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The Erdős–Rado Arrow for Singular Cardinals

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Abstract. We prove in ZFC that if $\mathrm{cf}(\lambda) > \aleph_0$ and $2^{\mathrm{cf}(\lambda)} < \lambda$, then $\lambda \to (\lambda, \omega + 1)^2$.

1 Introduction

For every finite cardinal κ, the Erdös–Dushnik–Miller theorem, [1, Theorem 11.1], states that $\kappa \to (\kappa,\ \omega)^2$. Erdös, Hajnal, Maté, and Rado proved that $\kappa \to (\kappa, \omega + 1)$ for every regular uncountable κ, (see [1, Theorem 11.3]). For singular cardinals, κ, they were only able to obtain the weaker result in [1, Theorem 11.1] that $\kappa \to (\kappa,\ \omega)^2$. It is not hard to see that if $\mathrm{cf}(\kappa) = \omega$, then $\kappa \not\to (\kappa,\ \omega+1)^2$. If $\mathrm{cf}\,(\kappa) > \omega$ and κ is a strong limit cardinal, then it follows from the General Canonization Lemma, [1, Lemma 28.1], that $\kappa \to (\kappa,\ \omega + 1)^2$. Question 11.4 of [1] is whether this holds without the assumption that κ is a strong limit cardinal, *e.g.*, whether, in ZFC,

$$\aleph_{\omega_1} \to (\aleph_{\omega_1},\ \omega + 1)^2.$$

In [5] it was proved that $\lambda \to (\lambda, \omega + 1)^2$ if $2^{\mathrm{cf}(\lambda)} < \lambda$ and there is a nice filter on κ (see [3, Ch.V]; it follows from suitable failures of SCH). Also proved there are consistency results when $2^{\mathrm{cf}(\lambda)} > \lambda$.

Here, continuing [5] but not relying on it, we eliminate the extra assumption, *i.e.,* we prove the following (in ZFC).

Theorem 1.1 *If* $\aleph_0 < \kappa = \mathrm{cf}(\lambda)$ *and* $2^\kappa < \lambda$ *then* $\lambda \to (\lambda,\ \omega + 1)^2$.

Before starting the proof, let us recall the well-known definition.

Definition 1.2 Let D be an $\aleph_1$-complete filter on Y, $f \in {}^Y\mathrm{Ord}$, and $\alpha \in \mathrm{Ord} \cup \{\infty\}$.

We define $\mathrm{rk}_D(f) = \alpha$ by induction on α (it is well known that $\mathrm{rk}_D(f) < \infty$): $\mathrm{rk}_D(f) = \alpha$ if and only if $\beta < \alpha \Rightarrow \mathrm{rk}_D(f) \neq \beta$ and for every $g \in {}^Y\mathrm{Ord}$ satisfying $g <_D f$, there is $\beta < \alpha$ such that $\mathrm{rk}_D(g) = \beta$.

Notice that we will use normal filters on $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, so the demand for $\aleph_1$-completeness in the definition is satisfied.

Recall also the following definition.

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Definition 1.3 Assume Y, D, f are as in Definition 1.2.

$$J[f, D] = \{Z \subseteq Y : Y \setminus Z \in D \text{ or } \mathrm{rk}_{D+Z}(f) > \mathrm{rk}_D(f)\}$$

Lastly, we quote the next claim (Definition 1.3 and Claim 1.4 are from [2], and explicitly [4, 5.8(2),5.9].

Claim 1.4 *Assume $\kappa > \aleph_0$ is regular, and D is a κ-complete (resp. normal) filter on Y.*

Then for any $f \in {}^Y\mathrm{Ord}$, $J[f, D]$ is a κ-complete (resp. normal) ideal on Y disjoint to D.

2 The Proof

In this section we prove Theorem 1.1, which, for convenience, we now restate.

Theorem 2.1 *If $\aleph_0 < \kappa = \mathrm{cf}(\lambda),\ 2^\kappa < \lambda$ then $\lambda \to (\lambda,\ \omega + 1)^2$.*

Proof

Stage A. Given that $\aleph_0 < \kappa = \mathrm{cf}(\lambda) < \lambda,\ 2^\kappa < \lambda$, we will show that $\lambda \to (\lambda,\ \omega+1)^2$. So, towards a contradiction, suppose that

(i) $c\colon [\lambda]^2 \to \{\text{red, green}\}$ but has no red set of cardinality λ and no green set of order type $\omega + 1$.

Choose $\bar{\lambda}$ such that:

(ii) $\bar{\lambda} = \langle \lambda_i : i < \kappa \rangle$ is increasing and continuous with limit λ, and for $i = 0$ or i a successor ordinal, λ_i is a successor cardinal. We also let $\Delta_0 = \lambda_0$ and for $i < \kappa,\ \Delta_{1+i} = [\lambda_i,\ \lambda_{i+1})$. For $\alpha < \lambda$ we will let $\mathbf{i}(\alpha)$ be the unique $i < \kappa$ such that $\alpha \in \Delta_i$.

We can clearly assume, in addition, that

(iii) $\lambda_0 > 2^\kappa$, for $i < \kappa,\ \lambda_{i+1} \geq \lambda_i^{++}$, and each Δ_i is homogeneously red for c.

The last is justified by the Erdös–Hajnal–Maté–Rado theorem for λ_{i+1}, *i.e.*, as $\lambda_{i+1} \to (\lambda_{i+1}, \omega + 1)^2$ because λ_{i+1} is regular.

Stage B. For $0 < i < \kappa$, we define Seq_i to be

$$\{\langle \alpha_0, \ldots, \alpha_{n-1} \rangle : \mathbf{i}(\alpha_0) < \cdots < \mathbf{i}(\alpha_{n-1}) < i\}.$$

For $\zeta \in \Delta_i$ and $\langle \alpha_0, \ldots, \alpha_{n-1} \rangle = \bar{\alpha} \in \mathrm{Seq}_i$, we say $\bar{\alpha} \in \mathcal{T}^\zeta$ if and only if $\{\alpha_0, \ldots, \alpha_{n-1},\ \zeta\}$ is homogeneously green for c. Note that an infinite $\triangleleft$-increasing branch in $\mathcal{T}^\zeta$ violates the non-existence of a green set of order type $\omega + 1$, so,

(iv) $\mathcal{T}^\zeta$ is well-founded, that is we cannot find $\eta_0 \triangleleft \eta_1 \triangleleft \cdots \triangleleft \eta_n \triangleleft \cdots$.

Therefore the following definition of a rank function, rk^ζ, on Seq_i can be carried out. If $\eta \in \mathrm{Seq}_i \setminus \mathcal{T}^\zeta$ then $\mathrm{rk}^\zeta(\eta) = -1$. We define $\mathrm{rk}^\zeta\colon \mathrm{Seq}_i \to \mathrm{Ord} \cup \{-1\}$ by induction on the ordinal ξ as follows. We have $\mathrm{rk}^\zeta(\bar{\alpha}) = \xi$ if and only if for all $\epsilon < \xi$, $\mathrm{rk}^\zeta(\bar{\alpha})$ was not defined as ϵ but there is a β such that $\mathrm{rk}^\zeta(\bar{\alpha}{}^\frown\langle\beta\rangle) \geq \epsilon$. Of course, if ξ is a successor ordinal, it is enough to check for $\epsilon = \xi - 1$, and for a limit ordinal, δ, if for all $\xi < \delta$, $\mathrm{rk}^{\zeta}(\bar{\alpha}) \geq \xi$, then $\mathrm{rk}^{\zeta}(\bar{\alpha}) \geq \delta$. In fact, it is clear that the range of rk^{ζ} is a proper initial segment of μ_i^+, where $\mu_i := \mathrm{card}(\bigcup \{\Delta_\epsilon : \epsilon < i\})$, and so, in particular, the range of rk^{ζ} has cardinality at most λ_i. Note that $\lambda_{i+1} \geq \lambda_i^{++} > \mu_i^+$.

Now we can choose B_i, an end-segment of Δ_i such that for all $\bar{\alpha} \in \mathrm{Seq}_i$ and all $-1 \leq \gamma < \mu_i^+$, if there is $\zeta \in B_i$ such that $\mathrm{rk}^{\zeta}(\bar{\alpha}) = \gamma$, then there are λ_{i+1} such ζ. Recall that Δ_i and therefore also B_i are of order type λ_{i+1}, which is a successor cardinal $> \mu_i^+ > |\mathrm{Seq}_i|$, hence such B_i exists. Everything is now in place for the main definition.

Stage C. $(\bar{\alpha}, Z, D, f) \in K$ if and only if

(a) D is a normal filter on κ,
(b) $f : \kappa \to \mathrm{Ord}$,
(c) $Z \in D$
(d) for some $0 < i < \kappa$ we have $\bar{\alpha} \in \mathrm{Seq}_i$, Z is disjoint to $i + 1$ and for every $j \in Z$ (hence $j > i$) there is $\zeta \in B_j$ such that $\mathrm{rk}^{\zeta}(\bar{\alpha}) = f(j)$ (so, in particular, $\bar{\alpha} \in \mathcal{T}^{\zeta}$).

Stage D. Note that $K \neq \varnothing$, since if we choose $\zeta_j \in B_j$, for $j < \kappa$, take $Z = \kappa \setminus \{0\}$, $\bar{\alpha} =$ the empty sequence, choose D to be any normal filter on κ and define f by $f(j) = \mathrm{rk}^{\zeta_j}(\bar{\alpha})$, then $(\bar{\alpha}, Z, D, f) \in K$.

Now clearly by Definition 1.2, among the quadruples $(\bar{\alpha}, Z, D, f) \in K$, there is one with $\mathrm{rk}_D(f)$ minimal. So, fix one such quadruple, and denote it by $(\bar{\alpha}^*, Z^*, D^*, f^*)$. Let D_1^* be the filter on κ dual to $J[f^*, D^*]$; so by Claim 1.4 it is a normal filter on κ extending D^*.

For $j \in Z^*$, set $C_j = \{\zeta \in B_j : \mathrm{rk}^{\zeta}(\bar{\alpha}^*) = f^*(j)\}$. Thus by the choice of B_j we know that $\mathrm{card}(C_j) = \lambda_{j+1}$, and for every $\zeta \in C_j$ the set $(\mathrm{Rang}(\bar{\alpha}^*) \cup \{\zeta\})$ is homogeneously green under the colouring c. Now suppose $j \in Z^*$. For every $\Upsilon \in Z^* \setminus (j+1)$ and $\zeta \in C_j$, let $C_\Upsilon^+(\zeta) = \{\xi \in C_\Upsilon : c(\{\zeta, \xi\}) = \mathrm{green}\}$. Also, let $Z^+(\zeta) = \{\Upsilon \in Z^* \setminus (j+1) : \mathrm{card}(C_\Upsilon^+(\zeta)) = \lambda_{\Upsilon+1}\}$.

Stage E. For $j \in Z^*$ and $\zeta \in C_j$, let $Y(\zeta) = Z^* \setminus Z^+(\zeta)$. Since $\lambda_0 > 2^\kappa$ and $\lambda_{j+1} > \lambda_0$ is regular, for each $j \in Z^*$ there are $Y = Y_j \subseteq \kappa$ and $C'_j \subseteq C_j$ with $\mathrm{card}(C'_j) = \lambda_{j+1}$ such that $\zeta \in C'_j \Rightarrow Y(\zeta) = Y_j$.

Let $\hat{Z} = \{j \in Z^* : Y_j \in D_1^*\}$. Now the proof splits into two cases.

Case 1. $\hat{Z} \neq \varnothing \bmod D_1^*$

Define $Y^* = \{j \in \hat{Z}$: for every $i \in \hat{Z} \cap j$, we have $j \in Y_i\}$. Notice that Y^* is the intersection of $\hat{Z}$ with the diagonal intersection of κ sets from D_1^* (since $i \in \hat{Z} \Rightarrow Y_i \in D_1^*$), hence (by the normality of D_1^*) $Y^* \neq \varnothing \bmod D_1^*$. But then, as we will see soon, by shrinking the C'_j for $j \in Y^*$, we can get a homogeneous red set of cardinality λ, which is contrary to the assumption toward contradiction.

We define $\hat{C}_j$ for $j \in Y^*$ by induction on j such that $\hat{C}_j$ is a subset of C'_j of cardinality λ_{j+1}. Now, for $j \in Y^*$, let $\hat{C}_j$ be the set of $\xi \in C'_j$ such that for every $i \in Y^* \cap j$ and every $\zeta \in \hat{C}_i$ we have $\xi \notin C_j^+(\zeta)$. So, in fact, $\hat{C}_j$ has cardinality λ_{j+1}, as it is the result of removing $< \lambda_{j+1}$ elements from C'_j where $|C'_j| = \lambda_{j+1}$ by its choice. Indeed, the number of such pairs (i, ζ) is $\leq \lambda_j$ and for $i \in Y^* \cap j$ and $\zeta \in \hat{C}_i$

(a) $j \in Y_i$ [by the definition of Y^* as $j \in Y^*$].

(b) $\zeta \in C'_i$ [as $\zeta \in \hat{C}_i$ and $\hat{C}_i \subseteq C'_i$ by the induction hypothesis].
(c) $Y(\zeta) = Y_i$ [as by (b) we have $\zeta \in C'_i$ and the choice of C'_i].
(d) $j \in Y(\zeta)$ [by (a)+(c)].
(e) $j \notin Z^+(\zeta)$ [by (d) and the choice of $Y(\zeta)$ as $Z^* \setminus Z^+(\zeta)$].
(f) $C^+_j(\zeta)$ has cardinality $< \lambda_{j+1}$ [by (e) and the choice of $Z^+(\zeta)$, as $j \in \hat{Z} \subseteq Z^*$].

So $\hat{C}_j$ is a well defined subset of C'_j of cardinality λ_{j+1} for every $j \in Y^*$. But then, clearly the union of the $\hat{C}_j$ for $j \in Y^*$, call it $\hat{C}$, satisfies the following.

(a) it has cardinality λ [as $j \in Y^* \Rightarrow |\hat{C}_j| = \lambda_{j+1}$ and $\sup(Y^*) = \kappa$ as $Y^* \neq \varnothing \bmod D^*_1$],
(b) $c\restriction[\hat{C}_j]^2$ is constantly red [as we are assuming (iii),
(c) if $i < j$ are from Y^* and $\zeta \in \hat{C}_i, \xi \in \hat{C}_j$ then $c\{\zeta, \xi\} = \text{red}$ [as $\xi \notin C^+_j(\zeta)$].

So $\hat{C}$ has cardinality λ and is homogeneously red. This concludes the proof of Case 1.

Case 2. $\hat{Z} = \varnothing \bmod D^*_1$.

In this case there are $i \in Z^*, \beta \in C_i$ such that $Z^+(\beta) \neq \varnothing \bmod D^*_1$

[Because $Z^* \in D^* \subseteq D^*_1$ and $\hat{Z} = \varnothing \bmod D^*_1$, hence $Z^* \setminus \hat{Z} \neq \varnothing$. Choose $i \in Z^* \setminus \hat{Z}$. By the definition of $\hat{Z}$, $Y_i \notin D^*_1$. So, if $\beta \in C'_i$ then $Y(\beta) = Y_i \notin D^*_1$ and choose $\beta \in C'_i$, so $Y(\beta) \notin D^*_1$ hence by the definition of $Y(\beta)$ we have $Z^* \setminus Z^+(\beta) = Y(\beta) \notin D^*_1$. Since $Z^* \in D^*_1$, we conclude that $Z^+(\beta) \neq \varnothing \bmod D^*_1$].

Let $\bar{\alpha}' = \bar{\alpha}^* {}^\frown \langle \beta \rangle, Z' = Z^+(\beta), D' = D^* + Z'$. It is a normal filter Claim 1.4, and by the previous sentence which makes sure that $Z' \neq \varnothing$, as $D^* \subseteq D^*_1$. Lastly we define $f' \in {}^\kappa\mathrm{Ord}$ by

(a) if $j \in Z'$ then $f'(j) = \mathrm{Min}\{\mathrm{rk}^\gamma(\bar{\alpha}') : \gamma \in C^+_j(\beta) \subseteq B_j\}$,
(b) otherwise $f'(j) = 0$.

Clearly

(a) $(\bar{\alpha}', Z', D', f') \in K$, and
(b) $f' <_{D'} f^*$
[Because, as $Z' \in D'$ and if $j \in Z'$ then for some $\gamma \in C^+_j(\beta)$ we have $f'(j) = \mathrm{rk}^\gamma(\bar{\alpha}') = \mathrm{rk}^\gamma(\bar{\alpha}^* {}^\frown \langle \beta \rangle)$ which by the definition of rk^γ is $< \mathrm{rk}^\gamma(\bar{\alpha}^*) = f^*(j)$, recalling (d) from Stage C.]
hence
(c) $\mathrm{rk}_{D'}(f') < \mathrm{rk}_{D'}(f^*)$
[By Definition 1.2].

But $\mathrm{rk}_{D'}(f^*) = \mathrm{rk}_{D^*}(f^*)$ as $Z' = Z^+(\beta) \neq \varnothing \bmod D^*_1$ by the definition of D^*_1 as extending the filter dual to $J[f^*, D^*]$, see Definition 1.3. Hence $\mathrm{rk}_{D'}(f') < \mathrm{rk}_{D^*}(f^*)$, so we get a contradiction to the choice of $(\bar{\alpha}^*, Z^*, D^*, f^*)$.

Clearly at least one of the two cases holds, so we are done. ■

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