DEFINABLE GROUPS FOR DEPENDENT AND 2-DEPENDENT THEORIES

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ABSTRACT. Let T be a (first order complete) dependent theory, $\mathfrak{C}$ a $\bar{\kappa}$-saturated model of T and G a definable subgroup which is abelian. Among subgroups of bounded index which are the union of $< \bar{\kappa}$ type-definable subsets there is a minimal one, i.e. their intersection has bounded index. See history in [19]. We then deal with definable groups for 2-dependent theories, a wider class of first order theories proving that for many pairs (M, N) of models, the minimal bounded subgroup definable over $M \cup N$ is the intersection of the minimal ones for M and for N.

0. INTRODUCTION

Assume that T is a dependent (complete first order) theory, $\mathfrak{C}$ is a $\bar{\kappa}$-saturated model of T (a monster), G is a type-definable (in $\mathfrak{C}$) group in $\mathfrak{C}$ (of course we consider only types of cardinality $< \bar{\kappa}$).

A subgroup H of G is called bounded if the index $(G : H)$ is $< \bar{\kappa}$. By [19], we know that among $\mathbf{H}_{\mathrm{tb}} = \{H : H$ is a type-definable subgroup of G of bounded, i.e. with index $< \bar{\kappa}\}$ there is a minimal one.

But what occurs for $\mathbf{H}_{\mathrm{stb}} = \{H : H$ is a union-type-definable (see below) subgroup of G of bounded, i.e. $< \bar{\kappa}$ index$\}$? is there a minimal one? Our main result is a partial positive answer: if G is an abelian group then in $\mathbf{H}_{\mathrm{stb}}$ there is a minimal one.

We call $C \subseteq \mathfrak{C}$ union-type-definable (over A) when for some sequence $\langle p_i(x) : i < \alpha \rangle$ we have $\alpha < \bar{\kappa}$, each $p_i(x)$ is a type (over A; of course A is of

cardinality $< \bar{\kappa}$) and $C = \cup\{p_i(\mathfrak{C}) : i < \alpha\}$ where $p_i(\mathfrak{C}) = \{b \in \mathfrak{C} : b$ realizes $p_i(\mathfrak{C})\}$; this is equivalent to being $\mathbb{L}_{\infty,\kappa}$-definable in $\mathfrak{C}$ for some $\kappa \leq \bar{\kappa}$.

In Definition 2.4 we recall the definition of 2-dependent T (where T is 2-independent when some $\langle \varphi(\bar{x}, \bar{b}_m, \bar{c}_n) : m, n < \omega \rangle$ is an independent sequence of formulas); see [18, §5 (H)]. Though a reasonable definition, can we say anything interesting on it? Well, we prove the following result related to [19].

Let G_A be the minimal type-definable over A subgroup of G, for suitable κ; as we fix A it always exists. Theorem 2.12 says that if M is κ-saturated and $|B| < \kappa$ then $G_{M \cup B}$ can be represented as $G_M \cap G_{A \cup B}$ for some $A \subseteq M$ of cardinality $< \kappa$. So though this does not prove "2-dependent is a dividing line", it seems enough for showing it is an interesting property.

The first theorem on this line for T stable is of Baldwin-Saxl [2]. Recently Hrushovski, Peterzil and Pillay [5] investigated definable groups, o-minimality and measures; an earlier work on dependent theories is [13]; and on definable subgroups in o-minimal T is Berarducci, Otero, Peterzil and Pillay [1] where the existence of the minimal type-definable bounded index theorem and more results are proved for o-minimal theories.

A natural question was whether there is a minimal type-definable bounded subgroup, when T is dependent. Assuming more (existence of measure) on the group, related suitably to the family of definable sets, this was proved in the original version of Hrushovski-Peterzil-Pillay [5]. Then [19] proves this for every dependent theory and definable group. (The final version of their paper [5] includes an exposition of the proof of [19].)

Recent works of the author on dependent theories are [15] (see §3,§4 on groups) [18] (e.g. the first order theory of the p-adics is strongly[1] dependent but not strongly[2] dependent, see end of §1; on strongly[2] dependent fields see §5) and [20] and later [21].

Hrushovski has pointed out the following application of §2 to Cherlin-Hrushovski [ChHr03]. They deal with complete first order theories T with few λ-complete 4-types such that every finite subset of T has a finite model. Now from the classification of finite simple groups several (first order) properties of such theories were deduced. From them they get back information on automorphism groups of finite structures. An interesting gain of that investigation is that if looked at this as a round trip from classification of finite simple groups to such T's, we get uniform bounds for some of the existence

results in the classification of finite simple groups. Note that they deal with the case the finite field is fixed, while the vector space over it varies. This is related to the properties being preserved by reducts (and interpretations) of first order theories hence we get the uniform bounds. Now by the present work some of those first order properties of such T's which they proved using finite simple groups, are redundant in the sense that they follows from the others though originally it was non-trivial to prove them. Why? Because (the relevant first order theories are 2-dependent by Observation 2.7 and so) Theorem 2.12 can be applied. More specifically, the main point is a version of modularity that looking at an expansion of $M = (V, B)$, where V is a vector space over a fixed finite field F, B a bilinear map from V to F, M an ultraproduct of finite structures, for suitable $M_0 \prec M_\ell \prec M$ for $\ell = 1, 2, acl(M_1 \cup M_2) = \{a_1 + a_2 : a_1 \in M_1, a_2 \in M_2\}$.

This raises further questions.

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The reader may ask

Question 0.1. Is the notion of 2-independent and more generally n-independent first order theory interesting?

In my view they certainly are. Why?

(A)   Their definition is simple and natural.

True this is a statement easy to argue about, hard to convince about, you may say it is a matter of personal taste. But we can look at history: earlier cases in which the author suggest such notions proving something on them. True, the interest in the (failure of the) strict order property (see [8]; [12, §5], Kikyo-Shelah [14], Shelah-Usvyatsov, [16]) and n-NSOP relatives (see [10, §2], Dzamonja-Shelah [11], Shelah-Usvyatsov [17]) have not so far been proved to skeptics. But other cases looked for a non-trivial number of years to be in a similar situation but eventually become quite popular: stable and superstable ([7]), the independence property itself ([8]), and a more strict version of the (failure of the) strict order property = simple theory (= failure of the tree property).

(B)   The theory of such classes is not empty; we prove in §2 that "the minimal bounded subgroup of G over A" behaves nicely. A skeptical reader

suggests that to justify our interest we have to point out "natural examples", i.e. from other branches of mathematics. We object to this, just as we object to a parallel view on mathematics as a whole. In both cases applications are a strong argument for and desirable, <u>but</u> not a necessary condition. In fact such cases were not present in the author's first works on the notions mentioned above.

Moreover the connection to Chelin-Hrushovski [3] already showed such connection.

In any case there are such examples, generally if in $\mathfrak{C} = \mathfrak{C}_T$ we can define a field K and vector spaces V_1, V_2, V_3 over it (or just abelian groups) and a bi-linear mapping $F : V_1 \times V_2 \to V_3$, then in the non-degenerated case the formulas $\{F(x, b) = 0_{V_3} : b \in V_2\}$ form an infinite family of independent formulas. As bi-linear maps appear in many examples, it makes sense even to a reasonable skeptic to look for a (real) parallel to the class of stable theories, including them and 2-dependent is the most natural one.

Concerning [19], note that

Claim 0.2. (T dependent).
Assume

(a) *G is a A^*-definable semi-group with cancellation*

(b) *$q(x, \bar{a})$ is a type, $q(\mathfrak{C}, \bar{a})$ a sub-semi-group of G*

(c) *for $\bar{a}'$ realizing $\mathrm{tp}(\bar{a}, A*)$ let $\mathrm{set}(\bar{a}') = \{b:$ for some $a, c \in q(\mathfrak{C}, \bar{a}')$ we have $abc \in q(\mathfrak{C}, \bar{a}')\}$; if G is a group, $q(\mathfrak{C}, \bar{a})$ a sub-group then $\mathrm{set}(\bar{a}') = q(\mathfrak{C}, \bar{a})$*

<u>then</u> *we can find q^* and $\langle \bar{a}_i : i < \alpha \rangle$ and B such that:*

(α) *$\alpha < \lambda^+$ where $\lambda = |T| + |\ell g(\bar{a})|$*

(β) *$\mathrm{tp}(\bar{a}_i, A^*) = \mathrm{tp}(\bar{a}, A^*)$ hence $q(\mathfrak{C}, \bar{a}_i)$ a sub-semi-group of G*

(γ) *$q^*(x) = \cup\{q(x, \bar{a}_i) : i < \alpha\}$*

(δ) *$B \subseteq \bigcup_{i<\alpha} q(G, \bar{a}_i) \subseteq G$ and $|B| \leq |\alpha|$; in fact, $B \subseteq \bigcup_i \bigcap_{j \in \alpha \setminus \{i\}} q(G, \bar{a}_i)$*

(ε) *if $\bar{a}'$ realizes $\mathrm{tp}(\bar{a}, A^*)$ and $B \subseteq q(\mathfrak{C}, \bar{a}')$ then $q^*(\mathfrak{C}) \subseteq \mathrm{set}(\bar{a}')$.*

Remark 0.3. This was part of [19], but it was claimed that it seems to be wrong: Note that we do not require "$B \subseteq q^*(\mathfrak{C})$". The following example may clarify the claim.

Let T be the theory of $(\mathbb{Q}, +, <, 0, 1)$ say, and G the monster model. Let a be infinitesimal, namely it realizes $p(x) := \{0 < x < 1/n : n = 1, 2, 3, \ldots\}$. Let $q(x, a)$ say that $|x| < a/n$ for $n = 1, 2, \ldots$. Then for any set $A \subseteq p(\mathfrak{C})$ (of cardinality $< \bar{\kappa}$) and set B (of cardinality $< \bar{\kappa}$) contained in $q(\mathfrak{C}, a')$ for all $a' \in A$, there is by compactness some d realizing $p(x)$ such that $d/n > b$ for all $b \in B$ and $n = 1, 2, \ldots$ and $d < a'/n$ for all $a' \in A$ and $n = 1, 2, \ldots$. However, what (ε) of Claim 0.2 says is: if a' realizes $p(x)$ (i.e. a' infinitesimal) and $B \subseteq q(\mathfrak{C}, a')$, i.e. $b \in B \wedge n < \omega \Rightarrow |b| < a'/n$ then $q^*(\mathfrak{C}) \subseteq q(\mathfrak{C}, a')$, so if $a \in B$ this holds.

Proof. We try to choose $\bar{a}_\alpha, b_\alpha$ by induction on $\alpha < (|T|^{\aleph_0})^+$ such that

$\circledast$ (a) $\bar{a}_\alpha$ realizes $\text{tp}(\bar{a}, A^*)$

     (b) $b_\alpha \notin q(\mathfrak{C}, \bar{a}_\alpha)$ and $b_\alpha \in G$ and moreover $a, c \in q(\mathfrak{C}, a_\alpha) \Rightarrow$
          $a b_\alpha c \notin q(\mathfrak{C}, a_\alpha)$

     (c) b_α realizes $q(x, \bar{a}_\beta)$ for $\beta < \alpha$

     (d) b_β realizes $q(x, \bar{a}_\alpha)$ for $\beta < \alpha$.

First, if we are stuck at some $\alpha < (|T|^{\aleph_0})^+$ then the desired result is exemplified by $\langle \bar{a}_i : i < \alpha \rangle$ and $B := \{b_i : i < \alpha\}$. Note that if $\alpha \geq 1$ then $B \subseteq \cup\{\cap\{q(\mathfrak{C}, \bar{a}_i) : i \in \alpha \backslash \{j\} : j < \alpha\}$ and if $\alpha = 0$ then $B = \emptyset$.

Second, if we succeed we get contradiction similarly to the proof in [19, §1] but we elaborate; let $\bar{y} = \langle y_i : i < \ell g(\bar{a}) \rangle$. Let $\varphi_\alpha(x, \bar{y}) \in q(x, \bar{a}_\alpha)$ be such that $\mathfrak{C} \models \neg\varphi_\alpha(b_\alpha, \bar{a}_\alpha)$, exists by clause (b) of $\circledast$. The number of possible $\varphi_\alpha(x, \bar{a}_\alpha)$ is $\leq \lambda$, hence for some $\varphi_*(x, \bar{y}) \in \mathbb{L}(\tau_T)$ the set $W := \{\alpha < \lambda^+ : \varphi_\alpha(x, \bar{y}) = \varphi_*(x, \bar{y})\}$ is a set of cardinality λ^+, so it is enough to prove that for every finite non-empty $w \subseteq W$ there is $b_w \in \cap\{\varphi_\alpha(\mathfrak{C}, a_\alpha) : \alpha \in W \backslash w\} \cap \bigcap\{\varphi_*(\mathfrak{C}, \bar{a}_\alpha) : \alpha \in w \backslash \{\max(w), \min(w)\}$.

Now letting $\alpha_0 < \ldots < \alpha_n$ list w, we can choose $b_w := b_{\alpha_0} \ldots b_{\alpha_n}$ (the product). Now on the one hand, $b_w \in \varphi_*(\mathfrak{C}, \bar{a}_\alpha)$ for $\alpha \in W \backslash w$ because $\varphi_*(\mathfrak{C}, \bar{a}_\alpha)$ is closed under products by the choice of φ_α and $\ell < n \Rightarrow b_{\alpha_\ell} \in \varphi_*(\mathfrak{C}, \bar{a}_\alpha)$. On the other hand if $\ell = 1, \ldots, n-1$ and $b_u \in \varphi_*(\mathfrak{C}, \bar{a}_{\alpha_\ell})$ by the "moreover" in clause (b) of $\circledast$ we are done. $\square_{0.2}$

1. $\mathbb{L}_{\infty, \bar{\kappa}}(\tau_T)$-DEFINABLE SUBGROUPS OF BOUNDED INDEX

The main result of this section is Theorem 1.12: if the monster model $\mathfrak{C}$ is $\bar{\kappa}$-saturated, $T = Th(\mathfrak{C})$ is dependent, G is a definable abelian group over

8   SAHARON SHELAH

$A_*, |A_*| < \bar{\kappa}$ <u>then</u> $H_* = \cap\{H : H$ is a union-type-definable [1] subgroup of G of bounded index, i.e. $< \bar{\kappa}\}$ <u>then</u> H_* is a union-type-definable over A_* and has index $\leq 2^{|T|+|A_*|}$. As usual $\bar{\kappa}$ is strongly inaccessible (strong limit of large enough cofinality such that $(\mathcal{H}(\bar{\kappa}), \in) \prec_{\Sigma_n} (\mathbf{V}, \in)$ for some n large enough suffices).

Context 1.1.

(a) $\mathfrak{C}$ is a monster ($\bar{\kappa}$-saturated) model of the complete first order theory T (we assume $\bar{\kappa}$ is strongly inaccessible $> |T|$; this just for convenience, we do not really need to assume there is such cardinal).

(b) $p_*(x)$ is a type and $(x, y) \mapsto x * y, x \mapsto x^{-1}$ and e_G are first order definable (in $\mathfrak{C}$) two-place function, one place function and element (with parameters $\subseteq \mathrm{Dom}(p_*)$ for simplicity) such that their restriction to $p_*(\mathfrak{C})$ gives it a group structure which we denote by $G = G_{p_*}^{\mathfrak{C}}$. Let $\mathrm{Dom}(p_*) = A_*$; we may write ab instead of $a * b$. When the group is Abelian we may use the additive notation.

Notation 1.2. For an m-type $p(\bar{x})$ let $p(B) = \{\bar{b} \in {}^{\ell g(\bar{x})}B : \bar{b}$ realizes $p(\bar{x})\}$, so if $p = p(x)$ then we stipulate $p(\mathfrak{C}) \subseteq \mathfrak{C}$. For a set $\mathbf{P}$ of $< \bar{\kappa}$ of m-types let $\mathbf{P}(B) = \cup\{p(B) : p \in \mathbf{P}\}$.

Definition 1.3. For any $\alpha < \bar{\kappa}$ and sequence $\bar{\mathbf{a}} \in {}^{\alpha}\mathfrak{C}$ and $n < \omega$ we define $q_{\bar{\mathbf{a}}}^n(x) = q_{\Gamma_{\bar{\mathbf{a}}}}^n(x)$ where

(a) $\Gamma_{\bar{\mathbf{a}}} = \{\varphi(x, \bar{\mathbf{a}}) : \varphi \in \mathbb{L}(\tau_T)\}$ and

(b) for Γ_1, Γ_2 sets of formulas (possibly with parameters) with one free variable x we let:

$$q_{\Gamma_1, \Gamma_2}^n(x) =: \Big\{(\exists y_0, \ldots, y_{2n-1})[\bigwedge_{\ell < 2n} \psi(y_\ell) \wedge \bigwedge_{\substack{\ell < n \\ k < m}} (\varphi_k(y_{2\ell}, \bar{a}) \equiv \varphi_k(y_{2\ell+1}, \bar{a}))$$
$$\wedge x = (y_0^{-1} * y_1) * \ldots * (y_{2n-2}^{-1} * y_{2n-1})] :$$
$$(a) \quad \psi(x) \text{ is a finite conjunction with}$$
no repetitions of members of $\Gamma_1 \cup p_*(x)$
$$(b) \quad m < \omega \text{ and } \varphi_k(x, \bar{a}) \in \Gamma_2 \text{ is a formula for } k < m$$
$$(\text{ so if } \Gamma_2 = \Gamma_{\bar{\mathbf{a}}}, \text{ we may replace } \bar{a} \text{ by } \bar{\mathbf{a}})\Big\}$$

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[1] recall that this means: is of the form $\cup\{p_i(\mathfrak{C}) : i < i_*\}$ where for some $A \subset \mathfrak{C}$ of cardinality $< \bar{\kappa}$, each p_i is a type over A.

(c) if $\Gamma_1 \subseteq p_*$ we may omit it. If $\Gamma_2 = \Gamma_{\bar{\mathbf{a}}}$ and $\Gamma_1 \subseteq p_*$ we may write $q^n_{\bar{\mathbf{a}}}$ instead of $q^n_{\Gamma_{\bar{\mathbf{a}}}}$.

Remark 1.4. We have used $x = w(y_0^{-1}y_1, y_2^{-1}y_3, \ldots, y_{2n-2}^{-1}y_{2n-1})$ with $w(x_0, \ldots, x_{n-1})$ the "word" $x_0 \ldots x_{n-1}$; but we may replace $x_0 x_1 \ldots x_{n-1}$ by any group word $w(x_0, \ldots, x_{n-1})$ and then may write $q^w_{\Gamma_1, \Gamma_2}, q^w_\Gamma(x), q^w_{\bar{\mathbf{a}}}(x)$ instead of $q^n_{\Gamma_1, \Gamma_2}, q^n_{\Gamma_2}, q^n_{\bar{\mathbf{a}}}(x)$ respectively. Of course, the subgroup of G generated by $q^1_{\bar{\mathbf{a}}}(\mathfrak{C})$ includes $q^w_{\bar{\mathbf{a}}}(\mathfrak{C})$ for any non-trivial group word w.

Observation 1.5. For $n < \omega$ and $\bar{\mathbf{a}} \in {}^\alpha\mathfrak{C}$ as above

(a) $q^n_{\bar{\mathbf{a}}}(x)$ is a 1-type

(b) $q^n_{\bar{\mathbf{a}}}(\mathfrak{C}) = \{c_0 \ldots c_{n-1}$: there are $d_\ell \in p_*(\mathfrak{C})$ for $\ell < 2n$ such that $c_\ell = d_{2\ell}^{-1} d_{2\ell+1}$ and $d_{2\ell}, d_{2\ell+1}$ realize the same type over $\bar{\mathbf{a}}$ for every $\ell < n\}$

(c) $|q^n_{\bar{\mathbf{a}}}|$ is equal to $|T| + \aleph_0 + |\ell g(\bar{\mathbf{a}})| + |p_*(x)|$

(d) $q^n_{\bar{\mathbf{a}}} = \cup\{q^n_{\bar{\mathbf{a}}\restriction u} : u \subseteq \ell g(\bar{\mathbf{a}})$ finite$\}$

(e) $q^n_\Gamma = \cup\{q^n_{\Gamma_1} : \Gamma_1 \subseteq \Gamma$ finite$\}$.

Proof. Straightforward. $\square_{1.5}$

Observation 1.6. 1) $e_G \in q^n_\Gamma(\mathfrak{C}) \subseteq q^{n+1}_\Gamma(\mathfrak{C})$.
2) $q^n_\Gamma(\mathfrak{C})$ is closed under $(-)^{-1}$ and e_G belongs to it so if $a \in q^k_\Gamma(\mathfrak{C})$ <u>then</u> $a^{-1} \in q^k_\Gamma(\mathfrak{C})$.
3) If $a_\ell \in q^{k(\ell)}_\Gamma(\mathfrak{C})$ for $\ell = 1, 2$ <u>then</u> $a_1 * a_2 \in q^{k(1)+k(2)}_\Gamma(\mathfrak{C})$.
4) $\cup\{q^n_\Gamma(\mathfrak{C}) : n < \omega\}$ is a subgroup of G.
5) We have $q^m_{\bar{\mathbf{a}}}(\mathfrak{C}) \supseteq q^n_{\bar{\mathbf{b}}}(\mathfrak{C})$ when $n \le m < \omega$ and $\mathrm{Rang}(\bar{\mathbf{a}}) \subseteq \mathrm{Rang}(\bar{\mathbf{b}})$.

Proof. Easy. $\square_{1.6}$

Observation 1.7. 1) If $\bar{\mathbf{a}}_\ell \in {}^{\alpha(\ell)}\mathfrak{C}$ for $\ell = 1, 2$ and $\mathrm{Rang}(\bar{\mathbf{a}}_1) \subseteq \mathrm{Rang}(\bar{\mathbf{a}}_2)$ <u>then</u> $q^n_{\bar{\mathbf{a}}_2} \vdash q^n_{\bar{\mathbf{a}}_1}$.
2) Assume $n < \omega$ and $H = p(\mathfrak{C})$ is a subgroup of G of bounded index (i.e. $(G : H) < \bar{\kappa}$) and $\alpha < \bar{\kappa}, \bar{\mathbf{a}} \in {}^\alpha\mathfrak{C}$ has representatives from each left H-coset and $\mathrm{Rang}(\bar{\mathbf{a}})$ include A_*. If $p(x)$ is a 1-type $\supseteq p_*(x)$ which is over $\mathrm{Rang}(\bar{\mathbf{a}}) = M \prec \mathfrak{C}$ and $p(\mathfrak{C}) = H$ <u>then</u> $q^n_{\bar{\mathbf{a}}}(x) \vdash p(x)$.
3) In part (1) if $\mathbf{P}$ is a set of $(< \bar{\kappa})$ 1-types $\supseteq p_*(x)$ which are over $\mathrm{Rang}(\bar{\mathbf{a}}) = M \prec \mathfrak{C}$ and $\mathbf{P}(\mathfrak{C}) = H$ <u>then</u> $q^n_{\bar{\mathbf{a}}}(\mathfrak{C}) \subseteq H$.

4) If $\bar{a} \in {}^\alpha\mathfrak{C}$ so $\alpha < \bar{\kappa}$ <u>then</u> we have $\bigcup_{k<\omega} q_{\bar{a}}^k(\mathfrak{C})$ is a union-type definable subgroup of G of bounded index, in fact of index $\leq 2^{|T|+|\alpha|}$.

5) If H is union-type-definable over $\bar{a}$ where $\bar{a} \in {}^\alpha\mathfrak{C}, \alpha = \ell g(\bar{y}) < \bar{\kappa}$ and H is a subgroup of G with bounded index, <u>then</u> $(G : H) \leq 2^{|T|+|\alpha|}$.

Claim 1.8. *Any subgroup of G which is union-type-definable (so using all together $< \bar{\kappa}$ parameters) and is of bounded index (in G, bounded means $< \bar{\kappa}$) contains a subgroup of the form $\bigcup_n q_{\bar{a}}^n(\mathfrak{C})$ for some $\bar{a} \in {}^{\bar{\kappa}>}\mathfrak{C}$ such that $A_* \subseteq \mathrm{Rang}(\bar{a})$.*

Proof. Let $\langle a_{2,i} : i < i(*) \rangle$ be a set of representatives of the left cosets of H in G. Let $\bar{a}_1$ be such that $H = \bigcup_i p_i(\mathfrak{C}), p_i(x)$ a type over $\bar{a}_1$ and let $\bar{a}_2 = \bar{a}_1 \char`\^ \langle a_{2,i} : i < i(*) \rangle$. Clearly if $a, b \in G$ realize the same type over $\bar{a}_2$ then $ab^{-1} \in H$.

By applying $1.6 + 1.7(1)$ we are done. $\qquad\qquad\qquad\qquad \square_{1.8}$

Main Claim 1.9. *Assume T is dependent and G is Abelian. If $k(1) + 2 < k(2) < \omega$ and $\alpha < \bar{\kappa}$, <u>then</u> there are $\lambda < \bar{\kappa}$ and $\bar{a}_\varepsilon \in {}^\alpha\mathfrak{C}$ for $\varepsilon < \lambda$ such that for every $\bar{a} \in {}^\alpha\mathfrak{C}$ satisfying $A_* \subseteq \mathrm{Rang}(\bar{a})$ we have $\cap\{q_{\bar{a}_\varepsilon}^{k(1)}(\mathfrak{C}) : \varepsilon < \lambda\} \subseteq q_{\bar{a}}^{k(2)}(\mathfrak{C})$, i.e. $\bigcup_{\varepsilon<\lambda} q_{\bar{a}_\varepsilon}^{k(1)} \vdash q_{\bar{a}}^{k(2)}$.*

Remark 1.10. 1) We may choose λ and $\bar{a}_\varepsilon$ (for $\varepsilon < \lambda$) independently of $k(1), k(2)$.

2) The proof says that $\lambda = 2^{2^{|\alpha|+|T|+|A_*|}}$ is enough.

Question 1.11. Is G abelian or the subgroup abelian necessary?

Proof. <u>Stage A</u>: Let $\lambda := (2^{2^{|\alpha|+|T|+|A_*|}})$, and assume that the desired conclusion fails for λ.

<u>Stage B</u>: As this fails, we can find $\langle \bar{a}_\varepsilon, c_\varepsilon : \varepsilon < \lambda^+ \rangle$ such that

   (a) $\bar{a}_\varepsilon \in {}^\alpha\mathfrak{C}$ and $A_* \subseteq \mathrm{Rang}(\bar{a}_\varepsilon)$

   (b) $c_\varepsilon \in \bigcap\{q_{\bar{a}_\zeta}^{k(1)}(\mathfrak{C}) : \zeta < \varepsilon\} \subseteq G$ equivalently $c_\varepsilon \in q_{\bar{a}_\zeta}^{k(1)}(\mathfrak{C})$ for $\zeta < \varepsilon$

   (c) $c_\varepsilon \notin q_{\bar{a}_\varepsilon}^{k(2)}(\mathfrak{C})$.

[Why? Choose $(\bar{a}_\varepsilon, c_\varepsilon)$ by induction on $\varepsilon < \lambda^+$.]

<u>Stage C</u>: Without loss of generality $\langle (\bar{a}_\varepsilon, c_\varepsilon) : \varepsilon < \lambda^+ \rangle$ is an indiscernible sequence over A_*.

[Why? We first use Erdös-Rado theorem to repalce clause (c) by

$(c)'$ $c_\varepsilon \in (\bigwedge_{l<m} \varphi_l(\mathfrak{C}, \bar{\mathbf{a}}_\varepsilon))$ for some m and $\varphi_l(x, \bar{\mathbf{a}}_\varepsilon) \in q_{\bar{\mathbf{a}}_\varepsilon}^{k(2)}$.

and then use Ramsey theorem (and compactness, i.e. saturation of $\mathfrak{C}$).]

<u>Stage D</u>: For $\varepsilon < \lambda^+$ let $c'_\varepsilon = c_{2\varepsilon}^{-1} * c_{2\varepsilon+1}$ and let $\bar{\mathbf{a}}'_\varepsilon = \bar{\mathbf{a}}_{2\varepsilon}$ and for any finite $u \subseteq \lambda^+$ let $c'_u = c'_{\varepsilon_0} * \ldots * c'_{\varepsilon_{m-1}}$ when $\varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_{m-1}$ list u.

<u>Stage E</u>: If $u \subseteq \lambda^+$ is finite and $\varepsilon \in \lambda^+ \backslash u$ then c'_u realizes $q_{\bar{\mathbf{a}}'_\varepsilon}^1$.
[Why? As G is abelian, using additive notation $c'_u = \Sigma\{c_{2\zeta+1} : \zeta \in u\} - \Sigma\{c_{2\zeta} : \zeta \in u\}$. By the indiscernibility the sequences $\langle c_{2\zeta+1} : \zeta \in u\rangle, \langle c_{2\zeta} : \zeta \in u\rangle$ realize the same type over $\bar{\mathbf{a}}'_\varepsilon = \bar{\mathbf{a}}_{2\varepsilon}$ as $\varepsilon \notin u$, (noting the specific sequences we use) hence $\Sigma\{c_{2\zeta+1} : \zeta \in u\}$ and $\Sigma\{c_{2\zeta} : \zeta \in u\}$ are members of G realizing the same type over $\bar{\mathbf{a}}'_\varepsilon = \bar{\mathbf{a}}_{2\varepsilon}$ hence $c'_u \in q_{\bar{\mathbf{a}}_\varepsilon}^1(\mathfrak{C})$. So the conclusion is clear.]

<u>Stage F</u>: There is a finite sequence $\bar{\bar{\vartheta}} = \langle \vartheta_\ell(x, \bar{\mathbf{y}}) : \ell < \ell(*)\rangle$ of formulas, $\ell g(\bar{\mathbf{y}}) = \alpha$ and $\bar{\varphi} = \langle \varphi_i(x, \bar{b}) : i < i(*)\rangle, \varphi_i(x, \bar{b}) \in p_*(x)$ for $i < i(*) < \omega, \bar{b} \subseteq A_*$ such that: for $\varepsilon < \lambda^+$, for no $d_0, \ldots, d_{2k(2)-1} \in \bigcap_{i<i(*)} \varphi_i(\mathfrak{C}, \bar{b}) \subseteq G$ do we have $c_\varepsilon = \sum_{\ell<k(2)}((-d_{2\ell}) + d_{2\ell+1})$ and $\ell < l(*) \wedge k < k(2) \Rightarrow \mathfrak{C} \models \vartheta_\ell(d_{2k}, \bar{\mathbf{a}}_\varepsilon) \equiv \vartheta_\ell(d_{2k+1}, \bar{\mathbf{a}}_\varepsilon)$, in other words $c_\varepsilon \notin \vartheta^*(\mathfrak{C}, \bar{\mathbf{a}}'_\varepsilon)$ where $\vartheta^*(x, \mathbf{a}'_\varepsilon) \in q_{\{\vartheta_\ell(x, \bar{\mathbf{a}}_\varepsilon):\ell<\ell(*)\}}^{k(2)}(\mathfrak{C})$ says the above, i.e. is $\neg(\exists z_0, \ldots, z_{2k(c)-1}) [(\bigwedge_{k<2k(2)} \bigwedge_{i<i(*)} \varphi_i (z_k, \bar{b})) \wedge \bigwedge_{\ell<k(2)}(\vartheta_\ell(z_{2k}, \bar{\mathbf{a}}'_\varepsilon) \equiv \vartheta_\ell(z_{2k+1}, \bar{\mathbf{a}}'_\varepsilon)]$.
[Why? Because $c_\varepsilon \notin q_{\bar{\mathbf{a}}_\varepsilon}^{k(2)}(\mathfrak{C})$, see (c) of stage (B). By compactness, we can choose finitely many formulas; by the indiscernibility their choice does not depend on ε, see stage (C).]

<u>Stage G</u>: If $u \subseteq \lambda^+$ is finite and $\varepsilon \in u$ <u>then</u> $c'_u \notin q_{\{\vartheta_\ell(x, \bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}^{k(*)}(\mathfrak{C})$; moreover, $\bar{c}'_u \notin \vartheta^*(\mathfrak{C}, \bar{\mathbf{a}}')$ for any $k(*) \leq k(2) - 1 - k(1)$.
[Why? Because

(a) $c'_u = \Sigma\{c'_\zeta : \zeta \in u\backslash\{\varepsilon\}\} - c_{2\varepsilon} + c_{2\varepsilon+1}$

hence

(b) $c_{2\varepsilon} = \Sigma\{c'_\zeta : \zeta \in u\backslash\{\varepsilon\}\} + c_{2\varepsilon+1} - c'_u$

(c) $\Sigma\{c'_\zeta : \zeta \in u\backslash\{\varepsilon\}\} \in q_{\{\vartheta_\ell(y, \bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}^1(\mathfrak{C})$.

[Why? As in stage E.]

(d) $c_{2\varepsilon+1} \in q_{\bar{\mathbf{a}}_{2\varepsilon}}^{k(1)}(\mathfrak{C}) \subseteq q_{\{\vartheta_\ell(x, \bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}^{k(1)}(\mathfrak{C})$.

[Why? First, the membership by clause (b) of Stage B. Second, the inclusion by monotonicity as $\bar{\mathbf{a}}'_\varepsilon = \bar{\mathbf{a}}_{2\varepsilon}$.]

(e) $c_{2\varepsilon} \notin q^{k(2)}_{\{\vartheta_\ell(x,\bar{\mathbf{a}}_{2\varepsilon}):\ell<\ell(*)\}}(\mathfrak{C}) = q^{k(2)}_{\{\vartheta_\ell(x,\bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}(\mathfrak{C})$; moreover, $c_{2\varepsilon} \notin \vartheta^*(\mathfrak{C},\bar{\mathbf{a}}_{2\varepsilon}) = \vartheta^*(\mathfrak{C},\bar{\mathbf{a}}'_\varepsilon)$.

[Why? By Stage F.]

Now consider the right side in clause (b).

The first summand is from $q^1_{\{\vartheta_\ell(x,\bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}(\mathfrak{C})$ by clause (c).

The second summand is from $q^{k(1)}_{\{\vartheta_\ell(x,\bar{a}'_\varepsilon):\ell<\ell(*)\}}(\mathfrak{C})$ by clause (d).

For awhile assume $c'_u \in \vartheta^*(\mathfrak{C},\bar{\mathbf{a}}'_\varepsilon)$ so $(-c'_u) \in q^{k(*)}_{\{\vartheta_\ell(x,\bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}(\mathfrak{C})$ then by 1.6(3) and the previous two sentence we deduce $\Sigma\{c'_\zeta : \zeta \in u\backslash\{\varepsilon\}\} + c_{2\varepsilon+1} + (-c'_u) \in q^{k(1)+1+k(*)}_{\{\vartheta_\ell(x,\bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}(\mathfrak{C})$ which by clause (b) means $c_{2\varepsilon} \in q^{k(1)+1+k(*)}_{\{\vartheta_\ell(x,\bar{a}'_\varepsilon):\ell<\ell(*)\}}(\mathfrak{C})$. Now by the assumption of the stage we have $k(2) \geq k(1) + 1 + k(*)$ but $q^n_\Gamma(\mathfrak{C})$ does $\subseteq$-increase with n hence from the previous sentence we get a contradiction to clause (e). So necessarily $c'_u \notin \vartheta^*(\mathfrak{C},\bar{\mathbf{a}}_\varepsilon)$, the desired conclusion of this stage.]

Stage H: Let $k(*) := k(2) - 1 - k(1)$.

By Stages (E)+(G) we have: for finite $u \subseteq \lambda^+, c'_u$ realizes $q^{k(*)}_{\{\vartheta_\ell(x,\bar{\mathbf{a}}'_\varepsilon):\ell<\ell(*)\}}$ iff $\varepsilon \in u$ iff $\mathfrak{C} \models \vartheta^*(c'_u,\bar{\mathbf{a}}'_\varepsilon)$. So $\vartheta^*(x,\bar{\mathbf{y}}) \in \mathbb{L}(\tau_T)$ has the independence property, contradiction. $\square_{1.9}$

Theorem 1.12. *Assume T is dependent and G is Abelian. There is $\mathbf{P} \subseteq \mathbf{S}(A_*)$ such that:*

(a) $\mathbf{P}(\mathfrak{C}) := \cup\{p(\mathfrak{C}) : p \in \mathbf{P}\}$ *is a subgroup of G of bounded index, i.e. $(G : \mathbf{P}(\mathfrak{C})) < \bar{\kappa}$*

(b) $(G : \mathbf{P}(G)) \leq 2^{|A_*|+|T|+\aleph_0}$

(c) $\mathbf{P}(\mathfrak{C})$ *is minimal, i.e., if $\alpha < \bar{\kappa}, \bar{\mathbf{a}} \in {}^\alpha\mathfrak{C}, \psi(x,\bar{\mathbf{y}}) \in \mathbb{L}_{\infty,\kappa}(\tau_T)$ and $\psi(\mathfrak{C},\bar{\mathbf{a}})$ is a subgroup of G <u>then</u> $\psi(\mathfrak{C},\bar{\mathbf{a}})$ is of bounded index iff $\psi(\mathfrak{C},\bar{\mathbf{a}}) \supseteq \mathbf{P}(\mathfrak{C})$.*

Proof. Note:

$\circledast_1$ there is $\mathbf{P} \subseteq \mathbf{S}(A_*)$ such that $\mathbf{P}(\mathfrak{C})$ is a subgroup of G of bounded index.

[Why? Use $\mathbf{P} = \{p \in \mathbf{S}(A_*) : p_* \subseteq p\}$.]

$\circledast_2$ the family of $\mathbf{P}$'s as in $\circledast_1$ is closed under intersection.

[Why? As $\bar{\kappa}$ is strongly inaccessible or just $\text{cf}(\bar{\kappa}) > \beth_3(|A_*| + |T| + \aleph_1) \geq 2^{|\{\mathbf{P}:\mathbf{P} \subseteq \mathbf{S}(A_*)\}|}$ hence the product of $\leq 2^{2^{|A_*|+|T|+\aleph_0}}$ cardinals $< \bar{\kappa}$ is $< \bar{\kappa}$.]

$\circledast_3$ Let $\mathbf{P}^* = \cap\{\mathbf{P} \subseteq \mathbf{S}(A_*) : \mathbf{P}(\mathfrak{C})$ is a subgroup of a bounded index$\}$.

We shall show that $\mathbf{P}^*$ is as required.
    So by $\circledast_2 + \circledast_3$

$\circledast_4$ clause (a) holds, i.e. $\mathbf{P}^* \subseteq \mathbf{S}(A_*)$ and $\mathbf{P}^*(\mathfrak{C})$ is a subgroup of G of bounded index

$\circledast_5$ clause (b), i.e. $(G : \mathbf{P}^*(\mathfrak{C})) \leq 2^{|A_*|+|T|+\aleph_0}$.

[Why? Follows from 1.7(5).]
    Recall

$\circledast_6$ <u>if</u> $1 \leq k < \omega$ and $\mathbf{P}$ is a set one-types (of cardinality $< \bar{\kappa}$) and $\mathbf{P}(\mathfrak{C})$ a subgroup of G of bounded index <u>then</u> $q_{\bar{\mathbf{a}}}^k(\mathfrak{C}) \subseteq \mathbf{P}(\mathfrak{C})$ for some $\bar{\mathbf{a}} \in {}^\alpha\mathfrak{C}$ for some $\alpha < \bar{\kappa}$.

[Why? See above 1.8.]
    Fix $\alpha < \bar{\kappa}$ and we shall prove that:

$\boxtimes$ if $\bar{\mathbf{a}}^* \in {}^\alpha\mathfrak{C}, A_* \subseteq \text{Rang}(\bar{\mathbf{a}}*)$ and $\mathbf{P}_{\bar{\mathbf{a}}^*} := \{p \in \mathbf{S}(\bar{\mathbf{a}}^*) : p$ extend $q_{\bar{\mathbf{a}}^*}^n$ for some $n\}$, so $\mathbf{P}_{\mathbf{a}^*}(\mathfrak{C}) = \bigcup_{n<\omega} q_{\bar{\mathbf{a}}^*}^n(\mathfrak{C})$ is a subgroup of G of bounded index <u>then</u> $\mathbf{P}^*(\mathfrak{C}) \subseteq \mathbf{P}_{\mathbf{a}^*}(\mathfrak{C})$.

This clearly suffices by $\circledast_4, \circledast_5$ and $\circledast_6$, i.e. $\boxtimes$ means that clause (c) of the conclusion holds by $\circledast_6$.
    Now comes the real point:

$\circledast_7$ for $k < \omega$ we can choose $\lambda_k < \bar{\kappa}$ and $\langle \bar{\mathbf{a}}_\varepsilon^k : \varepsilon < \lambda_k \rangle$ such that:
    (a) $\bar{\mathbf{a}}_\varepsilon^k \in {}^\alpha\mathfrak{C}$ for $\varepsilon < \lambda_k$
    (b) for every $\bar{\mathbf{a}} \in {}^\alpha\mathfrak{C}$ we have $(\bigcup_{\varepsilon<\lambda_k} q_{\bar{\mathbf{a}}_\varepsilon^k}^k) \vdash q_{\bar{\mathbf{a}}}^{k+3}$.

[Why is there such a sequence? By the main claim 1.9 so actually $\lambda_k = (2^{2^{|T|+|A_*|+|\ell g(\bar{a}^*)|}})$ suffice by the proof of 1.9.]
    Define

$\odot_1$ $X_k := \cap\{q_{\bar{\mathbf{a}}_\varepsilon^k}^k(\mathfrak{C}) : \varepsilon < \lambda_k\}$

$\odot_2$ $Y_k := \cap\{q_{\bar{\mathbf{a}}}^k(\mathfrak{C}) : \bar{\mathbf{a}} \in {}^\alpha(\mathfrak{C})\}$.

Then:

$\circledast_8$ $Y_k \subseteq X_k$.

[Why? As $\mathbf{a}_k^\varepsilon \in {}^\alpha\mathfrak{C}$.]

$\circledast_9$ $X_k \subseteq Y_{k+3}$.

[Why? By $\circledast_7$.]

$\circledast_{10}$ $\bigcup_{k<\omega} X_k = \bigcup_{k<\omega} Y_k$.

[Why? By $\circledast_8 + \circledast_9$.]

$\circledast_{11}$ $Y_k = \mathbf{P}_k(\mathfrak{C})$ for some $\mathbf{P}_k \subseteq \mathbf{S}(A_*)$.

[Why? As any automorphism F of $\mathfrak{C}$ over A_* maps Y_k onto itself as it maps $q_{\bar{\mathbf{a}}}^k(\mathfrak{C})$ to $q_{F(\bar{\mathbf{a}})}^k(\mathfrak{C})$.]

$\circledast_{12}$ $\bigcup_{k<\omega} Y_k$ is $(\bigcup_{k<\omega} \mathbf{P}_k)(\mathfrak{C})$.

[Why? By $\circledast_{11}$.]

$\circledast_{13}$ $\bigcup_{k<\omega} X_k$ is a subgroup of G of bounded index.

[Why? By 1.6(3) + 1.7(4).]
   Recall $\mathbf{P}_{\bar{\mathbf{a}}_*}$ is from $\boxtimes$ above.

$\circledast_{14}$ $Y_k \subseteq \mathbf{P}_{\bar{\mathbf{a}}^*}(\mathfrak{C})$.

[Why? By the definition of Y_k and $\circledast_7$ we have $Y_k \subseteq q_{\bar{\mathbf{a}}^*}^k(\mathfrak{C}) \subseteq \mathbf{P}_{\bar{\mathbf{a}}^*}(\mathfrak{C})$.]
   Let us sum up and prove $\boxtimes$ thus finishing: $\mathbf{P}_{\bar{\mathbf{a}}^*}(\mathfrak{C})$ include $\bigcup_{k<\omega} Y_k$ by $\circledast_{14}$ and $\bigcup_{k<\omega} Y_k$ is equal to $\bigcup_{k<\omega} X_k$ by $\circledast_{10}$, and is equal to $(\bigcup_{k<\omega} \mathbf{P}_k)(\mathfrak{C})$ by $\circledast_{12}$. Hence by $\circledast_{13}$ we know that $(\bigcup_{k<\omega} \mathbf{P}_k)(\mathfrak{C})$ is a subgroup of G of bounded index, hence by the definition of $\mathbf{P}^*$ in $\circledast_3$ we know that $\bigcup_{k<\omega} \mathbf{P}_k \subseteq \mathbf{P}^*$. Hence $(\bigcup_{k<\omega} \mathbf{P}_k)(\mathfrak{C})$ includes $\mathbf{P}^*(\mathfrak{C})$. So $\mathbf{P}_{\bar{\mathbf{a}}^*}(\mathfrak{C}) \supseteq \bigcup_{k<\omega} Y_k = (\bigcup_{k<\omega} \mathbf{P}_k)(\mathfrak{C}) \supseteq \mathbf{P}^*(\mathfrak{C})$ as required in $\boxtimes$.
   So we have proved $\boxtimes$ hence has proved the conclusion. $\square_{1.12}$

2. On 2-dependent T

We try to see what, from 2-dependence of T, we can deduce on definable groups. On n-dependent T see [18, §5 (H)].

Hypothesis 2.1.

(a) T be first order complete, $\mathfrak{C} = \mathfrak{C}_T$

(b) G is a type definable group over A_*, i.e. for some 1-type p_*, G has a set of elements $p_*(\mathfrak{C})$ and the functions $(x, y) \mapsto xy, x \mapsto x^{-1}, e_G$ which are definable over $A_* = \text{Dom}(p_*)$; this is irrelevant for 2.4 - 2.10.

Definition 2.2. For a set $B \subseteq \mathfrak{C}$ let

(a) $\mathbf{R}_B = \{q : q = q(x)$ is a 1-type over B and G_q a subgroup of G of index $< \bar{\kappa}\}$ where

(b) $G_q = G[q] = \{a \in G : a$ realizes $q\} = (p_* \cup q)(\mathfrak{C})$

(c) $q_B = q[B] = \cup\{q : q \in \mathbf{R}_B\}$ and $G_B = q_B(\mathfrak{C}) \cap G$ and, of course

(d) $\mathbf{R}_{\bar{b}} = \mathbf{R}_{\text{Rang}(\bar{b})}$ and $q_{\bar{b}} = q_{\text{Rang}(\bar{b})}$ and $G_{\bar{b}} = G_{\text{Rang}(\bar{b})}$.

Observation 2.3. 1) $q_B = \cup\{q_{\bar{b}} : \bar{b} \in {}^{\omega \geq}B\} = \cup\{q \in \mathbf{R}_B : q$ countable$\}$.
2) q_B is $\subseteq$-maximal in $\mathbf{R}_B$.
3) $G_B = \cap\{G_q : q \in \mathbf{R}_B\} = \cap\{G_q : q \in \mathbf{R}_B$ is countable$\}$ and is $\subseteq$-minimal in $\{G_q : q \in \mathbf{R}_B\}$.
4) If $q \in \mathbf{R}_B$ and $q' \subseteq q$ is countable then we can find a sequence $\langle \psi_n(x, \bar{a}) : n < \omega \rangle$ of finite conjunctions of members of q such that:

(α) $p_*(x) \cup \{\psi_n(x, \bar{a}) : n < \omega\} \vdash q'(x)$

(β) $\psi_{n+1}(x, \bar{a}) \vdash \psi_n(x, \bar{a})$

(γ) $\bar{a} \in {}^{\omega \geq}B$, (for notational simplicity we allow it to be infinite)

(δ) $p_*(x) \cup p_*(y) \cup \{\psi_{n+1}(x, \bar{a}), \psi_{n+1}(y, \bar{a})\} \vdash \psi_n(xy, \bar{a}) \wedge \psi_n(x^{-1}, \bar{a}) \wedge \psi_n(xy^{-1}, \bar{a})$.

5) In part (4), if we allow $\psi_n(x, \bar{a}_n)$ to be a finite conjunction of members of $q \cup p_*$ (e.g. if $p_* \subseteq q$) <u>then</u> we can omit p_* in clauses $(\alpha), (\delta)$ so $\bigcap_{n < \omega} \psi_n(\mathfrak{C}, \bar{a})$ is a group.

6) There is a countable $p'(x) \subseteq p_*(x)$ such that $p'(\mathfrak{C})$ is a group under the definable functions $(x, y) \mapsto xy, x \mapsto x^{-1}, e_G$, moreover there is a sequence $\langle \psi_n(x, \bar{a}) : n < \omega \rangle$ of finite conjunctions of members of $p_*(x)$ such that:

(α) $\bigwedge_{\ell \leq 2} \psi_0(x_\ell, \bar{a}) \vdash (x_0 x_1) x_2 = x_0 (x_1 x_2) \wedge x_0 e_G = e_G x_0 = x_0 \wedge x_0 x_0^{-1} = x_0^{-1} x_0 = e_G$, (implicitly this means that $x_0 x_1, (x_0 x_1) x_2, x_1 x_2, x_0$ (x_1, x_2) and x_0^{-1} are well defined)

(β) $\psi_{n+1}(x, \bar{a}) \vdash \psi_n(x)$

(γ) $\psi_{n+1}(x, \bar{a}) \wedge \psi_{n+1}(y, \bar{a}) \vdash \psi_n(xy, \bar{a}) \wedge \psi_n(x^{-1}, \bar{a}) \wedge \psi_n(xy^{-1}, \bar{a})$.

Proof. Obvious and as in [19]. $\qquad\qquad\qquad\qquad\qquad\qquad \square_{2.3}$

Definition 2.4. 1) We say T is 2-independent <u>when</u> we can find an independent sequence of formulas of the form $\langle \varphi(\bar{x}, \bar{b}_n, \bar{c}_m) : n, m < \omega \rangle$ in $\mathfrak{C} = \mathfrak{C}_T$ or just in some model of T.
2) T is 1/2-dependent means the negation of T is 2-independent (see [18, §5 (H)]).
3) We say $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{n-1})$ is n-independent (for T) <u>when</u> in $\mathfrak{C}_T$ we can, for each $\lambda < \bar{\kappa}$, find $\bar{a}_\alpha^\ell \in {}^{\ell g(\bar{y}_\ell)}(\mathfrak{C}_T)$ for $\alpha < \lambda, \ell < n$ such that the sequence $\langle \varphi(\bar{x}, \bar{a}_{\eta(0)}^0, \ldots, \bar{a}_{\eta(n-1)}^{n-1}) : \eta \in {}^n\lambda \rangle$ is an independent sequence of formulas.
4) T is n-independent <u>when</u> some formula $\varphi(\bar{x}, \bar{y}_0, \bar{y}_1, \ldots, \bar{y}_{n-1})$ is n-independent.
5) T is n-dependent <u>when</u> it is not n-independent.

Remark 2.5. 1) In fact T is n-independent iff some $\varphi(x, \bar{y}_0, \ldots, \bar{y}_{n-1})$ is n-independent (for T). We shall write it down in 2.6 below.
2) So $1/n$-independent means independent.

Claim 2.6. 1) *For a complete first order theory T, there is a 2-independent formula $\varphi(x, \bar{y}, \bar{z})$ in T <u>iff</u> T is 2-dependent, i.e. for some $m \geq 1$ there is a 2-independent formula $\varphi(\bar{x}_m, \bar{y}, \bar{z})$ with $\bar{x}_m = \langle x_\ell : \ell < m \rangle$ <u>iff</u> this holds for every $m \geq 1$.*
2) *Similarly for k-independent.*
3) *Moreover, if the formula $\varphi(\bar{x}_m, \bar{y}_0, \ldots, \bar{y}_{k-1})$ is k-independent, <u>then</u> for some $n < m$ and $\langle b_i : i < m, i \neq m \rangle$ the formula*

$$\varphi = \varphi(b_0, \ldots, b_{n-1}, x, b_{n+1}, \ldots, b_{m-1}, \bar{y}_0, \ldots, \bar{y}_{k-1})$$

is k-independent.

Proof. 1) By (2).

2) Easily the third statement implies the second, obviously the first statement implies the third as trivially as we can add dummy variables. For the "second implies the first" direction we prove it by induction on m; so assume $k < \omega, \bar{x}_m = \langle x_\ell : \ell < m \rangle$ and the formula $\varphi(\bar{x}_m, \bar{y}_0, \ldots, \bar{y}_{k-1})$ is k-independent. Let $n_\ell = \ell g(\bar{y}_\ell), n = \sum_{\ell=1}^{k} n_\ell$, of course, $m > 1$. This means that in $\mathfrak{C}_T$ we have $\bar{a}_{\ell,i} \in {}^{\ell g(\bar{y}_\ell)}\mathfrak{C}$ for $\ell < k, i < \omega$ such that the sequence $\langle \varphi(\bar{x}_m, \bar{a}_{0,\eta(0)}, \ldots, \bar{a}_{k-1,\eta(k-1)}) : \eta \in {}^k\omega \rangle$ of formulas is independent. Let $\mathrm{inc}_n(\omega) = \{\eta \in {}^n\omega : \eta \text{ increasing}\}$, similarly $\mathrm{inc}_{<n}(\omega)$.

So for any $R \subseteq \mathrm{inc}_k(\omega)$ there is $\bar{b}_R \in {}^m\mathfrak{C}$ such that

$$\mathfrak{C} \models \varphi[\bar{b}_R, \bar{a}_{0,\eta(0)}, \ldots, \bar{a}_{k-1,\eta(k-1)}]^{\mathrm{if}(\eta \in R)}$$

for $\eta \in \mathrm{inc}_n(\omega)$.

As we can add dummy variables without loss of generality $\bar{a}_{\ell,i} = \bar{a}_i$, i.e. it does not depend on ℓ and also $\langle \bar{a}_i : i < \omega \rangle$ is an indiscernible sequence.

Let $R_* \subseteq \mathrm{inc}_k(\omega)$ be random enough[2].

For $\eta \in \mathrm{inc}_{<\omega}(\omega)$, let $\bar{a}_\eta = \bar{a}_{\eta(0)} \char94 \bar{a}_{\eta(1)} \char94 \ldots \char94 \bar{a}_{\eta(k-1)}$. We say $u_1, u_2 \subseteq \omega$ are R_*-similar if $|u_1| = |u_2|$ and the one-to-one order preserving function h from u_1 onto u_1 is an isomorphism from $(u_1, R_* \restriction u_1)$ onto $(u_2, R_* \restriction u_2)$. Let I be the model $(\omega, <, R_*)$.

Without loss of generality (by Nessetril-Rodl theorem see e.g. [4]; on such uses see [6, Ch.III,§1] = [9], in particular [9, 1.26])

$(*)_1$ in $\mathfrak{C}$, the sequence $\langle \bar{a}_t : t \in I \rangle$ is indiscernible above $\bar{b}_{R_*}$ which means

$(*)_2$ if $j < \omega$ and $\eta_\ell \in \mathrm{inc}_j(\omega), u_\ell = \mathrm{Rang}(\eta_\ell)$ for $\ell = 1, 2$ and u_1, u_2 are R_*-similar then $\bar{a}_{\eta_1}, \bar{a}_{\eta_2}$ realizes the same type in $\mathfrak{C}$ over $\bar{b}_{R_*}$.

Let $\bar{b}_{R_*} = \bar{b}_1 \char94 \bar{b}_2$ where $\ell g(\bar{b}_1) < m, \ell g(\bar{b}_2) < m$.

So by $(*)_1$ we have

$(*)_3$ if $\eta \in \mathrm{inc}_k(\omega)$ <u>then</u> the value $\mathrm{tp}(\bar{a}_\eta, \bar{b}_2, \mathfrak{C})$ depends just on truth value of $\eta \in R_*$.

---

[2]which means that e.g. choose a countable $N \prec (\mathcal{H}(\aleph_1), \in)$ and R_* does not belong to any null Borel subset of $\mathcal{P}(\mathrm{inc}_k(\omega))$; the probability space is defined by considering R_* as the random subset and the events "$\eta \in R_*$" for $\eta \in \mathrm{inc}_k(\omega)$ are independent, each of probability $1/2$.

First assume $\{\mathrm{tp}(\bar{a}_\eta, \bar{b}_2, \mathfrak{C}) : \eta \in \mathrm{inc}_k(\omega)\}$ is constant, then letting $\bar{x}' = \bar{x}{\restriction}\ell g(\bar{b}_1)$ the formula $\varphi(\bar{x}', \bar{b}_2, \bar{y}_0, \ldots, \bar{y}_{k-1})$ is k-independent when $\langle \bar{a}_i : i < \omega \rangle$ is indiscernible over $\bar{b}_2$; pedantically the formula $\varphi(\bar{x}', (\bar{x}{\restriction}[\ell g(\bar{b}_2)\hat{}\,\bar{y}_0), \bar{y}_1, \ldots, \bar{y}_{k-1})$ is (the restriction to increasing sequences is not serious, see 2.7(4) below), so by the induction hypothesis we are done.

If $\langle \bar{a}_i : i < \omega \rangle$ is not an indiscernible sequence over $\bar{b}_\eta$, then let n be minimal such that we can choose $i_0 < \ldots i_{n-1} < \omega, j_0 < \ldots < j_{n-1} < \omega$ we have $\mathrm{tp}(\bar{a}_{i_0}\hat{}\,\ldots\hat{}\,\bar{a}_{i_{n-1}}, \bar{b}_2) \neq \mathrm{tp}(\bar{a}_{j_0}\hat{}\,\ldots\hat{}\,\bar{a}_{j_{n-1}}, \bar{b}_2)$. By $(*)_2$ the sets $\{i_\ell : \ell < n\}, \{j_\ell : \ell < n\}$ are not R_*-similar. Without loss of generality there is a unique $v \in [n]^k$ such that $\langle i_\ell : \ell \in v \rangle \in R_* \leftrightarrow \langle j_\ell : \ell \in v \rangle \notin R_*$. Now playing with indiscernible and using $\bar{b}_2, \{\bar{a}_i : i < n, i \in v\}$ as parameters, we can finish.

Second assume that $\{\mathrm{tp}(\bar{a}_\eta, \bar{b}_2, \mathfrak{C}) : \eta \in \mathrm{inc}_k(\omega)\}$ is not constant, by $(*)_1$ equivalently $(*)_2$ this means that there is a formula $\psi(\bar{x}_{\ell g(\bar{b}_1)}, \bar{y}_0, \ldots, \bar{y}_{k-1})$ such that:

$(*)_4$ if $\eta \in \mathrm{inc}_k(\omega)$ then $\mathfrak{C} \models \psi[\bar{b}_2, \bar{a}_{\eta(0)}, \ldots, \bar{a}_{\eta(k-1)}]$ iff $\eta \in R_*$.

This also suffices by $\langle \bar{a}_i : i < \omega \rangle$ being an indiscernible sequence.

3) As in part (2). $\square_{2.6}$

Observation 2.7. 1) T is k-dependent <u>when</u> : for every m, ℓ and $\varphi(\bar{x}_m, \bar{y}) \in \mathbb{L}(\tau_T)$ for infinitely many $n < \omega$ we have $|A| \leq n \Rightarrow |\mathbf{S}^m_{\{\varphi(\bar{x}_m, \bar{y})\}}(A)| < 2^{(n/\ell)^k}$.

2) In fact we can restrict ourselves to $m = 1$.

3) For any k, T is k-independent iff T^{eq} is k-independent.

3A) T is not k-independent <u>when</u> for every $\varphi(x, \bar{y}_0, \ldots, \bar{y}_{k-1}) \in \mathbb{L}(\tau(\mathfrak{C}^{\mathrm{eq}}_T))$, for infinitely many n we have $A \subseteq \mathfrak{C}^{\mathrm{eq}} \wedge |A| \leq kn \Rightarrow |\mathbf{S}_{\{\varphi(x, \bar{y}_0, \ldots, \bar{y}_{k-1})\}}(A)| < 2^{n^k}$.

4) In Definition 2.4(3) we can restrict ourselves to "increasing η", similarly in 2.4(1). In fact, $\varphi(\bar{x}, \bar{y}_0, \ldots, \bar{y}_{k-1})$ is k-independent <u>if</u> for every n there are $\bar{a}_{\ell,m} \in {}^{\ell g(\bar{y}_\ell)}\mathfrak{C}$ for $m < n, \ell < k$ such that $\langle \varphi(\bar{x}, \bar{a}_{0,\eta(0)}, \ldots, \bar{a}_{k-1,\eta(k-1)}) : \eta \in \mathrm{inc}_k(n) \rangle$ is an independent sequence of formulas.

Proof. 1) Straightforward and see [18, §5 (G)].

2) Similarly using 2.6 above.

3) Easy by the definition.

3A) By parts (1),(2).

4) It is enough to prove the second sentence; for every n we first find $\langle \bar{a}_{\ell,m} : m < nk, \ell < k \rangle$ as guaranteed there (for nk). Now let $\bar{a}'_{\ell,m} = \bar{a}_{\ell,\ell n+m}$ so $\langle \bar{a}'_{\ell,m} : m < n, \ell < k \rangle$ are as required in 2.4(3) for $\lambda = n$. By compactness equivalently, by "$\mathfrak{C}$ is $\bar{\kappa}$-saturated" we are done. $\square_{2.7}$

Example 2.8. Let $k \geq 1$, a natural k-independent but $1/(k+1)$-dependent theory, as simple as possible, is the model completion of the following theory (so for $k = 1$ this is a $(1/2)$-dependent, independent T):

   (A) the vocabulary is

   $\qquad P_\ell$ $(\ell < k+1)$, unary predicates

   $\qquad R$, a $(k+1)$-place predicate

   (B) M a model of T <u>iff</u>:

   $\qquad (a)$ $\langle P_\ell^M : \ell \leq k \rangle$ is a partition of $|M|$

   $\qquad (b)$ $R^M \subseteq \{(a_0, \ldots, a_k) : a_\ell \in P_\ell^M$ for $\ell = 0, \ldots, k\}$.

[Note first that clearly the model completion exists and has elimination of quantifiers. Second, the formula $R(x, y_0, \ldots, y_{k-1})$ exemplifies that T_k is k-independent. Third, T is $1/(k+1)$-dependent by 2.7(1) and the elimination of quantifiers.]

Example 2.9. Let T_n^* be the theory with the vocabulary $\{R\}$, R is n-place predicate such that T is saying R is symmetric and irreflexive (i.e. $\bigvee_{i<j<n} x_i = x_j \to \neg R(x_0, \ldots, x_{n-1})$ and $\bigwedge \{R(x_{\varkappa(0)}, \ldots, x_{\varkappa(n-1)}) \equiv R(x_{\pi(0)}, \ldots, x_{\pi(n-1)})$ for permutations $\varkappa, \pi$ of $\{0, \ldots, n-1\}$.

Let T_n be the model completion of n. Then T_n is $(n-1)$-independent but not n-independent (for n-independent use 2.7(1)).

Example 2.10. Any theory T of an infinite Boolean algebras is (independent, moreover) k-independent for every k.
[Why? Let for simplicity $k = 2$. Let $\varphi(x, y, z) := (y \cap z \leq x)$. Now for any n, let $\mathbb{B}$ be a Boolean sub-algebra of $\mathfrak{C}_T$ with pairwise disjoint non-zero elements $\langle a_{i,j} : i, j < n \rangle$ and let $b_i := \cup \{a_{i,j} : j < n\}$ and $c_j := \cup \{a_{i,j} : i < n\}$. Now $\langle \varphi(x, b_i, c_j) : i, j < n \rangle$ are independent because for $u \subseteq n \times n$ the element $a_u := \cup \{a_{i,j} : (i,j) \in u\}$ realizes the type $\{\varphi(x, b_i, c_j)^{\text{if}((i,j) \in u)} : (i,j) \in u\}$. For $n > 2$ we use $\varphi(x, y_0, \ldots, y_{n-1}) = $ "$(y_0 \cap y_1 \cap \ldots \cap y_{n-1}) \leq x$".]

Now comes the property concerning the definable group G which interests us.

Definition 2.11. We say that G has κ-based bounded subgroups <u>when</u>: for every κ-saturated $M \prec \mathfrak{C}$ which include A_* (hence $p_* \subseteq q_M$) and $\bar{b} \in {}^{\omega>}(\mathfrak{C})$ there is $r \in \mathbf{R}_{M \cup \bar{b}}$ of cardinality $< \kappa$ such that $q_{M \cup \bar{b}}$ is equivalent to $q_M \cup r$ (equivalently $q_M \cup r \vdash q_{M \cup \bar{b}}$).

The main result here is

Theorem 2.12. *If T is $1/2$-dependent and $\kappa = \beth_2(|T| + |p_*|)^+$ or just $\kappa = ((||T|| + |p_*|)^{\aleph_0})^+ + \beth_2^+$ (and G is as in 2.1) <u>then</u> G has κ-based bounded subgroups.*

Proof. Assume not and let $\theta = \aleph_0$. Let M and $\bar{b} \in {}^{\omega>}\mathfrak{C}$ form a counter-example. Then we choose the triple $(r_\alpha, \bar{c}_\alpha, d_\alpha)$ by induction on $\alpha < \kappa$ such that:

$\circledast_1$ (a) $\bar{c}_\alpha \in {}^{\omega \geq}M$

    (b) $r_\alpha = r_\alpha(x, \bar{c}_\alpha, \bar{b}) = \{\psi_n^\alpha(x, \bar{c}_\alpha, \bar{b}) : n < \omega\} \in \mathbf{R}_{\bar{c}_\alpha {}^\frown \bar{b}}$, see 2.2

    (c) $d_\alpha \in G_{\cup\{\bar{c}_\beta : \beta < \alpha\} \cup \bar{b}}$ or just (which follows) $d_\alpha \in G_{r_\beta}$ for $\beta < \alpha$

    (d) $d_\alpha \notin G_{r_\alpha}$ moreover $\mathfrak{C} \models \neg\psi_0^\alpha(d_\alpha, \bar{c}_\alpha, \bar{b})$.

    (e) $d_\alpha \in q_M(\mathfrak{C})$

    (f) $\{\psi_{n+1}^\alpha(x, \bar{c}_\alpha, \bar{b})\}, \psi_{n+1}^\alpha(y, \bar{c}_\alpha, \bar{b})\}$
           $\vdash \psi_n^\alpha(xy, \bar{c}_\alpha, \bar{b}) \wedge \psi_n^\alpha(x^{-1}, \bar{c}_\alpha, \bar{b}) \wedge \psi_n^\alpha(xy^{-1}, \bar{c}_\alpha, \bar{b})$.

[Why we can? By the assumption toward contradiction.]

    Now as $\mathrm{cf}(\kappa) > |T|^{\aleph_0}$ without loss of generality

$\circledast_2$ $\psi_n^\alpha = \psi_n$ for $\alpha < \kappa$.

Of course

$\circledast_3$ $(G : G_{r_\alpha}) \leq 2^{\aleph_0}$.

[Why? Otherwise let $a_\varepsilon \in G$ for $\varepsilon < (2^{\aleph_0})^+$ be such that $\langle a_\varepsilon G_{r_\alpha} : \varepsilon < (2^{\aleph_0})^+\rangle$ is without repetition. For each $\varepsilon < \zeta < (2^{\aleph_0})^+$ let $n_{\varepsilon,\zeta}$ be the minimal n such that $\mathfrak{C} \models \neg\psi_n(a_\varepsilon^{-1}a_\zeta, \bar{c}_\alpha, \bar{b})$, so by Erdös-Rado theorem for some $n(*)$ and infinite $\mathcal{U} \subseteq (2^{\aleph_0})^+$ we have $n_{\varepsilon,\zeta} = n(*)$ for $\varepsilon < \zeta$ from $\mathcal{U}$. By compactness we can find $a_\varepsilon \in G$ for $\varepsilon < \bar{\kappa}$ such that $\varepsilon < \zeta < \bar{\kappa} \Rightarrow \mathfrak{C} \models \neg\psi_{n(*)}[a_\varepsilon^{-1}a_\zeta, \bar{c}_\alpha, \bar{b}]$, contradiction to $(G : G_{r_\alpha}) < \bar{\kappa}$.]

    $\circledast_4$ there is $\mathcal{U} \in [\kappa]^\theta$ such that: if $\alpha < \beta < \gamma$ are from $\mathcal{U}$ then $d_\alpha^{-1}d_\beta \in G_{r_\gamma}$.

[Why? For each $\alpha < \kappa$ let $\langle a_{\alpha,\varepsilon} G_{r_\alpha} : \varepsilon < \varepsilon_\alpha \leq 2^{\aleph_0} \rangle$ be a partition of G. For $\alpha < \beta < \kappa$ let $\varepsilon = \varepsilon_{\alpha,\beta}$ be such that $d_\alpha \in a_{\beta,\varepsilon} G_{r_\beta}$. As $\kappa \to (\theta)^2_{2^{\aleph_0}}$ because $\beth_2^+ \to (\omega)^2_{2^{\aleph_0}}$ clearly, for some $\mathcal{U} \in [\kappa]^\theta$ and $\varepsilon_* < 2^{\aleph_0}$ we have: if $\alpha < \beta$ are from $\mathcal{U}$ then $\varepsilon_{\alpha,\beta} = \varepsilon_*$. So if $\alpha < \beta < \gamma$ are from $\mathcal{U}$ then $d_\alpha \in a_{\gamma,\varepsilon_*} G_{r_\gamma}$ and $d_\beta \in a_{\gamma,\varepsilon_*} G_{r_\gamma}$, so $d_\alpha = a_{\gamma,\varepsilon_*} a_1$ and $d_\beta = a_{\gamma,\varepsilon_*} a_2$ for some $a_1, a_2 \in G_{r_\gamma}$ hence $(d_\alpha^{-1} d_\beta) = a_1^{-1} a_{\gamma,\varepsilon_*}^{-1} a_{\gamma,\varepsilon_*} a_2 = a_1^{-1} a_2 \in G_{r_\gamma}$.]

⊛$_5$ Without loss of generality for $\alpha, \beta < \theta$ we have $d_\alpha \in G_{r_\beta} \Leftrightarrow \alpha \neq \beta$.

[Why? Let $\mathcal{U}$ be as in ⊛$_4$. Without loss of generality $\mathrm{otp}(\mathcal{U}) = \theta$ and let $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ list $\mathcal{U}$ in increasing order; let $d'_\varepsilon = d_{\alpha_{2\varepsilon}}^{-1} d_{\alpha_{2\varepsilon+1}}$ and let $r'_\varepsilon = r'_\varepsilon(x, \bar{c}'_\varepsilon, \bar{b}) = r_{\alpha_{2\varepsilon}}(x, \bar{c}_{\alpha_{2\varepsilon}}, \bar{b})$.

So if $\zeta < \varepsilon < \theta$ then $d_{\alpha_{2\varepsilon}}^{-1} d_{\alpha_{2\varepsilon+1}} \in G[r_{\alpha_{2\zeta}}]$ by ⊛$_1$(c) hence $d'_\varepsilon = d_{\alpha_{2\varepsilon}}^{-1} d_{\alpha_{2\varepsilon+1}} \in G_{r_{\alpha_{2\zeta}}} = G_{r'_\zeta}$. Also if $\varepsilon < \zeta < \theta$ then $d'_\varepsilon = d_{\alpha_{2\varepsilon}}^{-1} d_{\alpha_{2\varepsilon+1}} \in G_{r_{2\zeta}} = G_{r'_\zeta}$ by ⊛$_4$.

Also, if $\varepsilon = \zeta$ then $d_{\alpha_{2\varepsilon+1}} \in G_{r_{\alpha_{2\varepsilon}}}$ by ⊛$_1$(c) and $d_{\alpha_{2\varepsilon}} \notin G_{r_{\alpha_{2\varepsilon}}}$ by ⊛$_1$(d) hence $d'_\varepsilon = d_{\alpha_{2\varepsilon}}^{-1} d_{\alpha_{2\varepsilon+1}} \notin G_{r_{\alpha_{2\varepsilon}}} = G_{r'_{\alpha_\varepsilon}}$. Of course, $d_{\alpha_{2\varepsilon}}, d_{\alpha_{2\varepsilon+1}} \in G_{\cup\{\bar{c}_\beta : \beta < \alpha_{2\varepsilon}\} \cup \bar{b}}$ hence $d'_\varepsilon = d_{\alpha_{2\varepsilon}}^{-1} d_{\alpha_{2\varepsilon+1}} \in G_{\cup\{\bar{c}'_\zeta : \zeta < \varepsilon\} \cup \bar{b}}$.

Moreover $d'_\varepsilon \notin \psi_1(\mathfrak{C}, \bar{c}'_\varepsilon, \bar{b})$ as otherwise we recall that $d_{\alpha_{2\varepsilon}} = d_{\alpha_{2\varepsilon+1}}(d'_\varepsilon)^{-1}$ and $d_{\alpha_{2\varepsilon+1}} \in r_{\alpha_{2\varepsilon}}(\mathfrak{C}, \bar{c}_{\alpha_{2\varepsilon}}, \bar{b}) \subseteq \psi_1(\mathfrak{C}, \bar{c}_{\alpha_{2\varepsilon}}, \bar{b})$ and, we are now assuming $d'_\varepsilon \in \psi_1(\mathfrak{C}, \bar{c}'_\varepsilon, \bar{b}) = \psi_1(\mathfrak{C}, \bar{c}_{\alpha_{2\varepsilon}}, \bar{b})$ together by ⊛$_1$(f) we have $d_{\alpha_{2\varepsilon}} \in \psi_0(\mathfrak{C}, \bar{c}_{\alpha_{2\varepsilon}}, \bar{b})$, contradiction to ⊛$_1$(d). So letting $\psi'_n = \psi_{n+1}$, clearly renaming we are done as we shall not use $\alpha \geq \theta$.]

Now by induction on $\varepsilon < \kappa$ we choose $A_\varepsilon, \bar{b}_\varepsilon, \langle d_{\alpha,\varepsilon} : \alpha < \theta \rangle$ from the model M such that

⊛$_6$ (a) $\bar{b}_\varepsilon {}^\frown \langle d_{\alpha,\varepsilon} : \alpha < \theta \rangle$ realizes $\mathrm{tp}(\bar{b} {}^\frown \langle d_\alpha : \alpha < \theta \rangle, A_\varepsilon)$

(b) $A_\varepsilon = \cup\{\bar{c}_\alpha : \alpha < \theta\} \cup \bigcup\{\bar{b}_\zeta, \bar{d}_{\alpha,\zeta} : \alpha < \theta \text{ and } \zeta < \varepsilon\} \cup A_*$.

For $\alpha < \theta, \varepsilon < \kappa$, let $r_{\alpha,\varepsilon} := \{\psi_n(x, \bar{c}_\alpha, \bar{b}_\varepsilon) : n < \omega\}$, so $G_{r_{\alpha,\varepsilon}}$ is a subgroup of G of bounded index (even $\leq 2^{\aleph_0}$). Now for $\alpha, \varepsilon < \theta$ clearly $d_\alpha \in \cap\{G_{r_{\beta,\zeta}} : \beta < \theta, \zeta < \varepsilon\}$ by clause ⊛$_1$(e) (as $r_{\beta,\zeta} \in \mathbf{R}_M$). Hence by the choice of $d_{\alpha,\varepsilon}$ as realizing $\mathrm{tp}(d_\alpha, A_\varepsilon)$, see ⊛$_6$(a), as $A_\varepsilon \supseteq \cup\{\mathrm{Dom}(r_{\beta,\zeta}) : \beta < \theta, \zeta < \varepsilon\}$, clearly

⊛$_7$ $d_{\alpha,\varepsilon} \in \cap\{G_{r_{\beta,\zeta}} : \beta < \theta \text{ and } \zeta < \varepsilon\}$ by ⊛$_6$(b).

But $d_\alpha \in \cap\{G_{r_\beta} : \beta < \theta \text{ and } \beta \neq \alpha\}$ by ⊛$_5$ and $\bar{b}_\varepsilon {}^\frown \langle d_{\beta,\varepsilon} : \beta < \theta \rangle$ realizes $\mathrm{tp}(\bar{b} {}^\frown \langle d_\beta : \beta < \theta \rangle, A_\varepsilon)$ by ⊛$_6$, so

⊛$_8$ $d_{\alpha,\varepsilon} \in \cap\{G_{r_{\beta,\zeta}} : \beta < \theta, \beta \neq \alpha \text{ and } \zeta = \varepsilon\}$.

Also by $\circledast_6(a) + \circledast_1(d) + \circledast_2$ we have

$\circledast_9$ $d_{\alpha,\varepsilon} \notin G_{r_{\alpha,\varepsilon}}$ moreover $\mathfrak{C} \models \neg\psi_0(d_{\alpha,\varepsilon}, \bar{c}_\alpha, \bar{b}_\varepsilon)$.

Also by $\circledast_1(f) + \circledast_2$ we have

$\circledast_{10}$ if $d_1, d_2 \in \psi_{n+1}(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$ then $d_1 d_2, d_1^{-1}, d_1 d_2^{-1} \in \psi_n(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$.

Now forget M but retain

$(*)_1$ $\psi_n(n < \omega), b_\varepsilon, \bar{c}_\varepsilon, r_{\alpha,\varepsilon}, d_{\alpha,\varepsilon}, r_\alpha(\alpha < \theta, \varepsilon < \kappa)$ satisfy $\circledast_1(b) - (f), \circledast_7,$
       $\circledast_8, \circledast_9, \circledast_{10}$.

Now by Ramsey theorem and compactness, without loss of generality

$(*)_2$ $\big\langle \langle \bar{c}_\alpha : \alpha < \theta \rangle {}^\frown \bar{b}_\varepsilon {}^\frown \langle d_{\alpha,\varepsilon} : \alpha < \theta \rangle : \varepsilon < \kappa \big\rangle$ is an indiscernible sequence
       over A_*.

Let $d'_{\alpha,\varepsilon} = d^{-1}_{\alpha,2\varepsilon} d_{\alpha,2\varepsilon+1}$ and $\bar{b}'_\varepsilon = \bar{b}_{2\varepsilon}$ for $\varepsilon < \kappa, \alpha < \theta$ and let $r'_{\alpha,\varepsilon}(x) = r(x, \bar{c}_\alpha, b'_\varepsilon) = r_{\alpha,2\varepsilon}$.
     Now

$(*)_3$ $d'_{\alpha,\varepsilon} \in r(\mathfrak{C}, \bar{c}_\beta, \bar{b}'_\zeta) \cap G$ $\underline{\text{iff}}$ $(\alpha, \varepsilon) \neq (\beta, \zeta)$.

[Why? First assume $\varepsilon > \zeta$ then by $(*)_1 + \circledast_7$ we have $d_{\alpha,2\varepsilon}, d_{\alpha,2\varepsilon+1} \in G_{r_{\beta,2\zeta}}$ hence $d'_{\alpha,\varepsilon} = d^{-1}_{\alpha,2\varepsilon} d_{\alpha,2\varepsilon+1} \in G_{r_{\beta,2\zeta}} = G_{r'_{\beta,\zeta}}$.
     Second, assume $\varepsilon < \zeta$, then by the indiscernibility, i.e. $(*)_2$ easily $d_{\alpha,2\varepsilon}$
$G_{r_{\beta,2\zeta}} = d_{\alpha,2\varepsilon+1} G_{r_{\beta,2\zeta}}$ hence $d_{\alpha,2\varepsilon} G_{r'_{\beta,\zeta}} = d_{\alpha,2\varepsilon+1} G_{r'_{\beta,\zeta}}$ so $d'_{\alpha,\varepsilon} = d^{-1}_{\alpha,2\varepsilon} d_{\alpha,2\varepsilon+1}$
$\in G_{r'_{\beta,\zeta}}$ as required.
     Third, assume $\varepsilon = \zeta, \alpha \neq \beta$, then we have $d'_{\alpha,\varepsilon} \in r(\mathfrak{C}, \bar{c}_\beta, b'_\zeta)$ because:
$d_{\alpha,2\varepsilon} \in r(\mathfrak{C}, \bar{c}_\beta, \bar{b}'_\zeta)$ as $\bar{b}'_\zeta = b_{2\zeta} = b_{2\varepsilon}$ in the present case and as $\alpha \neq \beta$ using
$(*)_1 + \circledast_8$ and $d_{\alpha,2\varepsilon+1} \in r(\mathfrak{C}, \bar{c}_\beta, \bar{b}'_\zeta)$ as $\bar{b}'_\zeta = \bar{b}_{2\zeta} = \bar{b}_{2\varepsilon}$ and as $2\zeta = 2\varepsilon < 2\varepsilon+1$
in the present case , by $\circledast_7 + (*)_1$; of course, $d_{\alpha,2\varepsilon}, d_{\alpha,2\varepsilon+1} \in G$ hence $d'_{\alpha,\varepsilon} = d^{-1}_{\alpha,2\varepsilon} d_{\alpha,2\varepsilon+1} \in r(\mathfrak{C}, \bar{c}_\beta, \bar{b}'_\zeta) \cap G$.
     Fourth, assume $\varepsilon = \zeta, \alpha = \beta$. So by $\circledast_9 + (*)_1$, we know that $d_{\alpha,2\varepsilon} \notin r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}_{2\varepsilon})$ which means $d_{\alpha,2\varepsilon} \notin r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$. By $\circledast_7 + (*)_1$ we know that $d_{\alpha,2\varepsilon+1} \in r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}_{2\varepsilon}) = r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$ and of course $d_{\alpha,2\varepsilon}, d_{\alpha2\varepsilon+1} \in G$. Putting together the last two sentences and the choice of $d'_{\alpha,\varepsilon}$, as $r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon) \cap G$ is a subgroup of G we have $d'_{\alpha,\varepsilon} = (d_{\alpha,2\varepsilon})^{-1} d_{\alpha,2\varepsilon+1} \notin r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon) \cap G$ as required in $(*)_3$.]

$(*)_4$ $d'_{\alpha,\varepsilon} \in \psi_1(\mathfrak{C}, \bar{c}_\beta, \bar{b}'_\zeta)$ iff $(\alpha, \varepsilon) \neq (\beta, \zeta)$.

[Why? The "if" direction holds by $(*)_3$ because $r(x, \bar{c}_\beta, \bar{b}'_\zeta) = \{\psi_n(x, \bar{c}_\beta, \bar{b}_\zeta) : n < \omega\}$. For the other direction assume $(\alpha, \varepsilon) = (\beta, \zeta)$. As in the proof of $(*)_3$ we have $d_{\alpha,2\varepsilon} \notin \psi_0(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$. Also $d_{\alpha,2\varepsilon} = d_{\alpha,2\varepsilon+1} (d'_{\alpha,\varepsilon})^{-1}$ so $d_{\alpha,2\varepsilon+1}(d'_{\alpha,\varepsilon})^{-1} \notin \psi_0(\mathfrak{C}, \bar{c}_\alpha, \bar{b}_{2\varepsilon})$. Also as in the proof of $(*)_3$ we have $d_{\alpha,2\varepsilon+1} \in r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$ hence $d_{\alpha,2\varepsilon+1} \in \psi_1(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\alpha)$. By the last two sentences and $\circledast_{10} + (*)_1$ we have $d'_{\alpha,\varepsilon} \in \psi_1(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon) \Rightarrow d_{\alpha,2\varepsilon} \in \psi_0(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$ but we already note the conclusion fails so $d'_{\alpha,\varepsilon} \notin \psi_1(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon)$.

So we are done proving $(*)_4$.]

$(*)_5$ if $u \subseteq \theta \times \kappa$ then for some $d \in \mathfrak{C}$ realizes $\{\psi_3(x, \bar{c}_\alpha, \bar{b}'_\zeta)^{\text{if}((\alpha,\zeta)\in u)} : \alpha < \theta, \varepsilon < \kappa\}$.

[Why? By saturation without loss of generality u is co-finite, let $\langle(\alpha(\ell), \varepsilon(\ell)) : \ell < k\rangle$ list $\theta \times \kappa \setminus u$ with no repetitions and let $d := d'_{\alpha(0),\varepsilon(0)} d'_{\alpha(1),\varepsilon(1)} \cdots d'_{\alpha(k-1),\varepsilon(k-1)}$.

On the one hand by $(*)_3$ clearly

$$(\alpha, \varepsilon) \in u \;\; \Rightarrow (\alpha, \varepsilon) \in \theta \times \kappa \setminus \{(\alpha(\ell), \varepsilon(\ell)) : \ell < k\}$$
$$\Rightarrow \{d'_{\alpha(\ell),\varepsilon(\ell)} : \ell < k\} \subseteq r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}'_\varepsilon) \cap G$$
$$\Rightarrow d \in r(\mathfrak{C}, \bar{c}_\alpha, \bar{b}_\varepsilon) \Rightarrow \mathfrak{C} \models \psi_3[d, \bar{c}_\alpha, \bar{b}'_\varepsilon].$$

On the other hand if $\ell < k$ then let $e_1 = d'_{\alpha(0),\varepsilon(0)} \cdots d'_{\alpha(\ell-1),\varepsilon(\ell-1)}$ and let $e_2 = d'_{\alpha(\ell+1),\varepsilon(\ell+1)} \cdots d'_{\alpha(k-1),\varepsilon(k-1)}$ so $d = e_1 d'_{\alpha(\ell),\varepsilon(\ell)} e_2$ hence $d'_{\alpha(\ell),\varepsilon(\ell)} = e_1^{-1} d e_2^{-1}$. As above $e_1, e_2 \in r(\mathfrak{C}, \bar{c}_{\alpha(\ell)}, \bar{b}'_{\varepsilon(\ell)}) \cap G$ hence $e_1^{-1}, e_2^{-1} \in \psi_3(\mathfrak{C}, \bar{c}_{\alpha(\ell)}, \bar{b}_{\varepsilon(\ell)}) \cap G$. As $d \in G$ and $e^{-1}d \in G$, by $\circledast_{10}$ we get

$$d \in \psi_3(\mathfrak{C}, \bar{c}_{\alpha(\ell)}, \bar{b}'_{\varepsilon(\ell)}) \;\; \Rightarrow e_1^{-1}d \in \psi_2(\mathfrak{C}, \bar{c}_{\alpha(\ell)}, \bar{b}'_{\varepsilon(\ell)})$$
$$\Rightarrow e_1^{-1}de_2^{-1} \in \psi_1(\mathfrak{C}, \bar{c}_{\alpha(\ell)}, \bar{b}'_{\varepsilon(\ell)})$$
$$\Rightarrow d'_{\alpha(\ell),\varepsilon(\ell)} \in \psi_1(\mathfrak{C}, \bar{c}_{\alpha(\ell)}, \bar{b}'_{\varepsilon(\ell)}).$$

But this contradicts $(*)_4$.

Now $(*)_5$ gives "$\psi_3(x, \bar{z}, \bar{y})$ witness T is 2-independent" so we are done. $\square_{2.12}$

Claim 2.13. *If G is Abelian, then* 2.12 *can be proved also replacing $q_B(\mathfrak{C})$ by $\cap\{G' : G'$ is a subgroup of G of bounded index preserved by automorphisms of $\mathfrak{C}$ over $A \cup A_*\}$.*

Proof. We shall prove this elsewhere. $\square$

Discussion 2.14. 1) Is 1/2-dependence preserved by weak expansions (as in [15, §1])? Of course not, as if M is a model of T_2 from 2.8 then any $Y \subseteq \prod_{\ell < k} P_i^M$ is definable in such an expansion, and easily for some such Y we can interpret number theory (as number theory is interpretable in some bi-partite graph).

2) Is the following interesting? I think yes! It seems that we can prove the k-dimensional version of 2.12 for $1/k$-dependent T, i.e. for $k = 1$ it should give [19], for $k = 2$ it should give 2.12. E.g. think of having $|T| < \lambda_0 < \ldots < \lambda_k, \lambda_{\ell+2} = (\lambda_{\ell+1})^{\lambda_\ell}$ and we choose $M_\ell \prec \mathfrak{C}$ of cardinality λ_ℓ closed enough by downward induction on ℓ, i.e. we get a $\mathcal{P}^-(k)$-diagram. We shall try to deal with this elsewhere.

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