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Group metrics for graph products of cyclic groups [☆]

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A B S T R A C T

We complement the characterization of the graph products of cyclic groups $G(\Gamma, \mathfrak{p})$ admitting a Polish group topology of [9] with the following result. Let $G = G(\Gamma, \mathfrak{p})$, then the following are equivalent:

(i) there is a metric on Γ which induces a separable topology in which E_Γ is closed;
(ii) $G(\Gamma, \mathfrak{p})$ is embeddable into a Polish group;
(iii) $G(\Gamma, \mathfrak{p})$ is embeddable into a non-Archimedean Polish group.

We also construct left-invariant separable group ultrametrics for $G = G(\Gamma, \mathfrak{p})$ and Γ a closed graph on the Baire space, which is of independent interest.

1. Introduction

Definition 1. Let $\Gamma = (V, E)$ be a graph and $\mathfrak{p} : V \to \{p^n : p \text{ prime}, n \geq 1\} \cup \{\infty\}$ a graph colouring. We define a group $G(\Gamma, \mathfrak{p})$ with the following presentation:

$$\langle V \mid a^{\mathfrak{p}(a)} = 1,\ bc = cb : \mathfrak{p}(a) \neq \infty \text{ and } bEc \rangle.$$

We call the group $G(\Gamma, \mathfrak{p})$ the Γ-*product*[1] *of the cyclic groups* $\{C_{\mathfrak{p}(v)} : v \in \Gamma\}$, or simply the *graph product of* $(\Gamma, \mathfrak{p})$. These groups have received much attention in combinatorial and geometric group theory. In [9] the authors characterized the graph products of cyclic groups admitting a Polish group topology, showing that G has to have the form $G_1 \oplus G_2$ with G_1 a countable graph product of cyclic groups and G_2 a direct sum of finitely many continuum sized vector spaces over a finite field. In the present study we complement the work of [9] with the following results:

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[1] Notice that this is consistent with the general definition of graph products of groups from [6]. In fact every graph product of cyclic groups can be represented as $G(\Gamma, \mathfrak{p})$ for some Γ and $\mathfrak{p}$ as above.

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Theorem 2. *Let $\Gamma = (\omega^\omega, E)$ be a graph and $\mathfrak{p} : V \to \{p^n : p$ prime, $n \geq 1\} \cup \{\infty\}$ a graph colouring. Suppose further that E is closed in the Baire space ω^ω, and that $\mathfrak{p}(\eta)$ depends only on $\eta(0)$. Then $G = G(\Gamma, \mathfrak{p})$ admits a left-invariant separable group ultrametric extending the standard metric on the Baire space.*

Theorem 3. *Let $G = G(\Gamma, \mathfrak{p})$, then the following are equivalent:*

(a) there is a metric on Γ which induces a separable topology in which E_Γ is closed;
(b) G is embeddable into a Polish group;
(c) G is embeddable into a non-Archimedean Polish group;

Corollary 4. *Let $G = G(\Gamma, \mathfrak{p})$, then the following are equivalent:*

(a) there is a metric on Γ which induces a separable topology in which E_Γ is closed;
(b) G is embeddable into the automorphism group of the random graph;
(c) G is embeddable into the automorphism group of Hall's universal locally finite group.

The condition(s) occurring in Theorem 3 and Corollary 4 fail e.g. for the $\aleph_1$-half graph $\Gamma = \Gamma(\aleph_1)$, i.e. the graph on vertex set $\{a_\alpha : \alpha < \aleph_1\} \cup \{b_\beta : \beta < \aleph_1\}$ with edge relation defined as $a_\alpha E_\Gamma b_\beta$ if and only if $\alpha < \beta$.

Theorem 2 is of independent interest and generalizes results on left-invariant group metrics on free groups on continuum many generators, see [2], [3] and [4].

2. Proofs of the theorems

Convention 5. *In Definition 1 it is usually assumed that for every $a \in \Gamma$ we have $\{a, a\} \notin E_\Gamma$. In order to make our proofs more transparent we will diverge from this convention and assume that our graphs Γ are such that $a \in \Gamma$ implies $a E_\Gamma a$. This is of course irrelevant from the point of view of the group $G = G(\Gamma, \mathfrak{p})$, since an element $a \in G$ always commutes with itself.*

Proposition 6. *Let G be a separable topological group which is metrizable (resp. ultrametrizable) by the metric d and $V \subseteq G$. Then the metric (resp. ultrametric) $d \restriction V \times V$ makes V into a separable space such that for every group term σ the set $\{\bar{a} \in V^{|\sigma|} : G \models \sigma(\bar{a}) = e\}$ is closed in the induced topology.*

Proof. For every group term σ the map $\bar{a} \mapsto \sigma(\bar{a})$ is continuous. Thus the set $\{\bar{a} \in G^{|\sigma|} : G \models \sigma(\bar{a}) = e\}$ is closed in (G, d), and so the set:

$$\{\bar{a} \in V^{|\sigma|} : G \models \sigma(\bar{a}) = e\} = \{\bar{a} \in G^{|\sigma|} : G \models \sigma(\bar{a}) = e\} \cap V^{|\sigma|}$$

is closed in $(V, d \restriction V \times V)$. □

Notation 7.

(1) Given a graph $\Gamma = (V, E)$ and a set R, by a map $h : \Gamma \to R$ we mean a map with domain V. Furthermore, given a map $h : \Gamma \to R$ we let $h(E) = \{\{h(a), h(b)\} : \{a, b\} \in E\}$.
(2) Given $\eta \in X^\omega$, $n < \omega$ and $\nu \in X^n$, we write $\nu \triangleleft \eta$ to mean that $\eta \restriction n = \nu$.
(3) Given $\eta \neq \eta' \in X^\omega$, we let $\eta \wedge \eta'$ be the unique $\nu \in X^n$ such that $\nu \triangleleft \eta$, $\nu \triangleleft \eta'$ and n is maximal, and in this case we also let $lg(\eta \wedge \eta') = lg(\nu) = n$.
(4) Given a topological space X and $Y \subseteq X$, we denote by $\overline{Y}$ the topological closure of Y in X. Also, we denote by Δ_X the set $\{(x, x) : x \in X\}$.

Lemma 8. *Let Γ be a graph and $\mathfrak{p} : \Gamma \to \omega$ a graph colouring. Suppose that Γ admits a separable metric d which makes E_Γ closed in the induced topology. Then:*

(1) Γ admits an ultrametric d' with the same properties;
(2) there exists a one-to-one map $h : \Gamma \to \omega^\omega$ and a map $\mathfrak{p}^* : \omega^\omega \to \omega$ such that:
 (a) $\overline{h(E_\Gamma) \cup \Delta_{\omega^\omega}} \cap h(\Gamma \times \Gamma) = h(E_\Gamma)$;
 (b) $\mathfrak{p}(a) = \mathfrak{p}^*(h(a))$, for every $a \in \Gamma$;
 (c) $\eta_1(0) = \eta_2(0)$ if and only if $\mathfrak{p}^*(\eta_1) = \mathfrak{p}^*(\eta_2)$, for every $\eta_1, \eta_2 \in \omega^\omega$.

Proof. Let $(\Gamma, \mathfrak{p})$ and d be as in the statement of the lemma. If Γ is countable the lemma is clearly true. Assume then that Γ is uncountable. Let $D \subseteq \Gamma$ be a countable dense set of (Γ, d), and $\leq_D$ a well-order of D of order type ω. Renaming the elements of Γ we can assume that $D = \omega$ and $\leq_D$ is the usual order of the natural numbers. For $a \in \Gamma$ we define $\eta_a \in \omega^\omega$ by letting:

$$\eta_a(n) = \begin{cases} \mathfrak{p}(a) & \text{if } n = 0 \\ x(a, n) & \text{if } n > 0, \end{cases}$$

where:

(i) $x(a, n)$ is at distance $< 1/4^n$ from a;
(ii) $x(a, n)$ is minimal under the condition (i).

We define $d' : \Gamma \times \Gamma \to \mathbb{R}_{\geq 0}$ such that:

$$d'(a, b) = \frac{1}{lg(\eta_a \wedge \eta_b) + 2}.$$

Clearly d' is an ultrametric. We verify d' is as required.

$(*)_1$ (Γ, d') is separable.

For each $\nu \in \omega^{<\omega}$ choose a_ν such that $\nu \triangleleft \eta_{a_\nu}$, if possible, and arbitrarily otherwise. Let $D' = \{a_\nu : \nu \in \omega^{<\omega}\}$. We claim that D' is dense in (Γ, d'). This suffices, since obviously D' is a countable subset of Γ. Let then $b \in \Gamma$ and $\varepsilon > 0$, we shall find $a \in D'$ such that $d'(a, b) < \varepsilon$. Choose $n > 0$ such that $1/(n + 2) < \varepsilon$, and let $\nu = \eta_b \restriction n$. Now, by the choice of ν, $a_\nu \in D'$ and $\nu \triangleleft \eta_{a_\nu}$. Furthermore, clearly $\nu \trianglelefteq \eta_{a_\nu} \wedge \eta_b$, and so $lg(\eta_{a_\nu} \wedge \eta_b) \geq lg(\nu) = n$. Thus we have:

$$d'(a_\nu, b) = \frac{1}{lg(\eta_{a_\nu} \wedge \eta_b) + 2} \leq \frac{1}{n + 2} < \varepsilon.$$

$(*)_2$ E_Γ is closed in (Γ, d').

Let $a, b \in \Gamma$ and suppose that $\{a, b\} \notin E_\Gamma$. Since E_Γ is closed in (Γ, d), there is $\varepsilon \in (0, 1)$ such that:

$$a', b' \in \Gamma,\ d(a, a') < \varepsilon,\ d(b, b') < \varepsilon \ \Rightarrow\ \{a', b'\} \notin E_\Gamma. \tag{1}$$

Let $n < \omega$ be such that $n > 1$ and $1/n < \varepsilon$, we shall prove that:

$$a', b' \in \Gamma,\ d'(a, a') < \frac{1}{n + 2},\ d'(b, b') < \frac{1}{n + 2} \ \Rightarrow\ \{a', b'\} \notin E_\Gamma. \tag{2}$$

Now, for any a' as in (2) we have that $lg(\eta_a \wedge \eta_{a'}) > n$, and so $\eta_a(n) = \eta_{a'}(n)$. Hence:

$$d(a, a') \leq d(a, \eta_a(n)) + d(a', \eta_a(n)) < \frac{1}{4^n} + \frac{1}{4^n} < 1/n < \varepsilon.$$

Using the same argument we see that for any b' as in (2) we have that $d(b, b') < \varepsilon$, and so by (1) we conclude that $\{a', b'\} \notin E_\Gamma$, as wanted.

$(*)_3$ The map $h : \Gamma \to \omega^\omega$ such that $h(a) = \eta_a$ is one-to-one.

If $\eta_a = \eta_b$, then:

$$lim_{n\to\infty}\eta_a(n) = a = lim_{n\to\infty}\eta_b(n) = b.$$

$(*)_4$ $\overline{h(E_\Gamma)} \cap h(\Gamma \times \Gamma) = h(E_\Gamma)$.

Notice that for $(c_n)_{n<\omega} \in \Gamma^\omega$ and $c \in \Gamma$ we have:

$$lim_{n\to\infty}\eta_{c_n} = \eta_c \;\Rightarrow\; lim_{n\to\infty}c_n = c \;\text{ in } (\Gamma, d').$$

Thus, if we have:

$$lim_{n\to\infty}\eta_{a_n} = \eta_a,\; lim_{n\to\infty}\eta_{b_n} = \eta_b \text{ and } \bigwedge_{n<\omega} a_n E_\Gamma b_n,$$

then $aE_\Gamma b$, since E_Γ is closed in (Γ, d').

$(*)_5$ Let $\mathfrak{p}^* : \omega^\omega \to \omega$ be such that:

$$\mathfrak{p}^*(\eta) = \begin{cases} \eta(0) & \text{if } \exists \eta_a(\eta_a(0) = \eta(0)) \\ 1 & \text{otherwise.} \end{cases}$$

Then the map $\mathfrak{p}^*$ is clearly as wanted. □

We need some basic word combinatorics for $G(\Gamma, \mathfrak{p})$.

Definition 9. Let $(\Gamma, \mathfrak{p})$ be as usual and $G = G(\Gamma, \mathfrak{p})$.

(1) A word w in the alphabet Γ is a sequence $(a_1^{\alpha_1}, ..., a_k^{\alpha_k})$, with $a_i \neq a_{i+1} \in \Gamma$, for $i = 1, ..., k-1$, and $\alpha_1, ..., \alpha_k \in \mathbb{Z} - \{0\}$.
(2) We denote words simply as $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ instead of $(a_1^{\alpha_1}, ..., a_k^{\alpha_k})$.
(3) We call each $a_i^{\alpha_i}$ a syllable of the word $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$.
(4) We say that the word $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ spells the element $g \in G$ if $G \models g = a_1^{\alpha_1} \cdots a_k^{\alpha_k}$.
(5) We say that the word w is reduced if there is no word with fewer syllables which spells the same element of G.
(6) We say that the consecutive syllables $a_i^{\alpha_i}$ and $a_{i+1}^{\alpha_{i+1}}$ are adjacent if $a_i E_\Gamma a_{i+1}$.
(7) We say that the word w is a normal form for g if it spells g and it is reduced.

Fact 10 *([7, Lemmas 2.2 and 2.3]). Let $G = G(\Gamma, \mathfrak{p})$.*

(1) If the word $a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ spelling the element $g \in G$ is not reduced, then there exist $1 \leq p < q \leq k$ such that $a_p = a_q$ and a_p is adjacent to each vertex $a_{p+1}, a_{p+2}, ..., a_{q-1}$.
(2) If $w_1 = a_1^{\alpha_1} \cdots a_k^{\alpha_k}$ and $w_2 = b_1^{\beta_1} \cdots b_k^{\beta_k}$ are normal forms for $g \in G$, then w_1 can be transformed into w_2 by repeatedly swapping the order of adjacent syllables.

Definition/Proposition 11. Let $\Gamma = (\omega^\omega, E)$, with E closed in the Baire space, and $\mathfrak{p} : V \to \{p^n : p \text{ prime}, n \geq 1\} \cup \{\infty\}$ such that $\mathfrak{p}(\eta)$ depends only on $\eta(0)$. For $0 < n < \omega$, let:

$$E_n = \{(\eta, \nu) : \eta, \nu \in \omega^n \text{ and there are } (\eta', \nu') \in E \text{ such that } \eta \triangleleft \eta' \text{ and } \nu \triangleleft \nu'\},$$

and $G_n = G((\omega^n, E_n), \mathfrak{p}_n)$, where $\mathfrak{p}_n(\eta) = \mathfrak{p}(\eta')(0)$ for any $\eta \triangleleft \eta'$. For $g \in G(\Gamma, \mathfrak{p}) - \{e\}$ and $\eta_1^{\alpha_1} \cdots \eta_k^{\alpha_k}$ a word spelling g, we define $n(g)$ as the minimal $0 < n < \omega$ such that:

$$G_n \models (\eta_1 \restriction n)^{\alpha_1} \cdots (\eta_k \restriction n)^{\alpha_k} \neq e.$$

Finally, for $g \in G(\Gamma, \mathfrak{p}) - \{e\}$, we define $d(g) = 2^{-n(g)}$, and $d(e) = 0$.

Proof. We have to show that $n(g)$ does not depend on the choice of the word spelling g. So let $\eta_1^{\alpha_1} \cdots \eta_k^{\alpha_k}$ and $\theta_1^{\beta_1} \cdots \theta_m^{\beta_m}$ be words spelling $g \in G$, we want to show that, for every $0 < n < \omega$, the words $(\eta_1 \restriction n)^{\alpha_1} \cdots (\eta_k \restriction n)^{\alpha_k}$ and $(\theta_1 \restriction n)^{\beta_1} \cdots (\theta_m \restriction n)^{\beta_m}$ spell the same element $g' \in G_n$. By Fact 10 this is clear, since $\eta_1 E \eta_2$ implies $\eta_1 \restriction n E_n \eta_2 \restriction n$, and $\mathfrak{p}(\eta)$ depends only on $\eta(0)$. □

The following lemma proves Theorem 2.

Lemma 12. *Let $\Gamma = (\omega^\omega, E)$, with E closed in the Baire space, $\mathfrak{p} : V \to \{p^n : p$ prime, $n \geq 1\} \cup \{\infty\}$ such that $\mathfrak{p}(\eta)$ depends only on $\eta(0)$, and $G = G(\Gamma, \mathfrak{p})$. The function $d : G \times G \to [0, 1)_{\mathbb{R}}$ such that $d(g, h) = d(g^{-1}h)$, for $d : G \to [0, 1)_{\mathbb{R}}$ as in Definition/Definition/Proposition 11, is a left-invariant separable group ultrametric extending the usual metric on ω^ω.*

Proof. We show that the function $d : G \to [0, 1)_{\mathbb{R}}$ of Definition/Proposition 11 is an ultranorm, i.e. that it satisfies the following:

(i) $d(g) = 0$ iff $g = e$;
(ii) $d(gh) \leq max\{d(g), d(h)\}$, for every $g, h \in G$;
(iii) $d(g) = d(g^{-1})$, for every $g \in G$.

We prove (i). Let $g \neq e$ and $\eta_1^{\alpha_1} \cdots \eta_k^{\alpha_k}$ a normal form for g. Let $0 < m < \omega$ be such that for every $1 \leq i < j \leq k$ with $\eta_i \neq \eta_j$ we have $\eta_i E \eta_j$ iff $\eta_i \restriction m E_m \eta_j \restriction m$. Then $n(g) \leq m$ and so $2^{-m} \leq 2^{-n(g)} = d(g)$.

We prove (ii). Without loss of generality $g \neq e$ and $h \neq e$. Let $\eta_1^{\alpha_1} \cdots \eta_k^{\alpha_k}$ and $\theta_1^{\beta_1} \cdots \theta_p^{\beta_p}$ be normal forms for g and h, respectively, and let $t = min\{n(g), n(h)\}$. Then for every $0 < m < t < \omega$ we have:

$$G_m \models (\eta_1 \restriction m)^{\alpha_1} \cdots (\eta_k \restriction m)^{\alpha_k} (\theta_1 \restriction m)^{\beta_1} \cdots (\theta_p \restriction m)^{\beta_p} = ee = e.$$

Hence, $t \leq n(gh)$ and so $d(gh) \leq max\{d(g), d(h)\}$.

We prove (iii). Let $\eta_1^{\alpha_1} \cdots \eta_k^{\alpha_k}$ be a normal form for g. It suffices to show that for every $0 < n < \omega$ we have:

$$G_n \models (\eta_1 \restriction n)^{\alpha_1} \cdots (\eta_k \restriction n)^{\alpha_k} = e \Leftrightarrow G_n \models (\eta_k \restriction n)^{-\alpha_k} \cdots (\eta_1 \restriction n)^{-\alpha_1} = e,$$

but this is trivially true.

The fact that d extends the usual metric on ω^ω is immediate. Thus we are only left to show the separability of (G, d). For every $n < \omega$, define a relation R_n on G by letting aR_nb iff there exist normal forms:

$$a = \eta_{a,1}^{\alpha(a,1)} \cdots \eta_{a,k_a}^{\alpha(a,k_a)} \quad \text{and} \quad b = \eta_{b,1}^{\beta(b,1)} \cdots \eta_{b,k_b}^{\beta(b,k_b)}$$

such that $k_a = k_b$, $\alpha(a,\ell) = \beta(b,\ell)$ and $\eta_{a,\ell} \restriction n = \eta_{b,\ell} \restriction n$. Clearly R_n is an equivalence relation on G and it has $\leq \aleph_0$ equivalence classes. For every $n < \omega$, let X_n be a set of representatives of R_n equivalence classes. Then $X = \bigcup_{n<\omega} X_n$ is countable and dense in (G, d), and so it witnesses the separability of (G, d). □

We need two facts before proving Theorem 3.

Fact 13 *([5, Theorem 2.1.3]). Let G be a topological group with compatible left-invariant metric (resp. ultrametric) d. Let D be defined such that:*

$$D(g, h) = d(g, h) + d(g^{-1}, h^{-1}),$$

and $\hat{G}$ the completion of the metric space (G, D). Then the multiplication operation of G extends uniquely onto $\hat{G}$ making $\hat{G}$ into a topological group. Furthermore, there is a unique compatible left-invariant metric (resp. ultrametric) $\hat{d}$ on $\hat{G}$ extending d.

Definition 14. We say that a Polish group G is non-Archimedean if it has a neighbourhood base of the identity that consists of open subgroups.

Fact 15 *([1, Theorem 1.5.1]). Let G be Polish. The following are equivalent:*

(a) G is non-Archimedean;
(b) G is isomorphic to a closed subgroup of $Sym(\omega)$;
(c) G admits a compatible left-invariant ultrametric;
(d) G is isomorphic to the automorphism group of a countable first-order structure.

We finally prove Theorem 3 and Corollary 4.

Proof of Theorem 3. Suppose that $G(\Gamma, \mathfrak{p})$ is embeddable into a Polish group, then by Proposition 6 there is a separable metric on Γ such that E_Γ is closed in the induced topology. On the other hand, if there is a separable metric d on Γ which induces a topology in which E_Γ is closed, then using Lemma 8 we can embed $(\Gamma, \mathfrak{p})$ in a coloured graph on ω^ω which satisfies the assumptions of Lemma 12, and so using Facts 13 and 15 we are done. □

Proof of Corollary 4. As well-known, the automorphism group of the random graph embeds $Sym(\omega)$ (this also follows from the main result of [8]). Furthermore, in [10] it is proved that the automorphism group of Hall's universal locally finite group embeds $Sym(\omega)$. Thus, by Theorem 3 and Fact 15 we are done. □

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