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Note

Decomposing Uncountable Squares to Countably Many Chains

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We construct an ordered set I of cardinality $\aleph_1$, such that its square is the union of $\aleph_0$ chains (in the natural partial order).

The theorem mentioned in the abstract solved a problem of Countryman [1] and was announced in [3].

The construction generalizes a variant of the construction of Aronszajn trees (see, e.g., [2]) where the function is into subsets of ω rahter than rationals. Morley noticed that the Countryman problem is equivalent to an arithmetical statement; hence, there was a small hope for an independence result.

Notation. Let $I^n = \{\langle t_0, ..., t_{n-1}\rangle : t_i \in I\}$, and if $\bar{t} = \langle t_0, ..., t_{n-1}\rangle \in I^n$, let $\bar{t}(i) = t_i$. We write $\bar{t} \in I$ instead of $\bar{t} \in I^n$. If I is (partially) ordered by $<$, the natural partial ordering $<$ on I^n is defined by $\bar{t} \leqslant \bar{s} \Leftrightarrow (\forall i < n)[\bar{t}(i) \leqslant \bar{s}(i)]$. If I is partially ordered, a chain is a totally ordered subset of I. Among sequences of rationals (finite or infinite) $\lessdot$ is the partial order of being an initial segment (not necessarily proper).

Let i, j, n, m, k, l, r, p be natural numbers, and let $\alpha, \beta, \gamma, \delta$ be ordinals.

THEOREM 1. *There is an ordered set I of cardinality $\aleph_1$, such that for every n, I^n is the union of $\aleph_0$ chains (in the natural ordering).*

Proof. For $\alpha < \omega_1$ let S_α be the set of sequences of rationals of length α, and $S^\alpha = \bigcup_{\beta<\alpha} S_\beta$, $S = S^{\omega_1}$. S is partially ordered by $\lessdot$ and ordered by the lexicographic order $<$. We define by induction on $\alpha < \omega_1$ sets $T_\alpha \subseteq S_\alpha$ and sets $C(\bar{t})$ for $\bar{t} \in T^{\alpha+1}$ ($T^\alpha = \bigcup_{\beta<\alpha} T_\beta$) (i.e., $\bar{t} \in (T^{\alpha+1})^n$ for some $n < \omega$ such that:

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(1) T_α is countable, and $\neq \phi$.

(2) If $\beta < \alpha$, $s \in T_\beta$ then for some $t \in T_\alpha$, $s \lessdot t$.

(3) If $s \lessdot t$, $t \in T_\alpha$, $s \in S_\beta$ then $s \in T_\beta$.

(4) $C(\bar{t})$ is an infinite subset of ω, for $\bar{t} \in T^{\alpha+1}$.

(5) If $\bar{s}, \bar{t} \in (T^{\alpha+1})^n$; and $i < n$ implies $\bar{s}(i) \lessdot \bar{t}(i)$; and $i, j < n$, $\bar{s}(i) \in S_\beta$, $s(j) \in S_\gamma$, $\beta < \gamma$ implies $\bar{s}(i) = \bar{t}(i)$; then $C(\bar{t}) \subseteq C(\bar{s})$.

(6) If $\beta < \alpha$, $s_0, \ldots, s_n \in T^\beta$, $s_{n+1}, \ldots, s_m \in T_\beta$, and $k < \omega$ then there are $s'_{n+1}, \ldots, s_m'$ such that:

(A) $s_l \lessdot s_l'$, $s_l' \in T_\alpha$ for $n < l \leqslant m$

(B) let $s_l' = s_l$ for $l \leqslant n$; if $r(0), \ldots, r(p-1) \leqslant m$, $p \leqslant k$ then $k \cap C(\langle s_{r(0)}, \ldots, s_{r(p-1)} \rangle) = k \cap C(\langle s'_{r(0)}, \ldots, s'_{r(p-1)} \rangle)$.

(7) If $n < \omega$, $\bar{s}, \bar{t} \in (T^{\alpha+1})^n$, $C(\bar{s}) \cap C(\bar{t}) \neq \varnothing$ *then* (i) $\bar{s} \leqslant \bar{t}$ or $\bar{t} \leqslant \bar{s}$ and (ii) if $\bar{s} \neq \bar{t}$, $i < n$, $\bar{s}(i) = \bar{t}(i)$ then $\bar{s}(i) \in T^\alpha$.

The theorem is proved by $I = \bigcup_{\alpha<\omega_1} T_\alpha$ with the order $<$, and for each $n < \omega$ the decomposition of I^n to chains $I^n = \bigcup_{l<\omega} J_l^n$, $J_l^n = \{\bar{s} \in I^n: l = \min C(\bar{s})\}$. As $C(\bar{s})$ is nonempty this is a decomposition, and by condition (7) each J_l^n is a chain. ∎

Case A. $\alpha = 0$. Let T_0 be $S_0 = \{\langle\ \rangle\}$, and for every $\bar{s} \in T_0$, $C(\bar{s}) = \omega$.

Case B. $\alpha = \gamma + 1$. Let $T_\alpha = \{s: s \in S_\alpha, (\exists t \in T_\gamma)\ (t \lessdot s)\}$, so clearly (1), (2), (3) hold. Notice that it suffices to show that (6) holds only for $\beta = \gamma$. So let $k(i)$, $n(i)$, $m(i)$, $\bar{s}(i) = \langle s_0^i, \ldots, s_{m(i)}^i \rangle$ ($i < \omega$, i odd) be a list of all possible candidates for (6). Let $\bar{s}^i$ ($i < \omega$, i even) be a list of all $\bar{t} \in \bigcup_n [(T^{\alpha+1})^n - (T^\alpha)^n]$, each appearing ω times. For such $\bar{t}$ let $\bar{s} = g(\bar{t})$ be a sequence of the same length such that $\bar{t}(l) \in T^\alpha \Rightarrow \bar{s}(l) = \bar{t}(l)$ and $\bar{t}(l) \in T_\alpha \Rightarrow \bar{s}(l) \lessdot \bar{t}(l) \wedge \bar{s}(l) \in T_\gamma$.

We define by induction on i a finite set Γ_i of conditions of the form $l \in C(\bar{s})$, so that we do not contradict conditions (5), (7), if we later define $C(\bar{s}) = \{l: [l \in C(\bar{s})] \in \bigcup_{i<\omega} \Gamma_i\}$ for $\bar{s} \in T^{\alpha+1}$, $\bar{s} \notin T^\alpha$. Hence, it will be trivial to check that we prove Case B (condition (6) holds by (β), and (4) by (α) and (5), (7) by the construction).

Case $B(\alpha)$. i is even. As $C(g(\bar{s}^i))$ is infinite, there is $l_i \in C(g(\bar{s}^i))$ such that $k_j < l_i$ for odd $j < i$ and l_i do not appear in $\bigcup_{j<i} \Gamma_j$. Let $\Gamma_i = \{l \in C(\bar{s}^i)\}$.

Case $B(\beta)$. i is odd. Choose a rational number q so that for every $j < i$ and every t appearing in Γ_j, if $t \in T_\alpha$ then $t(\gamma) < q$. Let $\Gamma_i = \{l \in C(\langle t_{r(0)}, \ldots, t_{r(p-1)} \rangle)$: where $r(0), \ldots, r(p-1) \leqslant m(i)$, $p \leqslant k(i)$, $l < k(i)$,

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$l \in C(\langle s^i_{r(0)}, \ldots, s^i_{r(p-1)} \rangle)$ and $j \leqslant n(i) \to t_j = s_j^i$, $n(i) < j \leqslant m(i) \to t_j = s_j^i\langle q \rangle\}$.

Let us check condition (7), so suppose $\bar{s}, \bar{t} \in (T^{\alpha+1})^m$, and $n \in C(\bar{s})$ belong to $\bigcup_{j \leqslant i} \Gamma_j$ or holds in $T^{\alpha+1}$, and similarly for $n \in C(\bar{t})$. By the induction hypothesis on α and on i we can assume $[n \in C(\bar{s})] \in \Gamma_i$. Let $V = \{l < n: \bar{s}(i) \in T^\alpha\}$, so $l \notin V$ implies $\bar{s}(l) \in T_{\alpha+1}$, and the last element of $\bar{s}(l)$ is q.

Suppose $\bar{s}$, $\bar{t}$ contradicts 7(i), so for some k, l, $\bar{s}(l) < \bar{t}(l)$, $\bar{s}(l) > \bar{t}(l)$. Clearly $n \in C(g(\bar{s}))$ and $n \in C(g(\bar{t}))$ and $[g(\bar{s})](l) \leqslant [g(\bar{t})](l)$, $[g(\bar{s})](k) \geqslant [g(\bar{t})](l)$; hence, for some $p \in \{l, k\}$, $[g(\bar{s})](p) = [g(\bar{t})](p)$. Also $C(g(\bar{s})) \cap C(g(\bar{t})) \neq \varnothing$; by 7(ii) $g(\bar{s}) \neq g(\bar{t})$ implies that for each $r < m$,

$$[g(\bar{s})](r) = [g(\bar{t})](r) \Rightarrow [g(\bar{s})](r) \in T^\alpha.$$

Suppose $g(\bar{s}) \neq g(t)$; then necessarily $[g(\bar{s})](p) = [g(\bar{t})](p) \in T^\alpha$ but then $\bar{s}(p) = [g(\bar{s})](p)$, $\bar{t}(p) = [g(\bar{t})](p)$ (by g's definition) so $\bar{s}(p) = \bar{t}(p)$, contradiction.

Now suppose $g(\bar{s}) = g(\bar{t})$. Then $\bar{t} \leqslant \bar{s}$ by the choice of q.

It remains to check 7(ii); so assume $\bar{s}(l) = \bar{t}(l)$, $\bar{s}(l) \in T_{\alpha+1}$, then necessarily $[n \in C(\bar{t})] \in \Gamma_i$ and the checking is easy.

Case C. $\alpha = \delta$ is a limit ordinal. We choose $\alpha_n < \delta$ $(n < \omega)$, $\alpha_n < \alpha_{n+1}$, $\delta = \bigcup_{n<\omega} \alpha_n$. We can easily define by induction on $i < \omega$, $k_i < \omega$ and $s_0^i, \ldots, s^i_{m(i)} \in T^{\alpha_{i+1}}$ such that:

(i) if $l \leqslant m(i)$, $i < j$ *then*: $s_l^i \lessdot s_l^j$ and $s_l^i \in T^{\alpha_i}$ iff $s_l^j \in T^{\alpha_j}$ iff $s_l^i = s_l^j$

(ii) $k_i < k_{i+1}$, $m(i) < m(i+1)$

(iii) if $r(0), \ldots, r(p-1) \leqslant m(i)$, $p \leqslant k_i$ then

(α) $k_i \cap C(\langle s^i_{r(0)}, \ldots, s^i_{r(p-1)} \rangle) = k_i \cap C(\langle s^{i+1}_{r(0)}, \ldots, s^{i+1}_{r(p-1)} \rangle)$

(β) $\{l: k_i < l < k_{i+1}\} \cap C(\langle s^{i+1}_{r(0)}, \ldots, s^{i+1}_{r(p-1)} \rangle) \neq \varnothing$

(iv) for any $\beta, k, n, m, s_0, \ldots, s_m$ appropriate for (6) there is i so that: $k \leqslant k_i$, $\beta < \alpha_i$; $n < l \leqslant m \to s_l \lessdot s^{i+1}_{m(i)+l}$, and $s_l = s^{i+1}_{m(i)+l}$ for $l \leqslant n$; and for every $p \leqslant k$, $r(0), \ldots, r(p-1) \leqslant n$,

$$k \cap C(\langle s_{r(0)}, \ldots, s_{r(p-1)} \rangle) = k \cap C(\langle s^{i+1}_{m(i)+r(0)}, \ldots, s^{i+1}_{m(i)+r(p-1)} \rangle). \quad ∎$$

By the induction hypothesis (6), it easy to define k_i, $m(i)$, and s_j^i. Let s_j be the minimal member of $S^{\alpha+1}$ such that $j < i < \omega \to s_j^i \lessdot s_j$. Let $T_\alpha = \{s_j: j < \omega, s_j \in S_\alpha\}$. Let h_i be defined on $T^{\alpha+1}$: $h_i(s_j) = s_j^i$ for $j \leqslant i$, and $h_i(s_j) = s_0^0$ for $j > i$, but $h_i(s) = s$ for $s \in T^\alpha$. Let $C(\langle t_0, \ldots, t_{p-1} \rangle)$ be the set of $l < \omega$ such that for every i big enough

$l \in C(\langle h_i(t_0),..., h_i(t_{p-1})\rangle)$ (by (iii)(α) this is equivalent to: for arbitrarily large i). Clearly if $t_0 ,..., t_{p-1} \in T^\alpha$ we do not change the C we have. Now conditions (1), (3) hold trivially, (2) follows from (i), (iv); (4) follows from (iii)(β); (5), (7) follow from the definition of $C(\bar{s})$, and themselves as an induction assumption; (6) follows by (iv).

So we finish Case C, hence the induction, hence the proof.

Observations.

(1) By a similar construction we can prove that the weak Bethe theorem fails for $L_{\infty,\omega}$ (thus solving [14, problem 8]) and some similar theorems. The proofs will appear.

(2) If $\lambda = \sum_{\mu<\lambda} \lambda^\mu$ we can construct a similar tree for λ, λ^+ instead of $\aleph_0$, $\aleph_1$, by a similar construction or prove its existence by Chang's two-cardinal theorem (see, e.g., [5]).

(3) Clearly we can construct the tree so that it will be a special Aronszajn tree.

Notation. We write $*[I]$ if I is uncountable and I^2 is the union of $\aleph_0$ chains, usually denoted by J_n ($n < \omega$), which are w.l.o.q., pairwise disjoint. Orders I^1, I^2 are called near if they have isomorphic uncountable subsets; hereditarily near if any uncountable $I_1 \subset I^1$, $I_2 \subset I^2$ are near.

(4) If $*[I]$, then I is a Specker order, i.e., I is uncountable, but we cannot embed into it ω_1, $\omega_1{}^*$ and any uncountable set of reals. Hence, its cardinality is $\aleph_1$. (We leave the proof as an exercise; this was noticed already by Countryman [1].)

(5) If $*[I]$ then for each n, I^n is the union of $\aleph_0$ chains [$\langle s_0 ,..., s_{n-1}\rangle$, $\langle t_0 ,..., t_{n-1}\rangle$ will be in the same chain iff

$$(\forall k, l)(\forall m)(k < l < n \rightarrow \langle s_k , s_l\rangle \in J_m \equiv \langle t_k , t_l\rangle \in J_m)]$$

(This was observed by Galvin before Theorem 1 was proved, and then by the referee and the author.)

(6) If $*[I]$, then I cannot contain two anti-isomorphic uncountable subsets. (If $f: I_1 \rightarrow I_2$ is such an anti-isomorphism, $\{\langle s, f(s)\rangle : s \in I_1\}$ is an uncountable subset of I^2, no two members of which belong to a chain). In particular it follows that I, I^* are not near. (This was observed by Galvin before Theorem 1 was proved, and later by U. Avraham and the author.)

(7) It is easy to prove that if $\diamondsuit_{\aleph_1}$ (e.g., if $V = L$, see [6]) then there are $2^{\aleph_1}$ pairwise not near orders, satisfying $*[I]$.

(8) H. Friedman asked for the existence of an infinite complete order I, such that any open interval of I is isomorphic to I, but I is not

antiisomorphic to itself. The completion I^c of the I from Theorem 1 can serve as an example if we construct it with care. Another way is to define I_n by induction: $I_0 = I$, I_{n+1} is an extension of I_n by adding to the right of each element of I_n a copy of I; then the completion of $\bigcup_{n<\omega} I_n$ is an example.

(9) *Conjecture*. (A) "For every Specker order I there is a J near to it with $*[J]$" is consistent.

(B) "If $*[I]$ and $*[J]$ then I is near to J or to J^*" is consistent.

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