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# THE $\aleph_2$-SOUSLIN HYPOTHESIS

BY

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ABSTRACT. We prove the consistency with $CH$ that there are no $\aleph_2$-Souslin trees.

The $\aleph_2$-Souslin hypothesis, $SH_{\aleph_2}$, is the statement that there are no $\aleph_2$-Souslin trees. In Mitchell's model [5] from a weakly compact the stronger statement holds (Mitchell and Silver) that there are no $\aleph_2$-Aronszajn trees, a property which implies that $2^{\aleph_0} > \aleph_1$.

THEOREM. Con($ZFC$ + *there is a weakly compact cardinal*) *implies*

$$\mathrm{Con}(ZFC + 2^{\aleph_0} = \aleph_1 + SH_{\aleph_2}).$$

In the forcing extension, $2^{\aleph_1}$ is greater than $\aleph_2$, and can be arbitrarily large. Analogues of this theorem hold with $\aleph_2$ replaced by the successor of an arbitrary regular cardinal. Strengthenings and problems are given at the end of the paper.

Let $\mathfrak{M}$ be a ground model in which $\kappa$ is a weakly compact cardinal. The extension which models $SH_{\aleph_2}$ and $CH$ is obtained by iteratively forcing $\geqslant \kappa^+$ times with certain $\kappa cc$, countably closed partial orders, taking countable supports in the iteration. For $\alpha \geqslant 1$, $(\mathcal{P}_\alpha, \leqslant)$ is the ordering giving the first $\alpha$ steps in the iteration. $\mathcal{P}_\alpha$ is a set of functions with domain $\alpha$.

Let $L_{\aleph_1\kappa}$ be the Levy collapse by countable conditions of each $\beta \in [\aleph_1, \kappa)$ to $\aleph_1$ (so $\kappa$ is the new $\aleph_2$). Then $\mathcal{P}_1$ (isomorphic to $L_{\aleph_1\kappa}$) is $\{f\colon \operatorname{dom} f = 1, f(0) \in L_{\aleph_1\kappa}\}$, ordered by $f \leqslant g$ iff $f(0) \leqslant g(0)$. To define $\mathcal{P}_{\beta+1}$, choose a term $A_\beta$ in the forcing language of $\mathcal{P}_\beta$ for a countably closed partial ordering (to be described later) and let $\mathcal{P}_{\beta+1} = \{f\colon \operatorname{dom} f = \beta + 1, f \upharpoonright \beta \in \mathcal{P}_\beta, f \upharpoonright \beta \Vdash_{\mathcal{P}_\beta} f(\beta) \in A_\beta\}$, ordered by $f \leqslant g$ iff $f \upharpoonright \beta \leqslant g \upharpoonright \beta$ and $g \upharpoonright \beta \Vdash_{\mathcal{P}_\beta} f(\beta) \leqslant g(\beta)$. For $\alpha$ a limit ordinal, $\mathcal{P}_\alpha = \{f\colon \operatorname{dom} f = \alpha, f \upharpoonright \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and $f(\beta)$ is (the term for) $\varnothing$, the least element of $A_\beta$, for all but $\leqslant \aleph_0$ $\beta$'s$\}$, ordered by $f \leqslant g$ iff for all $\beta < \alpha, f \upharpoonright \beta \leqslant g \upharpoonright \beta$.

Each $\mathcal{P}_\alpha$ is countably closed. We are done as in Solovay-Tennenbaum [7] if the $A_\beta$'s can be chosen so that each $\mathcal{P}_\alpha$ has the $\kappa cc$, and therefore that every $\aleph_2$ $(= \kappa)$-Souslin tree which crops up gets killed by some $A_\beta$.

If $T$ is a tree then $(T)_\lambda$ is the $\lambda$th level of $T$, $(T)_{<\lambda} = \bigcup_{\mu<\lambda} T_\mu$. Regarding the previous problem, it is a theorem of Mitchell that if $CH$ and $\diamond\{\alpha < \omega_2\colon cf(\alpha) = \aleph_1\}$ hold, then there are countably closed $\aleph_2$-Souslin trees $T_n$, $n < \omega$, such that for

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each $m < \omega$, $\bigotimes_{n<m} T_n$ has the $\aleph_2 cc$, but $\bigotimes_{n<\omega} T_n$ does not have the $\aleph_2 cc$. We give for interest his proof modulo the usual Jensen methods. At stage $\mu < \omega_2$ construct each $(T_n)_\mu$ normally above $(T_n)_{<\mu}$. If $\mu = \nu + 1$ let each $x \in (T_n)_\nu$ have at least two successors in $(T_n)_\mu$. If $cf(\mu) = \omega$ let all branches in $(T_n)_{<\mu}$ go through. If $cf(\mu) = \omega_1$ make sure that the antichain given by the $\diamondsuit$-sequence for $\bigotimes_{n<m_\mu} T_n$ is taken care of, and choose $\langle c_{\mu n}: n < \omega\rangle \in \bigotimes_{n<\omega}(T_n)_\mu$ so that if $\mu' < \mu$, $cf(\mu') = \omega_1$, then $\langle c_{\mu' n}: n < \omega\rangle \nleqslant \langle c_{\mu n}: n < \omega\rangle$. We also carry along the following induction hypothesis: if $\nu < \mu$, $\langle x_n: n < \omega\rangle \in \bigotimes_{n<\omega}(T_n)_\nu$, $m < \omega$, $\langle y_n: n < m\rangle \in \bigotimes_{n<m}(T_n)_\mu$, $x_n < y_n$ $(n < m)$ and $\langle x_n: n < \omega\rangle \ngeqslant \langle c_{\lambda n}: n < \omega\rangle$, for all $\lambda \leqslant \nu$ with $cf(\lambda) = \omega_1$, then there are $y_n \in (T_n)_\mu$ $(m \leqslant n < \omega)$ with $x_n < y_n$, such that $\langle y_n: n < \omega\rangle \ngeqslant \langle c_{\lambda n}: n < \omega\rangle$, for all $\lambda \leqslant \mu$ with $cf(\lambda) = \omega_1$.

If $\delta$ is inaccessible, then forcing with $L_{\aleph_1\delta}$ (whence $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2 = \delta$, and $\diamondsuit\{\alpha < \omega_2: cf(\alpha) = \omega_1\}$ hold) followed by forcing with the $\bigotimes_{n<\omega} T_n$ constructed previously, gives a countably closed length $\omega$ iteration of countably closed, $\delta cc$ partial orderings which does not have $\delta cc$.

The previous theorem does not rule out that an iteration of $\aleph_2$-Souslin trees can give $CH$ and $SH_{\aleph_2}$; in this paper, though, the $\aleph_2$-Souslin trees are killed by a different method. Let $T$ be an $\aleph_2$-Souslin tree (we may assume without loss of generality that $T$ is normal and $\mathrm{Card}(T)_1 = \aleph_1$). The antichain partial order $A_T$ is defined to be $(\{x \subseteq T: x \text{ a countable antichain, root } T \notin x\}, \subseteq)$. Now $A_T$ need not have the $\aleph_2 cc$, as shown by the following result of the first author: Con($ZFC$) implies Con($ZFC$ + "there is an $\aleph_2$-Souslin tree $T$ and a sequence $\langle d_{\alpha n}: n < \omega\rangle$ from $(T)_\alpha$, for each $\alpha < \omega_2$, such that if $\alpha < \beta$, there is an $m < \omega$ with $d_{\alpha n} < d_{\beta n}$, for all $n > m$".) Namely, start with a model of $CH$. Determine in advance that, say, $(T)_\alpha = [\omega_1\alpha, \omega_1(\alpha + 1))$ and that $d_{\alpha n} = \omega_1\alpha + n$. Conditions are countable subtrees $S$ of $T$ such that if $S \cap (T)_\alpha \neq \varnothing$ then $\{d_{\alpha n}: n < \omega\} \subseteq S$, which meet the requirements on the $d_{\beta n}$'s.

Devlin [2] has shown that such a tree exists in $L$.

We show now that if each $A_\beta$ is an $A_T$, $T$ an $\aleph_2$-Souslin tree, then each $\mathcal{P}_\alpha$ has the $\kappa cc$, which will prove the theorem (we actually just use that Card $T <$ the cardinal designated as the new $2^{\aleph_1}$ and $T$ has no $\omega_2$-paths; see remarks at the end). This theorem was originally proved by the first author when $\kappa$ is measurable; that the assumption can be weakened to weak compactness of $\kappa$ is due to the second author.

We consider now only the case $\alpha \leqslant \kappa^+$ (which will suffice, assuming $2^\kappa = \kappa^+$ in $\mathfrak{M}$, for $CH + SH_{\aleph_2} + 2^{\aleph_1} = \aleph_3$); $\alpha$ arbitrary will be dealt with at the end.

Fix $\alpha$ for the rest of the proof. We assume by induction that

(1) For each $\beta < \alpha$, $\mathcal{P}_\beta$ has the $\kappa cc$.

(One more induction hypothesis is listed later.)

For $\beta < \alpha$, let $T_\beta$ be the $\beta$th $\aleph_2$-Souslin tree, so $\mathcal{P}_{\beta+1} = \mathcal{P}_\beta \tilde{\otimes} A_\beta$, where $A_\beta = A_{T_\beta}$. Assume without loss of generality that for each $\lambda < \kappa$,

$$(T_\beta)_\lambda \subseteq [\omega_1\lambda, \omega_1(\lambda + 1)).$$

An $f \in \mathcal{P}_\beta$, $\beta \leqslant \alpha$, is said to be determined if there is in $\mathfrak{M}$ a sequence $\langle z_\gamma: \gamma \in \mathrm{dom} f - \{0\}\rangle$ of countable sets of ordinals such that for all $\gamma \in \mathrm{dom} f - \{0\}$,

$f \upharpoonright \gamma \Vdash_{\mathcal{P}_\gamma} f(\gamma) = z_\gamma$. If $\langle f_n : n < \omega \rangle$ is a sequence of determined members of $\mathcal{P}_\beta$, with $f_n \leqslant f_{n+1}$, then the coordinatewise union $f_\omega$ of the $f_n$'s is seen to be a determined member of $\mathcal{P}_\beta$ extending each $f_n$. From this it may be seen, by induction on $\beta \leqslant \alpha$, that the set of determined members of $\mathcal{P}_\beta$ is cofinal in $\mathcal{P}_\beta$. Redefine each $\mathcal{P}_\beta$ then to consist just of the determined conditions. Clearly Card $\mathcal{P}_\beta \leqslant \kappa$, for all $\beta \leqslant \alpha$.

For $f, g \in \mathcal{P}_\beta, f \sim g$ means that $f$ and $g$ are compatible.

Fix for the rest of the proof a one-one enumeration $\alpha = \{\alpha_\mu : \mu \in S\}$, for some $S \subseteq \kappa$ (this induces a similar enumeration of each $\beta < \alpha$, the induction hypothesis (2) for $\beta$ below, is with respect to this induced enumeration). For notational simplicity we now assume that $S$ is some $\kappa' \leqslant \kappa$.

If $\lambda < \kappa$, $\beta \leqslant \alpha, f \in \mathcal{P}_\beta$, define $f|\lambda$ to be the function $h$ with domain $\beta$ such that $h(\gamma) = \varnothing$ unless $\gamma \in \{\alpha_\mu : \mu < \lambda\} \cap \beta$, in which case,

$$\gamma = 0 \Rightarrow h(\gamma) = f(\gamma) \upharpoonright (\omega_1 \times \lambda), \qquad \gamma > 0 \Rightarrow h(\gamma) = f(\gamma) \cap \lambda.$$

The function $f|\lambda$ need not be a condition, but for $g \in \mathcal{P}_\beta$, we will still write $f|\lambda \leqslant g$ to mean that $f|\lambda$ is coordinatewise a subset of $g$. Let $\mathcal{P}_\beta|\lambda = \{f \in \mathcal{P}_\beta : f|\lambda = f\}$.

Suppose $0 < \beta \leqslant \alpha, \lambda < \kappa$. Define

$$\#_\lambda^\beta(f, g, h) \Leftrightarrow f, g \in \mathcal{P}_\beta, f|\lambda = g|\lambda = h,$$

$$*_\lambda^\beta(f, h) \Leftrightarrow f \in \mathcal{P}_\beta, h \in \mathcal{P}_\beta|\lambda \text{ and for every } h' \geqslant h \text{ with } h' \in \mathcal{P}_\beta|\lambda, h' \sim f,$$

$$*_\lambda^\beta(f, g, h) \Leftrightarrow *_\lambda^\beta(f, h) \text{ and } *_\lambda^\beta(g, h).$$

For $P \subseteq Q$, $Q$ a partial ordering, $P \subseteq_{\text{reg}} Q$ means that $P$ is a regular subordering of $Q$, that is, any two members of $P$ compatible in $Q$ are compatible in $P$, and every maximal antichain of $P$ is a maximal antichain of $Q$. If $\mathcal{P}_\beta|\lambda \subseteq_{\text{reg}} \mathcal{P}_\beta$, then $*_\lambda^\beta(f, h)$ states that $h \Vdash_{\mathcal{P}_\beta|\lambda} [\![ f ]\!] \neq 0$.

Recall that the sets of the form $\{\lambda < \kappa : (R_\lambda, \in, A \cap R_\lambda) \models \Phi\}$, where $A \subseteq \kappa$, $\Phi$ is $\pi_1^1$, and $(R_\kappa, \in, A) \models \Phi$, belong to a normal uniform filter $\mathcal{F}_{wc}$, the weakly compact filter on $\kappa$ (see **[9]**, **[0]**). The second thing we assume by induction is

(2) for all $\beta < \alpha$, for $\mathcal{F}_{wc}$-almost all $\lambda < \kappa$, for all $f, g, h$, $\#_\lambda^\beta(f, g, h)$ implies that for some $h' \geqslant h$, $*_\lambda^\beta(f, g, h')$.

If $\beta < \alpha$, $\lambda < \kappa$, say that $(T_\beta)_{<\lambda}$ is determined by $\mathcal{P}_\beta|\lambda$ if for each $\theta, \tau$ in $(T_\beta)_{<\lambda}$ there is a $\mathcal{P}_\beta$-maximal antichain $R$ of conditions deciding the ordering between $\theta$ and $\tau$ in $T_\beta$, such that $R \subseteq \mathcal{P}_\beta|\lambda$.

LEMMA 1. *There is a closed unbounded set of* $\lambda < \kappa$ *such that for all* $\mu < \lambda$, $(T_{\alpha_\mu})_{<\lambda}$ *is determined by* $\mathcal{P}_{\alpha_\mu}|\lambda$.

PROOF. This is a consequence of the strong inaccessibility of $\kappa$ and the assumption that each $\mathcal{P}_\beta$, $\beta < \alpha$, has $\kappa cc$.

LEMMA 2. *For $\mathcal{F}_{wc}$-almost all $\lambda < \kappa$,*
(a) *$\lambda$ is strongly inaccessible.*
(b) *For all $\mu < \lambda$, $\mathcal{P}_{\alpha_\mu}|\lambda$ has the $\lambda cc$.*
(c) *For all $\mu < \lambda$, $\mathcal{P}_{\alpha_\mu}|\lambda \subseteq_{\text{reg}} \mathcal{P}_{\alpha_\mu}$.*
(d) *For all $\mu < \lambda, \Vdash_{\mathcal{P}_{\alpha_\mu}|\lambda} \lambda = \aleph_2$.*
(e) *For all $\mu < \lambda, \Vdash_{\mathcal{P}_{\alpha_\mu}|\lambda} (T_{\alpha_\mu})_{<\lambda}$ is an $\aleph_2$-Souslin tree.*

PROOF. By $\pi_1^1$ reflection and the normality of $\mathcal{F}_{wc}$.

LEMMA 3. *Let $\beta \leqslant \alpha$, $\lambda < \kappa$, $\mathcal{P}_\beta|\lambda \subseteq_{\text{reg}} \mathcal{P}_\beta$.*
(a) *If $f \in \mathcal{P}_\beta, j \in \mathcal{P}_\beta|\lambda$, and $f \sim j$, then there is an $h \geqslant j$ with $*_\lambda^\beta(f, h)$.*
(b) *If $*_\lambda^\beta(f, g, h)$ and $D$, $E$ are cofinal subsets of $\mathcal{P}_\beta$, then there exists $\langle f', g', h'\rangle \geqslant \langle f, g, h\rangle$ with $*_\lambda^\beta(f', g', h'), f' \in D, g' \in E, h \leqslant f', g'$.*

PROOF. These are standard facts about forcing.
The following is T. Carlson's version of the lemma we originally used here.

LEMMA 4. *Suppose $\lambda$ satisfies Lemma* 1 *and Lemma* 2(c), *$\mu < \lambda$, and $*_\lambda^{\alpha_\mu}(f, h)$. Then $f|\lambda \leqslant h$.*

PROOF. Otherwise there is a $\nu < \lambda$ with $\alpha_\nu < \alpha_\mu$, and a $\theta \in f(\alpha_\nu) \cap \lambda$ such that $\theta \notin h(\alpha_\nu)$. We have that $h \restriction \alpha_\nu \Vdash_{\mathcal{P}_{\alpha_\nu}|\lambda} \theta$ is $T_{\alpha_\nu}$-incomparable with each member of $h(\alpha_\nu)$; otherwise $*_\lambda^{\alpha_\mu}(f, h)$ would be contradicted. Pick an $h' \in \mathcal{P}_{\alpha_\nu}|\lambda$, $h' \geqslant h \restriction \alpha_\nu$, and a $\theta' < \lambda$ such that $h' \Vdash_{\mathcal{P}_{\alpha_\nu}|\lambda} \theta <_{T_{\alpha_\nu}} \theta'$. Let $\bar{h}$ be $h'^\frown\langle h(\alpha_\nu) \cup \{\theta'\}\rangle^\frown h \restriction [\alpha_\nu + 1, \alpha_\mu)$. Then $\bar{h} \in \mathcal{P}_{\alpha_\mu}|\lambda$, $h \leqslant \bar{h}$, and $\bar{h} \nsim f$, a contradiction.

LEMMA 5. *Suppose $\lambda$ satisfies Lemma* 1 *and Lemma* 2(c), *$\mu < \lambda$, and $*_\lambda^{\alpha_\mu}(f, g, h)$. Then there is an $(f', g', h') \geqslant (f, g, h)$ with $\#_\lambda^{\alpha_\mu}(f', g', h')$.*

PROOF. Choose $(f, g, h) = (f_0, g_0, h_0) \leqslant \cdots \leqslant (f_n, g_n, h_n) \leqslant \cdots$ so that $*_\lambda^{\alpha_\mu}(f_n, g_n, h_n)$, $h_n \leqslant f_{n+1}$, $h_n \leqslant g_{n+1}$. This is done by repeated applications of Lemma 3(a), (b). Then Lemma 4 implies that the coordinatewise union $(f', g', h')$ of the $(f_n, g_n, h_n)$'s is as desired.

DEFINITION. Suppose $\lambda < \kappa$, $\mu < \lambda$, $f, g \in \mathcal{P}_{\alpha_\mu}$, and suppose $\theta, \tau$ are nodes of $(T_{\alpha_\mu})_{\geqslant\lambda}$ ($\theta = \tau$ allowed). Then $\langle f, g\rangle$ is said to $\lambda$-separate $\langle \theta, \tau\rangle$ if there is a $\gamma < \lambda$ and $\theta', \tau' \in (T_{\alpha_\mu})_\gamma$, with $\theta' \neq \tau'$, such that

$$f \Vdash_{\mathcal{P}_{\alpha_\mu}} \theta' <_{T_{\alpha_\mu}} \theta, \qquad g \Vdash_{\mathcal{P}_{\alpha_\mu}} \tau' <_{T_{\alpha_\mu}} \tau.$$

LEMMA 6. *Suppose $\lambda$ satisfies Lemmas* 1 *and* 2, *$\mu < \lambda$, $*_\lambda^{\alpha_\mu}(f, g, h)$, $\{\theta, \tau\} \subseteq (T_{\alpha_\mu})_{\geqslant\lambda}$, with $\theta = \tau$ allowed. Then there is an $\langle f', g', h'\rangle \geqslant \langle f, g, h\rangle$ such that $*_\lambda^{\alpha_\mu}(f', g', h')$ and $\langle f', g'\rangle$ $\lambda$-separates $\langle \theta, \tau\rangle$.*

PROOF.
*Claim*. There are $f_0, f_1 \geqslant f$, $\bar{h} \geqslant h$, with $*_\lambda^{\alpha_\mu}(f_0, \bar{h})$, $*_\lambda^{\alpha_\mu}(f_1, \bar{h})$, such that $\langle f_0, f_1\rangle$ $\lambda$-separates $\langle \theta, \theta\rangle$ via a $\langle \theta_1, \theta_2\rangle \in (T_{\alpha_\mu})_\gamma$, for some $\gamma < \lambda$.

PROOF. Consider the result of taking a generic set $G_{\alpha_\mu}|\lambda$ over $\mathcal{P}_{\alpha_\mu}|\lambda$ which contains $h$. In $\mathfrak{M}[G_{\alpha_\mu}|\lambda]$, $(T_{\alpha_\mu})_{<\lambda}$ is a $\lambda$ ($= \aleph_2$)-Souslin tree. In the further extension $\mathfrak{M}[G_{\alpha_\mu}]$, $\theta$ determines a $\lambda$-path through $(T_{\alpha_\mu})_{<\lambda}$. Since this path is not in $\mathfrak{M}[G_{\alpha_\mu}|\lambda]$, there must be $\bar{h} \in G_{\alpha_\mu}|\lambda$, $\bar{h} \geqslant h$, $f_0, f_1 \geqslant f$, $\gamma < \lambda$, $\theta_0, \theta_1 \in (T_{\alpha_\mu})_\gamma$, $\theta_0 \neq \theta_1$, with

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$\bar{h} \Vdash f_0 \Vdash \theta_0 <_{T_{\alpha_\mu}} \theta$, $\bar{h} \Vdash f_1 \Vdash \theta_1 <_{T_{\alpha_\mu}} \theta$, such that $*_\lambda^{\alpha_\mu}(f_0, \bar{h})$ and $*_\lambda^{\alpha_\mu}(f_1, \bar{h})$. This gives the claim.

Now, by Lemma 3, choose $(g', h') \geqslant (g, h)$ and a $\tau' \in (T_{\alpha_\mu})_\gamma$ so that $*_\lambda^{\alpha_\mu}(g', h')$ and $g' \Vdash \tau' <_{T_{\alpha_\mu}} \tau$. Pick $i \in \{0, 1\}$ with $\tau' \neq \theta_i$. Let $f' = f_i$, $\theta' = \theta_i$. Then $(f', g', h')$ are as desired. This proves the lemma.

We claim that the induction hypotheses (1) and (2) automatically pass up to $\alpha$ if $cf(\alpha) > \omega$. Namely, (1) holds at $\alpha$ by a $\Delta$-system argument. For (2), suppose that for an $\mathcal{F}_{wc}$-positive set $W$ of $\lambda$'s there is a counterexample $\langle f_\lambda, g_\lambda, h_\lambda \rangle$. Let $N_\lambda = (\text{support } f_\lambda \cup \text{support } g_\lambda)$. If $cf(\alpha) \neq \kappa$ then for some $\beta < \alpha$ and $\mathcal{F}_{wc}$-positive $V \subseteq W$, $\lambda \in V$ implies $N_\lambda \subseteq \beta$, and we are done. If $cf(\alpha) = \kappa$, pick a closed unbounded set $C \subseteq \kappa$ such that $\langle \sup\{\alpha_\nu : \nu < \lambda\} : \lambda \in C \rangle$ is increasing, continuous and cofinal in $\alpha$ and an $\mathcal{F}_{wc}$-positive $V \subseteq W \cap C$ such that for some $\beta < \alpha$ and all $\lambda \in V$, $N_\lambda \cap \sup\{\alpha_\nu : \nu < \lambda\} \subseteq \beta$, then apply (2) at $\beta$.

Thus, we may assume for the rest of the proof that $\alpha$ is a successor ordinal or $cf(\alpha) = \omega$. Fix $\langle \mu_n : n < \omega \rangle$ such that if $\alpha = \beta + 1$ then each $\mu_n$ is the $\mu$ with $\alpha_\mu = \beta$, and if $cf(\alpha) = \omega$ then $\langle \alpha_{\mu_n} : n < \omega \rangle$ is an increasing sequence converging to $\alpha$.

LEMMA 7. *For $\mathcal{F}_{wc}$-almost all $\lambda$, the following holds: if $f, g \in \mathcal{P}_\alpha$, $h \in \mathcal{P}_\alpha | \lambda$ and $\#_\lambda^\alpha(f, g, h)$ then there exists $\langle f', g', h' \rangle \geqslant \langle f, g, h \rangle$ such that $\#_\lambda^\alpha(f', g', h')$ and such that for each $\mu < \lambda$ with $\alpha_\mu \neq 0$, and each $\theta \in f'(\alpha_\mu) - \lambda$, each $\tau \in g'(\alpha_\mu) - \lambda$, $\langle f' \restriction \alpha_\mu, g' \restriction \alpha_\mu \rangle$ $\lambda$-separates $\langle \theta, \tau \rangle$.*

PROOF. We prove the lemma for $\lambda$, assuming that $\lambda$ satisfies Lemmas 1 and 2, $\lambda > \mu_n$ $(n < \omega)$ and for each $n < \omega$, $\lambda$ is in the $\mathcal{F}_{wc}$ set given by induction hypothesis (2) for $\alpha_{\mu_n}$. Construct $\langle f_n, g_n, h_n \rangle$, $n < \omega$, so that

(a) $f_n, g_n \in \mathcal{P}_{\alpha_{\mu_n}}$, $\#_\lambda^{\alpha_{\mu_n}}(f_n, g_n, h_n)$,

(b) $\langle f \restriction \alpha_{\mu_n}; g \restriction \alpha_{\mu_n}, h \restriction \alpha_{\mu_n} \rangle \leqslant \langle f_n, g_n, h_n \rangle$,

(c) $\langle f_n, g_n, h_n \rangle \leqslant \langle f_{n+1}, g_{n+1}, h_{n+1} \rangle$,

(d) if, at stage $n > 1$, $\langle \theta_n, \tau_n \rangle$ is the $n$th pair (in the appropriate bookkeeping list for exhausting them) with $\theta_n \in f_n(\alpha_{\nu_n}) - \lambda$, $\tau_n \in g_n(\alpha_{\nu_n}) - \lambda$, $\nu_n < \lambda$, $\alpha_{\nu_n} \leqslant \alpha_{\mu_n}$, then

$$\langle f_n \restriction \alpha_{\nu_n}, g_n \restriction \alpha_{\nu_n} \rangle \text{ } \lambda\text{-separates } \langle \theta, \tau \rangle.$$

Let $f_0 = f \restriction \alpha_{\mu_0}$, $g_0 = g \restriction \alpha_{\mu_0}$, $h_0 = h \restriction \alpha_{\mu_0}$. Suppose $n > 1$ and $f_{n-1}, g_{n-1}, h_{n-1}$ have been constructed. Let

$$f_n' = f_{n-1} {}^\frown f \restriction [\alpha_{\mu_{n-1}}, \alpha_{\mu_n}), \qquad g_n' = g_{n-1} {}^\frown g \restriction [\alpha_{\mu_{n-1}}, \alpha_{\mu_n}),$$
$$h_n' = h_{n-1} {}^\frown h \restriction [\alpha_{\mu_{n-1}}, \alpha_{\mu_n}).$$

Then $\#_\lambda^{\alpha_{\mu_n}}(f_n', g_n', h_n')$. By induction hypothesis (2), there is an $\bar{h}_n \geqslant h_n'$ such that $*_\lambda^{\alpha_\mu}(f_n', g_n', \bar{h}_n)$. By Lemma 6, there is $\langle f_n'', g_n'', h_n'' \rangle \geqslant \langle f_n', g_n', \bar{h}_n \rangle$ such that $*_\lambda^{\alpha_{\mu_n}}(f_n'', g_n'', h_n'')$ and

$$\langle f_n'' \restriction \alpha_{\nu_n}, g_n'' \restriction \alpha_{\nu_n} \rangle \text{ separates } \langle \theta_n, \tau_n \rangle.$$

Finally, by Lemma 5 we may choose $\langle f_n, g_n, h_n \rangle \geqslant \langle f_n'', g_n'', h_n'' \rangle$ so that $\#_\lambda^{\alpha_{\mu_n}}(f_n, g_n, h_n)$.

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Taking $f'$, $g'$, $h'$ to be the coordinatewise unions of the $f_n$'s, $g_n$'s, $h_n$'s gives the lemma.

We now verify the two induction hypotheses.

(1) $\mathcal{P}_\alpha$ has the $\kappa cc$.

PROOF. Given $f_\lambda \in \mathcal{P}_\alpha$, $\lambda < \kappa$. For each $\lambda$ which satisfies Lemmas 1, 2 and 7, with $\lambda > \mu_n$ ($n < \omega$), apply Lemma 7 to the triple $\langle f_\lambda, f_\lambda, f_\lambda|\lambda\rangle$, obtaining a triple $\langle f^*_\lambda, f^{**}_\lambda, j_\lambda\rangle$ (so $f_\lambda \leqslant f^*_\lambda, f^{**}_\lambda, j_\lambda = f^*_\lambda|\lambda = f^{**}_\lambda|\lambda$).

Let

$$B_\lambda = (\text{support} f^*_\lambda \cup \text{support} f^{**}_\lambda) \cap \{\alpha_\mu : \mu < \lambda\}.$$

If $0 \neq \alpha_\mu \in B_\lambda$, write

$$f^*_\lambda(\alpha_\mu) - \lambda = \{\theta_{\mu\lambda n} : n < r_{\mu\lambda}\}, \qquad r_{\mu\lambda} \leqslant \omega,$$

$$f^{**}_\lambda(\alpha_\mu) - \lambda = \{\tau_{\mu\lambda m} : m < s_{\mu\lambda}\}, \qquad s_{\mu\lambda} \leqslant \omega.$$

To each pair $\langle\theta_{\mu\lambda n}, \tau_{\mu\lambda m}\rangle$, $n < r_{\mu\lambda}$, $m < s_{\mu\lambda}$, $\langle f^*_\lambda, f^{**}_\lambda\rangle$ assigns a separating pair $\langle\theta'_{\mu\lambda n}, \tau'_{\mu\lambda m}\rangle \in \lambda \times \lambda$.

Let $J_\lambda = (\text{dom} f^*_\lambda(0) \cup \text{dom} f^{**}_\lambda(0)) - (\omega_1 \times \lambda)$.

By the normality of $\mathcal{F}_{wc}$, there is an $\mathcal{F}_{wc}$-positive set $U$ such that on $U$, the sets $B_\lambda$, $r_{\mu\lambda}$, $s_{\mu\lambda}$, $\theta'_{\mu\lambda n}$, $\tau'_{\mu\lambda m}$, $J_\lambda$ are independent of $\lambda$, and such that if $\lambda, \lambda' \in U$, $\lambda < \lambda'$, then $(\text{support} f^*_\lambda \cup \text{support} f^{**}_\lambda) \cap (\text{support} f^*_{\lambda'} \cup \text{support} f^{**}_{\lambda'}) = B_\lambda$, and $J_\lambda \cap J_{\lambda'} = \varnothing$.

By induction on $\gamma \leqslant \alpha$ it is seen that if $\lambda, \mu \in U$ and $\lambda < \mu$, then $f^*_\lambda \sim f^{**}_\mu$. Namely, there is no trouble with coordinates in the support of at most one of these functions; coordinates in both supports, being in $B_\lambda$, are taken care of by the construction. Since $f_\lambda \leqslant f^*_\lambda$ and $f_\mu \leqslant f^{**}_\mu$, we are done.

The following strengthening of $\kappa cc$ for $P_\alpha$ has thus been proved: if for an $\mathcal{F}_{wc}$-positive set $W$ of $\lambda$'s, $\#^\alpha_\lambda(f_\lambda, g_\lambda, h_\lambda)$, then there is an $\mathcal{F}_{wc}$-positive $U \subseteq W$ and $(f'_\lambda, g'_\lambda, h'_\lambda)$, $\lambda \in U$, such that $\langle f_\lambda, g_\lambda, h_\lambda\rangle \leqslant \langle f'_\lambda, g'_\lambda, h'_\lambda\rangle$, $\#^\alpha_\lambda(f'_\lambda, g'_\lambda, h'_\lambda)$, and so that if $\lambda, \mu \in W$, $\lambda < \mu$, then $f'_\lambda \sim g'_\mu$ in the strong sense that the coordinatewise union of $f'_\lambda$ and $g'_\mu$ is a condition extending both $f'_\lambda$ and $g'_\mu$.

Lastly, we prove the second induction hypothesis for $\alpha$.

(2) For $\mathcal{F}_{wc}$-almost all $\lambda < \kappa$, for all $f, g, h$, $\#^\alpha_\lambda(f, g, h)$ implies that for some $h' \geqslant h$, $*^\alpha_\lambda(f, g, h')$.

PROOF. Otherwise for an $\mathcal{F}_{wc}$-positive set $W$ of $\lambda$'s there exists a counterexample $\langle f_\lambda, g_\lambda, h_\lambda\rangle$. We may assume that for each $\lambda \in W$, $\lambda > \mu_n$ ($n < \omega$) and $\lambda$ satisfies Lemmas 1, 2 and 7. Furthermore, since we have already proved that $\mathcal{P}_\alpha$ has the $\kappa cc$, we may assume that for each $\lambda \in W$, $\mathcal{P}_\alpha|\lambda \subseteq_{\text{reg}} \mathcal{P}_\alpha$ and $\mathcal{P}_\alpha|\lambda$ has the $\lambda cc$. If $f_\lambda$ or $g_\lambda$ equals $h_\lambda$ we are done, so assume, for each $\lambda \in W$, that $f_\lambda, g_\lambda \notin \mathcal{P}_\alpha|\lambda$.

Apply Lemma 7 to each $\langle f_\lambda, g_\lambda, h_\lambda\rangle$, $\lambda \in W$, getting $\langle f'_\lambda, g'_\lambda, h'_\lambda\rangle$. Now uniformize as in part (a) to get an $\mathcal{F}_{wc}$-positive $V \subseteq W$ such that if $\lambda, \mu \in V$ and $\lambda < \mu$ then $f'_\lambda \sim g'_\mu$. Since $\langle f_\lambda, g_\lambda, h_\lambda\rangle$ is a counterexample to (b), there is a maximal antichain $H_\lambda$ of $\{h \in \mathcal{P}_\alpha|\lambda : h \geqslant h_\lambda\}$ such that for each $h \in H_\lambda$, $h \nsim f_\lambda$ or $h \nsim g_\lambda$. Then $H_\lambda$ is a maximal antichain of $\{h \in \mathcal{P}_\alpha : h \geqslant h_\lambda\}$, and Card $H_\lambda < \lambda$. Pick an $\mathcal{F}_{wc}$-positive $U \subseteq V$ on which $H_\lambda = H$ is independent of $\lambda$ and such that for each $h \in H$, the questions, whether or not $h \sim f_\lambda$, $h \sim g_\lambda$, are independent of $\lambda$. Pick

$\lambda, \mu \in U, \lambda < \mu$, and let $j \geqslant f'_\lambda, g'_\mu$. Now $j \geqslant h_\lambda$, and $j \notin \mathcal{P}_\alpha|\lambda$ (whence $j \notin H$). But for each $h \in H$, either $h \nsim g_\lambda$ (whence $h \nsim g_\mu$) or $h \nsim f_\lambda$. In either case, $h \nsim j$ since $j \geqslant f_\lambda, g_\mu$, so $H$ is not maximal, a contradiction.

This completes the proof of the theorem.

Denote by an $\omega_2$-tree a tree $T$ of any cardinality with no paths of length $\omega_2$. An $\omega_2$-tree $T$ is special if there is an $f\colon T \to \omega_1$ such that $x <_T y$ implies $f(x) \neq f(y)$. By the previous methods, using countable specializing functions instead of countable antichains, the consistency of "$2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} > \aleph_2$, and every $\omega_2$-tree of cardinality $< 2^{\aleph_1}$ is special" is obtained – the analogous theorem for the $\aleph_1$ case being Baumgartner-Malitz-Reinhardt [**1**]. We can also get this model to satisfy the "generalized Martin's axioms" (which are consistent relative to just $ZFC$ but which do not imply $SH_{\aleph_2}$) that have been considered by the first author and by Baumgartner (see Tall [**8**]). Desirable, of course, would be the consistency of a generalized $MA$ which is both simple and powerful.

The partial orderings appropriate for the prior methods can be iterated an arbitrary number of times, giving generalized $MA$ models in which $2^{\aleph_1}$ is arbitrarily large. The ordering $\mathcal{R}_\alpha$ giving the first $\alpha$ steps of the iteration need not be of cardinality $\leqslant \kappa$, but, assuming each $\mathcal{R}_\beta$, $\beta < \alpha$, has $\kappa cc$, any sequence $\langle p_\lambda\colon \lambda < \kappa\rangle$ from $\mathcal{R}_\alpha$ is a subset of a sufficiently closed model of power $\kappa$, in which the proof that two $p_\lambda$'s are compatible can be carried out.

Regarding the analog of these results where $\aleph_2$ is replaced by $\gamma^+$–the relevant forcing is $\gamma$-directed closed, so by upward Easton forcing we may guarantee that, for example, $\gamma$ remains supercompact if it was in the ground model.

For results involving consequences of $SH_{\aleph_2}$: with $GCH$, see Gregory [**3**], [**4**] (Con($SH_{\aleph_2}$ and $GCH$) is open); with just $CH$, see a forthcoming paper by Stanley and the second author.

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