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GRAPHS WITH PRESCRIBED ASYMMETRY AND MINIMAL NUMBER OF EDGES

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§0. INTRODUCTION

We shall deal with non-directed graphs, without loops and double edges, and having a finite number of vertices.

A graph is symmetric if it has a non-trivial automorphism = a permutation of its vertices, such that a pair of vertices is connected iff their images are connected. The asymmetry of a graph is the minimal number of changes (i.e. adding and deleting of edges) which is necessary to make the graph symmetric. Erdős and Rényi [1] defined and investigated this notion, and defined, $F(n, k)$ $[C(n, k)]$ for $k \geqslant 1$, $n > 1$ as the minimal number of edges in a [connected] graph, with n vertices, whose asymmetric is k; if there is no such graph the value of the function will be ∞ (If n is too small, this happens). (see [1], §5 p. 311): They proved that $C(6, 1) = 6$, $C(1, 1) = 0$, $C(n, 1) = n - 1$ for $n \geqslant 7$; also $C(n, 2) > n + 1$ for

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$n \geqslant 7$ and $F(n,3) \geqslant 4n/3 - 3/2$. It is obvious that $F(n,k) \leqslant C(n,k)$. They also show that $C(n,1) = \infty$ for $1 < n \leqslant 5$, and $C(n,k) = \infty$ for $n < 2k+1$.

We shall compute $C(n,k)$ *and* $F(n,k)$ *for* $k > 1$ *and* n *sufficiently larger than* k. For $k > 2$, n sufficiently larger than k, we affirm the conjecture in [1] that $C(n,k) = F(n,k)$. (See [1] p. 314. before the remarks.) It will be interesting to know for any k, from what n our formulas are correct. From the proof a bound can be found, but seemingly it will be far from the exact value.

First we shall formulate the results. Then, in §1, we prove that $F(n,k)$ is not smaller than the values mentioned in the theorems, by generalizing a proof from [1]. In §2 we describe examples of connected graphs with n vertices and asymmetry k, whose number of edges is the number appearing in the theorems. Finally, in §3 we shall prove for the case $k \geqslant 41$, that the graphs described in §2, have the required asymmetry. (For $3 \leqslant k \leqslant 40$, the proof is messy and with the same central idea).

The results are the following:

Theorem 0.1. *For* n *sufficiently large*

$$F(n,2) = n+1, \quad C(n,2) = n+2.$$

Remark. This was independently found by Nesetril in his M. Sc. thesis.

Theorem 0.2. *For odd* $k > 2$, *and* n *sufficiently larger than* k

$$F(n,k) = C(n,k) = [(k+3)n/4 - 0.5[2n/(k+3)] + 1/2].$$

Theorem 0.3. *For even* $k > 2$ *and* n *sufficiently larger than* k

$$F(n,k) = C(n,k) = [(k+2)n/4 + 1/2].$$

Notations. Let G denote a graph, P, Q, R, S vertices of the graph, $N = N(G)$ the number of vertices of G, $E = E(G)$ the number of edges of G. Let ν_P be the valence of P (= the number of edges incident to P), and ν^i the valence of P_i. Also V_k will denote the number of vertices (in G) whose valence is k, $V_{\geq k}$ the number of vertices (in G) whose valence is $\geq k$, etc. Thus, $2E = \sum_P v_P = \sum_k kV_k$. Let m, n, k, l denote natural numbers, and i, j, r integers. $A(G)$ will stand for the asymmetry of G.

We say that $P_1, P_2, \ldots, P_m$ is a path, if $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m$ are edges, and $P_1P_2 \ldots P_m$ is a circle, if $P_1P_2, P_2P_3, \ldots, P_{m-1}P_m, P_mP_1$, are edges. $[x]$ is the integral part of x.

§ 1. PROOF OF THE LOWER BOUNDS

First we shall observe some facts, which, in fact, appear in [1].

If P, Q are vertices of G, which are not connected to any other vertices (but PQ may be an edge) then the permutation interchanging them is an automorphism of G. Hence

Observation 1. $A(G) \leq v_P + v_Q$ if P, Q are distinct vertices of G.

Observation 2. $A(G) \leq v_P + v_Q - 2$ if P, Q are vertices of G, and PQ is an edge.

If P, Q, R are vertices of G, such that RP, RQ are edges, and there are no other edges containing P or Q, except possibly PQ, then the permutation interchanging P and Q is an automorphism of G, hence G is symmetric.

Observation 3. $A(G) \leq v_P + v_Q - 2$, if P, Q, R are distinct vertices of G, and RP, RQ are edges of G.

Lemma 1.1. *$F(n, 2) \geq n + 1$ for $n > 7$, and $C(n, 2) \geq n + 2$ for $n \geq 7$.*

Proof. By [1], $C(n, 2) \geq n + 2$, for $n > 6$.

Suppose $N(G) = n$, $A(G) = 2$; we should prove $E(G) \geq n + 1$. Let $G_1, \ldots, G_k$ be the components of G. By [1] there are, up to isomorphism, only two asymmetric connected graphs G with $E(G) \leq N(G)$. One, G^1 is the graph with one point, and the other G^2 have six vertices and six edges. Now $E(G) = \sum_{l=1}^{k} E(G_l)$ and $N(G) = \sum_{l=1}^{k} N(G_l)$. Now clearly no two of the G_k's can be isomorphic, hence the worst case is when, say, $G_1 = G^1$, $G_2 = G^2$. As $N(G) > 7$, $k > 2$. Hence

$$E(G) = \sum_{l=1}^{k} E(G_l) = E(G_1) + E(G_2) + \sum_{l=3}^{l} E(G_l) \geqslant$$

$$\geqslant 0 + 6 + \sum_{l=3}^{k} (N(G_l) + 2) =$$

$$= 0 + 6 + \sum_{l=3}^{k} N(G_l) + 2(k-2) =$$

$$= 6 + N(G) - 7 + 2(k-2) =$$

$$= N(G) - 1 + 2(k-2) \geqslant N(G) + 1 .$$

Lemma 1.2. *If* $k > 2$ *is even, then*

$$F(n, k) \geqslant [(k+2)n/4 + 1/2] .$$

Proof. Let G be a graph with n vertices, $A(G) \geqslant k$. We should prove that $E = E(G) \geqslant [(k+2)n/4 + 1/2]$, or, as $E(G)$ is an integer, $E(G) \geqslant (k+2)n/4$, or $2E(G) \geqslant (k+2)n/2$.

If the valency of every vertex is $\geqslant (k+2)/2$, then

$$2E = \sum_{l} lV_l \geqslant ((k+2)/2) \sum_{l} V_l = (k+2)n/2 ,$$

so let R_0 be a vertex with valency $< (k+2)/2$, that is $\leqslant k/2$. Then, for any other vertex P we have $\nu_P \geqslant k/2$, because by observation 1

$$k \leqslant A(G) \leqslant \nu_P + \nu_Q \leqslant \nu_P + k/2 .$$

Now if $\nu_Q \leqslant k/2$, and PQ is an edge, then ν_P is $\geqslant k/2 + 2$ as by observation 2

$$k \leqslant A(G) \leqslant \nu_P + \nu_Q - 2 = \nu_P + k/2 - 2 .$$

Similarly if Q, P, S, are vertices of G, QS, PS are edges, then $\nu_Q \leqslant k/2$ implies $\nu_P \geqslant k/2 + 2$ (by observation 3).

Assume first $v_{R_0} = k/2$. As we have shown that for every other P, $v_P \geqslant k/2$, clearly $V_{<k/2} = 0$. As every vertex of valence $\leqslant k/2$ is connected only with vertices of valence $\geqslant k/2 + 2$, and no vertex is connected with two vertices of valency $k/2$, *clearly* $V_{\geqslant(k/2+2)} \geqslant V_{k/2}$.

Hence

$$2E = \sum l V_l \geqslant (k/2)V_{k/2} + (k/2+1)V_{k/2+1} +$$

$$+ (k/2+2)V_{\geqslant(k/2+2)} =$$

$$= (k/2)V_{k/2} + (k/2+1)(n - V_{k/2} - V_{\geqslant(k/2+2)}) +$$

$$+ (k/2+2)V_{\geqslant(k/2+2)} =$$

$$= - V_{k/2} + (k/2+1)n + V_{\geqslant(k/2+2)} \geqslant$$

$$\geqslant (k/2+1)n = (k+2)n/2 .$$

Now assume $v_{R_0} < k/2$. Then by observation 1, the valency of any other vertex P is $\geqslant (k+2)/2$, as $k \leqslant A(G) \leqslant V_P + k/2 - 1$. If $v_{R_0} \neq \neq 0$, and P is connected with R_0, then $v_P \geqslant k - v_{R_0} + 2$. Hence

$$2E = \sum_Q v_Q \geqslant (k/2+1)(n-2) + v_{R_0} + v_P \geqslant$$

$$\geqslant (k+2)n/2 - (k+2) + v_{R_0} + k - v_{R_0} + 2 = (k+2)n/2 .$$

So we are left with the case $v_{R_0} = 0$. Then for every $P \neq R_0$, $v_P \geqslant k$ (by observation 1). Hence

$$2E \geqslant \sum_Q v_Q \geqslant k(n-1) = kn - k =$$

$$= (k+2)n/2 + (k-2)n/2 - k \geqslant$$

$$\geqslant (k+2)n/2 + (4-2)n/2 - k =$$

$$= (k+2)n/2 + n - k \geqslant (k+2)n/2$$

(we use the assumption that k is even and > 2, hence $\geqslant 4$; and that $n \geqslant 2k + 1 > k$ (for $n < 2k + 1 > k$ implies $F(n, k) = \infty$).

Lemma 1.3. *If k is odd and > 2, then*

$$F(n, k) \geqslant [(k+3)n/4 - 0.5[2n/(k+3)] + 1/2] .$$

Remark. For $k = 3$, this slightly improves Th. 8 p. 314 [1].

Proof. Let $l = (k+1)/2$. Let $N(G) = n$, $A(G) \geqslant k$.

If R is a vertex of G with valency $< l$, then for every other vertex P we have $v_P \geqslant l$, because by observation 1

$$k \leqslant A(G) \leqslant v_P + v_R \leqslant v_P + l - 1$$

$$v_P \geqslant k - (l-1) = k - (k+1)/2 + 1 =$$

$$= k/2 - 1/2 + 1 = k/2 + 1/2 = l .$$

Hence there is at most one vertex with valency $< l$. Now if $v_P \leqslant l$, and P, Q are connected, then $v_Q \geqslant l + 1$ (by observation 2), and similarly if PR, QR are edges then $v_Q \geqslant l + 1$ (by observation 3). Hence if $v_P \leqslant l$ and PQ are connected, then $v_Q \geqslant l + 1$, and P is the only vertex connected with Q with a valency $\leqslant l$. Hence $V_{\geqslant (l+1)} \geqslant \sum_{m \leqslant l} mV_m \geqslant lV_l$.

Case I. Let us assume first $V_{<l} = 0$.

Then $n = V_l + V_{>l} \geqslant V_l + lV_l = (l+1)V_l$, or $V_l \leqslant n/(l+1)$.

$$2E(G) = \sum mV_m \geqslant lV_l + (l+1)V_{\geqslant (l+1)} =$$

$$= lV_l + (l+1)(n - V_l) = (l+1)n - V_l \geqslant$$

$$\geqslant (l+1)n - [n/(l+1)] = (k+3)n/2 - [2n/(k+3)] .$$

So, if $V_{<l} = 0$, the lemma holds. Suppose $V_{<l} \neq 0$, hence $V_{<l} = 1$, as noted in the beginning of the proof, and let R_0 be the only vertex with valency $< l$.

Case II. Assume now $v_{R_0} = 0$. Then, by observation 1, every $P \neq R_0$ has valency $\geqslant k$. Hence $2E \geqslant k(n-1)$. For $k > 3$, as $k \geqslant 5$

$$2E - (k+3)n/2 \geq k(n-1) - (k+3)n/2 =$$

$$= kn - k - kn/2 - 3n/2 = n(k - k/2 - 3/2) - k =$$

$$= n(k/2 - 3/2) - k \geq n - k > 0 .$$

This clearly implies the required inequality. For $k = 3$, $n \geq 9$, the required inequality also holds.

$$2E \geq k(n-1) = 3n - 3 = (k+3)n/2 - 3 =$$

$$= (k+3)n/2 - [2 \cdot 9/6] \geq (k+3)n/2 - [2n/(k+3)] .$$

As $\infty = F(n, k)$ for $n < 2k + 1 = 7$, the remaining cases are $k = 3$, $n = 7$, $k = 3$, $n = 8$. If we remove R_0, we get a graph G_1, $N(G_1) = n - 1$, $E(G_1) = E(G)$, $A(G_1) \geq 3$, and the valency of every vertex is ≥ 3. For $n = 7$, we get a graph with six vertices and asymmetry 3, contradicting Theorem 1.1 in [1], according to which

$$A(G) \leq (N(G) - 1)/2 .$$

So we are left with the case $n = 8$. As $\sum v_P$ is even, there is in G_1 at least one vertex with valency ≥ 4. If there are two such vertices, or one with valency > 4, we get $E(G) = E(G_1) \geq 12$ which is the required inequality. So let P be the only vertex of valency four, Q_1, Q_2, Q_3, Q_4 the vertices connected with it, and S_1, S_2 the two other vertices. As S_1 has valency three, and it is not connected with P, it is connected with two of the Q's, say Q_1, Q_2. Now clearly in order to make the permutation interchanging Q_1 and Q_2 to an automorphism of G, it is sufficient to remove two edges. This is a contradiction. So have finished the case $v_{R_0} = 0$.

Case III. $l > v_{R_0} > 0$

By observations 2 and 3 it is clear that if P is connected with R_0, or connected with a vertex which is connected with R_0, then $v_P \geq m = k - v_{R_0} + 2$, and hence $V_{\geq m} \geq m$. Clearly $0 < v_{R_0} < l = (k+1)/2$ implies $m > (k+3)/2 = l + 1$. As noted before in Case I

$V_{\geqslant (l+1)} \geqslant \sum_{m \leqslant l} mV_m \geqslant lV_l$, hence

$$n = 1 + V_l + V_{\geqslant l} \geqslant 1 + V_l + lV_l = 1 + (l+1)V_l ,$$

$$V_l \leqslant (n-1)/(l+1) \leqslant n/(l+1) - 1$$

whence

$$V_l \leqslant [n/(l+1)] - 1 = [2n/(k+3)] - 1 .$$

Now

$$2E = \sum_r rV_r \geqslant v_{R_0} \cdot 1 + lV_l +$$

$$+ (l+1)(V_{\geqslant (l+1)} - V_{\geqslant m}) + mV_{\geqslant m} =$$

$$= v_{R_0} + lV_l + (l+1)V_{\geqslant (l+1)} + (m-l-1)V_{\geqslant m} =$$

$$= v_{R_0} + lV_l + (l+1)(n-1-V_l) + (m-l-1)V_{\geqslant m} =$$

$$= v_{R_0} - V_l + (l+1)n - (l+1) + (m-l-1)V_{\geqslant m} \geqslant$$

$$\geqslant (v_{R_0} - V_l) + (l+1)n - (l+1) +$$

$$+ (m-l-1)m \geqslant$$

$$\geqslant -[2n/(k+3)] + (k+3)n/2 +$$

$$+ (-(l+1) + (m-l-1)m) \geqslant$$

$$\geqslant (k+3)n/2 - [2n/(k+3)]$$

(the last inequality holds, as $m > l+1$, implies

$$(m-l-1)m \geqslant m > (l+1)) .$$

So the required inequality holds in case III, and Lemma 1.3 is proved.

§2. THE DESCRIPTIONS OF THE EXAMPLES

Example 2.1. We will show that $C(n, 2) \leq n + 2$.

Description. Let $n_1, \dots, n_6$ be different natural numbers $> 0'$. The vertices of G will be $R_1, \dots, R_4$ (of valency three), and P^i_k, $i = 1, \dots, 6$, $k = 1, \dots, n_i$ (of valency 2). Now

$$R_1 P^1_1 \dots P^1_{n_1} R_2, R_1 P^2_1 \dots P^2_{n_2} R_3, R_1 P^3_1 \dots$$

$$R_1 P^3_1 \dots P^3_{n_3} R_4, R_2 P^4_1 \dots P^4_{n_4} R_3,$$

$$R_2 P^5_1 \dots P^5_{n_5} R_4, R_3 P^6_1 \dots P^6_{n_6} R_4$$

will be paths, (and every edge of G appears in one of them)

Example 2.2. Proof of $F(n, 2) \leq n + 1$.

It is the same as the previous one, if we add one isolated vertex.

Example 2.3. We show the upper bound for $F(n, k)$, for even $k > 2$.

Remark. Every pair of vertices which will not be said to be connected, will be considered unconnected. We shall concentrate on the case $k > 40$.

Construction. Clearly, by Lemma 1.2, every vertex will have a valency $l = (k + 2)/2$, except one vertex if n is odd. Let us choose numbers $r_1, \dots, r_l$ such that

1. r_i is odd, and $0 < r_1 < -r_2 < r_3 < -r_4 < \dots < (-1)^l r_l$.
2. If $r_{i_1} + r_{i_2} = r_{m_1} + r_{m_2}$ then $\{i_1, i_2\} = \{m_1, m_2\}$.
3. If $r_{i_1} + r_{i_2} + r_{i_3} = r_{m_1} + r_{m_2} + r_{m_3}$ then $\{i_1, i_2, i_3\} = \{m_1, m_2, m_3\}$, and $i_1 = i_2 = m_1$ implies $m_1 = m_2$ or $m_1 = m_3$.
4. No sum of ≤ 5 of the numbers $\{\pm r_i: 1 \leq i \leq l\}$ is > 0 and $\leq k$.

Clearly $r_i = (-2)^{i+1}(k + 1) + 1$ satisfy the conditions, but we can easily find much smaller r_i's: for example defining r_i by induction as the

$V_{\geqslant (l+1)} \geqslant \sum_{m<l} m V_m \geqslant l V_l$, hence

$$n = 1 + V_l + V_{\geqslant l} \geqslant 1 + V_l + l V_l = 1 + (l+1) V_l ,$$

$$V_l \leqslant (n-1)/(l+1) \leqslant n/(l+1) - 1$$

whence

$$V_l \leqslant [n/(l+1)] - 1 = [2n/(k+3)] - 1 .$$

Now

$$\begin{aligned} 2E = \sum_r r V_r &\geqslant v_{R_0} \cdot 1 + l V_l + \\ &+ (l+1)(V_{\geqslant (l+1)} - V_{\geqslant m}) + m V_{\geqslant m} = \\ &= v_{R_0} + l V_l + (l+1) V_{\geqslant (l+1)} + (m - l - 1) V_{\geqslant m} = \\ &= v_{R_0} + l V_l + (l+1)(n - 1 - V_l) + (m - l - 1) V_{\geqslant m} = \\ &= v_{R_0} - V_l + (l+1)n - (l+1) + (m - l - 1) V_{\geqslant m} \geqslant \\ &\geqslant (v_{R_0} - V_l) + (l+1)n - (l+1) + \\ &+ (m - l - 1)m \geqslant \\ &\geqslant -[2n/(k+3)] + (k+3)n/2 + \\ &+ (-(l+1) + (m - l - 1)m) \geqslant \\ &\geqslant (k+3)n/2 - [2n/(k+3)] \end{aligned}$$

(the last inequality holds, as $m > l + 1$, implies

$$(m - l - 1)m \geqslant m > (l + 1)) .$$

So the required inequality holds in case III, and Lemma 1.3 is proved.

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$$R_1 P_1^1 \dots P_{n_1}^1 R_2, R_1 P_1^2 \dots P_{n_2}^2 R_3, R_1 P_1^3 \dots$$

$$R_1 P_1^3 \dots P_{n_3}^3 R_4, R_2 P_1^4 \dots P_{n_4}^4 R_3,$$

$$R_2 P_1^5 \dots P_{n_5}^5 R_4, R_3 P_1^6 \dots P_{n_6}^6 R_4$$

will be paths, (and every edge of G appears in one of them)

Example 2.2. Proof of $F(n, 2) \leqslant n + 1$.

It is the same as the previous one, if we add one isolated vertex.

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Construction. Clearly, by Lemma 1.2, every vertex will have a valency $l = (k + 2)/2$, except one vertex if n is odd. Let us choose numbers $r_1, \dots, r_l$ such that

1. r_i is odd, and $0 < r_1 < -r_2 < r_3 < -r_4 < \dots < (-1)^l r_l$.

2. If $r_{i_1} + r_{i_2} = r_{m_1} + r_{m_2}$ then $\{i_1, i_2\} = \{m_1, m_2\}$.

3. If $r_{i_1} + r_{i_2} + r_{i_3} = r_{m_1} + r_{m_2} + r_{m_3}$ then $\{i_1, i_2, i_3\} = \{m_1, m_2, m_3\}$, and $i_1 = i_2 = m_1$ implies $m_1 = m_2$ or $m_1 = m_3$.

4. No sum of $\leqslant 5$ of the numbers $\{\pm r_i\colon 1 \leqslant i \leqslant l\}$ is > 0 and $\leqslant k$.

Clearly $r_i = (-2)^{i+1}(k + 1) + 1$ satisfy the conditions, but we can easily find much smaller r_i's; for example defining r_i by induction as the first satisfying the conditions.

Let us first define a graph $G(n, k)$.

Case I. n is even.

The vertices are $P_1, \ldots, P_n$; and let $P(i) = P_i$, and $P_i = P_j$ if $i = j \pmod n$. Now for even i, P_i is connected with $P(i + r_j)$ for $j = 1, \ldots, l$.

Case II. n is odd.

The points will be $P_1, \ldots, P_{n-1}$ and Q. As before $P_i = P(i)$ $P_i = P_j$ if $i = j \pmod{n-1}$. For even i, P_i is connected with $P(i + r_j)$ $j = 1, \ldots, l$; except if $j = 1$ and i belongs to $\{2r\colon 1 \leqslant r \leqslant$ $\leqslant (l+1)/2\}$. Q is connected to $P_{2r}, P_{2r + r_1}$ for $1 \leqslant r \leqslant (l+1)/2$.

Let $L_1 = 6|r_l|$, $L = 12|r_l|$, $(|r_l| = (-1)^l r_l)$. (For $k > 40$ this is more then sufficient, but for $3 \leqslant k \leqslant 40$, greater values can be more convenient).

Now we shall define the required graph $G^*(n, k)$, by slightly modifying $G(n, k)$. For $m = 1, \ldots, k + 15$ we omit the edges

$$P(2[n/4] + 2mL)P(2[n/4] + 2mL + r_1)$$

and $P(2[n/4] + 2mL + 2L_1 + 2m)P(2[n/4] + 2mL + 2L_1 + 2m + r_1)$ and add the edges $P(2[n/4] + 2mL)P(2[n/4] + 2mL + 2L_1 + 2m + r_1)$ and $P(2[n/4] + 2mL + r_1)P(2[n/4] + 2mL + 2L_1 + 2m)$.

Notice that the r_i's and L, L_1 depend only on k and not on n.

Example 2.4. Proof of the upper bound of $F(n, k)$ for odd $k > 3$.

Clearly here the valencies of the vertices will be $l = (k+1)/2$ or $l + 1$. Let n_1 be such that $((l+1)n_1 + l(n - n_1))/2$ is the number appearing in Theorem 0.3; clearly there is such a number. We define the r_i's as in example 2.3, and also L, L_1. Clearly $ln_2 \leqslant n_1$, where $n_2 = n - n_1$. Now we define $G(n, k)$.

Case I. n_1 is even.

The vertices will be $P_1, \dots, P_{n_1}, R_1, \dots, R_{n_2}$, where $n_2 = n - n_1$. As before $P_i = P(i)$, and $P_i = P_j$ if $i = j \pmod{n_1}$. For even i, we connect P_i with $P(i + r_j)$ for $j = 1, \dots, l$. We also connect R_1 with $P_1, \dots, P_l$; R_2 with $P_{l+1}, \dots, P_{2l}; \dots; R_m$ with $P_{lm-l+1}, \dots$ $\dots, P_{lm}; \dots$. Finally, if $n_1 > ln_2$ we connect P_{ln_2+1} with $P_{ln_2+2}, \dots, P_{ln_2+2m}$ with $P_{ln_2+2m-1}, \dots$. (Note that this is possible as $n_1 - ln_2$ is even, because n_1 is even, and as $(l+1)n_1 + ln_2$ is even, ln_2 is even). Notice that the valency of the P_i's is $l + 1$, and that of the R_i's is l.

Case II. n_1 is odd.

Note that as $(l+1)n_1 + ln_2$ is even, l cannot be even. So l is odd and n_2 is even. Now the vertices will be $P_1, \dots, P_{n_1-1}, R_1, \dots$ $\dots, R_{n_2}$ and Q. As usual $P_i = P(i)$ and $P_i = P_j$ if $i = j \pmod{n_1 - 1}$. For even i we connect P_i with $P(i + r_j)$, $j = 1, \dots, l$. We alco connect for $m = 1, \dots, n_2$, P_m with $P(ml - l + j)$ for $j = 1, \dots, l$. Now for $m = 1, \dots, (l+1)/2$, we "disconnect" $P(2Lm)P(2Lm + r_1)$ and connect each of them with Q. Note that the P_i's and Q have valency $l + 1$, whereas the R_i's have valency l. Now the definition of $G^*(n, k)$ from $G(n, k)$ is the same as in example 2.3.

Example 2.5. Proof of the upper bound for $F(n, 3)$.

Let n_1 be a number such that $(3n_1 + 2(n - n_1))/2$ is the number in Theorem 0.3, and $n_2 = n - n_1$. Clearly n_1 is even, and $2n_2 \leq n_1$; and so $n_1 - 2n_2$ is even, hence it is zero or two. Let us define $G(n, k)$.

Case I. $n_1 = 2n_2$.

The vertices of the graph will be $P_1, \dots, P_{n_1}, R_1, \dots, R_{n_2}$. $P_1, \dots, P_{n_1}$ will be a circle. R_m will be connected with P_{2m} and $P_{(2m+17)}$ (where as usual $P_i = P_j$ if $i = j \pmod{n_1}$).

Case II. $n_1 = 2n_2 + 2$.

The vertices will be $P_1, \dots, P_{n_1-2}$, $R_1, \dots, R_{n_2}, S_1, S_2$.

$S_1P_1 \ldots P_{L-1}S_2P_L \ldots P_{n_1-2}$ is a circle, S_1S_2 is an edge, and R_m is connected with P_{2m}, P_{2m+17}.

The definition of $G^*(n, k)$ is as in 2.3, taking $r_1 = 1$, $L_1 = 100$, $L = 200$.

§ 3. PROOF THAT THE EXAMPLES HAVE THE REQUIRED ASYMMETRY

We shall prove it only for $k > 40$. Let us mention some properties of the graphs $G^*(n, k)$ we shall need. We assume implicitly that n is always sufficiently large.

Property A. There is no square in the graph.

Proof. By property (4) of the r_i's, a square cannot contain as a vertex one of the R_i's in 2.4, nor Q in 2.3 II. By the definition of L it cannot contain Q from 2.4 II. By the definition of $G^*(n, k)$ and L, L_1, it suffices to prove that in $G(n, k)$, there is no circle $P(i_1)P(i_2)P(i_3)P(i_4)$. Now if there is a such a circle, i_1 is odd iff i_2 is even. So assume i_1 is even, hence i_2, i_4 are odd, i_3 is even. Moreover; $i_2 - i_1, i_4 - i_1, i_2 - i_3, i_4 - i_3 \in \{r_m : m = 1, \ldots, l\}$. As $(i_4 - i_1) + (i_2 - i_3) = (i_2 - i_1) + (i_4 - i_3)$ by property (2) of the r_i's

$$\{i_4 - i_1, i_2 - i_3\} = \{i_2 - i_1, i_4 - i_3\}.$$

Hence $i_4 - i_1 = i_2 - i_1$ or $i_4 - i_1 = i_4 - i_3$ so $i_4 = i_2$ or $i_1 = i_3$, and so this is not a square.

Property B. If $P(i_1) \ldots P(i_6)$ is a circle in $G^*(n, k)$ then $i_2 - i_1 = i_4 - i_5$; and similarly $i_3 - i_2 = i_5 - i_6$, $i_4 - i_3 = i_6 - i_1$. Moreover $i_2 = i_4 = i_6$, $i_1 = i_3 = i_5 \pmod 2$.

The proof is similar to that of (A).

Property C. For every P_i, the number of vertices among $\{P(i-j): r_1 < j \leqslant r_l\}$ adjacent to P_i is $\geqslant [(l-1)/2]$.

Proof. Suppose i is even. Then clearly P_i is connected with nected with $P(i - r_{2m})$ for $m = 1, \ldots, [(l-1)/2]$.

Now we shall prove the theorem itself.

Theorem 3.1. *$A[G^*(n, k)] = k$, for $k > 40$, and n sufficiently large relative to k. (Clearly it is $\leqslant k$).*

Proof. We shall prove a stronger result: if θ is a permutation of the vertices of $G^*(n, k)$, then the number of edges PQ, such that $\theta(P), \theta(Q)$ are not connected, is $\geqslant k$.

Suppose G was obtained from $G^*(n, k)$ by $< k$ changes, and θ is an automorphism of G. We should prove θ is the identity.

We shall first prove

(*) there are m_1, m_2, r such that $(r_l)^2 \leqslant m_2$ and for every i, $m_1 \leqslant i \leqslant m_1 + m_2$, $\theta(P_i) = P(i + r)$, or for every i, $m_1 \leqslant i \leqslant m_1 + m_2$, $\theta(P_i) = P(-i + r)$.

Proof of (*). Let A be the set of all vertices of $G^*(n, k)$ satisfying at least one of the following conditions:

(1) An edge which contains it, was removed or added in the change of $G(n, k)$ to $G^*(n, k)$ or from $G^*(n, k)$ to G.

(2) It is connected to Q or is Q (when there is a vertex named Q in the graph).

(3) Its image by θ satisfies (1) or (2).

Now the number of vertices satisfying (1) is $\leqslant 4(k + 15) + 2(k - 1) \leqslant \leqslant 8k$, the number of vertices satisfying (2) is $\leqslant 1 + (l + 1) \leqslant k/2 + 4 \leqslant \leqslant 2k$; and the number of vertices satisfying (3) cannot be more than $8k + 2k$. So $|A| \leqslant 20k$. Hence there are m_1, m_2; $(r_l)^2 < m_2$ such that: for every i, $m_1 \leqslant i \leqslant m_1 + m_2$, P_i and $\theta(P_i)$ do not belong to A. (If there are $> 20k(r_l)^2$ P_i's, this clearly holds, and we have assumed n is sufficiently large). Now clearly P_i and $\theta(P_i)$ have the same valency in $G^*(n, k)$ (as they both $\notin A$) and also they are not Q. So $\theta(P_i)$ is not

Q, and not an R_i, hence it is a P_j let $\theta(P_j) = P_{\theta(j)}$. Clearly for $m_1 \leqslant i$, $j \leqslant m_1 + m_2$, P_i, P_j are connected iff $P_{\theta(i)} P_{\theta(j)}$ are connected. If i is even, $P_i P_j$ are connected if $j - i \in \{r_m : 1 \leqslant m \leqslant l\}$, and similarly if $\theta(i)$ is even, $P_{\theta(i)} P_{\theta(j)}$ are connected iff $\theta(j) - \theta(i) \in \{r_m : 1 \leqslant m \leqslant l\}$. Between the ordered pairs from $\{i\colon m_1 \leqslant i \leqslant m_1 + m_2\}$ we define a relation E_1: $(j_1, j_2) E_1 (j_3, j_4)$ holds iff there are k_1, k_2 in this interval such that $P_{j_1}, P_{j_2} P_{k_1} P_{j_4} P_{j_3} P_{k_2}$ is a circle. Let E be the minimal equivalence relation which extends E_1. By property (B), $(j_1, j_2) E_1 (j_3, j_4)$ implies $j_2 - i_1 = j_4 - j_3$, and $j_1 = j_3$ (mod 2), $j_2 = j_4$ (mod 2). Clearly also $(j_1, j_2) E (j_3, j_4)$ implies the same. If we restrict ourselves to pairs (i, j) such that P_i, P_j is an edge, we have exactly $2l$ equivalence classes ($\langle i, j\rangle$ and $\langle j, i\rangle$ belongs to different equivalence classes), $\langle j_1 j_2\rangle E \langle i_1, i_2\rangle$ holds iff $j_1 = j_2$ (mod 2) and $j_1 - j_2 = i_1 - i_2 \in \{\pm r_m : 1 \leqslant m \leqslant l\}$. We can define similarly E'_1 and E' among $\{\theta(i)\colon m_1 \leqslant i \leqslant m_1 + m_2\}$. On the one hand we can easily see that E'_1 and E' are the images of E_1 and E under θ. That is $\langle j_1, j_2\rangle E \langle j_3, j_4\rangle$ holds iff $\langle\theta(j_1), \theta(j_2)\rangle E' \langle\theta(j_3), \theta(j_4)\rangle$ holds, where $j_1, j_2, j_3, j_4 \in \{i\colon m_1 \leqslant i \leqslant m_1 + m_2\}$. On the other hand repeating the proof for E, and using the fact that E', restricted to pairs $\langle i, j\rangle$ for which $P_i P_j$ is an edge, has exactly $2l$ equivalence classes (as an image of E) *we can conclude* that: $\langle j_1, j_2\rangle E' \langle j_3, j_4\rangle$ holds iff $j_1 = j_3$ (mod 2) and $j_1 - j_2 = j_3 - j_4 \in \{\pm r_m : 1 \leqslant l\}$ (where of course $j_1, j_2, j_3, j_4 \in \{\theta(i)\colon m_1 \leqslant i \leqslant m_1 + m_2\}$.

For $2 \leqslant m \leqslant l$, let $\theta(r_m) = \theta(j + r_m) - \theta(j)$ for any even j_1, $m_1 \leqslant j \leqslant m_1 + m_2$. (Clearly $\theta(r_m)$ is independent of the choice of j). It is easy to prove that either for every m, $\theta(r_m) \in \{r_i\colon 1 \leqslant i \leqslant l\}$, or for every m, $\theta(r_m) \in \{-r_i\colon 1 \leqslant i \leqslant l\}$. Now we shall prove that in the first case $\theta(r_m) = r_m$ and in the second $\theta(r_m) = -r_m$. This is done by considering circles whose edges are from four equivalence classes. Clearly, this implies (*).

Now we shall prove that

(**) either for every i, $\theta(P_i) = P(i + r)$ or for every i. $\theta(P_i) = = P(-i + r)$.

Suppose this is not true. We prove first that there are less than nine

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the vertices $\{P(i_j - m): r, < m < r_l\}$ in $G^*(n, k)$ and $G(n, k)$. It is easy to observe that there are no $k_1, k_2 < i_j$, $k_1, k_2 \neq i_1, \ldots, i_{j-1}, i_j - r_1$ such that $P_{k_1}P_{i_j}, P_{k_2}P_{i_j}$ are connected in $G^*(n, k)$ and $G(n, k)$, and also their images are connected in $G^*(n, k)$ and $G(n, k)$. (Otherwise $P(\theta(k_1))P(\theta(k_2))P(\pm i_j + r)$ is a square in $G^*(n, k)$, contradiction.) Hence the number of edges (in $G^*(n, k)$) $P_iP_{i_j}$, $i \leqslant i_j$ such that $P_{\theta(i)}P_{\theta(ij)}$ is not an edge in $G^*(n, k)$, is $\geqslant [(l-1)/2] - j$. Hence the number of changes is

$$\geqslant \sum_{j=1}^{9} ([(l-1)/2] - j) =$$

$$= 9[l-1)/2] - \sum_{j=1}^{9} j \geqslant 9(l-2)/2 - 10 \cdot 9/2 =$$

$$= 9l/2 - 9 - 5 \cdot 9 = 9l/2 - 54 \geqslant 9(k+1)/4 - 54 =$$

$$= 9k/4 - 9/4 - 54 = k + 5k/4 - 51.75 =$$

$$= (k-1) + (5k - 203)/4 > k - 1$$

(if $k > 203/5 = 40.6$) contradiction.

So we have proved that (**) holds except perhaps for $\leqslant 8P_i$'s. Noticing the edges we add to $G(n, k)$ to create $G^*(n, k)$, clearly, for every i, except possibly the nine mentioned above, $\theta(P_i) = P_i$.

Suppose there are $m > 2$, i's for which $\theta(P_i) \neq P_i$, and let $i_1, \ldots, i_m$ be such indices. Then as before, for each i_j, if $\theta(P_i) = P_i$, $P_iP_{i_j}$ are connected then $\theta(P_i)\theta(P_{i_j})$ are not connected, except possibly one i. Hence the number of edges $P_iP_{i_j}$ such that $\theta(P_i)\theta(P_{i_j})$ are not connected, is $\geqslant l - m$. Hence the number of changes made in $G^*(n, k)$ to create G is $\geqslant m(l-m)$; a contradiction, for $2 < m < 9$, $k > 40$.

So except possibly for two i's $\theta(P_i) = P_i$. Now it is not hard to see that also the number of vertices not transferred to themselves is $\leqslant 2$.

So if θ is not the identity, it interchanges only two vertices. As in $G^*(k, n)$ there is no square or triangle, clearly this also leads to contradiction. So we proved theorem 3.1.

Hint to the proof for $k < 41$. The same way, as we choose $i_1 < \ldots < i_9$, we can choose $i^1 > \ldots > i^9$, which are $< m_1$ and are the greatest nine i's for which $(**)$ fails. Also we can have several intervals satisfying $(*)$, and we can prove their number is not too large, and that the number of i's not in any of them is also not too large. Then we should examine many cases separately, until at least if follows that $(**)$ holds, and the rest is similar to the proof that appears here. For $k = 3$ $(*)$ should include the R_j's as well as the P_i's.

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