models of size λ . As shown in [29, II.1], any AEC is uniquely determined by its restriction $\mathfrak{K}_{\leq \text{LS}(\mathfrak{K})}$.

When we say that $M \in \mathfrak{K}$ is an *amalgamation base*, we mean (as in [32]) that it is an amalgamation base in $\mathfrak{K}_{\|M\|}$, i.e. we do *not* require that larger models can be amalgamated.

For $M_0 \in \mathfrak{K}$, we say that M is *universal over M_0* if $M_0 \leq_{\mathfrak{K}} M$ and for any $N \in \mathfrak{K}$ with $M_0 \leq_{\mathfrak{K}} N$, if $\|N\| \leq \|M_0\| + \text{LS}(\mathfrak{K})$, there exists $f : N \xrightarrow{M_0} M$ (usually we will require also that $\|M\| = \|M_0\|$). We say that M is (λ, δ) -*brimmed over M_0* (often also called (λ, δ) -*limit* e.g. in [32,16]) if $\delta < \lambda^+$ is a limit ordinal, $M_0 \in \mathfrak{K}_\lambda$, and there exists an increasing continuous chain $\langle N_i : i \leq \delta \rangle$ of members of \mathfrak{K}_λ such that N_0 is universal over M_0 , $N_\delta = M$, and N_{i+1} is universal over N_i for all $i < \delta$. We say that M is *brimmed over M_0* if it is $(\|M\|, \delta)$ -brimmed over M_0 for some limit $\delta < \|M\|^+$. We say that M is *brimmed* if it is brimmed over some M_0 .

The following key concept appears in [29, I.3.3]:

Definition 2.1. We say that $M \in \mathfrak{K}$ is *superlimit* if, letting $\lambda := \|M\|$, we have that $\lambda \geq \text{LS}(\mathfrak{K})$, M is universal in \mathfrak{K}_λ (i.e. any $M' \in \mathfrak{K}_\lambda$ embeds into M), M is not maximal, and whenever $\delta < \lambda^+$ is limit, $\langle M_i : i < \delta \rangle$ is increasing with $M_i \cong M$ for all $i < \delta$, then $\bigcup_{i < \delta} M_i \cong M$.

The following notion of types already appears in [28]. It is called Galois types by many, but we prefer the term *orbital types* here. They are the same types that are defined in [29, II.1.9], but we also define them over sets. As pointed out in [38, Section 2], this causes no additional difficulties. The following technical point is important: when the AEC does *not* have amalgamation, we may want to compute orbital types only in the subclass of *amalgamation bases* in \mathfrak{K} (as in [32]). Thus we allow orbital types to be computed in a subclass of \mathfrak{K} in the definition.

Definition 2.2. Fix an AEC \mathfrak{K} and a subclass \mathfrak{K}^* of \mathfrak{K} , closed under isomorphisms.

- (1) We say $(A, N_1, \bar{b}_1) E_{\text{at}}^{\mathfrak{K}^*} (A, N_2, \bar{b}_2)$ if:
 - (a) For $\ell = 1, 2$, $N_\ell \in \mathfrak{K}^*$, $A \subseteq |N_\ell|$, and $\bar{b}_\ell \in {}^{<\infty}N_\ell$.
 - (b) There exists $N \in \mathfrak{K}^*$ and $f_\ell : N_\ell \xrightarrow{A} N$, $\ell = 1, 2$, such that $f_1(\bar{b}_1) = \bar{b}_2$.
- (2) $E_{\text{at}}^{\mathfrak{K}^*}$ is a reflexive and symmetric relation. Let $E^{\mathfrak{K}^*}$ be its transitive closure.
- (3) Let $\text{ortp}_{\mathfrak{K}^*}(\bar{b}/A; N)$ be the $E^{\mathfrak{K}^*}$ -equivalence class of (\bar{b}, A, N) .

- (4) Define $\mathcal{S}_{\mathfrak{K}^*}(A, N)$, $\mathcal{S}_{\mathfrak{K}^*}(M)$, $\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(M)$, $\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset)$, etc. for the Stone spaces of orbital types, *computed inside* \mathfrak{K}^* . This is defined as in [38, 2.20]. For example, $\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset) = \{\text{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; N) \mid N \in \mathfrak{K}^*, \bar{b} \in {}^{<\omega}N\}$.
- (5) When $\mathfrak{K}^* = \mathfrak{K}$, we omit it.

Let us say that an AEC \mathfrak{K} is *stable in* λ if for any $M \in \mathfrak{K}_\lambda$, $|\mathcal{S}(M)| \leq \lambda$. This makes sense in any AEC, and is quite well-behaved assuming amalgamation and no maximal models (since then it is known that one can build universal extensions). We will often work in the following axiomatic setup, a slight weakening where full amalgamation is not assumed. This comes from the context derived in [32]:

Definition 2.3. Let \mathfrak{K} be an AEC and let λ be a cardinal. We say that \mathfrak{K} is *nicey stable in* λ (or *nicey λ -stable*) if:

- (1) $\text{LS}(\mathfrak{K}) \leq \lambda$.
- (2) $\mathfrak{K}_\lambda \neq \emptyset$.
- (3) \mathfrak{K} has joint embedding in λ .
- (4) Density of amalgamation bases: For any $M \in \mathfrak{K}_\lambda$, there exists $N \in \mathfrak{K}_\lambda$ such that $M \leq_{\mathfrak{K}} N$ and N is an amalgamation base (in \mathfrak{K}_λ).
- (5) Existence of universal extensions: For any amalgamation base $M \in \mathfrak{K}_\lambda$, there exists an amalgamation base $N \in \mathfrak{K}_\lambda$ such that $M <_{\mathfrak{K}} N$ and N is universal over M .
- (6) Any brimmed model in \mathfrak{K}_λ is an amalgamation base.

We say that \mathfrak{K} is *very nicey stable in* λ if in addition it has amalgamation in λ .

Remark 2.4. An AEC \mathfrak{K} is very nicey stable in λ if and only if $\text{LS}(\mathfrak{K}) \leq \lambda$, $\mathfrak{K}_\lambda \neq \emptyset$, \mathfrak{K} is stable in λ , and \mathfrak{K}_λ has amalgamation, joint embedding, and no maximal models. In particular, stability is a consequence of the existence of universal extensions in Definition 2.3.

We will repeatedly use the following fact [32, 1.3.6].

Fact 2.5. Let \mathfrak{K} be nicey stable in λ and let $M_0, M_1, M_2 \in \mathfrak{K}_\lambda$. Let $\delta_1, \delta_2 < \lambda^+$ be limit ordinals such that $\text{cf}(\delta_1) = \text{cf}(\delta_2)$.

- (1) If M_ℓ is (λ, δ_ℓ) -brimmed over M_0 , for $\ell = 1, 2$, then $M_1 \cong_{M_0} M_2$.
- (2) If M_ℓ is (λ, δ_ℓ) -brimmed, for $\ell = 1, 2$, then $M_1 \cong M_2$.

Proof. The first is a straightforward back and forth argument and the second follows from the first using joint embedding. \square

Remark 2.6. Uniqueness of brimmed models when $\text{cf}(\delta_1) \neq \text{cf}(\delta_2)$ is a much harder property to establish, akin to superstability. See for example [32,33,16,34]. However when $\lambda = \aleph_0$ we automatically have that $\text{cf}(\delta_1) = \text{cf}(\delta_2) = \omega$.

Good frames were first defined by the first author in his paper number 600, which eventually appeared as Chapter II of [29]. The idea is to provide a localized (i.e. only for base models of a given size λ) axiomatization of a forking-like notion for a “nice enough” set of 1-types. These axioms are similar to the properties of first-order forking in a superstable theory. Jarden and the first author (in [19]) later gave a slightly more general definition, not assuming the existence of a superlimit model and dropping some of the redundant clauses. We will make use of good frames for types of finite length (not just length one). Their definition is

just like for types of length one, we call them *good* $(< \omega, \lambda)$ -frames. For the convenience of the reader, we give the full definition from [7, 3.8] here:

Definition 2.7. Let λ be an infinite cardinal. A *good* $(< \omega, \lambda)$ -frame is a triple $(\mathfrak{K}, \perp, \mathcal{S}_s^{\text{bs}})$ satisfying, where:

- (1) \mathfrak{K} is an abstract elementary class with $\lambda \geq \text{LS}(\mathfrak{K})$, $\mathfrak{K}_\lambda \neq \emptyset$.
- (2) $\mathcal{S}_s^{\text{bs}} \subseteq \bigcup_{M \in \mathfrak{K}_\lambda} \mathcal{S}^{<\omega}(M)$. Moreover, if $\text{ortp}(\langle a_i : i < n \rangle / M; N) \in \mathcal{S}_s^{\text{bs}}(M)$, then $a_i \notin M$ for all $i < n$.
- (3) \perp is a relation on quadruples of the form (M_0, M_1, \bar{a}, N) , where $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} N$, $\bar{a} \in {}^{<\omega}N$, and M_0, M_1, N are all in K_λ . We write $\perp(M_0, M_1, \bar{a}, N)$ or $\bar{a} \perp_{M_0}^N M_1$ instead of $(M_0, M_1, \bar{a}, N) \in \perp$.
- (4) The following properties hold:
 - (a) Invariance: If $f : N \cong N'$ and $\bar{a} \perp_{M_0}^N M_1$, then $f(\bar{a}) \perp_{f[M_0]}^{N'} f[M_1]$. If $\text{ortp}(\bar{a}/M_1; N) \in \mathcal{S}_s^{\text{bs}}(M_1)$, then $\text{ortp}(f(\bar{a})/f[M_1]; N') \in \mathcal{S}_s^{\text{bs}}(f[M_1])$.
 - (b) Monotonicity: If $\bar{a} \perp_{M_0}^N M_1$, \bar{a}' is a subsequence of \bar{a} , $M_0 \leq_{\mathfrak{K}} M'_0 \leq_{\mathfrak{K}} M'_1 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} N' \leq_{\mathfrak{K}} N \leq_{\mathfrak{K}} N''$ with $\bar{a}' \in N'$, and $N'' \in K_{\mathcal{F}}$, then $\bar{a}' \perp_{M'_0}^{N'} M'_1$ and $\bar{a}' \perp_{M'_0}^{N''} M'_1$. If $\text{ortp}(\bar{a}/M_1; N) \in \mathcal{S}_s^{\text{bs}}(M_1)$ and \bar{a}' is a subsequence of \bar{a} , then $\text{ortp}(\bar{a}'/M_1; N) \in \mathcal{S}_s^{\text{bs}}(M_1)$. [This property and the previous one show that \perp is really a relation on types. Thus if $p \in \mathcal{S}^{<\omega}(M_1)$ is a type, we say p does not fork over M_0 if $\bar{a} \perp_{M_0}^N M_1$ for some (equivalently any) \bar{a} and N such that $p = \text{ortp}(\bar{a}/M_1; N)$. Note that this depends on \mathfrak{s} , but \mathfrak{s} will always be clear from context.]
 - (c) Nonforking types are basic: If $\bar{a} \perp_M^N M$, then $\text{ortp}(\bar{a}/M; N) \in \mathcal{S}_s^{\text{bs}}(M)$.
 - (d) K_λ has amalgamation, joint embedding, and no maximal models.
 - (e) bs-Stability: $|\mathcal{S}_s^{\text{bs}}(M)| \leq \|M\|$ for all $M \in K_\lambda$.
 - (f) Density of basic types: If $M <_{\mathfrak{K}} N$ are in K_λ , then there is $a \in N$ such that $\text{ortp}(a/M; N) \in \mathcal{S}_s^{\text{bs}}(M)$.
 - (g) Existence of nonforking extension: If $m \leq n < \omega$, $p \in \mathcal{S}_s^{\text{bs}}(M) \cap \mathcal{S}^m(M)$, $N \geq_{\mathfrak{K}} M$ is in K_λ , then there is some $q \in \mathcal{S}_s^{\text{bs}}(N) \cap \mathcal{S}^n(M)$ that does not fork over M and extends p .
 - (h) Uniqueness: If $p, q \in \mathcal{S}^{<\omega}(N)$ do not fork over M and $p \upharpoonright M = q \upharpoonright M$, then $p = q$.
 - (i) Symmetry: If $\bar{a}_1 \perp_{M_0}^N M_2$, $\bar{a}_2 \in {}^{<\alpha}M_2$, and $\text{ortp}(\bar{a}_2/M_0; N) \in \mathcal{S}_s^{\text{bs}}(M_0)$, then there is M_1 containing \bar{a}_1 and $N' \geq_{\mathfrak{K}} N$ such that $\bar{a}_2 \perp_{M_0}^{N'} M_1$.
 - (j) Local character: If δ is a regular cardinal, $\langle M_i \in K_\lambda : i \leq \delta \rangle$ is increasing continuous, and $p \in \mathcal{S}_s^{\text{bs}}(M_\delta)$ is such that $\ell(p) < \delta$, then there exists $i < \delta$ such that p does not fork over M_i .
 - (k) Continuity: If δ is a limit ordinal, $\langle M_i \in K_\lambda : i \leq \delta \rangle$ and $\langle \alpha_i < \alpha : i \leq \delta \rangle$ are increasing and continuous, and $p_i \in \mathcal{S}_s^{\text{bs}}(M_i)$ for $i < \delta$ are such that $j < i < \delta$ implies $p_j = p_i \upharpoonright M_j$, then there is some $p \in \mathcal{S}_s^{\text{bs}}(M_\delta)$ such that for all $i < \delta$, $p_i = p \upharpoonright M_i$. Moreover, if each p_i does not fork over M_0 , then neither does p .
 - (l) Transitivity: If $M_0 \leq_{\mathfrak{K}} M_1 \leq_{\mathfrak{K}} M_2$, $p \in \mathcal{S}(M_2)$ does not fork over M_1 and $p \upharpoonright M_1$ does not fork over M_0 , then p does not fork over M_0 .

A *good* λ -frame is defined similarly, except we require all types to be types of singletons (i.e. they are in $\mathcal{S}(M)$ instead of $\mathcal{S}^{<\omega}(M)$). We say that an AEC \mathfrak{K} has a *good* $(< \omega, \lambda)$ -frame if there is a good $(< \omega, \mathcal{F})$ -frame where K is the underlying AEC.

Notation 2.8. If $\mathfrak{s} = (\mathfrak{K}, \perp, \mathcal{S}_\mathfrak{s}^{\text{bs}})$ is a good- (ω, λ) -frame (or a good λ -frame), write $\perp := \perp$. Also write $\mathfrak{K}^\mathfrak{s}$ for \mathfrak{K} and $\mathfrak{K}_\mathfrak{s} = \mathfrak{K}_\lambda$. We will also write $M \leq_\mathfrak{s} N$ as a shortcut for $M \leq_\mathfrak{K} N$ and $M, N \in \mathfrak{K}_\mathfrak{s} (= \mathfrak{K}_\lambda)$.

Remark 2.9. The reader might wonder about the reasons for having a special class of basic types. Following [29, Definition III.9.2], let us call a good frame *type-full* if the basic types are all the nonalgebraic types. There are no known examples of a good λ -frame which cannot be extended to a type-full one. However a construction of good frames of the first author [29, II.3] builds a non type-full good frame and it is not clear that it can be extended to a type-full one until a lot more machinery has been developed. Thus it can be easier to build a good frame than to build a type-full one, and most results about frames already hold in the non-type-full context. That being said, readers would not miss the essence of the present paper if they assumed that all the frames here were type-full.

Remark 2.10. Any good λ -frame (i.e. for types of length one) extends to a good (ω, λ) -frame (using independent sequences, see [29, III.9.4]) or [7, 5.8]. This frame will however *not* be type-full.

From now on until the end of Section 5, “nonforking” will refer to nonforking in a fixed frame \mathfrak{s} (usually clear from context).

3. Weak nonforking amalgamation

In this section, we work in a good λ -frame and study a natural weak version of nonforking amalgamation, $\text{LWNF}_\mathfrak{s}$ (LWNF stands for “left weak nonforking amalgamation”). The goal is to obtain a natural criteria for proving the existence of a superlimit in \aleph_1 and also prepare the ground for the proof of symmetry in the good frame built in Section 5. The main results are the existence property (Theorem 3.10) and how the symmetry property of $\text{LWNF}_\mathfrak{s}$ is connected to \mathfrak{s} being good^+ (Theorem 3.14). Throughout this section, we assume:

Hypothesis 3.1.

- (1) $\mathfrak{s} = (\mathfrak{K}, \perp, \mathcal{S}_\mathfrak{s}^{\text{bs}})$ is a fixed good (ω, λ) -frame, except that it may not satisfy the symmetry axiom.
- (2) \mathfrak{K} is categorical in λ .

Remark 3.2. In this section, λ is allowed to be uncountable. However the case $\lambda = \aleph_0$ is the one that will interest us in the next sections.

The reason for not assuming symmetry is that we will use some of the results of this section to *prove* that the symmetry axiom holds of a certain nonforking relation in Section 5.

We will use:

Fact 3.3 (II.4.3 in [29]). Let $\delta < \lambda^+$ be a limit ordinal divisible by λ . Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in $\mathfrak{K}_\mathfrak{s}$. If for any $i < \delta$ and any $p \in \mathcal{S}_\mathfrak{s}^{\text{bs}}(M_i)$, there exists λ -many $j \in [i, \delta)$ such that the nonforking extension of p to M_j is realized in M_{j+1} , then M_δ is brimmed over M_0 .

To understand the definition below, it may be helpful to think of \mathfrak{s} as type-full. Then $\text{LWNF}_\mathfrak{s}(M_0, M_1, M_2, M_3)$ holds if and only if the type of any finite subsequences of M_1 over M_2 does not fork over M_0 (M_3 is the ambient model). Thus $\text{LWNF}_\mathfrak{s}$ is an attempt to extend nonforking to types of infinite sequences so that it keeps a strong finite character property. In the present paper, $\text{LWNF}_\mathfrak{s}$ will be a helpful technical tool but it is not clear that it has the uniqueness property (in contrast with the relation NF from [29, §II.6])

or [19, §5], which will have the uniqueness property but requires more assumptions on the good frame). If LWNF_s does have the uniqueness property, this has strong consequence on the structure of the frame, see Theorem 3.17.

Definition 3.4. Define the following 4-ary relations on \mathfrak{K}_s :

- (1) $\text{LWNF}_s(M_0, M_1, M_2, M_3)$ if and only if $M_0 \leq_s M_\ell \leq_s M_3$ for $\ell = 1, 2$ and for any $\bar{b} \in {}^{<\omega}M_1$, if $\text{ortp}(\bar{b}, M_2, M_3)$ and $\text{ortp}(\bar{b}, M_0, M_3)$ are basic then $\text{ortp}(\bar{b}, M_2, M_3)$ does not fork over M_0 .
- (2) $\text{RWNF}_s(M_0, M_1, M_2, M_3)$ if and only if $\text{LWNF}_s(M_0, M_2, M_1, M_3)$ [RWNF stands for “right weak non-forking amalgamation”].
- (3) $\text{WNF}_s(M_0, M_1, M_2, M_3)$ if and only if both $\text{LWNF}_s(M_0, M_1, M_2, M_3)$ and $\text{RWNF}_s(M_0, M_1, M_2, M_3)$ [WNF stands for “weak nonforking amalgamation”].

When s is clear from context, we write LWNF , RWNF , and WNF .

The following result often comes in handy.

Lemma 3.5. *Let $\delta < \lambda^+$ be a limit ordinal. Let $\langle M_i : i \leq \delta \rangle$, $\langle N_i : i \leq \delta \rangle$ be increasing continuous in \mathfrak{K}_s . Assume that for each $i \leq j < \delta$, we have that $\text{LWNF}(M_i, N_i, M_j, N_j)$. If for each $i < \delta$, N_i realizes all the basic types over M_i , then N_δ realizes all the basic types over M_δ .*

Proof. Let $p \in \mathcal{S}_s^{\text{bs}}(M_\delta)$. By local character, there exists $i < \delta$ such that p does not fork over M_i . In particular, $p \upharpoonright M_i$ is basic. Since N_i realizes all the basic types over M_i , there exists $a \in |N_i|$ such that $p \upharpoonright M_i = \text{ortp}(a, M_i, N_i)$. Because for all $j \in [i, \delta)$, $\text{LWNF}(M_i, N_i, M_j, N_j)$, we have by continuity that $\text{ortp}(a, M_\delta, N_\delta)$ does not fork over M_i , hence by uniqueness it must be equal to p . Therefore a realizes p , as needed. \square

Next, we investigate the properties of LWNF . We are especially interested in the symmetry property: whether LWNF is equal to RWNF . To understand it better, we consider the following ordering, defined similarly to $\leq_{\lambda^+}^*$ from [29, II.7.2]:

Definition 3.6. For $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$, define a relation \leq_R on \mathfrak{K}_{λ^+} as follows. For $M^0, M^1 \in \mathfrak{K}_{\lambda^+}$, $M^0 \leq_R M^1$ if and only if there exists increasing continuous resolutions $\langle M_i^\ell \in \mathfrak{K}_\lambda : i < \lambda^+ \rangle$ of M^ℓ for $\ell = 0, 1$ such that for all $i < j < \lambda^+$, $R(M_i^0, M_i^1, M_j^0, M_j^1)$.

The following is a straightforward “catching your tail argument”, see the proof of [39, 4.6] (this assumes that all types are basic, but the argument goes through without this restriction). Roughly, it says that if $M \leq_{\mathfrak{K}} N$ ($\leq_{\mathfrak{K}}$ is the usual order on \mathfrak{K}), then we can find a resolution of M and N so that the pieces are in *left* weak nonforking amalgamation.

Fact 3.7. Let $M, N \in \mathfrak{K}_{\lambda^+}$. If $M \leq_{\mathfrak{K}} N$, then $M \leq_{\text{LWNF}} N$.

Whether $M \leq_{\text{RWNF}} N$ can be concluded as well seems to be a much more complicated question, and in fact is equivalent to s being good^+ (Theorem 3.14), a weakening of symmetry. We now observe that an increasing union of a \leq_{RWNF} -increasing chain of saturated models is saturated:

Lemma 3.8. *Let $\delta < \lambda^{++}$ be a limit ordinal. If $\langle M_i : i < \delta \rangle$ is a \leq_{RWNF} -increasing sequence of saturated models in \mathfrak{K}_{λ^+} , then $\bigcup_{i < \delta} M_i$ is saturated.*

Proof. If $\text{cf}(\delta) \geq \lambda^+$, then any $\bigcup_{i < \delta} M_i$ will be λ^+ -saturated on general grounds. Thus assume without loss of generality that $\delta = \text{cf}(\delta) < \lambda^+$. Let $M_\delta := \bigcup_{i < \delta} M_i$. We build $\langle M_{i,j} : i \leq \delta, j \leq \lambda^+ \rangle$ such that:

- (1) For any $i \leq \delta$, $M_{i,\lambda^+} = M_i$.
- (2) For any $i < \delta, j < \lambda^+$, $M_{i,j} \in \mathfrak{K}_\mathfrak{s}$.
- (3) For any $i \leq \delta$, $\langle M_{i,j} : j < \lambda^+ \rangle$ is increasing and continuous.
- (4) For any $j \leq \lambda^+$, $\langle M_{i,j} : i < \delta \rangle$ is increasing and $M_{\delta,j} = \bigcup_{i < \delta} M_{i,j}$.
- (5) For any $i_1 < i_2 \leq \delta, j_1 < j_2 \leq \lambda^+$, M_{i_2,j_2} realizes all the types in $\mathcal{S}_\mathfrak{s}^{\text{bs}}(M_{i_1,j_1})$.

This is easy to do. Now for each $i_1 < i_2 < \delta$, we have by assumption that $M_{i_1} \leq_{\text{RWNF}} M_{i_2}$. Thus the set C_{i_1,i_2} of $j < \lambda^+$ such that for all $j' \in [j, \lambda^+)$, $\text{RWNF}(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$ is a club (that it is closed follows from the local character and continuity axioms of good frames). Therefore $C := \bigcap_{i_1 < i_2 < \delta} C_{i_1,i_2}$ is also a club. Hence by renaming without loss of generality for all $i_1 < i_2 < \delta$ and all $j \leq j' < \lambda^+$, $\text{RWNF}(M_{i_1,j}, M_{i_2,j}, M_{i_1,j'}, M_{i_2,j'})$.

Now let $N \leq_{\mathfrak{K}} M_\delta$ be such that $N \in \mathfrak{K}_\lambda$. We want to see that any type over N is realized in M_δ . By Fact 3.3, it is enough to show that any *basic* type over N is realized in M_δ .

Let $j < \lambda^+$ be big-enough such that $N \leq_{\mathfrak{K}} M_{\delta,j}$. It is enough to see that any basic type over $M_{\delta,j}$ is realized in $M_{\delta,j+1}$. To see this, use Lemma 3.5 with $\langle M_i : i \leq \delta \rangle, \langle N_i : i \leq \delta \rangle$ there standing for $\langle M_{i,j} : i \leq \delta \rangle, \langle M_{i,j+1} : i \leq \delta \rangle$ here. We know that for each $i \leq i' < \delta$, $\text{RWNF}(M_{i,j}, M_{i',j}, M_{i,j+1}, M_{i',j+1})$ and therefore $\text{LWNF}(M_{i,j}, M_{i,j+1}, M_{i',j}, M_{i',j+1})$. Thus the hypotheses of Lemma 3.5 are satisfied. \square

The next fact will be used to prove the existence property of LWNF. Its proof is a direct limit argument similar to e.g. [16, 5.2]. Roughly, the nonforking relation there is given by “there exists a smaller submodel over which the type does not *split*”; in fact, these smaller submodels have to be kept as part of the data of the tower. This is not needed here. The argument is also similar to [19, 3.1.8]. However there the symmetry axiom is needed: there is an extra requirement on the type of a certain element b , but here we do *not* make that requirement so do *not* need symmetry.

Fact 3.9. Let $\alpha < \lambda^+$. Let $\langle M_i : i \leq \alpha \rangle$ be $\leq_\mathfrak{s}$ -increasing continuous (in \mathfrak{K}_λ) and let $\langle \bar{a}_i : i < \alpha \rangle$ be given such that $\bar{a}_i \in {}^{<\omega}M_{i+1}$ for all $i < \alpha$ and $\text{ortp}(\bar{a}_i, M_i, M_{i+1}) \in \mathcal{S}_\mathfrak{s}^{\text{bs}}(M_i)$ (we allow the \bar{a}_i 's to have different length).

There exists $\langle N_i : i \leq \alpha \rangle \leq_\mathfrak{s}$ -increasing continuous such that:

- (1) $M_i <_\mathfrak{s} N_i$ for all $i \leq \alpha$.
- (2) $\text{ortp}(\bar{a}_i, N_i, N_{i+1})$ does not fork over M_i .

We can now list and then prove some basic properties of weak nonforking amalgamation. For the convenience of the reader, we repeat Hypothesis 3.1.

Theorem 3.10. Let $\mathfrak{s} = (\mathfrak{K}, \perp, \mathcal{S}_\mathfrak{s}^{\text{bs}})$ be a fixed good $(< \omega, \lambda)$ -frame, except that it may not satisfy the symmetry axiom. Assume that \mathfrak{K} is categorical in λ . Let $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$.

- (1) *Invariance:* If $R(M_0, M_1, M_2, M_3)$ and $f : M_3 \cong M'_3$, then $R(f[M_0], f[M_1], f[M_2], M'_3)$.
- (2) *Monotonicity:* If $R(M_0, M_1, M_2, M_3)$ and $M_0 \leq_\mathfrak{s} M'_\ell \leq_\mathfrak{s} M_\ell$ for $\ell = 1, 2$, then $R(M_0, M'_1, M'_2, M_3)$.
- (3) *Ambient monotonicity:* If $R(M_0, M_1, M_2, M_3)$ and $M_3 \leq_\mathfrak{s} M'_3$, then $R(M_0, M_1, M_2, M'_3)$. If $M'_3 \leq_\mathfrak{s} M_3$ contains $|M_1| \cup |M_2|$, then $R(M_0, M_1, M_2, M''_3)$.
- (4) *Continuity:* If $\delta < \lambda^+$ is a limit ordinal and $\langle M_i^\ell : i \leq \delta \rangle$ are increasing continuous for $\ell < 4$ with $R(M_i^0, M_i^1, M_i^2, M_i^3)$ for each $i < \delta$, then $R(M_\delta^0, M_\delta^1, M_\delta^2, M_\delta^3)$.

- (5) *Long transitivity:* If $\alpha < \lambda^+$ is an ordinal, $\langle M_i : i \leq \alpha \rangle$, $\langle N_i : i \leq \alpha \rangle$ are increasing continuous and $\text{LWNF}(M_i, N_i, M_{i+1}, N_{i+1})$ for all $i < \alpha$, then $\text{LWNF}(M_0, N_0, M_\alpha, N_\alpha)$.
- (6) *Existence:* If $R \neq \text{WNF}$, $M_0 \leq_s M_\ell$, $\ell = 1, 2$, then there exists $M_3 \in \mathfrak{K}_\lambda$ and $f_\ell : M_\ell \xrightarrow{M_0} M_3$ such that $R(M_0, f_1[M_1], f_2[M_2], M_3)$.

Proof. Invariance and the monotonicity properties are straightforward to prove. Continuity and long transitivity follow directly from the local character, continuity, and transitivity properties of good frames. We prove existence via the following claim:

Claim: There exists $N_0, N_1, N_2, N_3 \in \mathfrak{K}_s$ such that $\text{LWNF}(N_0, N_1, N_2, N_3)$ and N_ℓ is brimmed over N_0 for $\ell = 1, 2$.

Existence easily follows from the claim: given $M_0 \leq_s M_\ell$, $\ell = 1, 2$, there is (by categoricity in λ) an isomorphism $f : M_0 \cong N_0$ and (by universality of brimmed models) embeddings $f_\ell : M_\ell \rightarrow N_\ell$ extending f for $\ell = 1, 2$. After some renaming, we obtain the desired LWNF-amalgam. To obtain an RWNF-amalgam, reverse the role of M_1 and M_2 .

Proof of Claim: The idea of the proof is as follows: for some suitable ordinal α , we want to build $\langle M_i : i \leq \alpha \rangle$, $\langle a_i \in M_{i+1} : i < \alpha \rangle$ with the following property: whenever $\langle N_i : i \leq \alpha \rangle$ is as described by Fact 3.9 (plus slightly more), we must have that $\text{LWNF}(M_0, M_\alpha, N_0, N_\delta)$, M_α is brimmed over M_0 , and N_0 is brimmed over M_0 . To achieve this, we simply start with an arbitrary $\langle M_i : i \leq \alpha \rangle$, $\langle a_i : i < \alpha \rangle$ and, if it fails the property, take a witness to the failure, add some more a_j 's to make it more brimmed, and start again to consider whether this witness satisfies the property. After doing this for sufficiently many steps, we eventually succeed to build the desired object. This is somewhat similar to the construction of a reduced tower in [32,16], although here we are dealing with nonforking independence and not just set-theoretic disjointness.

We now start with the proof. Let $\delta := \lambda \cdot \lambda$. We choose $(\bar{M}^\alpha, \bar{a}^\alpha)$ by induction on $\alpha \leq \delta$ such that:

- (1) $\bar{M}^\alpha = \langle M_i^\alpha : i \leq \alpha \rangle$ is \leq_s -increasing continuous.
- (2) $\bar{a}^\alpha = \langle \bar{a}_i : i < \alpha \rangle$, and $\bar{a}_i \in M_{i+1}^\alpha$ for all $i < \alpha$.
- (3) For all $i < \alpha$, $\text{ortp}(\bar{a}_i^\alpha, M_i^\alpha, M_{i+1}^\alpha) \in \mathcal{S}_s^{\text{bs}}(M_i^\alpha)$.
- (4) For each $i \leq \delta$, $\langle M_i^\alpha : \alpha \in [i, \delta] \rangle$ is \leq_s -increasing continuous.
- (5) For each $i < \delta$ and each $\alpha \in (i, \delta]$, $\text{ortp}(\bar{a}_i, M_i^\alpha, M_{i+1}^\alpha)$ does not fork over M_i^i .
- (6) If $p \in \mathcal{S}_s^{\text{bs}}(M_i^\alpha)$ for $i \leq \alpha < \delta$, then for λ -many $\beta \in [\alpha, \delta)$, $\text{ortp}(\bar{a}_\beta, M_\beta^{\beta+1}, M_{\beta+1}^{\beta+1})$ is a nonforking extension of p .
- (7) If $i < \alpha < \delta$ and $\text{ortp}(\bar{a}, M_0^\alpha, M_{i+1}^\alpha) \in \mathcal{S}_s^{\text{bs}}(M_0^\alpha)$, then for some $\beta \in (\alpha, \delta)$ exactly one of the following occurs:
 - (a) $\text{ortp}(\bar{a}, M_0^{\beta+1}, M_{i+1}^{\beta+1})$ forks over M_0^α .
 - (b) There is no $\langle M_j^* : j \leq i+1 \rangle \leq_s$ -increasing continuous such that:
 - (i) $M_j^\beta \leq_s M_j^*$ for all $j \leq i+1$.
 - (ii) $\text{ortp}(\bar{a}_j, M_j^*, M_{j+1}^*)$ does not fork over M_j^β for all $j < i+1$.
 - (iii) $\text{ortp}(\bar{a}, M_0^*, M_{i+1}^*)$ forks over M_0^β .

This is possible: Along the construction, we also build an enumeration $\langle (\bar{b}_j^\gamma, k_j^\gamma, i_j^\gamma, \alpha_j^\gamma) : j < \lambda, \gamma < \lambda \rangle$ such that for any $\gamma \in (0, \lambda)$, any $\alpha < \lambda \cdot \gamma$, any $i < \alpha$, any $k \leq i$, and any $\bar{a} \in {}^{<\omega}M_{i+1}^\alpha$, if $\text{ortp}(\bar{a}, M_k^\alpha, M_{i+1}^\alpha) \in \mathcal{S}_s^{\text{bs}}(M_k^\alpha)$, then there exists $j < \lambda$ so that $\bar{b}_j^\gamma = \bar{a}$, $i_j^\gamma = i$, $k_j^\gamma = k$, and $\alpha_j^\gamma = \alpha$. We require that always $k_j^\gamma \leq i_j^\gamma < \alpha_j^\gamma < \lambda \cdot \gamma$ and the triple $(\bar{b}_j^\gamma, M_{k_j^\gamma}^{\alpha_j^\gamma}, M_{i_j^\gamma+1}^{\alpha_j^\gamma})$ represents a basic type. We

make sure that at stage $\lambda \cdot (\gamma + 1)$ of the construction below, $\bar{b}_j^{\gamma'}, k_j^{\gamma'}, i_j^{\gamma'}, \alpha_j^{\gamma'}$ are defined for all $j < \lambda$, $\gamma' \leq \gamma$.

For $\alpha = 0$, take any $M_0^0 \in \mathfrak{K}_s$. For α limit, let $M_i^\alpha := \bigcup_{\beta \in [i, \alpha)} M_i^\beta$ for $i < \alpha$ and $M_\alpha^\alpha := \bigcup_{i < \alpha} M_i^\alpha$. Now assume that $\bar{M}^\alpha, \bar{a}^\alpha$ have been defined for $\alpha < \delta$. We define $\bar{M}^{\alpha+1}$ and \bar{a}_α . Fix ρ and $j < \lambda$ such that $\alpha = \lambda \cdot \rho + j$. We consider two cases.

- Case 1: ρ is zero or a limit: Use Fact 3.9 to get $\langle M_i^{\alpha+1} : i \leq \alpha \rangle <_s$ -increasing continuous such that $M_i^\alpha <_s M_i^{\alpha+1}$ for all $i \leq \alpha$, and for all $i < \alpha$, $\text{ortp}(\bar{a}_i, M_i^{\alpha+1}, M_{i+1}^{\alpha+1})$ does not fork over M_i^α . Pick any $M_{\alpha+1}^{\alpha+1}$ with $M_\alpha^{\alpha+1} <_s M_{\alpha+1}^{\alpha+1}$ and any $\bar{a}_\alpha \in <^\omega M_{\alpha+1}^{\alpha+1}$ such that $\text{ortp}(\bar{a}_\alpha, M_\alpha^{\alpha+1}, M_{\alpha+1}^{\alpha+1}) \in \mathcal{S}_s^{\text{bs}}(M_\alpha^{\alpha+1})$.
- Case 2: ρ is a successor: Say $\rho = \gamma + 1$. Let $\bar{a} := \bar{b}_j^\gamma$, $\alpha_0 := \alpha_j^\gamma$, $k_0 := k_j^\gamma$, $i_0 := i_j^\gamma$. There are two subcases.

* Subcase 1: Either $k_0 \neq 0$, or $k_0 = 0$ and (7b) holds with i, α, β there standing for i_0, α_0, α here.

In this case, we proceed as in Case 1 to define $\langle M_i^{\alpha+1} : i \leq \alpha \rangle$. Then we pick $\bar{a}_\alpha, M_{\alpha+1}^{\alpha+1}$ such that $\text{ortp}(\bar{a}_\alpha, M_\alpha^\alpha, M_{\alpha+1}^{\alpha+1})$ is the nonforking extension of $\text{ortp}(\bar{a}, M_{i_0}^{\alpha_0}, M_{i_0+1}^{\alpha_0})$.

* Subcase 2: $k_0 = 0$ and (7b) fails.

In this case, let $\langle M_j^* : j \leq i_0 + 1 \rangle$ witness the failure and set $M_j^{\alpha+1} := M_j^*$ for $j \leq i_0 + 1$. Then continue as in Case 1 and define $\bar{a}_\alpha, M_{\alpha+1}^{\alpha+1}$ as before.

This is enough:

We proceed via a series of subclaims:

Subclaim 1: If $p \in \mathcal{S}_s^{\text{bs}}(M_i^\delta)$ for $i < \delta$, then for λ -many $\beta \in [i, \delta)$, $\text{ortp}(\bar{a}_\beta, M_\beta^\delta, M_{\beta+1}^\delta)$ is a nonforking extension of p .

Proof of subclaim 1: Pick $i' \in (i, \delta)$ such that p does not fork over $M_{i'}^{i'}$. By (6), we know that for λ -many $\beta \in [i', \delta)$, the nonforking extension of $p \upharpoonright M_{i'}^{i'}$ to $M_\beta^{\beta+1}$ is realized in $M_\beta^{\beta+1}$ by \bar{a}_β . But by (5) we also have that $\text{ortp}(\bar{a}_\beta, M_\beta^\delta, M_{\beta+1}^\delta)$ does not fork over M_β^β . In particular by uniqueness \bar{a}_β also realizes $p \upharpoonright_{\text{subclaim 1}}$.

Subclaim 2: M_δ^δ is brimmed over M_0^δ .

Proof of subclaim 2: Apply Fact 3.3 to the chain $\langle M_i^\delta : i \leq \delta \rangle$, using the previous step. $\upharpoonright_{\text{subclaim 2}}$

We now choose $\bar{M}^* = \langle M_i^* : i \leq \delta \rangle$ increasing continuous such that M_0^* is brimmed over M_0^δ , $M_i^\delta \leq_s M_i^*$ for all $i \leq \delta$, and $\text{ortp}(\bar{a}_i, M_i^*, M_{i+1}^*)$ does not fork over M_i^δ . This is possible, see case 1 above. Now let $(N_0, N_1, N_2, N_3) := (M_0^\delta, M_\delta^\delta, M_0^*, M_\delta^*)$. We have just said that M_0^* is brimmed over M_0^δ , and by subclaim 2, M_δ^δ is brimmed over M_0^δ . Thus N_ℓ is brimmed over N_0 for $\ell = 1, 2$. It remains to see:

Subclaim 3: $\text{LWNF}(M_0^\delta, M_\delta^\delta, M_0^*, M_\delta^*)$.

Proof of subclaim 3: Pick $\bar{a} \in <^\omega M_\delta^\delta$ such that $\text{ortp}(\bar{a}, M_0^\delta, M_\delta^\delta)$ is basic. By local character, there exists $\alpha < \delta$ such that $\text{ortp}(\bar{a}, M_0^\delta, M_\delta^\delta)$ does not fork over M_0^α . Further, we can increase α if necessary and pick $i < \alpha$ such that $\bar{a} \in <^\omega M_{i+1}^\alpha$. We now apply Clause (7). We know that (7a) fails for all $\beta \in (\alpha, \delta)$ by the choice of α , therefore (7b) must hold for all $\beta \in (\alpha, \delta)$. Now if $\text{ortp}(\bar{a}, M_0^*, M_\delta^*)$ forks over M_0^δ , then it must fork over M_0^β for all high-enough β , but then $\langle M_j^* : j \leq i + 1 \rangle$ would contradict Clause (7b). Therefore $\text{ortp}(\bar{a}, M_0^*, M_\delta^*)$ does not fork over M_0^δ , as desired. $\upharpoonright_{\text{subclaim 3}}$ $\upharpoonright_{\text{Claim}}$ \square

The following properties of LWNF may or may not hold in general (we have no examples for the failure of symmetry, but uniqueness fails in the last good frame of the Hart–Shelah example, see [10,8]):

Definition 3.11. Let $R \in \{\text{LWNF}, \text{RWNF}, \text{WNF}\}$.

- (1) We say that R has the *symmetry property* if $R(M_0, M_1, M_2, M_3)$ implies $R(M_0, M_2, M_1, M_3)$.

- (2) We say that R has the *uniqueness property* if whenever $R(M_0, M_1, M_2, M_3)$ and $R(M_0, M_1, M_2, M'_3)$, there exists M''_3 with $M'_3 \leq_s M''_3$ and $f : M_3 \xrightarrow{|M_1| \cup |M_2|} M''_3$.

The following are trivial observations about the definitions:

Remark 3.12.

- (1) WNF has the symmetry property, and LWNF has the symmetry property if and only if RWNF has the symmetry property if and only if LWNF = RWNF = WNF.
 (2) LWNF has the uniqueness property if and only RWNF has it.

Recall from [29, III.1.3]:

Definition 3.13. \mathfrak{s} is *good*⁺ when the following is *impossible*:

There exists an increasing continuous $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, a basic type $p \in \mathcal{S}_{\mathfrak{s}}^{\text{bs}}(M_0)$, and $\langle \bar{a}_i : i < \lambda^+ \rangle$ such that for any $i < \lambda^+$:

- (1) $M_i \leq_s N_i$.
 (2) $\bar{a}_{i+1} \in |M_{i+2}|$ and $\text{ortp}(\bar{a}_{i+1}, M_{i+1}, M_{i+2})$ is a nonforking extension of p , but $\text{ortp}(\bar{a}_{i+1}, N_0, N_{i+2})$ forks over M_0 .
 (3) $\bigcup_{j < \lambda^+} M_j$ is saturated.

We now show that being *good*⁺ is a consequence of symmetry for LWNF. Moreover, *good*⁺ allows us to build a superlimit in λ^+ .

Theorem 3.14. (1) \Rightarrow (2) \Leftrightarrow (3) \Rightarrow (4), *where*:

- (1) LWNF has the symmetry property.
 (2) \mathfrak{s} is *good*⁺.
 (3) For $M, N \in \mathfrak{K}_{\lambda^+}$ both saturated, $M \leq_{\mathfrak{R}} N$ implies $M \leq_{\text{WNF}} N$.
 (4) There is a superlimit model in \mathfrak{K}_{λ^+} .

Proof.

- (3) implies (4): This follows from Lemma 3.8 and the fact that the saturated model in λ^+ is universal and has a proper extension [29, II.4.13].
- $\neg(2)$ implies $\neg(3)$: Fix a witness $\langle M_i : i < \lambda^+ \rangle$, $\langle N_i : i < \lambda^+ \rangle$, $\langle \bar{a}_i : i < \lambda^+ \rangle$, p to the failure of being *good*⁺. Write $M_{\lambda^+} := \bigcup_{i < \lambda^+} M_i$, $N_{\lambda^+} := \bigcup_{i < \lambda^+} N_i$. By assumption, M_{λ^+} is saturated. Clearly, increasing the N_i 's will not change that we have a witness so without loss of generality N_{λ^+} is also saturated. We claim that $M_{\lambda^+} \not\leq_{\text{RWNF}} N_{\lambda^+}$. We show this by proving that for any $i < \lambda^+$ and any $j \leq i + 1$, $\neg \text{RWNF}(M_j, N_j, M_{i+2}, N_{i+2})$. Indeed, $\text{ortp}(\bar{a}_{i+1}, N_j, N_{i+2})$ forks over M_j : if not, then by transitivity $\text{ortp}(\bar{a}_{i+1}, N_j, N_{i+2})$ does not fork over M_0 , and hence $\text{ortp}(\bar{a}_{i+1}, N_0, N_{i+2})$ does not fork over M_0 , and we know that this is not the case of the witness we selected.
- $\neg(3)$ implies $\neg(2)$: Fix M, N saturated in \mathfrak{K}_{λ^+} such that $M \leq_{\mathfrak{R}} N$ but $M \not\leq_{\text{RWNF}} N$.

Claim: For any $A \subseteq |M|$ of size λ , there exists $M_0 \leq_s M_1 \leq_{\mathfrak{R}} M$ and $N_0 \leq_s N_1 \leq_{\mathfrak{R}} N$ such that $M_0 \leq_s N_0$, $M_1 \leq_s N_1$, $A \subseteq |M_0|$, but $\neg \text{RWNF}(M_0, N_0, M_1, N_1)$.

Proof of Claim: If not, we can use failure of the claim and continuity of RWNF to build increasing continuous resolution $\langle M_i : i \leq \lambda^+ \rangle$, $\langle N_i : i \leq \lambda^+ \rangle$ of M and N respectively such that $\text{RWNF}(M_i, N_i, M_j, N_j)$ for all $i < j < \lambda^+$. Thus $M \leq_{\text{RWNF}} N$, contradicting the assumption. \dagger_{Claim}

Build $\langle M_i^* : i \leq \lambda^+ \rangle$, $\langle N_i^* : i \leq \lambda^+ \rangle$ increasing continuous resolutions of M , N respectively such that for all $i < \lambda^+$, $M_i^* \leq_s N_i^*$ and $\neg \text{RWNF}(M_{i+1}^*, N_{i+1}^*, M_{i+2}^*, N_{i+2}^*)$. This is possible by the claim. Let $\bar{a}_{i+1} \in |M_{i+2}^*|$ witness the RWNF-forking, i.e. $\text{ortp}(\bar{a}_{i+1}, N_{i+1}^*, N_{i+2}^*)$ forks over M_{i+1}^* . By Fodor's lemma, local character, and stability, there exists a stationary set S , $i_0 < \lambda^+$ and $p \in \mathcal{S}_s^{\text{bs}}(M_{i_0}^*)$ such that for all $i \in S$, $\text{ortp}(\bar{a}_{i+1}, M_i^*, M_{i+2}^*)$ is the nonforking extension of p . Without loss of generality, i_0 is limit and all elements of S are also limit ordinals.

Now build an increasing continuous sequence of ordinals $\langle j_i : i < \lambda^+ \rangle$ as follows. Let $j_0 := i_0$. For i limit, let $j_i := \sup_{k < i} j_k$. For i successor, pick any $j_i \in S$ with $j_i > j_{i-1}$.

Now for i not the successor of a limit, let $M_i := M_{j_i}^*$, $N_i := N_{j_i}^*$, $\bar{a}_i := \bar{a}_{j_i}^*$. For $i = k + 1$ with k a limit, set $M_i := M_{j_k}^*$, $N_i := N_{j_k}^*$, $\bar{a}_i := \bar{a}_{j_k}^*$. This gives a witness to the failure of being good $^+$.

- (1) implies (3): If LWNF has the symmetry property, then by Remark 3.12, $\text{LWNF} = \text{RWNF} = \text{WNF}$. By Fact 3.7, it follows that $M \leq_{\mathfrak{R}} N$ implies $M \leq_{\text{WNF}} N$ for any $M, N \in \mathfrak{K}_{\lambda^+}$, so (3) holds. \square

Question 3.15. Are the conditions in Theorem 3.14 all equivalent?

Question 3.16. Is there a good λ -frame \mathfrak{s} such that $\text{LWNF}_{\mathfrak{s}}$ does *not* have the symmetry property?

The next result shows that the uniqueness property has strong consequences. The first author has given conditions under which when $\lambda = \aleph_0$, failure of uniqueness implies nonstructure [30, VII.4.16].

Theorem 3.17. *Assume that \mathfrak{s} is a good $(< \omega, \lambda)$ -frame (so it satisfies symmetry). If LWNF has the uniqueness property, then LWNF has the symmetry property and \mathfrak{s} is successful good $^+$ (see [29, III.1.1]).*

Proof. By [39, 3.11] (used with the pre- $(\leq \lambda, \lambda)$ -frame induced by LWNF, recalling Fact 3.7) \mathfrak{s} is weakly successful. This implies that there is a relation $\text{NF} = \text{NF}_{\mathfrak{s}}$ that is a nonforking relation respecting \mathfrak{s} (see [29, II.6.1], in particular it has all the properties listed in Theorem 3.10, as well as uniqueness and symmetry). Now as NF respects \mathfrak{s} , we must have that $\text{NF}(M_0, M_1, M_2, M_3)$ implies $\text{LWNF}(M_0, M_1, M_2, M_3)$. Since LWNF has the uniqueness property and NF has the existence property, it follows from [9, 4.1] that $\text{LWNF} = \text{NF}$. In particular, LWNF has the symmetry property.

To see that \mathfrak{s} is successful good $^+$, it is enough to show that for $M, N \in \mathfrak{K}_{\lambda^+}$, $M \leq_{\mathfrak{R}} N$ implies $M \leq_{\text{NF}} N$ (where \leq_{NF} is defined as in Definition 3.6). This is immediate from Fact 3.7 and $\text{LWNF} = \text{NF}$. \square

To prepare for the proof of symmetry in the $\lambda = \aleph_0$ case, we end this section by introducing yet another notion of nonforking amalgamation (VWNF stands for “very weak nonforking amalgamation”). In this case, we look at finite sequences both on the left *and* the right hand side. We show that if \mathfrak{s} is a good frame, then VWNF has the symmetry property and locality of types implies that $\text{VWNF} = \text{LWNF}$. Thus in this case LWNF has the symmetry property too.

Definition 3.18.

- (1) For $M \leq_s N$, $B \subseteq |N|$, $\bar{a} \in {}^{<\omega} N$, we say that $\text{ortp}(\bar{a}, B, N)$ does not fork over M if there exist M' , N' with $N \leq_s N'$, $M \leq_s M' \leq_s N'$, and $B \subseteq |M'|$ such that $\text{ortp}(\bar{a}, M', N')$ does not fork over M_0 .
- (2) We define a 4-ary relation $\text{VWNF}_{\mathfrak{s}} = \text{VWNF}$ on $\mathfrak{K}_{\mathfrak{s}}$ by $\text{VWNF}(M_0, M_1, M_2, M_3)$ if and only if $M_0 \leq_s M_\ell \leq_s M_3$, $\ell = 1, 2$ and for any $\bar{a} \in {}^{<\omega} M_1$ and any finite $B \subseteq |M_2|$, if $\text{ortp}(\bar{a}, M_0, M_3)$ and $\text{ortp}(\bar{a}, M_2, M_3)$ are both basic, then $\text{ortp}(\bar{a}, B, M_3)$ does not fork over M_0 .

Theorem 3.19. *Assume that \mathfrak{s} is a type-full good $(< \omega, \lambda)$ -frame.*

- (1) *VWNF has the symmetry property: $\text{VWNF}(M_0, M_1, M_2, M_3)$ if and only if $\text{VWNF}(M_0, M_2, M_1, M_3)$.*
- (2) *If for any $M \in \mathfrak{K}_{\mathfrak{s}}$ and any $p \neq q \in \mathcal{S}_{\mathfrak{s}}^{\text{bs}}(M)$ there exists $B \subseteq |M|$ finite such that $p \upharpoonright B \neq q \upharpoonright B$, then $\text{VWNF} = \text{LWNF}$. In particular, LWNF has the symmetry property.*

Proof.

- (1) By the symmetry axiom of good frames.
- (2) This is observed in [37, 4.5]. In details, it suffices to show that for $M \leq_{\mathfrak{s}} N$, $p \in \mathcal{S}_{\mathfrak{s}}^{\text{bs}}(N)$ does not fork over M if and only if $p \upharpoonright B$ does not fork over M for all finite $B \subseteq |N|$. Let $q \in \mathcal{S}_{\mathfrak{s}}^{\text{bs}}(N)$ be the nonforking extension of $p \upharpoonright M$. For any finite $B \subseteq |N|$, we have that $q \upharpoonright B = p \upharpoonright B$, by the uniqueness property for (the extended notion of) forking, see [9, 5.4]. Therefore by the assumption we must have $p = q$, as desired. \square

4. Building a superlimit

In this section, we work in \aleph_0 and show assuming \aleph_0 -stability and amalgamation that \mathfrak{K} is PC_{\aleph_0} (Theorem 4.2) and has a superlimit (Corollary 4.6).

Hypothesis 4.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $\text{LS}(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

We will use without comments Fact 2.5 and Remark 2.6. The essence of it is that since $\lambda = \aleph_0$ all brimmed models have the same length, and hence are isomorphic (and the isomorphism fixes any common base they may have).

First note that if \mathfrak{K} is stable and has few models, we can say something about its definability:

Theorem 4.2. *Assume that $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$.*

- (1) *The set $\{M \in K_{\aleph_0} : |M| \subseteq \omega\}$ is Borel.*
- (2) *If \mathfrak{K} has amalgamation in \aleph_0 and is stable in \aleph_0 , then the set $\{(M, N) : M \leq_{\mathfrak{K}} N \text{ and } |N| \subseteq \omega\}$ is Σ_1^1 .*

In particular if \mathfrak{K} has amalgamation in \aleph_0 and is stable in \aleph_0 , then \mathfrak{K} is a PC_{\aleph_0} -representable AEC.

Proof.

- (1) Fix $M \in \mathfrak{K}_{\aleph_0}$. By Scott's isomorphism theorem, there exists a formula ϕ_M of $\mathbb{L}_{\aleph_1, \aleph_0}(\tau_{\mathfrak{K}})$ such that $N \models \phi_M$ if and only if $M \cong N$. Now observe that the set

$$\{N : N \text{ is a } \tau_{\mathfrak{K}}\text{-structure with } |N| \subseteq \omega \text{ and } N \models \phi_M\}$$

is Borel and use that $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$.

- (2) For $M, N \in \mathfrak{K}_{\aleph_0}$ with $M \leq_{\mathfrak{K}} N$, let us say that N is *almost brimmed over* M if either N is brimmed over M , or N is $\leq_{\mathfrak{K}}$ -maximal. Using amalgamation, it is easy to check that if N, N' are both almost brimmed over M , then $N \cong_M N'$ (as in Fact 2.5, recalling that the chains witnessing the brimmedness must have cofinality ω). Moreover there always exists an almost brimmed model over any $M \in \mathfrak{K}_{\aleph_0}$. Fix $\langle M_n^* : n < \omega \rangle$ such that for any $M \in \mathfrak{K}_{\aleph_0}$ there exists $n < \omega$ such that $M \cong M_n^*$ (possible as $\mathbb{I}(\mathfrak{K}, \aleph_0) \leq \aleph_0$). For each $n < \omega$, fix $N_n^* \in \mathfrak{K}_{\aleph_0}$ almost brimmed over M_n^* . We have:

⊗₁ For $M, N \in \mathfrak{K}_{\aleph_0}$:

(a) There is $n < \omega$ and an isomorphism $f : M_n^* \cong M$.

(b) If N is almost brimmed over M , then any such f extends to $g : N_n^* \cong N$.

⊗₂ For $M_1, M_2 \in \mathfrak{K}_{\aleph_0}$, $M_1 \leq_{\mathfrak{K}} M_2$ if and only if $M_1 \subseteq M_2$ and for some $n_1, n_2 < \omega$, for some (N, f_1, f_2) we have: $M_1 \subseteq M_2 \subseteq N$ and for $\ell = 1, 2$, f_ℓ is an isomorphism from $(M_{n_\ell}^*, N_{n_\ell}^*)$ onto (M_ℓ, N) .

[Why? The implication “if” holds by the coherence axiom of AECs. The implication “only if” holds as there is $N \in \mathfrak{K}_{\aleph_0}$ which is almost brimmed over M_2 (and so $M_2 \leq_{\mathfrak{K}} N$) hence N is almost brimmed over M_1 and use ⊗₁ above.]

The result now follows from ⊗₂.

By [2, 3.3], it follows that \mathfrak{K} is PC_{\aleph_0} . □

We now study homogeneous models and show that they coincide with brimmed models. Note that the homogeneity here is with respect to a set D of orbital types.

Definition 4.3. Let D be a set of orbital types over the empty set and let $M \in \mathfrak{K}$. We say that M is (D, \aleph_0) -homogeneous if it realizes all the types in D and whenever $p \in D$ is the type of an $(n+m)$ -elements sequence and $\bar{a} \in {}^n M$ realizes p^n (the restriction of p to its first n “variables”), there exists a sequence $\bar{b} \in {}^m M$ such that $\bar{a}\bar{b}$ realizes p .

The next result characterizes the countable brimmed model in AECs that are nicely stable in \aleph_0 (recall Definition 2.3).

Theorem 4.4. Assume that \mathfrak{K} is nicely stable in \aleph_0 and let \mathfrak{K}^* be the class of amalgamation bases in \mathfrak{K}_{\aleph_0} . Let $M \in \mathfrak{K}_{\aleph_0}$. The following are equivalent:

- (1) M is brimmed.
- (2) M is $(\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$ -homogeneous (see Definition 2.2(4)).

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$. First we show:

Claim: If M is brimmed, $\bar{a}, \bar{b} \in M$, then $\text{ortp}_{\mathfrak{K}^*}(\bar{a}/\emptyset; M) = \text{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; M)$ if and only if there is an automorphism f of M sending \bar{a} to \bar{b} .

Proof of Claim: The right to left direction is clear. Now assume that $\text{ortp}_{\mathfrak{K}^*}(\bar{a}/\emptyset; M) = \text{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; M)$. Say $\langle M_i : i < \omega \rangle$ witness that M is brimmed, and without loss of generality assume that $\bar{a}, \bar{b} \in M_0$. Then $\text{ortp}_{\mathfrak{K}^*}(\bar{a}/\emptyset; M_0) = \text{ortp}_{\mathfrak{K}^*}(\bar{b}/\emptyset; M_0)$. Since \mathfrak{K}^* has amalgamation, there exists $M'_0 \in \mathfrak{K}^*$ with $M_0 \leq_{\mathfrak{K}} M'_0$ and $f : M_0 \rightarrow M'_0$ so that $f(\bar{a}) = \bar{b}$. Since M_1 is universal over M_0 , we can assume without loss of generality that $M'_0 = M_1$. Now extend f to an automorphism of M using a back and forth argument.

From the claim, it follows directly that if M is brimmed, then it is $(\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$ -homogeneous. Conversely, the countable $(\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$ -homogeneous model is unique (standard back and forth argument) and so it must also be brimmed. □

Remark 4.5. By adding constants to the language, we can also characterize brimmed models over M_0 as those that are homogeneous for orbital types of finite sequences over M_0 .

Corollary 4.6. If \mathfrak{K} is nicely stable in \aleph_0 , then there is a superlimit model of cardinality \aleph_0 .

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$ be brimmed (it exists by nice stability in \aleph_0). We claim that M is superlimit. To see this, we check the conditions of Definition 2.1. On general grounds, brimmed models are universal in \mathfrak{K}_{\aleph_0} , are

not maximal (from the definition of nice stability), and there is a unique brimmed model of cardinality \aleph_0 . Still, it is not obvious that if $\langle M_i : i < \delta \rangle$ is an increasing chain of brimmed models in \mathfrak{K}_{\aleph_0} and $\delta < \omega_1$, then $\bigcup_{i < \delta} M_i$ is brimmed. To see this, we use Theorem 4.4: each M_i is $((\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$ -homogeneous, and it is clear from the definition that an increasing union of such homogeneous models is homogeneous. Thus M_δ is $(\mathcal{S}_{\mathfrak{K}^*}^{<\omega}(\emptyset), \aleph_0)$ -homogeneous. By Theorem 4.4 again, M_δ is brimmed, as desired. \square

We have justified assuming amalgamation in the following sense:

Corollary 4.7. *If \mathfrak{K} is nicely stable in \aleph_0 , then there exists an AEC $\mathfrak{K}' = (K', \leq_{\mathfrak{K}'})$ such that:*

- (1) $LS(\mathfrak{K}') = \aleph_0$.
- (2) $\mathfrak{K}'_{<\aleph_0} = \emptyset$.
- (3) $\tau_{\mathfrak{K}'} = \tau_{\mathfrak{K}}$.
- (4) $K' \subseteq K$ and for $M, N \in K'$, $M \leq_{\mathfrak{K}'} N$ if and only if $M \leq_{\mathfrak{K}} N$.
- (5) For any $M \in \mathfrak{K}_{\aleph_0}$ there exists $M' \in \mathfrak{K}'_{\aleph_0}$ with $M \leq_{\mathfrak{K}} M'$.
- (6) \mathfrak{K}' is categorical in \aleph_0 .
- (7) \mathfrak{K}' is very nicely stable in \aleph_0 . In particular it has amalgamation in \aleph_0 .
- (8) \mathfrak{K}' is PC_{\aleph_0} .

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$ be superlimit (exists by Corollary 4.6). Let $K'_{\aleph_0} := \{N \in K : N \cong M\}$. Now let \mathfrak{K}' be the AEC generated by $(K'_{\aleph_0}, \leq_{\mathfrak{K}'})$ (in the sense of [29, II.23]). One can easily check that \mathfrak{K}' is nicely stable in \aleph_0 ; from this and \aleph_0 -categoricity we get amalgamation in \aleph_0 , hence (7) holds. As for (8), it follows from Theorem 4.2. \square

5. Building a good \aleph_0 -frame

The aim of this section is to build a good \aleph_0 -frame from nice \aleph_0 -stability. By Corollary 4.7, we may restrict the class to a superlimit so that it is categorical in \aleph_0 . As before, we assume:

Hypothesis 5.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $LS(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

The nonforking relation of the frame will be nonsplitting:

Definition 5.2. For $M \in \mathfrak{K}_{\aleph_0}$ and $A \subseteq |M|$, $p \in \mathcal{S}^{<\omega}(M)$ splits over A if whenever $p = \text{ortp}(\bar{a}/M; N)$, there exists $\bar{b}_1, \bar{b}_2 \in M$ such that $\text{ortp}(\bar{b}_1/A; M) = \text{ortp}(\bar{b}_2/A; M)$ but $\text{ortp}(\bar{a}\bar{b}_1/A; N) \neq \text{ortp}(\bar{a}\bar{b}_2/A; N)$.

The following is proven in [29, I.5.6]. Similar proofs appear in [18, 3.16] or [5, 4.2].

Fact 5.3. Assume that \mathfrak{K} is nicely stable in \aleph_0 and categorical in \aleph_0 . If $M \in \mathfrak{K}_{\aleph_0}$ and $p \in \mathcal{S}^{<\omega}(M)$, then there exists $A \subseteq |M|$ finite such that p does not split over A .

The following result about nonsplitting will also come in handy. It appears in various forms in the literature, see e.g. [6, 4.8].

Lemma 5.4 (Weak uniqueness). *Assume that \mathfrak{K} is nicely stable in \aleph_0 and categorical in \aleph_0 . Let $M \leq_{\mathfrak{K}} N$ both be in \mathfrak{K}_{\aleph_0} , $p, q \in \mathcal{S}^{<\omega}(N)$. If both p and q do not split over a finite subset of M and $p \upharpoonright A = q \upharpoonright A$ for all finite $A \subseteq |M|$, then $p \upharpoonright B = q \upharpoonright B$ for all finite $B \subseteq |N|$.*

Proof. Let N' be brimmed over N . Let $\bar{b}_1, \bar{b}_2 \in N'$ realize p and q respectively. Fix $A \subseteq |M|$ finite such that p and q do not split over A . Let $B \subseteq |N|$ be finite and let \bar{b} be an enumeration of B .

Since M is brimmed, there exists $\bar{b}' \in M$ such that $\text{ortp}(\bar{b}/A; N) = \text{ortp}(\bar{b}'/A; N)$. By nonsplitting, $\text{ortp}(\bar{b}_\ell \bar{b}/A; N') = \text{ortp}(\bar{b}_\ell \bar{b}'/A; N')$ for $\ell = 1, 2$. Now since $\bar{b}' \in M$ we have by assumption that $p \upharpoonright A\bar{b}' = q \upharpoonright A\bar{b}'$. Therefore $\text{ortp}(\bar{b}_1 \bar{b}'/A; N') = \text{ortp}(\bar{b}_2 \bar{b}'/A; N')$. Putting these equalities together, $\text{ortp}(\bar{b}_1 \bar{b}/A; N') = \text{ortp}(\bar{b}_2 \bar{b}/A; N')$, so $p \upharpoonright B = q \upharpoonright B$, as desired. \square

Definition 5.5. Let \mathfrak{K} be nicely stable in \aleph_0 and categorical in \aleph_0 . We define a pre- $(< \omega, \lambda)$ -frame $\mathfrak{s} = (\mathfrak{K}^{\mathfrak{s}}, \perp, \mathcal{S}_s^{\text{bs}})$ by:

- (1) $\mathfrak{K}^{\mathfrak{s}} = \mathfrak{K}$.
- (2) For $M_0 \leq_{\mathfrak{K}} M \leq_{\mathfrak{K}} N$ all in \mathfrak{K}_{\aleph_0} , $n < \omega$, $\bar{a} = \langle a_i : i < n \rangle \in {}^n N$, $\bar{a} \perp_{M_0}^N M$ holds if and only if $a_i \notin M$ for all $i < n$ and there exists a finite $A \subseteq |M_0|$ so that $\text{ortp}(\bar{a}/M; N)$ does not split over A .
- (3) For $M \in \mathfrak{K}_{\aleph_0}$, $\mathcal{S}_s^{\text{bs}}(M)$ is the set of all types of finite sequences $\langle a_i : i < n \rangle$ over M such that for all $i < n$, $a_i \notin M$.

In order to prove that \mathfrak{s} is a good \aleph_0 -frame, we will make an additional locality hypothesis. See Example 1.1 and the next section for setups where it holds.

Definition 5.6. \mathfrak{K} is $(< \aleph_0, \aleph_0)$ -local if for any $M \in \mathfrak{K}$ $p, q \in \mathcal{S}^{< \omega}(M)$, $p \upharpoonright A = q \upharpoonright A$ for all finite $A \subseteq |M|$ implies $p = q$. We say that \mathfrak{K} is weakly $(< \aleph_0, \aleph_0)$ -local if this holds for a superlimit M .

Remark 5.7. The definition of locality includes types of any finite length, not just of length one. This will be used to prove the symmetry property of $\text{LWNF}_{\mathfrak{s}}$, via Theorem 3.19.

We now prove, assuming nice stability, categoricity, and locality, that the pre-frame defined above is a good \aleph_0 -frame.

Theorem 5.8. Assume that \mathfrak{K} is nicely stable in \aleph_0 and categorical in \aleph_0 . If \mathfrak{K} is $(< \aleph_0, \aleph_0)$ -local, then \mathfrak{s} (Definition 5.5) is a type-full good $(< \omega, \aleph_0)$ -frame. Moreover $\text{LWNF}_{\mathfrak{s}}$ has the symmetry property (recall Definitions 3.4 and 3.11). In particular, \mathfrak{s} is good⁺.

Proof. Once we have shown that \mathfrak{s} is a type-full good frame, the moreover part follows from Theorem 3.19. The last sentence is by Theorem 3.14.

Now except for symmetry, the axioms of good frames are easy to check (see the proof of [29, II.3.4]). For example:

- Local character: Let $\langle M_i : i \leq \delta \rangle$ be increasing continuous in \mathfrak{K}_s . Let $p \in \mathcal{S}_s^{\text{bs}}(M_\delta)$. By Fact 5.3, there exists a finite $A \subseteq |M_\delta|$ such that p does not split over A . Pick $i < \delta$ such that $A \subseteq |M_i|$. Then p does not fork over M_i .
- Uniqueness: by Lemma 5.4 and locality.
- Extension: follows on general grounds, see [41, 3.5].

Symmetry is the hardest to prove, and is done as in [29, I.5.30]. We give a full proof for the convenience of the reader.

Suppose that $\text{ortp}(\bar{b}, N_2, N_3)$ does not fork over N_0 and let $\bar{c} \in {}^{< \omega} N_2 \setminus N_1$. We want to find N_1, N'_3 such that $N_0 \leq_s N_1 \leq_s N'_3$, $N_3 \leq_s N'_3$, $\bar{b} \in {}^{< \omega} N_1$ and $\text{ortp}(\bar{c}, N_1, N'_3)$ does not fork over N_0 . Assume for a contradiction that there is no such N_1 . Using existence for $\text{LWNF}_{\mathfrak{s}}$ (see Theorem 3.10), as well as the extension property for nonforking, we can increase N_2 and N_3 if necessary and find N_1 such that

$\text{LWNF}_s(N_0, N_1, N_2, N_3)$, N_ℓ is brimmed over N_0 , and N_3 is brimmed over N_ℓ for $\ell = 1, 2$. By assumption, $p := \text{ortp}(\bar{c}, N_1, N_3)$ forks over N_0 .

Claim 1: Let I be the linear order $[0, \infty) \cap \mathbb{Q}$. There exists an increasing chain $\langle M_s : s \in I \rangle$ such that for any $s < t$ in I , M_s, M_t are in \mathfrak{K}_{\aleph_0} and M_t is brimmed over M_s .

Proof of Claim 1: Let $\phi \in \mathbb{L}_{\omega_1, \omega}$ be a Scott sentence for the model in \mathfrak{K}_{\aleph_0} . Let $\psi \in \mathbb{L}_{\omega_1, \omega}$ be a Scott sentence for a pair $M, N \in \mathfrak{K}_{\aleph_0}$ such that N is brimmed over M . Now let K^* be the class of sequences $\langle M_j : j \in J \rangle$ such that J is a linear order, $M_j \models \phi$ for all $j \in J$, and $(M_j, M_k) \models \psi$ for all $j < k$ in J . It is easy to see that K^* is axiomatizable by a sentence in $\mathbb{L}_{\omega_1, \omega}$. Moreover, for each $\alpha < \omega_1$, there is a sequence $\langle M_i : i < \alpha \rangle$ in K^* . By [20, Theorem 12(i)], this implies that K^* contains an I -indexed member, as desired. $\dagger_{\text{Claim 1}}$

Fix $I, \langle M_s : s \in I \rangle$ as in Claim 1. Fix N'_0 such that N_0 is brimmed over N'_0 and $p \upharpoonright N_0$ does not fork over N'_0 .

For any fixed infinite $J \subseteq I$, write $M_J := \bigcup_{s \in J} M_s$. Assume now that M_I is brimmed over M_J . Let $N_0^J := M_J, N_1^J := M_I$. Let N_3^J be brimmed over N_1^J . By categoricity and uniqueness of brimmed models, there exists $f_0 : N'_0 \cong M_0, f_0^J : N_0 \cong N_0^J, f_1^J : N_1 \cong N_1^J$, and $f_3^J : N_3 \cong N_3^J$ such that $f_0 \subseteq f_0^J \subseteq f_1^J \subseteq f_3^J$. Let $f_2^J := f_3^J \upharpoonright N_2$ and let $N_2^J := f_2^J[N_2]$. Note that $\text{LWNF}_s(N_0^J, N_1^J, N_2^J, N_3^J)$ holds.

Let $p_J := \text{ortp}(f_3^J(\bar{c}), f_3^J[N_1], f_3^J[N_3]) = \text{ortp}(f_3^J(\bar{c}), M_I, N_3^J)$. Since we are assuming that $\text{ortp}(\bar{c}, N_1, N_3)$ forks over N_0 , we have that p_J forks over N_0^J . Moreover $p_J \upharpoonright N_0^J$ does not fork over M_0 .

Claim 2: If J has no last elements, $I \setminus J$ has no first elements, and $t \in I \setminus J$, then $p_J \upharpoonright M_t$ forks over N_0^J .

Proof of Claim 2: Suppose that $p_J \upharpoonright M_t$ does not fork over N_0^J . Note that M_t is brimmed over M_J . Find N'_1 such that $N_0 \leq_s N'_1 \leq_s N_1$, N'_1 is brimmed over N_1 , and $f_1^J : N'_1 \cong M_t$. Let $\bar{b}' \in {}^{<\omega} N'_1$ be such that $\text{ortp}(\bar{b}', N_0, N'_1) = \text{ortp}(\bar{b}, N_0, N_1)$. Since $\text{LWNF}_s(N_0, N_1, N_2, N_3)$, we know that $\text{ortp}(\bar{b}', N_2, N_3)$ does not fork over N_0 , hence by uniqueness $\text{ortp}(\bar{b}, N_2, N_3) = \text{ortp}(\bar{b}', N_2, N_3)$. But we have assumed that $\text{ortp}(\bar{c}, N'_1, N_3)$ does not fork over N_0 and $\bar{b}' \in {}^{<\omega_1} N'_1$, hence by a simple renaming we obtain a contradiction to our hypothesis that symmetry failed. $\dagger_{\text{Claim 2}}$

Claim 3: If $J_1 \subsetneq J_2$ are both proper initial segments of I with no last elements and $J_2 \setminus J_1$ has no first elements, then $p_{J_1} \neq p_{J_2}$.

Proof of Claim 3: Fix $t \in J_2 \setminus J_1$. By Claim 2, $p_{J_1} \upharpoonright M_t$ forks over $N_0^{J_1}$. We claim that $p_{J_2} \upharpoonright M_t$ does not fork over $N_0^{J_1}$. Indeed recall that $N_0^{J_2} = M_{J_2}$ and by assumption $p_{J_2} \upharpoonright N_0^{J_2}$ does not fork over M_0 . Therefore by monotonicity also $p_{J_2} \upharpoonright M_t$ does not fork over $M_{J_1} = N_0^{J_1}$. $\dagger_{\text{Claim 3}}$

To finish, observe that there are 2^{\aleph_0} cuts of I as in Claim 3. Therefore stability fails, a contradiction. \square

The next corollary does not assume categoricity, but uses amalgamation in \aleph_0 , rather than just density of amalgamation bases.

Corollary 5.9. *If \mathfrak{K} is very nicely stable in \aleph_0 and weakly ($< \aleph_0, \aleph_0$)-local, then \mathfrak{K} has a superlimit of cardinality \aleph_1 .*

Proof. By Corollary 4.6, \mathfrak{K} has a superlimit N_0 in \aleph_0 . Let \mathfrak{K}' be the class generated by this superlimit, as described by the proof of Corollary 4.7. Then \mathfrak{K}' is categorical in \aleph_0 and nicely stable in \aleph_0 , hence we can apply Theorem 5.8 and get a type-full good⁺ \aleph_0 -frame with underlying class \mathfrak{K}'_{\aleph_0} . By Theorem 3.14, \mathfrak{K}' has a superlimit model in \aleph_1 . This is also a superlimit in \mathfrak{K} : the only nontrivial property to check is universality. Let $M \in \mathfrak{K}_{\aleph_1}$. Fix any $M_0 \in \mathfrak{K}'_{\aleph_0}$ with $M_0 \leq_{\mathfrak{K}} M$. By universality of N_0 , there exists $f : M_0 \rightarrow N_0$. Now let $N \in \mathfrak{K}'$ be superlimit in \aleph_1 with $N_0 \leq_{\mathfrak{K}} N$. Using amalgamation (amalgamation in \aleph_0 suffices for this, see [29, I.2.11]), we can find $g : M \rightarrow N$ extending f , as needed. \square

6. Locality from supersimplicity

In this section, we give a sufficient condition for locality. As before, we assume:

Hypothesis 6.1. $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with $\text{LS}(\mathfrak{K}) = \aleph_0$ (and countable vocabulary).

In the context of a nicely \aleph_0 -stable AEC, the following definition generalizes that of a supersimple homogeneous model [11, 2.5(iv)]. The idea is that we want to have a nice notion of nonforking available for all finite sets (not only models). However we do not require that forking over finite sets satisfies any uniqueness requirement. Thus it is not a-priori clear that supersimplicity implies the existence of a good frame (although this will follow from \aleph_0 -categoricity and Theorems 5.8, 6.10).

Throughout this section expressions such as “nonforking” or “not fork” will refer to the relation defined in Definition 6.2(3) below. We give examples after the definition.

Definition 6.2. Assume that \mathfrak{K} is nicely \aleph_0 -stable and categorical in \aleph_0 . We say that \mathfrak{K} is *supersimple* if there exists a 4-ary relation \downarrow such that:

- (1) $\downarrow(A, B, C, N)$ implies that $N \in \mathfrak{K}_{\aleph_0}$, $A \cup B \cup C \subseteq |N|$ and A, B, C are all finite or countable. We write $B \downarrow_A^N C$ instead of $\downarrow(A, B, C, N)$. Below, we may abuse notation and write e.g. $A_1 A_2$ instead of $A_1 \cup A_2$, or $\bar{b} \downarrow_{\bar{a}}^N \bar{c}$ instead of $B \downarrow_A^N C$, where A, B, C stand for the ranges of \bar{a}, \bar{b} , and \bar{c} respectively.
- (2) Normality: $B \downarrow_A^N C$ if and only if $AB \downarrow_A^N AC$.
- (3) Invariance under \mathfrak{K} -embeddings: If $f : N \rightarrow N'$ and $A \cup B \cup C \subseteq |N|$, then $B \downarrow_A^N C$, if and only if $f[B] \downarrow_{f[A]}^{N'} f[C]$. This shows that \downarrow is really a relation on types, so we say **ortp**($\bar{b}/C; N$) *does not fork over A* if $\bar{b} \downarrow_A^N C$.
- (4) Monotonicity: If $B \downarrow_A^N C$ and $A \subseteq A' \subseteq B' \subseteq B$ then $B' \downarrow_{A'}^N C$.
- (5) Symmetry: If $B \downarrow_A^N C$, then $C \downarrow_A^N B$.
- (6) Local character: If $M \in \mathfrak{K}_{\aleph_0}$ and $p \in \mathcal{S}^{<\omega}(M)$, then there exists $A \subseteq |M|$ finite such that p does not fork over A .
- (7) Extension: If $p \in \mathcal{S}^{<\omega}(C; N)$ does not fork over $A \subseteq C$, then there is $q \in \mathcal{S}^{<\omega}(N)$ such that q extends p and q does not fork over A .
- (8) Transitivity: If $B \downarrow_A^N C$ and $B \downarrow_C^N D$ with $A \subseteq C \subseteq D$, then $B \downarrow_A^N D$.
- (9) Relationship with splitting: If $M \leq_{\mathfrak{K}} N$ are both in \mathfrak{K}_{\aleph_0} , $p \in \mathcal{S}^{<\omega}(N)$, and p does not fork over M , then there is $A \subseteq |M|$ finite such that p does not split over A .

Remark 6.3. It may be helpful to compare Definition 6.2 with Definition 2.7. The idea of 6.2 is to give a sort of analog of good frames but to allow types over sets. Note that the statement of symmetry in 2.7 is more technical, precisely because types over sets are not allowed. However the idea is the same. Another difference is that local character in 6.2 is stated as “every type does not fork over a finite set”. In 2.7, it is stated as “every type over the union of an increasing chain does not fork over a previous element of the chain”. Again, the lack of types over sets makes it impossible to state the former in good frames.

Example 6.4.

- (1) Working inside a supersimple homogeneous model N (in the sense of [11, 2.5(iv)]), we can define $B \underset{A}{\downarrow}^N C$ to hold if and only if B is \aleph_0 -free from C over A (in the sense of [11, 2.2]). The first four conditions of Definition 6.2 are then easy to check. Symmetry is [11, 2.14] and transitivity is [11, 2.15]. Local character and extension are given by the definition in [11, 2.5]. Now if in addition N is \aleph_0 -stable (in the sense of [11, 5.1] or equivalently in the sense given here), then by [11, 5.2] the last axiom of Definition 6.2 holds.
- (2) Let \mathfrak{K} be a FUR class [17, 2.17] (this includes in particular all quasiminimal pregeometry classes). Then letting \downarrow be defined as in [17, 2.38], we can also check that it satisfies Definition 6.2.

We first show that if a type p over a finite set does not fork over a subset A of a countable model M , then the type is realized inside M .

Lemma 6.5. *Assume that \mathfrak{K} is nicely \aleph_0 -stable, supersimple, and categorical in \aleph_0 . Let $M \leq_{\mathfrak{K}} N$ both be in \mathfrak{K}_{\aleph_0} , and let $B \subseteq |N|$ be finite. Let $p \in \mathcal{S}^{<\omega}(B; N)$. If p does not fork over $B \cap |M|$, then p is realized in M .*

Proof. Extending N if necessary, we can assume without loss of generality that N is brimmed over M . Let $q \in \mathcal{S}^{<\omega}(N)$ be a nonforking extension of p . Let \bar{b} be an enumeration of B . Fix $A' \subseteq |M|$ finite and big-enough such that $B \cap |M| \subseteq A'$, $\text{ortp}(\bar{b}/M; N)$ does not split and does not fork over A' and q does not split over A' (recall Fact 5.3). Let $M_0 \leq_{\mathfrak{K}} M$ be such that M is brimmed over M_0 and M_0 contains A' . Let $\bar{d} \in M$ be such that \bar{d} realizes $q \upharpoonright M_0$. We have that $\bar{b} \underset{M_0}{\downarrow}^N M$ so by symmetry and monotonicity, $\bar{d} \underset{M_0}{\downarrow}^N \bar{b}$.

By extension, we can pick M'_0 such that $M_0 \leq_{\mathfrak{K}} M'_0 \leq_{\mathfrak{K}} N$, M'_0 contains \bar{b} , and $\bar{d} \underset{M_0}{\downarrow}^N M'_0$. Now as $q \upharpoonright M'_0$ does not split over a finite subset of M_0 and \bar{d} realizes $q \upharpoonright M_0$, we must have by Lemma 5.4 that \bar{d} realizes $q \upharpoonright B = p$, as desired. \square

We will prove locality in supersimple \aleph_0 -stable AECs by a back and forth argument. More precisely, we start with $M \leq_{\mathfrak{K}} N$, N brimmed over M , and elements $\bar{a}_1, \bar{a}_2 \in N$ whose types over every finite subset of M match. First, we will do a back and forth argument to find an automorphism of N sending \bar{a}_1 to \bar{a}_2 and fixing M *setwise* (Lemma 6.8). We will then use this automorphism and nonsplitting to build another automorphism that fixes M *pointwise* (Theorem 6.10).

The next lemma starts setting up the stage by making sure that we can map an element of M to an element of M .

Lemma 6.6. *Assume that \mathfrak{K} is nicely \aleph_0 -stable, supersimple, and categorical in \aleph_0 . Let $M \leq_{\mathfrak{K}} N$ both be in \mathfrak{K}_{\aleph_0} . Let $\bar{b}_1, \bar{b}_2 \in N$, $\bar{a}_1, \bar{a}_2, \bar{c}_1 \in M$ be such that $\text{ortp}(\bar{a}_1 \bar{b}_1; N) = \text{ortp}(\bar{a}_2 \bar{b}_2; N)$. If $\bar{b}_1 \underset{\bar{a}_1}{\downarrow}^N \bar{c}_1$, then there exists $\bar{c}_2 \in M$ such that $\text{ortp}(\bar{a}_1 \bar{b}_1 \bar{c}_1; N) = \text{ortp}(\bar{a}_2 \bar{b}_2 \bar{c}_2; N)$.*

Proof. Extending N if necessary, we can assume without loss of generality that N is brimmed over M . By symmetry, $\bar{c}_1 \underset{\bar{a}_1}{\downarrow}^N \bar{b}_1$. Let f be an automorphism of N sending $\bar{a}_1 \bar{b}_1$ to $\bar{a}_2 \bar{b}_2$. By invariance, $f(\bar{c}_1) \underset{\bar{a}_2}{\downarrow}^N \bar{b}_2$ (but we do *not* know that $f(\bar{c}_1) \in M$). Let $q := \text{ortp}(f(\bar{c}_1)/\bar{a}_2 \bar{b}_2; N)$. We have to show that q is realized in M . Since q does not fork over $\bar{a}_2 \in M$, this is exactly what Lemma 6.5 tells us. \square

Our main lemma in the back and forth argument will be:

Lemma 6.7 (The back and forth lemma). Assume that \mathfrak{K} is nicely \aleph_0 -stable, supersimple, and categorical in \aleph_0 . Let $M \leq_{\mathfrak{K}} N$ both be in \mathfrak{K}_{\aleph_0} with N brimmed over M . Let $\bar{b}_1, \bar{b}_2 \in N$, $\bar{a}_1, \bar{a}_2 \in M$ be such that $\text{ortp}(\bar{a}_1 \bar{b}_1; N) = \text{ortp}(\bar{a}_2 \bar{b}_2; N)$ and $\bar{b}_\ell \downarrow_{\bar{a}_\ell}^N M$ for $\ell = 1, 2$. If $\bar{d}_1 \in N$, $\bar{d}'_1 \in M$, there exists $\bar{c}_1, \bar{c}_2, \bar{d}'_2 \in M$ and $\bar{d}_2 \in N$ such that:

- (1) $\text{ortp}(\bar{a}_1 \bar{b}_1 \bar{c}_1 \bar{d}'_1 \bar{d}_1; N) = \text{ortp}(\bar{a}_2 \bar{b}_2 \bar{c}_2 \bar{d}'_2 \bar{d}_2; N)$.
- (2) $\bar{b}_\ell \bar{d}_\ell \downarrow_{\bar{a}_\ell \bar{c}_\ell \bar{d}'_\ell}^N M$ for $\ell = 1, 2$.

Proof. Using Lemma 6.6, we can enlarge \bar{a}_1 and \bar{a}_2 if necessary to assume without loss of generality that \bar{d}'_1 is empty. Now using local character, fix $\bar{c}_1 \in M$ such that $\bar{b}_1 \bar{d}_1 \downarrow_{\bar{a}_1 \bar{c}_1}^N M$. By Lemma 6.6, there exists $\bar{c}_2 \in M$ such that $\text{ortp}(\bar{a}_1 \bar{b}_1 \bar{c}_1; N) = \text{ortp}(\bar{a}_2 \bar{b}_2 \bar{c}_2; N)$. Let f be an automorphism of N witnessing this. By extension and monotonicity, pick $\bar{d}_2 \in N$ such that $\text{ortp}(\bar{d}_2 / \bar{a}_2 \bar{b}_2 \bar{c}_2; N) = \text{ortp}(f(\bar{d}_1) / \bar{a}_2 \bar{b}_2 \bar{c}_2; N)$ and $\bar{d}_2 \downarrow_{\bar{a}_2 \bar{b}_2 \bar{c}_2}^N M \bar{b}_2$.

It remains to see that $\bar{b}_2 \bar{d}_2 \downarrow_{\bar{a}_2 \bar{c}_2}^N M$. We do this using a standard nonforking calculus argument: by normality, $\bar{b}_2 \bar{d}_2 \downarrow_{\bar{a}_2 \bar{b}_2 \bar{c}_2}^N M \bar{b}_2$ and we also know from the hypotheses of the lemma that $\bar{b}_2 \downarrow_{\bar{a}_2}^N M$, so by monotonicity and normality $\bar{a}_2 \bar{b}_2 \bar{c}_2 \downarrow_{\bar{a}_2 \bar{c}_2}^N M$. Now using transitivity, monotonicity, and symmetry, $\bar{b}_2 \bar{d}_2 \downarrow_{\bar{a}_2 \bar{c}_2}^N M$, as desired. \square

We can now build the desired automorphism which fixes M setwise.

Lemma 6.8. Assume that \mathfrak{K} is nicely \aleph_0 -stable, supersimple, and categorical in \aleph_0 . Let $M \in \mathfrak{K}_{\aleph_0}$, $A \subseteq |M|$ finite, $p, q \in \mathcal{S}^{<\omega}(M)$ such that p and q do not fork over A . If $p \upharpoonright A = q \upharpoonright A$, then there exists an automorphism f of M fixing A such that $f(p) = q$.

Proof. Let $N \in \mathfrak{K}_{\aleph_0}$ be brimmed over M . Say $p = \text{ortp}(\bar{b}_1 / M; N)$, $q = \text{ortp}(\bar{b}_2 / M; N)$. Let \bar{a} be an enumeration of A and let $\bar{a}_\ell := \bar{a}$, $\ell = 1, 2$. Now apply Lemma 6.7 repeatedly in a back and forth argument to build an automorphism g of N fixing A such that $g(\bar{b}_1) = \bar{b}_2$ and $g[M] = M$. Let $f := g \upharpoonright M$. \square

We now show that we can actually build an automorphism fixing M pointwise. The next lemma is the main argument for this:

Lemma 6.9. Assume that \mathfrak{K} is nicely \aleph_0 -stable, supersimple, and categorical in \aleph_0 . Let $M \in \mathfrak{K}_{\aleph_0}$ and let N be brimmed over M . Let $\bar{b}_1, \bar{b}_2 \in N$ be such that $\text{ortp}(\bar{b}_1 / A; N) = \text{ortp}(\bar{b}_2 / A; N)$ for all finite $A \subseteq |M|$. For any $\bar{d}_1 \in N$, there exists $\bar{d}_2 \in N$ such that $\text{ortp}(\bar{b}_1 \bar{d}_1 / A; N) = \text{ortp}(\bar{b}_2 \bar{d}_2 / A; N)$ for all finite $A \subseteq |M|$.

Proof. Let $p := \text{ortp}(\bar{b}_1 / M; N)$, $q := \text{ortp}(\bar{b}_2 / M; N)$. Fix $A \subseteq |M|$ finite such that both p and q do not fork over A and $\text{ortp}(\bar{b}_1 \bar{d}_1 / M; N)$ does not split over A . By Lemma 6.8, there exists an automorphism f of M fixing A such that $f(p) = q$. Let g be an automorphism of N extending f such that $g(\bar{b}_1) = \bar{b}_2$ and let $\bar{d}_2 := g(\bar{d}_1)$. We claim that this works. Fix $C \subseteq |M|$ finite and let \bar{c}_2 be an enumeration of C . Let $\bar{c}_1 := g^{-1}(\bar{c}_2)$. We know that $\text{ortp}(\bar{c}_1 / A; N) = \text{ortp}(\bar{c}_2 / A; N)$, so by nonsplitting, $\text{ortp}(\bar{b}_1 \bar{d}_1 \bar{c}_1 / A; N) = \text{ortp}(\bar{b}_1 \bar{d}_1 \bar{c}_2 / A; N)$. Applying g , we have that $\text{ortp}(\bar{b}_1 \bar{d}_1 \bar{c}_1 / A; N) = \text{ortp}(\bar{b}_2 \bar{d}_2 \bar{c}_2 / A; N)$. Putting the two equalities together, $\text{ortp}(\bar{b}_1 \bar{d}_1 \bar{c}_2 / A; N) = \text{ortp}(\bar{b}_2 \bar{d}_2 \bar{c}_2 / A; N)$, so $\text{ortp}(\bar{b}_1 \bar{d}_1 / C; N) = \text{ortp}(\bar{b}_2 \bar{d}_2 / C; N)$, as desired. \square

We have arrived to the main theorem of this section:

Theorem 6.10. *If \mathfrak{K} is nicely \aleph_0 -stable, supersimple, and categorical in \aleph_0 , then \mathfrak{K} is $(< \aleph_0, \aleph_0)$ -local (recall Definition 5.6).*

Proof. Let $M \in \mathfrak{K}_{\aleph_0}$ and let $p, q \in \mathcal{S}^{<\omega}(M)$ be such that $p \upharpoonright A = q \upharpoonright A$ for all finite $A \subseteq |M|$. Let N be brimmed over M and let $\bar{b}_1, \bar{b}_2 \in N$ realize p and q respectively. Now apply Lemma 6.9 in a back and forth argument to get an automorphism of N fixing M taking \bar{b}_1 to \bar{b}_2 . \square

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