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On a question about families of entire functions

by

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Abstract. We show that the existence of a continuum sized family $\mathcal{F}$ of entire functions such that for each complex number z, the set $\{f(z) : f \in \mathcal{F}\}$ has size less than continuum is undecidable in ZFC plus the negation of CH.

1. Introduction. In [2], Erdős asked the following (for some history on this, see [3]):

Question 1.1. *Is there a continuum sized family $\mathcal{F}$ of analytic functions from $\mathbb{C}$ to $\mathbb{C}$ such that for each $z \in \mathbb{C}$, $\{f(z) : f \in \mathcal{F}\}$ has size less than continuum?*

In the same paper, answering a question of Wetzel, Erdős showed that CH is equivalent to the following: There is an uncountable family $\mathcal{F}$ of analytic functions from $\mathbb{C}$ to $\mathbb{C}$ such that for each $z \in \mathbb{C}$, $\{f(z) : f \in \mathcal{F}\}$ is countable. We show here that the answer to Question 1.1 is undecidable in ZFC plus the negation of CH.

2. No such family in the Cohen real model. The following theorem implies that there is no such family in the Cohen real model which is obtained by adding $\aleph_2$ Cohen reals to L.

Theorem 2.1. *Suppose $V \models \mathfrak{c} = \lambda \geq \mathrm{cf}(\lambda) > \kappa = \omega_1$. Let $\mathbb{P}$ add κ Cohen reals. Then in $V^{\mathbb{P}}$, whenever $\mathcal{F}$ is a continuum sized family of entire functions, there exists $z \in \mathbb{C}$ such that $|\{f(z) : f \in \mathcal{F}\}| = \mathfrak{c}$.*

Proof. Let $r \in {}^{\kappa}2$ be the Cohen generic sequence added by $\mathbb{P}$. Clearly, $V[r] \models \mathfrak{c} = \lambda$. Suppose $\langle f_\alpha : \alpha < \lambda\rangle$ is a sequence of pairwise distinct

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entire functions in $V[r]$. Note that each f_α is coded in $V[r\restriction\xi_\alpha]$ for some $\xi_\alpha < \kappa$. As $\mathrm{cf}(\lambda) > \kappa$, we can choose $X \in [\lambda]^\lambda$ and $\xi_\star < \kappa$ such that for each $\alpha \in X$, f_α is coded in $V[r\restriction\xi_\star]$. Let $z_\star \in \mathbb{C}$ be Cohen over $V[r\restriction\xi_\star]$ so that it avoids every meager subset of the complex plane coded in $V[r\restriction\xi_\star]$. Since two distinct entire functions only agree on a countable set, it follows that $\langle f_\alpha(z_\star) : \alpha \in X\rangle$ are pairwise distinct. ∎

3. Consistency with failure of CH. We now show that a positive answer to 1.1 is also consistent with the failure of CH.

THEOREM 3.1. *It is consistent with ZFC plus the negation of CH that there is a family $\mathcal{F}$ of entire functions such that $|\mathcal{F}| = \mathfrak{c}$ and for every $z \in \mathbb{C}$, $|\{f(z) : z \in \mathbb{C}\}| < \mathfrak{c}$.*

Before we begin the proof of Theorem 3.1, let us recall Erdős' construction in [2] under CH. Let $\{z_i : i < \omega_1\} = \mathbb{C}$. Inductively construct $\langle f_i : i < \omega_1\rangle$ such that each $f_i : \mathbb{C} \to \mathbb{C}$ is entire and for every $j < i < \omega_1$, $f_i \neq f_j$ and $f_i(z_j)$ is a rational complex number. This is possible because for every countable $X \subseteq \mathbb{C}$, there is a non-constant entire function sending X into the set of rational complex numbers.

We adopt a slightly different strategy that exploits the singularity of continuum as follows. Starting with a model where $\mathfrak{c} = \omega_{\omega_1}$, we perform a finite support iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \omega_1\rangle$ such that, at each stage $i < \omega_1$, via a ccc forcing $\mathbb{Q}_i$ of size ω_{i+1}, we add a family $\mathcal{F}_i$ of entire functions such that $|\mathcal{F}_i| = \omega_{i+1}$ and for every $j \le i$, letting W_j be the set of first ω_{j+1} members of $V^{\mathbb{P}_i} \cap \mathbb{C}$ in some fixed enumeration, we have $(\forall z \in W_j)(|\{f(z) : f \in \mathcal{F}_i\}| \le \omega_{j+1})$. So $\mathcal{F} = \bigcup\{\mathcal{F}_i : i < \omega_1\}$ will be the required family in $V^{\mathbb{P}}$. The possible set of values for $\{f(z) : f \in \mathcal{F}_i\}$ is not fixed beforehand but added generically together with $\mathcal{F}$—this is the major point of difference with Erdős' construction. The main problem then is to ensure that $\mathbb{Q}_i$ is ccc. We do this by requiring that the finite approximations to members of $\{f(z) : z \in W_i\}$ can be chosen quite independently of those for $\{g(z) : z \in W_i\}$, for $f \neq g \in \mathcal{F}_i$. This is materialized by using strongly almost disjoint families in $[\omega_{i+1}]^{\omega_{i+1}}$. The next lemma says that such families can consistently exist.

LEMMA 3.2. *The following is consistent:*

(a) $\mathfrak{c} = \omega_{\omega_1}$.
(b) *There is a family $\{A_\alpha : \alpha < \omega_{\omega_1}\}$ such that each A_α is in $[\omega_{\omega_1}]^{\omega_{\omega_1}}$.*
(c) *For every $\alpha < \beta < \omega_{\omega_1}$, $A_\alpha \cap A_\beta$ is finite.*
(d) *For every $i < \omega_1$ and $\alpha < \omega_{\omega_1}$, $|A_\alpha \cap \omega_{i+1}| = \omega_{i+1}$.*

Proof. We use Baumgartner's thinning out forcing [1, Theorem 6.1]. Let $V \models \mathrm{GCH}$. Set $\lambda = \omega_{\omega_1}$ and $\lambda_i = \omega_{i+1}$. For each $1 \le i < \omega_1$, define $\mathbb{P}_i$ as follows. Let $K_i = \{\nu \in [\omega_2, \lambda_i] : \nu = \mathrm{cf}(\nu)\}$. Then $p \in \mathbb{P}_i$ iff:

(i) $p = \langle p_\nu : \nu \in K_i \rangle$.
(ii) Each p_ν is a function with $\mathrm{dom}(p_\nu) \in [\lambda]^{<\nu}$.
(iii) For each $\alpha \in \mathrm{dom}(p_\nu)$, $p_\nu(\alpha) \in [\lambda_i]^{<\nu}$.
(iv) If $\nu < \nu'$, then $\mathrm{dom}(p_\nu) \subseteq \mathrm{dom}(p_{\nu'})$, and for each $\alpha \in \mathrm{dom}(p_\nu)$, $p_\nu(\alpha) \subseteq p_{\nu'}(\alpha)$.

For $p, q \in \mathbb{P}_i$, write $p \leq_i q$ iff:

- For each $\nu \in K_i$, $\mathrm{dom}(p_\nu) \subseteq \mathrm{dom}(q_\nu)$.
- For each $\alpha, \beta \in \mathrm{dom}(p_\nu)$, $p_\nu(\alpha) \subseteq q_\nu(\alpha)$, and if $\alpha \neq \beta$, then $p_\nu(\alpha) \cap p_\nu(\beta) = q_\nu(\alpha) \cap q_\nu(\beta)$.

Let $\mathbb{P} = \prod\{\mathbb{P}_i : i < \kappa\}$ be the full support product of $\{\mathbb{P}_i : i < \kappa\}$. So $p \in \mathbb{P}$ iff $p = \langle p(i) : i < \kappa \rangle$ and $p(i) \in \mathbb{P}_i$ for every $i < \kappa$. For $p, q \in \mathbb{P}$, $p \leq q$ iff $p(i) \leq_i q(i)$ for every $i < \kappa$.

CLAIM 3.3. *$\mathbb{P}$ preserves all regular cardinals below λ.*

Proof of Claim 3.3. The proof is almost identical to that of [1, Lemma 6.6] but we provide a sketch. Let G be $\mathbb{P}$-generic over V. Let $\tau < \lambda$ be a regular cardinal in V and suppose $V[G] \models \tau > \mathrm{cf}(\tau) = \mu$. Note that $\mathbb{P}$ is ω_2-closed, so $\mu \geq \omega_2$. Fix $1 \leq i_\star < \omega_1$ such that $\mu = \lambda_{i_\star}$.

Let $\mathbb{Q} = \{\langle p(i) \restriction [\lambda_{i_\star+1}, \infty) : i < \omega_1 \rangle : p \in \mathbb{P}\}$ and $H = \{\langle p(i) \restriction [\lambda_{i_\star+1}, \infty) : i < \omega_1 \rangle : p \in G\}$. Then $\mathbb{Q}$ is $\lambda_{i_\star+1}$-closed and H is $\mathbb{Q}$-generic over V. In $V[H]$, for $i_\star < i < \omega_1$ and $\alpha < \lambda$, let $E_{i,\alpha} = \bigcup\{p(i)(\lambda_{i_\star+1})(\alpha) : p \in H\}$ and for $i \leq i_\star$ and $\alpha < \lambda$, let $E_{i,\alpha} = \lambda_i$. Let

$$\mathbb{Q}' = \{\langle p(i) \restriction [0, \lambda_{i_\star}] : i < \omega_1 \rangle : p \in \mathbb{P} \wedge (\forall \alpha \in \mathrm{dom}(p(i)(\lambda_{i_\star})))(p_i(\lambda_{i_\star})(\alpha) \subseteq E_{i,\alpha})\}$$

and $K = \{\langle p(i) \restriction [0, \lambda_{i_\star}] : i < \omega_1 \rangle : p \in G\}$. Then it is easily verified that K is $\mathbb{Q}'$-generic over $V[H]$ and $V[G] = V[H][K]$. As $\mathbb{Q}$ is $\lambda_{i_\star+1}$-closed, $\mathrm{cf}(\tau) \geq \lambda_{i_\star+1}$ in $V[H]$. Since $\lambda_{i_\star} \geq \omega_2$, a Δ-system argument shows that $V[H] \models \mathbb{Q}'$ satisfies $\lambda_{i_\star+1}$-c.c. (see [1, Lemma 6.3]), hence $V[G] = V[H][K] \models \mathrm{cf}(\tau) \geq \lambda_{i_\star+1} > \mu$, a contradiction. $\blacksquare_{3.3}$

Let G be $\mathbb{P}$-generic over V and $V_1 = V[G]$. In V_1, for $\alpha < \lambda$, let $F_\alpha = \bigcup\{F_{i,\alpha} \cap [\omega_i, \omega_{i+1}) : i < \omega_1\}$ where $F_{i,\alpha} = \bigcup\{q_{\omega_2}(\alpha) : (\exists p \in G)(q = p(i))\}$. Then each F_α is unbounded in ω_{i+1} for $1 \leq i < \omega_1$ and their pairwise intersections have sizes $\leq \omega_1$.

In V_1, define $\mathbb{P}_1$ by $p \in \mathbb{P}_1$ iff p is a function, $\mathrm{dom}(p) \in [\lambda]^{<\aleph_1}$ and $p(\alpha) \in [F_\alpha \cup \omega_1]^{<\aleph_1}$ for each $\alpha \in \mathrm{dom}(p)$. For $p, q \in \mathbb{P}_1$, $p \leq q$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $\alpha, \beta \in \mathrm{dom}(p)$, $p(\alpha) \subseteq q(\alpha)$ and if $\alpha \neq \beta$, then $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$. As CH holds in V_1, a Δ-system argument shows that $\mathbb{P}_1$ satisfies $\aleph_2$-cc. Since it is also countably closed, all cofinalities from V_1 are preserved.

Let G_1 be $\mathbb{P}_1$-generic over V_1 and $V_2 = V_1[G_1]$. For $\alpha < \lambda$, set $F'_\alpha = \bigcup\{p(\alpha) : p \in G_1\}$. Then each F'_α is unbounded in ω_{i+1} for $i < \omega_1$ and their pairwise intersections are countable.

In V_2, define $\mathbb{P}_2$ by $p \in \mathbb{P}_2$ iff p is a function, $\mathrm{dom}(p) \in [\lambda]^{<\aleph_0}$ and for each $\alpha \in \mathrm{dom}(p)$, $p(\alpha) \in [F'_\alpha]^{<\aleph_0}$. For $p, q \in \mathbb{P}_1$, $p \leq q$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all $\alpha, \beta \in \mathrm{dom}(p)$, $p(\alpha) \subseteq q(\alpha)$ and if $\alpha \neq \beta$, then $p(\alpha) \cap p(\beta) = q(\alpha) \cap q(\beta)$. A Δ-system argument shows that $\mathbb{P}_2$ satisfies ccc, so all cofinalities are preserved.

Let G_2 be $\mathbb{P}_2$-generic over V_2 and $V_3 = V_2[G_2]$. For $\alpha < \lambda$, set $A_\alpha = \bigcup\{p(\alpha) : p \in G_2\}$. Then each A_α is unbounded below ω_{i+1} for each $i < \omega_1$ and their pairwise intersections are finite. As $\{A_\alpha \cap \omega_1 : \alpha < \lambda\}$ is a mod finite almost disjoint family, $V_3 \models \mathfrak{c} \geq \lambda$. The other inequality follows from a name counting argument using $V_2 \models \lambda^{\aleph_0} = \lambda$. $\blacksquare_{3.2}$

Proof of Theorem 3.1. Let V be a model satisfying the clauses of Lemma 3.2. We will construct a finite support iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \omega_1 \rangle$ of ccc forcings with limit $\mathbb{P}$ satisfying the following:

- $|\mathbb{P}| = \omega_{\omega_1}$.
- $\Vdash_{\mathbb{P}_i} \langle \mathring{z}_{i,\alpha} : \alpha < \omega_{\omega_1} \rangle$ lists $\mathbb{C}$ and $\mathring{Z}_i = \{\mathring{z}_{j,\alpha} : j \leq i,\, \alpha < \omega_{i+1}\}$.
- $\langle \mathring{y}_\alpha : \alpha < \omega_{i+1} \rangle \in V^{\mathbb{P}_i}$ is such that $\Vdash_{\mathbb{P}_i} \langle \mathring{y}_\alpha : \alpha < \omega_{i+1} \rangle$ is a one-one listing of $\mathring{Z}_i$, so $\{\mathring{y}_\alpha : \alpha < \omega_{\omega_1}\} = \mathbb{C} \cap V^{\mathbb{P}}$.
- In $V^{\mathbb{P}_i}$, $\mathbb{Q}_i$ is a ccc forcing of size λ_i that adds a family $\mathcal{F}_i$ of entire functions of size ω_{i+1} such that for every $j \leq i$, $\Vdash_{\mathbb{Q}_i} |\{f(y_\alpha) : \alpha < \omega_{j+1}\}| \leq \omega_{j+1}$.

Set $\mathcal{F} = \bigcup_{i<\omega_1} \mathcal{F}_i$. If $\mathring{z} \in V^{\mathbb{P}} \cap \mathbb{C}$, then for some $i_\star < \omega_1$ and $\alpha < \omega_{i_\star+1}$, we have $\mathring{z} = y_\alpha$. Hence

$$\begin{aligned}|\{f(\mathring{z}) : f \in \mathcal{F}\}| &\leq \Big|\bigcup_{i<i_\star} \{f(\mathring{z}) : f \in \mathcal{F}_i\}\Big| + \Big|\bigcup_{i>i_\star} \{f(y_\alpha) : f \in \mathcal{F}_i\}\Big| \\ &\leq \omega_{i_\star+1} + \omega_1 \cdot \omega_{i_\star+1} = \omega_{i_\star+1} < \mathfrak{c}.\end{aligned}$$

The following lemma shows that $\mathbb{Q}_i$'s can be constructed.

LEMMA 3.4. *Suppose κ is regular uncountable. Let $\langle A_\alpha : \alpha < \kappa \rangle$ be such that for every $\alpha < \beta < \kappa$ and uncountable cardinal $\mu \leq \kappa$, $A_\alpha \cap \mu \in [\mu]^\mu$ and $A_\alpha \cap A_\beta$ is finite (so $\kappa \leq \mathfrak{c}$). Let $\langle y_\alpha : \alpha < \kappa \rangle$ be a sequence of distinct complex numbers. Then there exists a ccc forcing $\mathbb{Q}$ of size κ such that the following hold in $V^{\mathbb{Q}}$:*

(a) *There is a family $\mathcal{F}$ of entire functions of size κ.*
(b) *For every uncountable cardinal $\mu \leq \kappa$, $|\{f(y_\alpha) : \alpha < \mu, f \in \mathcal{F}\}| = \mu$.*

Proof of Lemma 3.4. For $\xi < \kappa$, let $h_\xi : \kappa \to A_\xi$ be such that $h_\xi(\alpha)$ is the αth member of A_ξ. Note that $h_\xi[\mu] = A_\xi \cap \mu$ for every regular uncountable

$\mu < \kappa$. Define $\mathbb{Q}$ as follows: $p \in \mathbb{Q}$ iff

$$p = (n_p, m_p, u_p, v_p, w_p, \langle m^p_{\xi,\alpha} : \xi \in u_p, \alpha \in v_p\rangle, \langle f^p_\xi : \xi \in u_p\rangle, \langle B^p_{\gamma,m} : \gamma \in w_p,\ m < m_p\rangle)$$

where:

- $1 \le n_p < \omega$, $1 \le m_p < \omega$.
- $u_p, v_p, w_p \in [\kappa]^{<\aleph_0}$ and $|y_\alpha| < n_p$ for every $\alpha \in v_p$.
- $w_p \supseteq \{h_\xi(\alpha) : \xi \in u_p,\ \alpha \in v_p\}$.
- $m^p_{\xi,\alpha} < m_p$ for all $\xi \in u_p$ and $\alpha \in v_p$.
- For each $\xi \in u_p$, $f^p_\xi = f^p_\xi(x, x'_\alpha, x''_\alpha)_{\alpha\in v_p} = f^p_\xi(x, x'_\alpha, x''_\alpha : \alpha \in v_p)$ is a rational function in the $2|v_p| + 1$ variables $\{x\} \cup \{x'_\alpha, x''_\alpha : \alpha \in v_p\}$ over the rational complex field (complex numbers whose real and imaginary parts are rational) which can be expressed as a polynomial in x whose coefficients are rational functions of $\{x'_\alpha, x''_\alpha : \alpha \in v_p\}$ such that $f^p_\xi(x'_\beta, x'_\alpha, x''_\alpha)_{\alpha\in v_p} = x''_\beta$ for every $\beta \in v_p$.
- For every $\gamma \in w_p$ and $m < m_p$, $B^p_{\gamma,m}$ is a closed disk in the complex plane with rational complex center and rational radius that satisfies: If $z_{\alpha,\xi,m} \in B^p_{h_\xi(\alpha),m}$ for some $\xi \in u_p$, $\alpha \in v_p$ and $m < m_p$, then $f_\xi(x, y_\alpha, z_{\alpha,\xi,m^p_{\xi,\alpha}})_{\alpha\in v_p}$ is well defined (no vanishing denominators).

Informally, p promises that for $\xi \in u_p$, the ξth entire function $\mathring{f}_\xi$ added by $\mathbb{Q}$ is approximated by $f^p_\xi(x, y_\alpha, z_\alpha)_{\alpha\in v_p}$ uniformly on the disk $\{x \in \mathbb{C} : |x| \le n_p\}$ with an error $\le 2^{-n_p}$ where z_α is an arbitrary point in $B^p_{h_\xi(\alpha),m^p_{\xi,\alpha}}$. It also promises that $\mathring{f}_\xi$ will map y_α (for $\alpha \in v_p$) into $B^p_{h_\xi(\alpha),m^p_{\xi,\alpha}}$. The parameter m in $B^p_{\gamma,m}$ allows us a countable amount of freedom to choose $\mathring{f}_\xi(y_\alpha)$ (this is useful to increase v_p, see Claim 3.5(c) below).

For $p, q \in \mathbb{Q}$, define $p \le q$ iff:

- $n_p \le n_q$, $m_p \le m_q$.
- $u_p \subseteq u_q$, $v_p \subseteq v_q$ and $w_p \subseteq w_q$.
- If $\xi \in u_p, \alpha \in v_p$, then $m^q_{\xi,\alpha} = m^p_{\xi,\alpha}$.
- $B^q_{\gamma,m} \subseteq B^p_{\gamma,m}$ for all $\gamma \in w_p$ and $m < m_p$.
- Whenever $|z| < n_p$, $\xi \in u_p$, $z_{\xi,\alpha} \in B^q_{h_\xi(\alpha),m^q_{\xi,\alpha}}$ with $\alpha \in v_q$, we have

$$|f^p_\xi(z, y_\alpha, z_{\xi,\alpha})_{\alpha\in v_p} - f^q_\xi(z, y_\alpha, z_{\xi,\alpha})_{\alpha\in v_q}| \le 1/2^{n_p} - 1/2^{n_q}.$$

Claim 3.5. *The following are dense in* $\mathbb{Q}$:

(a) $\{p \in \mathbb{Q} : \xi \in u_p\}$ *for* $\xi < \kappa$.
(b) $\{p \in \mathbb{Q} : (\gamma \in w_p) \wedge (n_p, m_p \ge N)\}$ *for* $N < \omega$ *and* $\gamma < \kappa$.
(c) $\{p \in \mathbb{Q} : \beta \in v_p\}$ *for* $\beta < \kappa$.
(d) $\{p \in \mathbb{Q} : (\forall m < m_p)(\forall \gamma \in w_p)(\mathrm{diam}(B^p_{\gamma,m}) < 2^{-N})\}$ *for* $N < \omega$.
(e) $\{p \in \mathbb{Q} : (\forall \gamma_1, \gamma_2 \in w_p)(\forall m_1, m_2 < m_p)((\gamma_1, n_1) \ne (\gamma_2, n_2) \Rightarrow B^p_{\gamma_1,m_1} \cap B^p_{\gamma_2,m_2} = \emptyset)\}$.

Proof of Claim 3.5. Clauses (b), (d) and (e) should be clear. Let us check (a) and (c).

(a) Suppose $q \in \mathbb{Q}$ with $\xi_\star \in \kappa \setminus u_q$. If $v_q = \emptyset$, then we can add $\xi_\star$ to u_q and set $f^q_{\xi_\star}(x) = 0$. So assume $v_q = \{\alpha_i : 1 \le i \le k\}$. Define $g_i = g_i(x, x'_{\alpha_j}, x''_{\alpha_j})_{1 \le j \le i}$ for $1 \le i \le k$ recursively as follows:

$$g_1 = x + x''_{\alpha_1} - x'_{\alpha_1},$$

$$g_{i+1} = g_i + \Big(\prod_{1 \le j \le i} (x - x'_{\alpha_j})\Big) \frac{x''_{\alpha_{i+1}} - g_i(x'_{\alpha_{i+1}}, x'_{\alpha_j}, x''_{\alpha_j})_{1 \le j \le i}}{\prod_{1 \le j \le i}(x'_{\alpha_{i+1}} - x'_{\alpha_j})}.$$

Define $p \ge q$ as follows. Set $n_p = n_q$, $m_p = m_q$, $u_p = u_q \cup \{\xi_\star\}$, $v_p = v_q$, $w_p = w_q \cup \{h_{\xi_\star}(\alpha) : \alpha \in v_q\}$, $f^p_{\xi_\star} = g_k$ and $f^p_\xi = f^q_\xi$ for $\xi \in u_q$. Set $m^p_{\xi,\alpha} = m^q_{\xi,\alpha}$ and $B^p_{\gamma,m} = B^q_{\gamma,m}$ if already defined; otherwise choose them arbitrarily.

(c) Suppose $q \in \mathbb{Q}$ and $\beta \in \kappa \setminus v_q$. By increasing n_q, we can assume $|y_\beta| < n_q$. For each $\xi \in u_q$, define f^p_ξ by

$$f^p_\xi = f^q_\xi + \Big(\prod_{\alpha \in v_q} (x - x'_\alpha)\Big) \frac{x''_\beta - f^q_\xi(x'_\beta, x'_\alpha, x''_\alpha)_{\alpha \in v_q}}{\prod_{\alpha \in v_q}(x'_\beta - x'_\alpha)},$$

where we take a product over the empty index set to be 1.

Let $\varepsilon = \min\{|y_\beta - y_\alpha| : \alpha \in v_q\}$ if $v_q \ne \emptyset$ and $\varepsilon = 1$ otherwise. Set $u_p = u_q$, $v_p = v_q \cup \{\beta\}$, $w_p = w_q \cup \{h_\xi(\beta) : \xi \in u_q\}$, $n_p = n_q + 1$, $m_p = m_q + |u_q|$. For each $\xi \in u_q$, choose $m^p_{\xi,\beta} \ge m_q$ such that $\xi_1 \ne \xi_2$ implies $m^p_{\xi_1,\beta} \ne m^p_{\xi_2,\beta}$. We need to choose $B^p_{\gamma,m}$'s such that whenever $|z| < n_q$, $\xi \in u_q$ and $z_{\xi,\alpha} \in B^p_{h_\xi(\alpha), m^p_{\xi,\alpha}}$ for $\alpha \in v_p$, we have

$$\left| \Big(\prod_{\alpha \in v_q} (z - y_\alpha)\Big) \frac{z_{\xi,\beta} - f^q_\xi(y_\beta, y_\alpha, z_{\xi,\alpha})_{\alpha \in v_q}}{\prod_{\alpha \in v_q}(y_\beta - y_\alpha)} \right| \le \frac{1}{2^{n_q}} - \frac{1}{2^{n_q+1}}.$$

For this it is enough to have

$$|z_{\xi,\beta} - f^q_\xi(y_\beta, y_\alpha, z_{\xi,\alpha})_{\alpha \in v_q}| \le \frac{\varepsilon^k}{(2n_q)^k 2^{n_q+1}}$$

where $k = |v_q|$. But this is easily arranged by first shrinking $B^q_{h_\xi(\alpha), m^q_{\xi,\alpha}}$'s for $\alpha \in v_q$ and then choosing $B^p_{h_\xi(\beta), m^p_{\xi,\beta}}$ accordingly. $\blacksquare_{3.5}$

Let G be $\mathbb{Q}$-generic over V. For $\gamma < \kappa$ and $m < \omega$, let $a_{\gamma,m}$ be the unique member of $\bigcap\{B^p_{\gamma,m} : p \in G\}$.

For $\xi < \lambda$, define $f_\xi : \mathbb{C} \to \mathbb{C}$ as follows. Choose $\{p_k : k < \omega\} \subseteq G$ such that $\xi \in u_{p_k}$ and $n_{p_k} \ge k$ for every $k < \omega$, and set $f_\xi(z) = \lim_k f^{p_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{p_k}_{\xi,\alpha}})_{\alpha \in v_{p_k}}$. Since we have uniform convergence on compact sets, f_ξ is analytic. Note that the definition of f_ξ is independent of the choice of $\{p_k : k < \omega\} \subseteq G$. For suppose $\{q_k : k < \omega\} \subseteq G$ is such that $\xi \in u_{q_k}$ and $n_{q_k} \geq k$ for every $k < \omega$. Let $r_k \in G$ be a common extension of p_k, q_k. Then, for every $z \in \mathbb{C}$ with $|z| < k$, we have

$$|f^{p_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{p_k}_{\xi,\alpha}})_{\alpha \in v_{p_k}} - f^{q_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{q_k}_{\xi,\alpha}})_{\alpha \in v_{q_k}}| \leq 2^{-k+1}$$

since it is at most

$$\begin{aligned}|f^{p_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{p_k}_{\xi,\alpha}})_{\alpha \in v_{p_k}} &- f^{r_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{r_k}_{\xi,\alpha}})_{\alpha \in v_{r_k}}| \\ &+ |f^{q_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{q_k}_{\xi,\alpha}})_{\alpha \in v_{q_k}} - f^{r_k}_\xi(z, y_\alpha, a_{h_\xi(\alpha), m^{r_k}_{\xi,\alpha}})_{\alpha \in v_{r_k}}|,\end{aligned}$$

and hence the two limits must be the same.

Set $\mathcal{F} = \{f_\xi : \xi < \kappa\}$. For $\xi, \alpha < \kappa$, let $m_{\xi,\alpha}$ be such that for some $p \in G$ we have $\xi \in u_p$, $\alpha \in v_p$ and $m^p_{\xi,\alpha} = m_{\xi,\alpha}$. Note that, for every $\xi, \alpha < \kappa$, by considering a sequence $\{p_k : k < \omega\} \subseteq G$ with $\alpha \in v_{p_k}$, we can infer that $f_\xi(y_\alpha) = a_{h_\xi(\alpha), m_{\xi,\alpha}}$. Next suppose $\xi_1 < \xi_2 < \kappa$. Choose $\alpha < \kappa$ such that $h_{\xi_1}(\alpha) \neq h_{\xi_2}(\alpha)$. Then $f_{\xi_1}(y_\alpha) = a_{h_{\xi_1}(\alpha), m_{\xi_1,\alpha}} \neq a_{h_{\xi_2}(\alpha), m_{\xi_2,\alpha}} = f_{\xi_2}(y_\alpha)$. So f_ξ's are pairwise distinct. Finally, for every uncountable $\mu \leq \kappa$, we have

$$\begin{aligned}|\{f_\xi(y_\alpha) : \alpha < \mu,\, \xi < \kappa\}| &\leq |\{a_{h_\xi(\alpha), m_{\xi,\alpha}} : \xi < \kappa,\, \alpha < \mu\}| \\ &\leq |\{a_{\gamma,m} : \gamma < \mu,\, m < \omega\}| = \mu.\end{aligned}$$

So it suffices to show that $\mathbb{Q}$ is ccc. Suppose $A \subseteq \mathbb{Q}$ is uncountable. Choose $S \subseteq A$ uncountable such that the following hold:

- $n_p = n_\star$, $m_p = m_\star$, $|u_p| = n^1_\star$ and $|v_p| = n^2_\star$ do not depend on $p \in S$.
- $\langle u_p : p \in S\rangle$ is a Δ-system with root $u_\star$, and $\langle v_p : p \in S\rangle$ is a Δ-system with root $v_\star$.
- If $\xi_1 \neq \xi_2$ are from $u_\star$ and $h_{\xi_1}(\alpha_1) = h_{\xi_2}(\alpha_2)$, then $\{\alpha_1, \alpha_2\} \cap (v_p \setminus v_\star) = \emptyset$ for every $p \in S$. This uses the fact that $A_{\xi_1} \cap A_{\xi_2}$ is finite (countable suffices).
- By possibly extending $p \in S$, we can assume $1 \leq |v_\star| < n^2_\star$ (so v_p and $v_p \setminus v_\star$ are non-empty).
- $u_p = \{\xi_{p,j} : j < n^1_\star\}$ and $v_p = \{\alpha_{p,k} : k < n^2_\star\}$ list members in increasing order, and $r^1_\star \subseteq n^1_\star$ and $r^2_\star \subseteq n^2_\star$ are such that $u_\star = \{\xi_{p,k} : j \in r^1_\star\}$ and $v_\star = \{\alpha_{p,k} : k \in r^2_\star\}$.
- For all $j < n^1_\star$, $k < n^2_\star$ and $m < m_\star$, we have $f^p_{\xi_{p,j}} = f_j(x, x'_{\alpha_{p,k}}, x''_{\alpha_{p,k}})_{k<n^2_\star}$, $m^p_{\xi_{p,j},\alpha_{p,k}} = m_{j,k}$ and $B^p_{h_{\xi_{p,j}}(\alpha_{p,k}),m} = B_{j,k,m}$ where f_j, $m_{j,k}$, $B_{j,k,m}$ do not depend on $p \in S$.
- $0 < \varepsilon_1 < 2^{-(n_\star+1)}$, ε_1 is smaller than the radius of every $B_{j,k,m}$, and $|y_{\alpha_{p,k_1}} - y_{\alpha_{p,k_2}}| > \varepsilon_1$ for all $p \in S$ and $k_1 < k_2 < n^2_\star$.
- Each point of $X = \{\langle y_{\alpha_{p,k}} : k < n^2_\star\rangle : p \in S\}$ is a condensation point of $X \subseteq \mathbb{C}^{n^2_\star}$.

Suppose $p, p' \in S$ and we would like to find a common extension q. This boils down to constructing f^q_ξ for $\xi \in u_p \cup u_{p'}$. For $\xi \in (u_p \cup u_{p'}) \setminus u_\star$, this is similar to the proof of Claim 3.5(c). To construct f^q_ξ for $\xi \in u_\star$, we will make use of the following lemma.

LEMMA 3.6. *Suppose:*

(i) $1 \leq n_\star < \omega$, $0 < \varepsilon_1 < 0.5$.
(ii) $f = f(z, x_k, y_k)_{k<k_\star}$ *is a rational function in the variables* $\{z\} \cup \{x_k, y_k : k < k_\star\}$ *over the rational complex field which can be expressed as a polynomial in* z *whose coefficients are rational functions of* x_k, y_k *for* $k < k_\star$ *over the rational complex field, satisfying* $f(x_l, x_k, y_k)_{k<k_\star} = y_l$ *for every* $l < k_\star$.
(iii) $a_k, b_k \in \mathbb{C}$ *for* $k < k_\star$, $|a_k| < n_\star$ *for* $k < k_\star$, *and* $|a_{k_1} - a_{k_2}| > \varepsilon_1$ *for every* $k_1 < k_2 < k_\star$.
(iv) *If* $|a'_k - a_k| < \varepsilon_1$ *and* $|b'_k - b_k| < \varepsilon_1$ *for* $k < k_\star$, *then* $f(z, a'_k, b'_k)_{k<k_\star}$ *is well defined (no vanishing denominators).*
(v) $v_\star \subseteq k_\star$, $v_\star \notin \{\emptyset, k_\star\}$.

Then there exist $0 < \varepsilon_2 < \varepsilon_1/8$ *and* $g = g(z, x_l, y_l, x^1_k, x^2_k, y^1_k, y^2_k)_{l\in v_\star, k\in k_\star\setminus v_\star}$ *such that whenever* $|a^2_k - a_k| < \varepsilon_2$ *for* $k \in k_\star \setminus v_\star$, *letting* $b^2_k = f(a^2_k, a_j, b_j)_{j<k_\star}$ *we have* $|b^2_k - b_k| < \varepsilon_1 - 2\varepsilon_2$ *for* $k \in k_\star \setminus v_\star$ *and the following hold:*

(a) g *is a polynomial in* z *whose coefficients are rational functions of the other variables over the rational complex field satisfying* $z = x_l$ *implies* $g = y_l$ *for* $l \in v_\star$ *and* $z = x^j_k$ *implies* $g = y^j_k$ *for* $j = 1, 2$ *and* $k \in k_\star \setminus v_\star$
(b) *Letting* $a^1_k = a_k$, $b^1_k = b_k$ *for* $k \in k_\star \setminus v_\star$ *we have the following. For every* c_l, c^j_k *satisfying* $|c_l - b_l| < \varepsilon_2$, $|c^j_k - b^j_k| < \varepsilon_2$ *for* $l \in v_\star$, $j = 1, 2$, $k \in k_\star \setminus v_\star$, *we have*

$$|f(z, a_l, a^j_k, c_l, c^j_k)_{l\in v_\star,\, k\in k_\star\setminus v_\star} - g(z, a_l, c_l, a^1_k, a^2_k, c^1_k, c^2_k)_{l\in v_\star,\, k\in k_\star\setminus v_\star}| < \varepsilon_1,$$

for all $|z| < n_\star$ *and* $j = 1, 2$.

Proof of Lemma 3.6. Set

$$g = f(z, x_l, x^1_k, y_l, y^1_k)_{l\in v_\star, k\in k_\star\setminus v_\star} + \sum_{j\in k_\star\setminus v_\star} G_j$$

where

$$G_j = \frac{F_j(z)[y^2_j - f(x^2_j, x_l, x^1_k, y_l, y^1_k)_{l\in v_\star, k\in k_\star\setminus v_\star}]}{F_j(x^2_j)},$$

$$F_j(z) = \prod_{\substack{k\in k_\star\setminus v_\star \\ k\neq j}} (z - x^2_k) \prod_{l\in v_\star} (z - x_l) \prod_{k\in k_\star\setminus v_\star} (z - x^1_k)$$

Clause (a) is easily verified. We need to find $0 < \varepsilon_2 < \varepsilon_1/8$ such that clause (b) holds. Note that for all sufficiently small $\varepsilon_2 < \varepsilon_1/8$, if $|a^2_j - a_j| < \varepsilon_2$, then $|b_j^2 - b_j| = |f(a_j^2, a_k, b_k)_{k<k_\star} - f(a_j, a_k, b_k)_{k<k_\star}| < 3\varepsilon_1/4 < \varepsilon_1 - 2\varepsilon_2$. Fix c_l, c_k^j as in clause (b) and consider

$$|f(z, a_l, a_k^j, c_l, c_k^j)_{l\in v_\star,\, k\in k_\star\setminus v_\star} - g(z, a_l, c_l, a_k^1, a_k^2, c_k^1, c_k^2)_{l\in v_\star,\, k\in k_\star\setminus v_\star}|.$$

This is at most

$$|f(z, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star,\, k\in k_\star\setminus v_\star} - f(z, a_l, a_k^2, c_l, c_k^2)_{l\in v_\star,\, k\in k_\star\setminus v_\star}| + \sum_{j\in k_\star\setminus v_\star} |G_j(z, a_l, c_l, a_k^1, a_k^2, c_k^1, c_k^2)_{l\in v_\star,\, k\in k_\star\setminus v_\star}|.$$

The former term is easily bounded by $\varepsilon_1/2$ by choosing sufficiently small ε_2. For the latter, notice that

$$\left|\frac{F_j(z)}{F_j(a_j^2)}\right| < \left(\frac{4n_\star}{\varepsilon_1}\right)^{2k_\star}.$$

So it suffices to ensure that

$$|c_j^2 - f(a_j^2, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star,\, k\in k_\star\setminus v_\star}| < \frac{\varepsilon_1^{2k_\star+1}}{k_\star(4n_\star)^{2k_\star}}.$$

The expression on the left side is at most

$$|c_j^2 - b_j^2| + |b_j^2 - f(a_j^2, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star,\, k\in k_\star\setminus v_\star}|.$$

Recalling our choice of b_j^2, this is bounded by

$$\varepsilon_2 + |f(a_j^2, a_l, a_k^1, b_l, b_k^1)_{l\in v_\star, k\in k_\star\setminus v_\star} - f(a_j^2, a_l, a_k^1, c_l, c_k^1)_{l\in v_\star, k\in k_\star\setminus v_\star}|.$$

It is clear that this can be made arbitrarily small by choosing sufficiently small ε_2. $\blacksquare_{3.6}$

Fix $p \in S$. For each $j \in r_\star^1$, using Lemma 3.6, we get $\varepsilon_2 = \varepsilon_{2,j}$ and $g = g_j$ for $f = f_j$, $a_k = y_{\alpha_{p,k}}$, $b_k =$ the center of $B_{j,k,m_{j,k}}$ and $v_\star = r_\star^2$. Let $\varepsilon_3 = \min\{\varepsilon_{2,j} : j \in r_\star^1\}$. Choose $p' \neq p$ from S such that $|y_{\alpha_{p,k}} - y_{\alpha_{p',k}}| < \varepsilon_3$ for each $k < n_\star^2$. We will construct a common extension q of p, p'.

Set $n_q = n_\star + 1$, $m_q = m_\star + n_\star^1 n_\star^2$, $u_q = u_p \cup u_{p'}$, $v_q = v_p \cup v_{p'}$ and $w_q = w_p \cup w_{p'} \cup \{h_\xi(\alpha) : \xi \in u_q,\ \alpha \in v_q\}$. Choose $m_{\xi,\alpha}^q$'s such that $\{m_{\xi,\alpha}^q : (\xi \in u_p \setminus u_\star \wedge \alpha \in v_p \setminus v_\star) \text{ or } (\xi \in u_{p'} \setminus u_\star \wedge \alpha \in v_{p'} \setminus v_\star)\}$ are pairwise distinct integers in $[m_\star, m_p)$. Next choose f_ξ^q, $B_{\gamma,m}^q$ for $\xi \in u_q$, $\gamma \in w_q$ and $m < m_q$ as follows:

- If $\xi \in u_p \setminus u_\star$, let f_ξ^q be as in the proof of Claim 3.5(c) applying the process $|v_{p'} - v_\star|$ times. Define $B_{h_\xi(\alpha), m_{\xi,\alpha}^q}^q$ for $\alpha \in v_p$ by shrinking $B_{h_\xi(\alpha), m_{\xi,\alpha}^p}^p$ and choose $B_{h_\xi(\alpha), m_{\xi,\alpha}^q}^q$ for $\alpha \in v_{p'} \setminus v_\star$ accordingly.
- If $\xi \in u_{p'} \setminus u_\star$, we define f_ξ^q and $B_{h_\xi(\alpha), m_{\xi,\alpha}^q}^q$ analogously.
- If $\xi \in u_\star$, choose $j \in r_\star^1$ such that $\xi_{p,j} = \xi$ and set $f_\xi^q = g_j$. Obtain b_k^2 for $k \in n_\star^2 \setminus r_\star^2$ as in Lemma 3.6 for $a_k^2 = y_{\alpha_{p',k}}$. For $k \in n_\star^2$, choose $B_{h_\xi(\alpha_{p,k}), m_{j,k}}^q$

to be a rational disk contained in a disk inside $B_{j,k,m_{j,k}}$ with center b_k and radius less than ε_3. For $k \in n^2_\star \setminus r^2_\star$, choose $B^q_{h_\xi(\alpha_{p',k}),m_{j,k}}$ to be a rational disk contained in a disk with center b^2_k and radius less than ε_3 (so it is contained in $B_{j,k,m_{j,k}}$). Notice that if $\xi_1 \neq \xi_2$ are from $u_\star$ and $\{\alpha_1,\alpha_2\} \cap (v_q \setminus v_\star) \neq \emptyset$, then $h_{\xi_1}(\alpha_1) \neq h_{\xi_2}(\alpha_2)$ so there is no conflict in doing this. $\blacksquare_{3.4}$

4. Regular continuum. We conclude with the following.

QUESTION 4.1. *Is a positive answer to Question* 1.1 *consistent with* $2^{\aleph_0} = \aleph_2$*?*

One way to get this would be to construct a model where $2^{\aleph_0} = \aleph_2$ and for some $A \in [\mathbb{C}]^{\aleph_1}$, for every $X \in [\mathbb{C}]^{\aleph_1}$, there is a non-constant entire function sending X into A. We do not know if this is possible.

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