II [Sh 355]

$\aleph_{\omega+1}$ HAS A JONSSON ALGEBRA

§0 Introduction

We advance our knowledge on the cofinality of products of regular cardinals and give several applications; for example there is a Jonsson algebra on $\aleph_{\omega+1}$ and $\aleph_{\omega+1}$-c.c. is not productive.

This chapter introduces pp (using pcf's) as our substitute to exponentiation and advances our understanding of pcf to have a considerable number of applications. For singular λ, $\mathrm{pp}(\lambda)$, [or $\mathrm{pp}_\Gamma(\lambda)$] is the supremum of tcf $\prod \mathfrak{a}/I$ where $\lambda = \sup \mathfrak{a}$ and $J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq I$ and $|\mathfrak{a}| = \mathrm{cf}\lambda$ [or $I \in \Gamma$]; particularly important are $\Gamma(\theta,\sigma) = \{I : \sigma\text{-complete ideal on a set of cardinality} < \theta\}$, $\Gamma(\tau) = \Gamma(\tau^+,\tau)$ (so τ regular) (on basic properties see 2.3). Now pp's are hard to change by forcing, to say $\mathrm{pp}(\lambda) \geq \mu$ is a strong way to say "$\mathcal{S}_{\leq \mathrm{cf}\lambda}(\lambda)$ is large", but we can show in a wide family of cases that they capture $\lambda^{\mathrm{cf}\lambda}$, and saying $\lambda^{\mathrm{cf}\lambda} \leq \mu$ is a strong way to say $\mathcal{S}_{\leq \mathrm{cf}\lambda}(\lambda)$ is small; so together we get much. For this a sufficient condition on λ is

$$\lambda > \mathrm{cf}\lambda > \aleph_0 \ \& \ \bigwedge_{\mu<\lambda} \mu^{\mathrm{cf}\lambda} < \lambda$$

(see §5). However, also if $\lambda < 2^{\aleph_0}$, we get a similar theorem replacing $\lambda^{\mathrm{cf}\lambda}$ by $\mathrm{cf}\,(\mathcal{S}_{\leq \mathrm{cf}\lambda}(\lambda), \subseteq)$; generalizations of this to $\mathrm{cov}(\lambda,\mu,\theta,\sigma)$ are the subject of §5.

Note a recurrent difficulty: for countable cofinality theorems are harder and rarer (but forcing arguments easier — two sides of the same phenomena).

The crucial advance made in §1 is showing that the set on which we take supremum in the definition of $\mathrm{pp}_\Gamma(\lambda)$ is an initial segment of the set of regular cardinals $> \lambda$. This means that there is no point to define $\mathrm{pp}(\lambda)$ as a set of cardinals (as we have done in $\mathrm{pcf}(\mathfrak{a})$). We also complete our understanding why in the definition of the $\mathrm{pcf}(\mathfrak{a})$ we use the cofinality of $\prod \mathfrak{a}/D$ for D an ultrafilter by showing $\mathrm{cf}(\prod \mathfrak{a}) = \max \mathrm{pcf}(\mathfrak{a})$ (and similarly for $\mathrm{cf}(\prod \mathfrak{a}/I)$, — see §3).

We now turn to the applications and the history.
The simplest case of 1.5 is that for some ultrafilter D on ω, $\prod_{n<\omega} \aleph_n/D$ has cofinality $\aleph_{\omega+1}$; this (and a more general case) was asked in [Sh68] and

proved under the additional assumption $2^{\aleph_0} < \aleph_\omega$. We use in its proof 1.3 which tries to answer: does an $<_I$-increasing sequence of $f_\alpha \in {}^\kappa\mathrm{Ord}(\alpha < \delta)$ have an exact upper bound when $\mathrm{cf}\delta > \kappa^+$. Previous versions of this Lemma appear in [Sh68], [Sh71], [Sh111], [Sh-b,XIII,§5] all with $2^\kappa < \mathrm{cf}\delta$, and [Sh282,14] which we represent in [Sh345a,2.6A].

See more in [Sh430,6.1].

The question "can θ be represented as the true cofinality of $\prod \mathfrak{a}/J^{\mathrm{bd}}_{\mathfrak{a}}$", which appeared in [Sh282], has some positive answers (λ^+ when $\mathrm{cf}(\lambda) > \lambda$ and $\forall \mu < \lambda$, $\mu^{\mathrm{cf}\lambda} < \lambda$) but did not seem to have significance, which it acquired by [Sh345] (see below). Here we get a strong positive answer for λ^+, when $\lambda > \mathrm{cf}\lambda > \aleph_0$ in 2.1: there is an increasing continuous sequence $\langle \lambda_i : i < \mathrm{cf}\lambda \rangle$ such that $\prod_{i<\mathrm{cf}\lambda} \lambda_i^+/J^{\mathrm{bd}}_{\mathrm{cf}\lambda}$ has true cofinality λ^+; (this will be used in the proof of pp $\aleph_\omega < \aleph_{\omega_4}$ in [Sh400,§2]). We return to this theme in [Sh371,§1].

An application, presented mainly in §6, concerns the property $\mathrm{NPT}(\lambda,\kappa)$ which means that there is a family of λ sets, each of cardinality $\leq \kappa$, which has no transversal, but every subfamily with $< \lambda$ members has a transversal (a transversal is a one to one choice function; we deal also with some variants of it, for example, for any subfamily of $< \lambda$ sets we can omit from each $< \kappa$ elements to make them pairwise disjoint; in some articles each member was required to be of cardinality $< \kappa$). This property in some sense says λ is not compact, and has a long history, (see below), its negation is denoted by $\mathrm{PT}(\lambda,\kappa)$. We prove that if $\mathrm{pp}_\kappa(\chi) > \lambda = \mathrm{cf}\lambda > \chi > \kappa$, $\mathrm{cf}\chi \leq \kappa$ then $\mathrm{NPT}(\lambda,\kappa)$ (in 1.5A getting a stronger version as above), this is a case where the negation of (a variant of) GCH has a consequence. This shows that, if we have a universe with a supercompact (or just compact) cardinal κ, and we force a failure of SCH (the Singular Cardinal Hypothesis $= \lambda^\kappa \leq \lambda^+ + 2^\kappa$) above κ, or just for some singular $\lambda > \kappa$ we have $\mathrm{pp}(\lambda) > \lambda^+$, then we cannot resurrect the supercompactness of κ without collapsing λ^+ (see 2.2B). This also gives a generalization of Solovay's theorem that SCH holds above a compact cardinal. Also $\mathrm{pp}(\lambda) > \lambda^+$ contradicts appropriate instances of Chang's conjecture, see 2.2.

The question when does $\mathrm{PT}(\lambda,\kappa)$ hold was first asked by Gustin and mentioned in Erdös Hajnal's list of problems [EH] as problem 42C. By [Sh40] if $\kappa < \lambda$, $\mathrm{cf}(\lambda) = \aleph_0$ then $\mathrm{PT}(\lambda,\kappa)$ holds and by Milner and Shelah [MlSh41] for regular λ, $\mathrm{NPT}(\lambda,\kappa)$ implies $\mathrm{NPT}(\lambda^+,\kappa)$ hence $\mathrm{NPT}(\aleph_n,\aleph_0)$. It is clear (see [EH1,p.279]) that if λ is a regular cardinal and has a non-reflecting stationary subset of members of cofinality κ then $\mathrm{NPT}(\lambda,\kappa)$. Later the author notes that this is similar to the problem of the existence of a non-free abelian group (or group) of cardinality λ which is λ-free, those problems have a long history of their own, and then to general varieties, see the book of Eklof and Mekler [EM] (the similarity is great indeed as by [Sh161] $\mathrm{NPT}(\lambda,\aleph_0)$ is equivalent to the existence of a non-free, λ-free abelian group). By [Sh52] $\lambda > \mathrm{cf}(\lambda) + \kappa$ implies $\mathrm{PT}(\lambda,\kappa)$. By [Sh108], if for example $\aleph_\omega$ is strong limit then $NPT(\aleph_{\omega+1},\aleph_0)$ and we continued in [BD]: it is consistent that for every λ $PT(\lambda, 2^{\aleph_0})$. By Magidor and Shelah [MgSh204] it is consistent that (GCH and) $PT(\aleph_{\omega^2+1},\aleph_0)$, moreover, letting χ be the first fixed point (i.e. cardinal χ such that $\chi = \aleph_\chi > \aleph_0$), it is consistent that for every $\lambda \geq \chi$ we have for example $PT(\lambda,\aleph_0)$. On the other hand by [MgSh204], we know $NPT(\aleph_{\omega_1+1},\aleph_0)$ and in fact $NPT(\lambda,\aleph_0)$ for arbitrarily large $\lambda < \chi$ (i.e. those results are provable in ZFC). Those results were first proved assuming weak versions of GCH, but after the advances made here were improved to ZFC results (and so they will appear in the final version). See somewhat more in [Sh523]

In 4.1-4.1D, 4.7, 4.8 we get quite strong colouring theorems on successor of singular. For example,

$\otimes_1$ for λ singular $Pr_1(\lambda^+, cf\lambda)$, i.e. there is a two place symmetric function from λ^+ to $cf\lambda$, such that: if $u_i(i<\lambda^+)$ are pairwise disjoint subsets of λ^+, $|u_i| < cf\lambda$ and $\gamma < cf\lambda$ then for some $i<j$, c is constantly γ on $u_i \times u_j$.

Relying on the later chapter here [Sh365] (or the earlier [Sh280], [Sh327])

$\otimes_2$ for $\lambda > \aleph_1$, $Pr_0(\lambda^+,\aleph_0)$. (Pr_0 is stronger than Pr_1, see Appendix §1).

An example of a conclusion is

$\otimes_3$ if λ is singular then the product of two topological spaces with cellularity λ may have cellularity $> \lambda$; equivalently λ^+-c.c. of Boolean algebras is not productive (i.e. for some λ^+-c.c. Boolean algebra B, the λ^+-c.c. fails for $B \times B$).

On the history see Appendix 1.

In 4.3-4.6 we deal with Jonsson algebras. We prove that

$\otimes_4$ on $\aleph_{\omega+1}$ there is a Jonsson algebra. The first regular Jonsson cardinal is a limit cardinal.
[a Jonsson algebra M is one such that for every subalgebra N,

$$N \neq M \Rightarrow \|N\| < \|M\|;$$

if not said otherwise M has $\leq \aleph_0$ functions; M is on λ if this is its set of elements, λ a Jonsson cardinal if no Jonsson algebra on λ exists].

Keisler and Rowbottom [KR] proved that if $V = L$ then in every (infinite) cardinal there is a Jonsson algebra. Erdös and Hajnal [EH2] proved that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ implies there is a Jonsson algebra in $\aleph_{\alpha+1}$, also they proved that in $\aleph_n$ there is a Jonsson algebra.

By [Sh68] if $2^{\aleph_0} \leq \aleph_{\omega+1}$, then there is a Jonsson algebra in $\aleph_{\omega+1}$, and generally under weak assumptions on cardinal arithmetic the induction of the proof that successor cardinals have Jonsson algebras does not stop in successors.

Independently, Tryba [Tr] and (earlier but unpublished) Woodin proved that for λ successor of regular there is a Jonsson algebra on λ^+.

In 4.9-4.14 we deal quite extensively with entangled linear orders (and very far linear orders) [a linear order $\mathcal{I}$ is entangled if for any $m<n<\omega$ and distinct $t^\ell_\alpha \in \mathcal{I}(\ell<n, \alpha<|\mathcal{I}|)$ we can find $\alpha < \beta$ such that

$$\bigwedge_{\ell<n} [t^\ell_\alpha < t^\ell_\beta \equiv \ell < m]].$$

We prove for example that if $\lambda \leq 2^{\aleph_0}$ is singular then there is one in λ^+, and for λ singular there is a sequence of $cf\lambda$ linear orders of cardinality λ^+ which is "entangled" (i.e. exemplifies $Ens(\lambda^+, cf\lambda)$) (if $pp\lambda$ is large we get such examples in regulars in $(\lambda, pp^+(\lambda))$. For historical notes see Appendix 2.

In 5.11 we prove for λ regular, if $2^{<\lambda} < 2^\lambda$, and for no $\mu \in (\lambda, 2^{\leq\lambda}]$, $cf\mu = \lambda$, $pp_{\Gamma(\lambda)}(\mu) =^+ 2^\lambda$ then for any regular $\chi \leq 2^\lambda$, there is a tree with λ nodes and $\geq \chi$ λ-branches. Also

$\otimes_5$ if $cf\lambda \leq \kappa < \lambda_0 < \lambda$ and $\lambda < \mu < pp^+_\kappa(\lambda)$
then there are $\sigma = cf\sigma \leq \kappa$ and a tree T with $\leq \lambda$ nodes and $\geq \mu$ σ-branches; moreover, for some strictly increasing sequence $\langle \lambda_i : i < \sigma\rangle$ of regular cardinals $< \lambda$, we have $T \subseteq \bigcup_{\alpha<\delta} \prod_{i<\alpha} \lambda_i$, the α-th level of the tree is $\subseteq \prod_{i<\alpha} \lambda_i$ and has cardinality $< \lambda_\alpha$.

[Let $\lambda_1 \in (\lambda_0, \lambda]$ be the minimal singular cardinal of cofinality $\leq \kappa$ with $pp^+_\kappa(\lambda_1) > \mu$, equivalently, $> \lambda$-by 2.3. Similarly

$$[\lambda_2 \in (\lambda_0, \lambda_1) \ \& \ cf\lambda_2 \leq \kappa \Rightarrow pp_\kappa(\lambda_2) < \lambda_1].$$

Let $\mathfrak{a}$ exemplifies $pp^+_\kappa(\lambda_1) > \mu$ and use 3.4; well, we need

$$\theta \in \mathfrak{a} \Rightarrow \theta > \max pcf(\mathfrak{a} \cap \theta).$$

So either use [Sh345a, 1.12] to change $\mathfrak{a}$, or quote [Sh371,§1] to get

$$pp^+_{J^{bd}_{cf\lambda_1}}(\lambda_1) > \lambda.$$

Turning to history, consider the relation $A(\chi,\lambda,\mu,\kappa)$ where $\chi > \lambda \geq \mu \geq \kappa$, meaning there is $\mathcal{P} \subseteq [\lambda]^\mu$ of cardinality χ, such that

$$[A \neq B \in \mathcal{P} \Rightarrow |A \cap B| < \kappa].$$

The investigation of this began in Sierpinski [Sr1] and Tarski [Ta], continued in Baumgartner [Ba] (and earlier [Ba1]). He completed its solution in the case GCH holds.

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Let $D(\chi,\lambda)$ [let $D(\chi,\lambda,\kappa)$] mean that there is a tree with λ nodes and $\geq \chi$ branches [of order type κ] (there is an equivalent form speaking on density of linear orders). He proved for example that

$$\text{if } 2^{\aleph_0} < 2^{\aleph_1} \ \& \ 2^{\aleph_0} < \aleph_{\omega_1} \text{ then } A\left(2^{\aleph_1},\aleph_1,\aleph_1,\aleph_1\right) \text{ and } D\left(2^{\aleph_1},\aleph_1,\aleph_1\right).$$

Mitchell [Mi] has independence results concerning D, and Baumgartner [Ba1] — concerning A. For example it is consistent to have

$$2^{\aleph_0} = \aleph_{\omega_1},\ 2^{\aleph_1} = \aleph_{\omega_1},\ \neg D(\aleph_2,\aleph_1,\aleph_1),\ \neg D\left(\aleph_{\omega_1+1},\aleph_1\right),\ \neg A(\aleph_2,\aleph_1,\aleph_1,\aleph_1).$$

By [Sh12, 3.2 and 3.3(C)] respectively:

(a) $A(2^\lambda,\lambda,\lambda,\aleph_0) \Rightarrow \exists$ regular $\aleph_\alpha \leq \lambda[2^\lambda = 2^{|\alpha|} + 2^{\aleph_0}]$,

(b) if $\lambda = \aleph_{\delta+\gamma}$, $\bigwedge_{\beta<\delta} \aleph_\beta^\kappa < \aleph_\delta$, $\bigwedge_{\beta\leq\gamma} \aleph_{\delta+\beta} \geq \kappa + (\text{cf}\delta)^+ + |\beta|^+$, $\mu \geq \kappa + \text{cf}(\delta)^+ + |\gamma|^+$ then $\neg A(\lambda^+,\lambda,\mu,\kappa)$

rephrased this means:

(b′) if $A(\lambda^+,\lambda,\mu,\kappa)$, $\lambda = \aleph_{\delta+\gamma}$, $\bigwedge_{\beta<\delta} \aleph_\beta^\kappa < \aleph_\delta$, κ regular, $\aleph_0 \leq \text{cf}(\delta) < \aleph_\delta$ then there is β such that $\beta = \aleph_{\delta+\beta} \leq \lambda$ (i.e. a fix point in $(\aleph_\delta,\lambda]$).

We shall return to those problems in [Sh410,4.3,4.4 and 6.1], [Sh430,3.4].

In §5 we consider a generalization $\text{cov}(\lambda,\mu,\theta,\sigma)$ of $\text{cf}(\mathcal{S}_{<\theta}(\lambda),\subseteq)$, it is $\min\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{<\mu}$ and every $a \in [\lambda]^{<\theta}$ is included in a union of $< \sigma$ members of $\mathcal{P}\}$.

The main result (5.4) is characterizing it when $\sigma > \aleph_0$,

$\otimes_6$ if $\lambda \geq \mu > \theta = \text{cf}\theta > \sigma = \text{cf}\sigma > \aleph_0$ then $\lambda + \text{cov}(\lambda,\mu,\theta,\sigma) = \lambda + \sup\{\text{pp}_{\Gamma(\theta,\sigma)}(\chi) : \chi \in [\mu,\lambda]$, and $\text{cf}(\chi) \in [\sigma,\theta]\}$.

In §7 we prove that if λ is singular of cofinality $> \aleph_1$, then there are models of cardinality λ which are $L_{\infty,\lambda}$-equivalent not isomorphic, and if $\text{cf}(\lambda) = \aleph_1 < \lambda$ it is true in most cases (maybe all).

This has a long history. Scott [Sc1] proved that a countable model (with a countable vocabulary) is characterized up to isomorphism by a single sentence from $L_{\omega_1,\omega}$. Karp [Ka] generalized the Ehrenfeucht-Fraisse games to $L_{\infty,\omega}$, and Benda [Be], Calais [Ca] and Shelah [Sh11,Lemma 4] independently generalized them to $L_{\infty,\lambda}$, as quoted here in 7.2, and presented in [Di], [Definition 4.2.3, Theorem 4.3.1, pp.352,3]; so 7.2 can serve as a definition of $L_{\infty,\lambda}$-equivalence of models.

The problem arises when $(*)_\lambda$ holds, where:

$(*)_\lambda$ if the models M,N are $L_{\infty,\lambda}$-equivalent of cardinality λ then M,N are isomorphic.

Morley constructs a counterexample to $(*)_\lambda$ for λ regular uncountable, using trees; (see [Ch p.45]). Chang [Ch] proved that if λ has cofinality $\aleph_0$, then $(*)_\lambda$ holds, so the case left open was $\lambda > \text{cf}(\lambda) > \aleph_0$. By [Sh188] if $\lambda = \lambda^{\aleph_0}$ then $\neg(*)_\lambda$, so under GCH the problem was resolved. Here we prove in §7 that $\lambda > \text{cf}\lambda > \aleph_1 \Rightarrow \neg(*)_\lambda$ (see more [NS], [Sh220], [Di]).

We can also mention an inverse monotonicity of pp (see more in 2.3)

$\otimes_7$ if $\lambda < \mu \leq \text{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ and λ,μ are singulars with cofinality in the interval $[\sigma,\theta)$ and $\text{cf}(\theta) = \theta \vee \text{cf}\theta < \sigma$ then $\text{pp}^+_{\Gamma(\theta,\sigma)}(\mu) \leq \text{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ (we also have appropriate behavior in limit).

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Notation: An ideal I here is a family of subsets of its domain, $\text{Dom}(I)$, closed under union and subsets; usually I is proper i.e. $\text{Dom}(I) \notin I$.

For an ideal I let

$\text{gen}(I) = \min\{|J| : J \subseteq I,\ J$ generates I (as an ideal) $\}$

$\lim_I \lambda_i = \min\{\sup\{\lambda_i : i \in A\} :$ for some $B \in I$, $A = \text{Dom}(I) \setminus B\}$

$\text{tlim}_I \lambda_i = \lambda$ if for every ideal J extending I (which is proper) $\lim_J \lambda_i = \lambda$

I, J denote ideals (usually over a cardinal)

if f, g are functions from $\text{Dom}(I)$ to the ordinals, then $f <_I g$ and $f/I < g/I$ and $f < g \bmod I$ all mean

$$\{t \in \text{Dom}(I) : f(t) \geq g(t)\} \in I$$

for a set A of ordinals with no last element,

$J^{\text{bd}}_A = \{B : B$ is a bounded subset of $A\}$.

For a partial order P,

$\text{cf}(P) = \text{cf}P = \min\{|A| : A \subseteq P,\ (\forall p \in P)(\exists q \in A)[p \leq q]\}$

$\text{tcf}(P)$ is κ when there are $p_i \in P$ for $i < \kappa$ such that $\kappa = \text{cf}\,\kappa$, $\bigwedge_{i<j} p_i < p_j$ and $(\forall p \in P) \bigvee_i p \leq p_i$.

$J_{<\lambda}[\mathfrak{a}]$ is $\{\mathfrak{b} : \mathfrak{b} \subseteq \mathfrak{a}, \max\text{pcf}(\mathfrak{b}) < \lambda\}$ (see [Sh345a,1.2])

it is called $J^0_{<\lambda}[\mathfrak{a}]$ in [Sh345, Def 5.2(2)].

$I + A = \{B : B \subseteq \text{Dom(I)}$ and $B \setminus A \in I\}$

$I^+ = \mathcal{P}(\text{Dom } I) \setminus I = \{A : A \subseteq \text{Dom } I, A \notin I\}$

$H(\chi)$ is the family of sets with transitive closure of cardinality $< \chi$

$<^*_\chi$ is some well ordering of $H(\chi)$

$\Gamma(\theta,\sigma) =: \{I :$ for some cardinal $\theta_I < \theta$, I is a σ-complete (proper) ideal on $\theta_I\}$

(but we use it also for the class of ideals isomorphic to such ideals according to convenience)

$\Gamma(\sigma) =: \Gamma(\sigma^+,\sigma)$

$\mathcal{S}_{<\kappa}(\lambda)$ is $\{a \subseteq \lambda : |a| < \kappa\}$, also called $[\lambda]^{<\kappa}$.

§1 Existence of lub in products, and representations of λ^+ as true cofinality

This is a central section in the book: we introduce the pseudo power $\mathrm{pp}_\kappa(\lambda)$, this can be thought of as a fine measure of $\mathcal{S}_{\leq\kappa}(\lambda) = \{a \subseteq \lambda : |a| \leq \kappa\}$. A major theme in the theory is expressing the other relevant measures by it (and some variants). If pcf was a replacement to product, pp is replacement for power; $\mathrm{pp}_\kappa(\lambda)$, for $\lambda > \kappa$ singular of cofinality $\leq \kappa$, is the supremum of the true cofinalities of $\prod \mathfrak{a}/J$ with $\lambda = \sup \mathfrak{a}$, $|\mathfrak{a}| \leq \kappa$, and J containing all bounded subsets of $\mathfrak{a}$. The reader may reproach me for not following the idea of the previous chapter; i.e. defining the set

$$PP_\kappa(\mu) = \{\mathrm{tcf}\Pi\mathfrak{a}/J : |\mathfrak{a}| \leq \kappa,\ \mu = \sup\ \mathfrak{a},\ J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq J\}.$$

This is a good question but there is a good answer:

$\otimes_1$ $PP_\kappa(\mu) = \{\theta : \mu < \theta = \mathrm{cf}\theta \leq^+ \mathrm{pp}_\kappa(\mu)\}$

(we have a problem if the sup is not obtained, this is the meaning of the "+" in $\leq^+$). To prepare for this is the main aim of the section.
Note the following simple consequence: as easily for μ singular, $PP_{\mathrm{cf}(\mu)}(\mu)$ is not empty, necessarily μ^+ belongs to it. Now even the case $\mu = \aleph_\omega$, i.e. "is there an ultrafilter D on ω such that the (true) cofinality of $\prod_{n<\omega} \aleph_n/D$ is $\aleph_{\omega+1}$" was not known.

In fact our theorems give stronger results than necessary for computing PP, which are good for other things:

$\otimes_2$ suppose I is an ideal on $\mathfrak{a}$, extending $J^{\mathrm{bd}}_{\mathfrak{a}}$, $\mu = \sup \mathfrak{a} > |\mathfrak{a}|$; and $\mu < \lambda = \mathrm{cf}\lambda < \mathrm{tcf}\ \prod \mathfrak{a}/I$ (or just $\prod \mathfrak{a}/I$ is λ^+-directed). Then we can find regular $\lambda_\theta < \theta$ for $\theta \in \mathfrak{a}$ such that:

(a) $\lim_I \lambda_\theta = \mu$ (i.e. $\mu' < \mu \Rightarrow \{\theta \in \mathfrak{a} : \lambda_\theta < \mu'\} \in I$)

(b) $\prod_{\theta\in\mathfrak{a}} \lambda_\theta/I$ has true cofinality λ

(c) $\prod \mathfrak{a}/I$ has an $<_I$-increasing cofinal sequence which is μ^+-free in the sense that: if $A \subseteq \lambda$, $|A| \leq \mu$ then we can find $\mathfrak{c}_\alpha \in I$ for $\alpha \in A$ such that $f_\alpha \restriction (\mathfrak{a}\backslash\mathfrak{c}_\alpha)$ for $\alpha \in A$ is really increasing (i.e. $\theta \in \mathfrak{a}\backslash\mathfrak{c}_\alpha\backslash\mathfrak{c}_\beta$ & $\alpha \in A$ & $\beta \in A$ & $\alpha < \beta \Rightarrow f_\alpha(\theta) < f_\beta(\theta)$)

(d) so every $<_I$-increasing cofinal sequence $\langle f'_\alpha : \alpha < \lambda\rangle$ in $\prod_{\theta\in\mathfrak{a}} \lambda_\theta/I$ has this property when restricted to some unbounded subset of λ.

In fact, the main proof (1.3) goes by constructing such a sequence of f_α ($\alpha < \lambda$) in $\prod \mathfrak{a}$, finding an exact upper bound f then "replacing" each $f(\theta)$ by its cofinality (so $\lambda_\theta = \mathrm{cf}[f(\theta)]$) and changing accordingly the f_α's. For this we need to know that exact upper bound exists (in 1.2), and to use the "silly square". The silly square $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$ looks like a very serious demand at the first glance: members of $\mathcal{P}_\alpha$ are closed subsets of α, $\alpha \in C \in \mathcal{P}_\beta \Rightarrow C \cap \alpha \in \mathcal{P}_\alpha$, $\mathcal{P}_\alpha$ has in it a club of order type $\mathrm{cf}\alpha$ unbounded in α; but its existence is trivial as we allow $|\mathcal{P}_\alpha| = \lambda$.

Note that, as usual, we also deal with some variants of $\mathrm{pp}(\lambda)$, the most important are $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ when we restrict ourselves to a σ-complete ideal of cardinality $< \theta$ (so $\sigma \leq \mathrm{cf}\lambda < \theta$) and $\mathrm{pp}_{\Gamma(\theta)}(\lambda) = \mathrm{pp}_{\Gamma(\theta^+,\theta)}(\lambda)$ (so $\mathrm{cf}\lambda = \theta$); in fact $\mathrm{pp}_{\Gamma(\mathrm{cf}\lambda)}(\lambda)$ has, in my mind at least, a strong claim to be the natural power operation.

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Definition 1.1 (1) For λ a limit cardinal, $\kappa < \lambda$, I an ideal on κ let

$\mathrm{pp}^*_I(\lambda) = \sup\{\mathrm{tcf}(\prod_{i<\kappa} \lambda_i, <_I) : \lambda_i = \mathrm{cf}(\lambda_i) < \lambda = \sup_{i<\kappa} \lambda_i$ and for each $\mu < \lambda$, $\{i : \lambda_i < \mu\} \in I$ and $(\prod_{i<\kappa} \lambda_i, <_I)$ has true cofinality$\}$.

(Note that without loss of generality $\lambda_i > \kappa$).

$\mathrm{pp}_I(\lambda) = \sup\{\mathrm{pp}^*_J(\lambda) : I \subseteq J$ and $\mathrm{Dom}\ J = \mathrm{Dom}\ I\}$.

(2) In (1) if Γ is a property of ideals (or a family of ideals), we let:

$\mathrm{pp}(\lambda, \Gamma) = \mathrm{pp}_\Gamma(\lambda) =$
$= \sup\{\mathrm{pp}^*_I(\lambda) : \kappa \leq \lambda$, I is an ideal on κ satisfying $\Gamma\}$

$\mathrm{pp}(\lambda, \kappa, \Gamma) = \mathrm{pp}_{\kappa,\Gamma}(\lambda) = \mathrm{pp}_{\Gamma_1}(\lambda)$ where Γ_1 is: I is an ideal on κ satisfying Γ

(3) $\mathrm{pp}_\kappa(\lambda) = \mathrm{pp}(\lambda, \{I : |\mathrm{Dom}\ I| \leq \kappa\})$, $\mathrm{pp}(\lambda) = \mathrm{pp}_{\mathrm{cf}(\lambda)}(\lambda)$.

(4) Those sups are not necessarily obtained (for example if λ is singular strong limit, 2^λ singular).[3]
Then for example $\chi \leq^+ \mathrm{pp}_I(\lambda)$ will mean: $\chi < \mathrm{pp}_I(\lambda)$ or $\chi = \mathrm{pp}(\lambda)$ and for some $\langle \lambda_i : i < \kappa\rangle$ and J we have $\chi = \mathrm{tcf}(\prod \lambda_i, <_J)$ (and $I \subseteq J$, J an ideal, $\mathrm{tlim}_J \lambda_i = \lambda$). Similarly $\mathrm{pp}_I(\lambda) <^+ \chi$ will mean: not $\chi \leq^+ \mathrm{pp}_I(\lambda)$.

(5) Alternatively let

$\mathrm{pp}^+_\Gamma(\lambda) = \min\{\mu : \mu = \mathrm{cf}\mu$ and for every ideal I satisfying Γ, and sequence $\langle \lambda_t : t \in \mathrm{Dom}\ I\rangle$ of regulars such that $\mathrm{tlim}\langle \lambda_t : t \in \mathrm{Dom}\ I\rangle = \lambda$ and $(\prod_t \lambda_t/I)$ has true cofinality we have $\mathrm{tcf}(\prod_t \lambda_t/I) < \mu\}$.

Question: Can $\mathrm{pp}_\kappa(\lambda) = \mu$ but $\mathrm{pp}_\kappa(\lambda) \neq^+ \mu$ (so μ is (weakly) inaccessible)? See §5 and [Sh400] for partial positive answers.

Claim 1.2 Assume $\mathrm{cf}(\delta) > \kappa^+$, I is an ideal on κ and suppose $\langle f_\alpha : \alpha < \delta\rangle$ is a $<_I$-increasing sequence of members of ${}^\kappa\mathrm{Ord}$. Then exactly one of the following holds:

(i) for some ultrafilter D on κ disjoint from I we have:
$(*)_D$ there are sets $s_i \subseteq \mathrm{Ord}$, $|s_i| \leq \kappa$ for $i < \kappa$ and $\langle \alpha_\zeta : \zeta < \mathrm{cf}(\delta)\rangle$ increasing continuous with limit δ, such that for each $\zeta < \mathrm{cf}(\delta)$ for

[3] This is known to be consistent: make some κ measurable with 2^κ singular and then make κ singular.

some $h_\zeta \in \prod_{i<\kappa} s_i$ we have:
$f_{\alpha_\zeta}/D < h_\zeta/D < f_{\alpha_{\zeta+1}}/D$

(ii) $(**)_I$ some $f \in {}^\kappa\mathrm{Ord}$ is a $<_I$ –eub of $\langle f_\alpha : \alpha < \delta\rangle$, (i.e. f satisfies $(\alpha) + (\beta)$ below) and (γ) holds:
(α) for $\alpha < \delta$, $f_\alpha <_I f$
(β) if $g \in {}^\kappa\mathrm{Ord}$, $g <_I f$ then for some α, $g <_I f_\alpha$
and
(γ) $\mathrm{cf}[f(i)] > \kappa$ for $i < \kappa$

(iii) condition (i) fails and
$(***)_I$ for some unbounded $A \subseteq \delta$ and $t_\alpha \subseteq \kappa$ for $\alpha \in A$ and $g \in {}^\kappa\mathrm{Ord}$ we have:
(α) for $\alpha < \beta$ in A, $t_\beta \backslash t_\alpha \in I$ but $t_\alpha \backslash t_\beta \notin I$
(i.e. $\langle t_\alpha/I : \alpha \in A\rangle$ is strictly decreasing in $\mathcal{P}(\kappa)/I$).
(β) $t_\alpha = \{i < \kappa : f_\alpha(i) \le g(i)\}$.

Remark 1.2A (1) See slightly more (and more details) in 1.6.
(2) Suppose for simplicity that I is a maximal proper ideal, $\kappa = \omega$; then what kinds of Dedekind cuts, with the cofinality of the lower part being $> \aleph_1$, does Ord^κ/I have? Some are Dedekind cuts of ω^ω/D (where D is the ultrafilter on ω dual to I) or "copies of it", and about them we cannot say much. Others are defined by one element.
(3) If $2^\kappa < \mathrm{cf}\delta$ necessarily in 1.2, possibility (ii) holds (why? case (i) is impossible as for the chosen $\langle s_i : i < \kappa\rangle$ there are $\le 2^\kappa$ possible h_ζ's and $[\zeta < \xi \Rightarrow h_\zeta \ne h_\xi]$ and case (iii) is impossible as the number of possible t_α's is again $\le 2^\kappa$).
(4) What if $\langle f_\alpha : \alpha < \delta\rangle$ is only $\le_I$-increasing? Well, there are three cases.
Case 1: $\bigwedge_{\alpha<\delta} \bigvee_{\beta<\delta} f_\alpha <_I f_\beta$.
Then for some club $E \subseteq \delta$, $\langle f_\alpha : \alpha \in E\rangle$ is $<_I$-increasing and we can apply 1.2.
Case 2: For some $\alpha < \delta$, $\bigwedge_{\beta<\delta} \neg f_\alpha <_I f_\beta$, (so not case 1) and

$$\bigwedge_{\beta<\delta} \bigvee_{\gamma\in(\delta\backslash\beta)} \neg f_\gamma =_I f_\beta.$$

Then for some club E of δ, for $\alpha < \beta$ in E, $\neg f_\alpha =_I f_\beta$ & $\neg f_\alpha <_I f_\beta$. For $\alpha \in E$ let $t_\alpha =: \{i < \kappa : f_\alpha(i) = f_{\min E}(i)\}$. They are as required in $(***)_I$ of 1.2(iii) so (iii) holds.
Case 3: For some α^*, $\alpha^* < \beta < \delta \Rightarrow f_\beta =_I f_{\alpha^*}$. Then f_{α^*} is eub of $\langle f_\alpha : \alpha < \delta\rangle$ mod I, similar to (ii) of 1.2.

Proof: First Stage: We prove that (i) and (ii) are contradictory. Suppose for $\langle f_\alpha : \alpha < \delta\rangle$ we have $\langle \alpha_\zeta, h_\zeta : \zeta < \mathrm{cf}\delta\rangle$, D and $\langle s_i : i < \kappa\rangle$ which exemplify (i) and f which exemplifies (ii); by (γ) of (iii) $\bigwedge_i f(i) > 0$. Let s_i' be the closure of $s_i \cup \{0\}$, so $|s_i'| \le \kappa$, and let $f' \in {}^\kappa\mathrm{Ord}$ be defined by $f'(i) = \sup(s_i' \cap f(i))$. As $\mathrm{cf}[f(i)] > \kappa$, clearly $\bigwedge_{i<\kappa} f'(i) < f(i)$; on the other hand for $\zeta < \mathrm{cf}(\delta)$, $h_\zeta \le_D f'$ (as $h_\zeta <_D f$, $h_\zeta \in \prod_{i<\kappa} s_i$) so $f_{\alpha_\zeta} <_D h_\zeta \le_D f'$; but for every $\alpha < \delta$ for some ζ, $\alpha < \alpha_\zeta$ hence $f_\alpha <_D f_{\alpha_\zeta} <_D f' <_D f$, contradicting the choice of f. So at most one of the conditions (i) and (ii) holds. Conditions (i), (iii) are contradictory trivially.

Second Stage:
Now we prove that the conditions (ii), (iii) are contradictory. Suppose not and f exemplifies condition (ii) and g, $\langle t_\alpha : \alpha \in A\rangle$ exemplify condition (iii). Define $g' \in {}^\kappa\mathrm{Ord}$:

$$g'(i) = \begin{cases} g(i) & \text{if } g(i) < f(i) \\ 0 & \text{otherwise.} \end{cases}$$

As for $i < \kappa$, $\mathrm{cf}(f(i)) > \kappa$ hence $f(i) > 0$ clearly $g' < f$, hence (by $(**)_I(\beta)$) for some α, $g' < f_\alpha$ mod I; without loss of generality $\alpha \in A$.
Choose β, $\alpha < \beta \in A$; as $g' < f_\alpha < f_\beta$ mod I clearly

$$s =: \{i < \kappa : \text{ not } g'(i) < f_\alpha(i) < f_\beta(i) < f(i)\} \in I,$$

and by the definition of g' clearly

$$i \in \kappa\backslash s \Rightarrow [f_\alpha(i) \le g(i) \Leftrightarrow f_\beta(i) \le g(i)]$$

hence $t_\alpha \backslash t_\beta \subseteq s$ (see (β) of $(***)_I$) but as $s \in I$ this contradicts (α) of $(***)_I$.

We have proved that in 1.2, at most one of the conditions holds. So it suffices to see that at least one of the conditions holds. Assume (i), (iii) fail and we shall prove (ii).
Third Stage: It suffices to find $f \in {}^\kappa\mathrm{Ord}$ satisfying $(\alpha) + (\beta)$ of (ii). Why? If $A =: \{i < \kappa : \mathrm{cf}[f(i)] \le \kappa\} \in I$, let

$$f'(i) = \begin{cases} f(i) & \text{if } \mathrm{cf}[f(i)] > \kappa \\ \kappa^+ & \text{otherwise.} \end{cases}$$

As $A \in I$, $f' = f$ mod I hence f' is as required in (ii). So assume $A \notin I$, hence there is an ultrafilter D on κ disjoint from I for which $A \in D$; let s_i be a set of $\le \kappa$ ordinals such that if $i \in A$ then $f(i) =: \sup(s_i)$ and if $i \notin A$, $s_i =: \kappa$. Now D, $\langle s_i : i < \kappa\rangle$, f will yield $\langle \alpha_\zeta, h_\zeta : \zeta < \mathrm{cf}(\delta)\rangle$ as required in (i).
Fourth Stage: Note that it suffices to find $f \in {}^\kappa\mathrm{Ord}$ such that:
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Then for some club E of δ, for $\alpha < \beta$ in E, $\neg f_\alpha =_I f_\beta$ & $\neg f_\alpha <_I f_\beta$. For $\alpha \in E$ let $t_\alpha =: \{i < \kappa : f_\alpha(i) = f_{\min E}(i)\}$. They are as required in $(***)_I$ of 1.2(iii) so (iii) holds.

<u>Case 3:</u> For some α^*, $\alpha^* < \beta < \delta \Rightarrow f_\beta =_I f_{\alpha^*}$. Then f_{α^*} is eub of $\langle f_\alpha : \alpha < \delta\rangle$ mod I, similar to (ii) of 1.2.

Proof: <u>First Stage:</u> We prove that (i) and (ii) are contradictory. Suppose for $\langle f_\alpha : \alpha < \delta\rangle$ we have $\langle \alpha_\zeta, h_\zeta : \zeta < \mathrm{cf}\delta\rangle$, D and $\langle s_i : i < \kappa\rangle$ which exemplify (i) and f which exemplifies (ii); by (γ) of (iii) $\bigwedge_i f(i) > 0$. Let s'_i be the closure of $s_i \cup \{0\}$, so $|s'_i| \leq \kappa$, and let $f' \in {}^\kappa\mathrm{Ord}$ be defined by $f'(i) = \sup(s'_i \cap f(i))$. As $\mathrm{cf}[f(i)] > \kappa$, clearly $\bigwedge_{i<\kappa} f'(i) < f(i)$; on the other hand for $\zeta < \mathrm{cf}(\delta)$, $h_\zeta \leq_D f'$ (as $h_\zeta <_D f$, $h_\zeta \in \prod_{i<\kappa} s_i$) so $f_{\alpha_\zeta} <_D h_\zeta \leq_D f'$; but for every $\alpha < \delta$ for some ζ, $\alpha < \alpha_\zeta$ hence $f_\alpha <_D f_{\alpha_\zeta} <_D f' <_D f$, contradicting the choice of f. So at most one of the conditions (i) and (ii) holds. Conditions (i), (iii) are contradictory trivially.

<u>Second Stage:</u>
Now we prove that the conditions (ii), (iii) are contradictory. Suppose not and f exemplifies condition (ii) and g, $\langle t_\alpha : \alpha \in A\rangle$ exemplify condition (iii). Define $g' \in {}^\kappa\mathrm{Ord}$:

$$g'(i) = \begin{cases} g(i) & \text{if } g(i) < f(i) \\ 0 & \text{otherwise.} \end{cases}$$

As for $i < \kappa$, $\mathrm{cf}(f(i)) > \kappa$ hence $f(i) > 0$ clearly $g' < f$, hence (by $(**)_I(\beta)$) for some α, $g' < f_\alpha$ mod I; without loss of generality $\alpha \in A$.
Choose β, $\alpha < \beta \in A$; as $g' < f_\alpha < f_\beta$ mod I clearly

$$s =: \{i < \kappa : \text{ not } g'(i) < f_\alpha(i) < f_\beta(i) < f(i)\} \in I,$$

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$$i \in \kappa \backslash s \Rightarrow [f_\alpha(i) \leq g(i) \Leftrightarrow f_\beta(i) \leq g(i)]$$

hence $t_\alpha \backslash t_\beta \subseteq s$ (see (β) of $(***)_I$) but as $s \in I$ this contradicts (α) of $(***)_I$.

We have proved that in 1.2, at most one of the conditions holds. So it suffices to see that at least one of the conditions holds. Assume (i), (iii) fail and we shall prove (ii).

<u>Third Stage:</u> It suffices to find $f \in {}^\kappa\mathrm{Ord}$ satisfying $(\alpha) + (\beta)$ of (ii). Why? If $A =: \{i < \kappa : \mathrm{cf}[f(i)] \leq \kappa\} \in I$, let

$$f'(i) = \begin{cases} f(i) & \text{if } \mathrm{cf}[f(i)] > \kappa \\ \kappa^+ & \text{otherwise.} \end{cases}$$

As $A \in I$, $f' = f$ mod I hence f' is as required in (ii). So assume $A \notin I$, hence there is an ultrafilter D on κ disjoint from I for which $A \in D$; let s_i be a set of $\leq \kappa$ ordinals such that if $i \in A$ then $f(i) =: \sup(s_i)$ and if $i \notin A$, $s_i =: \kappa$. Now D, $\langle s_i : i < \kappa\rangle$, f will yield $\langle \alpha_\zeta, h_\zeta : \zeta < \mathrm{cf}(\delta)\rangle$ as required in (i).

<u>Fourth Stage:</u> Note that it suffices to find $f \in {}^\kappa\mathrm{Ord}$ such that:
$(\alpha)'$ $f_\alpha < f$ mod I for $\alpha < \delta$

$(\beta)'$ if $g \in {}^\kappa\mathrm{Ord}$ and $f_\alpha < g$ mod I for every $\alpha < \delta$ then $f \le g$ mod I.
Why? We try to show that f is as required in $(**)_I$ of (ii); now (α) of $(**)_I$ holds, so by the third stage we can assume that (β) of $(**)_I$ fails; i.e. there is $g \in {}^\kappa\mathrm{Ord}$, $g <_I f$ but for no $\alpha < \delta$ do we have $g < f_\alpha$ mod I. Let $t_\alpha =: \{i < \kappa : f_\alpha(i) \le g(i)\}$. Clearly $\alpha < \beta \Rightarrow t_\alpha \supseteq t_\beta$ mod I (as $f_\alpha < f_\beta$ mod I); if $\bigwedge_\alpha \bigvee_\beta[\alpha < \beta \;\&\; t_\beta \backslash t_\alpha \notin I]$ we can satisfy $(***)_I$ of (iii) (with g, $\{\beta < \delta : \bigwedge_{\alpha<\beta} t_\alpha \ne t_\beta \bmod I\}$ standing for g, A respectively). So without loss of generality for some $\alpha(*) < \delta$ we have

$$[\alpha(*) \le \alpha < \delta \Rightarrow t_{\alpha(*)} = t_\alpha \bmod I].$$

Now let $g' \in {}^\kappa\mathrm{Ord}$ be defined as $g{\restriction}t_{\alpha(*)} \cup f{\restriction}(\kappa\backslash t_{\alpha(*)})$.
So

$$\alpha < \delta \Rightarrow f_\alpha{\restriction}t_{\alpha(*)} < f_{\alpha+1}{\restriction}t_{\alpha(*)} \le g{\restriction}t_{\alpha(*)} = g'{\restriction}t_{\alpha(*)} \bmod I.$$

Hence $\alpha < \delta \Rightarrow f_\alpha < g'$ mod I.
So by $(\beta)'$ $f \le g'$ mod I, but this implies $t_{\alpha(*)} \in I$ (as $g < f$ mod I) contradiction.
So it suffices to find f satisfying $(\alpha)' + (\beta)'$.
Fifth Stage: We define, by induction on $\zeta < \kappa^+$, a function $g_\zeta \in {}^\kappa\mathrm{Ord}$ such that: $[\xi < \zeta \Rightarrow g_\zeta \le_I g_\xi]$, $[\zeta = \xi + 1 \Rightarrow \neg g_\xi =_I g_\zeta]$ and $[\alpha < \delta \Rightarrow f_\alpha \le_I g_\zeta]$.
We let g_0 be defined by $g_0(i) = \bigcup_{\alpha<\delta}(f_\alpha(i) + 1)$. If $\langle g_\xi : \xi \le \zeta\rangle$ are defined, and there is a $g \in {}^\kappa\mathrm{Ord}$, $f_\alpha \le_I g$ for $\alpha < \delta$, $\neg(g =_I g_\zeta)$ and $g \le_I g_\zeta$ then choose g as $g_{\zeta+1}$; if we cannot, we have gotten "$(\alpha)' + (\beta)'$ are satisfied by g_ζ", as desired [i.e. we should show that $(\alpha)' + (\beta)'$ holds for g_ζ taking the role of f. Why does $(\alpha)'$ hold? As $f_\alpha <_I f_{\alpha+1} \le_I g_\zeta$. Why does $(\beta)'$ hold? Suppose $g' \in {}^\kappa\mathrm{Ord}$, $f_\alpha < g'$ mod I for every $\alpha < \delta$ and we should prove $g_\zeta \le g'$ mod I: if this fails let $g'' \in {}^\kappa\mathrm{Ord}$ be defined by $g''(i) = \min\{g'(i), g_\zeta(i)\}$; clearly $f_\alpha < g''$ mod I (as $f_\alpha < g'$ mod I by assumption) and $f_\alpha < g_\zeta$ mod I (by (α')) and $\neg g'' =_I g_\zeta$ (otherwise $g_\zeta \le g'$ mod I which we assume fails); so g'' satisfies the requirement on $g_{\zeta+1}$.]

If ζ is a limit ordinal, $\zeta \le \kappa^+$, let $s^\zeta_i = \{g_\xi(i) : \xi < \zeta\}$, so s^ζ_i is a set of $\le |\zeta|$ ordinals. For $\alpha < \delta$, let $f^\zeta_\alpha \in {}^\kappa\mathrm{Ord}$ be defined by

$$f^\zeta_\alpha(i) = \min\{\gamma \in s^\zeta_i : f_\alpha(i) \le \gamma\}$$

(well defined as $f_\alpha(i) < g_0(i) \in s^\zeta_i$). If $\zeta < \kappa^+$ and there is $\alpha_\zeta < \delta$ such that $\oplus$ below holds then $f^\zeta_{\alpha_\zeta}$ can serve as g_ζ and we choose it, where:

$\oplus$ $\alpha_\zeta \le \alpha < \delta \Rightarrow f^\zeta_\alpha = f^\zeta_{\alpha_\zeta}$ mod I.

Clearly, for any limit $\zeta \le \kappa^+$:

$\oplus_1$ $\alpha < \delta \Rightarrow f_\alpha \le f^\zeta_\alpha$

and also

$\oplus_2$ $[\alpha < \beta < \delta \Rightarrow f^\zeta_\alpha \le_I f^\zeta_\beta]$

$\oplus_3$ $\alpha < \beta < \delta \;\&\; f^\zeta_\alpha <_I f^\zeta_\beta \Rightarrow f_\alpha \le f^\zeta_\alpha <_I f_\beta \le f^\zeta_\beta$.

Let $\zeta \le \kappa^+$ be minimal such that g_ζ is not well defined (note: if $\zeta = \kappa^+$ then g_ζ cannot be well defined). Clearly ζ is a well defined limit ordinal $\le \kappa^+$.

Case I: $\zeta < \kappa^+$ and $(\forall\alpha < \delta)(\exists\beta < \delta)[\alpha < \beta \;\&\; f^\zeta_\alpha < f^\zeta_\beta \bmod I]$. Then for every ultrafilter D disjoint to I, $(*)_D$ of (i) holds by $\oplus_3$.

Case II: $\zeta < \kappa^+$ and for some $\alpha(*) < \delta$:

$(*)_1$ $(\forall\beta < \delta)[$ not $f^\zeta_{\alpha(*)} < f^\zeta_\beta$ mod I.$]$

but

$(*)_2$ $\forall\alpha < \delta, \exists\beta < \delta[\alpha < \beta \;\&$ not "$f^\zeta_\beta \le f^\zeta_\alpha$ mod I"$]$.

Let for $\alpha \le \beta < \delta$,

$$s_{\alpha,\beta} =: \{i < \kappa : f_\beta(i) < f_\alpha(i)\} \in I \text{ and}$$

$$t_{\alpha,\beta} =: \{i < \kappa : f_\beta(i) \le f^\zeta_\alpha(i)\}.$$

Clearly

$\oplus_4$ $\alpha_1 \le \alpha_2 \le \alpha_3 \le \alpha_4 < \delta$ implies $t_{\alpha_1,\alpha_4} \subseteq t_{\alpha_2,\alpha_3}$ mod I.

Now when $\alpha(*) \le \alpha \le \beta < \delta$ we have: ($t_{\alpha,\beta} \subseteq \kappa$ and) $t_{\alpha,\beta} \notin I$ [why? as $i \in \kappa\backslash s_{\alpha(*),\alpha} \Rightarrow f_\alpha(i) \ge f_{\alpha(*)}(i) \Rightarrow f^\zeta_\alpha(i) \ge f^\zeta_{\alpha(*)}(i)$ (second implication by the definition of f^ζ_α, $f^\zeta_{\alpha(*)}$); hence

$$i \in \kappa\backslash t_{\alpha,\beta}\backslash s_{\alpha(*),\alpha} \Rightarrow f_\beta(i) > f^\zeta_\alpha(i) \;\&\; i \notin s_{\alpha(*),\alpha} \Rightarrow$$

$$f^\zeta_\beta(i) > f^\zeta_\alpha(i) \;\&\; i \notin s_{\alpha(*),\alpha} \Rightarrow f^\zeta_\beta(i) > f^\zeta_{\alpha(*)}(i),$$

so $t_{\alpha,\beta} \in I$ contradicts the assumption $(*)_1$ of the case II (remember $s_{\alpha(*),\alpha} \in I$)].

For $\alpha \in (\alpha(*), \delta)$ the sequence $\bar t = \langle t_{\alpha,\beta} : \alpha < \beta < \delta\rangle$ is decreasing mod I (as $\alpha < \beta \Rightarrow f_\alpha < f_\beta$ mod I).
Now we split to subcases.

Subcase IIa: For some α, $\alpha(*) \le \alpha < \delta$ and $\langle t_{\alpha,\beta} : \beta \ge \alpha, \beta < \delta\rangle$ is not eventually constant mod I.

So $g =: f^\zeta_\alpha$, $A =: \{\beta : \alpha < \beta < \delta$ and $\bigwedge_{\gamma<\beta} t_{\alpha,\beta} \ne t_{\alpha,\gamma} \bmod I\}$ exemplify $(***)_I$ from condition (iii).

Subcase IIb: For some $\alpha < \delta$, $\bigwedge_{\beta>\alpha} t_{\alpha,\beta} = \kappa$ mod I.

So $\beta > \alpha \Rightarrow f_\beta \le f^\zeta_\alpha$ mod I hence by the definition of f^ζ_β (and as $f^\zeta_\alpha(i) \in s^\zeta_i$) we have

$$\beta > \alpha \Rightarrow f^\zeta_\beta \le f^\zeta_\alpha \text{ mod } I$$

contradiction to second assumption of case II.

Subcase IIc: Neither IIa nor IIb.
By "not Subcase IIa", for every $\alpha \in [\alpha(*), \delta)$, there is $\beta_\alpha \in (\alpha, \delta)$ such that: $\beta_\alpha \le \beta < \delta \Rightarrow t_{\alpha,\beta} = t_{\alpha,\beta_\alpha}$ mod I. By $\oplus_4$ clearly

$$\alpha(*) \le \alpha(1) \le \alpha(2) < \delta \Rightarrow t_{\alpha(1),\beta_{\alpha(1)}} \subseteq t_{\alpha(2),\beta_{\alpha(2)}} \text{ mod } I$$

and by "not Subcase IIb", (as $\langle t_{\alpha,\beta} : \alpha \le \beta < \delta\rangle$ is decreasing mod I) clearly $t_{\alpha,\beta_\alpha} \ne \kappa$ mod I. Hence there is an ultrafilter D on κ such that $I \cap D = \emptyset$ and $t_{\alpha,\beta_\alpha} \notin D$ for $\alpha \in (\alpha(*), \delta)$.
Now as D is an ultrafilter disjoint to I, $\langle f_\alpha/D : \alpha < \delta\rangle$ and $\langle f^\zeta_\alpha/D : \alpha < \delta\rangle$ are non-decreasing. Also $f_\alpha \le f^\zeta_\alpha$ hence $f_\alpha/D \le f^\zeta_\alpha/D$. Lastly

$$\alpha \in (\alpha(*), \delta) \Rightarrow \{i < \kappa : f_{\beta_\alpha}(i) \le f^\zeta_\alpha(i)\} = t_{\alpha,\beta_\alpha} \notin D$$
$$\Rightarrow f^\zeta_\alpha < f_{\beta_\alpha} \text{ mod } D$$
$$\Rightarrow f^\zeta_\alpha < f_{\beta_\alpha} \le f^\zeta_{\beta_\alpha} \text{ mod } D.$$

Together, $(*)_D$ (from (i)) holds, contradiction.

Case III: $\zeta < \kappa^+$ and both previous cases fail.
So for some $\alpha = \alpha_\zeta$ we have $(\forall \beta < \delta)[\alpha \le \beta \Rightarrow f^\zeta_\alpha =_I f^\zeta_\beta]$.
We let g_ζ be $f^\zeta_{\alpha_\zeta}$; i.e. we can continue the induction on ζ, contradicting the choice of ζ.

Case IV: $\zeta = \kappa^+$.
So $\langle g_\xi : \xi < \kappa^+\rangle$ are defined and for limit $\xi < \kappa^+$, $\langle f^\xi_\alpha : \alpha < \delta\rangle$ is defined and α_ξ is defined and $g_\xi = f^\xi_{\alpha_\xi}$.
Note that $\alpha^* =: \cup\{\alpha_\xi : \xi < \kappa^+\}$ is $< \delta$ (because $\mathrm{cf}(\delta) > \kappa^+$).
Note that for each $\alpha < \delta$, $i < \kappa$, $f^\zeta_\alpha(i) \in s^\zeta_i = \bigcup_{\xi<\kappa^+} s^\xi_i$. [Note: $\bigcup\limits_{\xi<\kappa^+} s^\xi_i$ is not necessarily closed, but $\min(s^\xi_i \backslash f_\alpha(i))$ is decreasing in ξ as the set s^ξ_i is increasing in ξ, hence $\langle \min(s^\xi_i \backslash f_\alpha(i)) : \xi < \kappa^+\rangle$ is eventually constant, and this value is $\min(s^\zeta_i \backslash f_\alpha(i))$]. So for some $\xi = \xi(\alpha, i) < \kappa^+$, $f^\zeta_\alpha(i) \in s^{\xi(\alpha,i)}_i$. Let $\xi(\alpha) = \sup_{i<\kappa} \xi(\alpha, i)$, it is $< \kappa^+$, and as $\langle s^\xi_i : \xi < \kappa^+\rangle$ is increasing, $f^\zeta_\alpha(i) \in s^{\xi(\alpha)}_i$. Clearly then, for example, $f^{\xi(\alpha)+\omega}_\alpha = f^{\xi(\alpha)+\omega+\omega}_\alpha$; hence if $\alpha \ge \alpha^*$, $g_{\xi(\alpha)+\omega} = g_{\xi(\alpha)+\omega+\omega}$ mod I hence $g_{\xi(\alpha)+\omega+1} \le g_{\xi(\alpha)+\omega}$ mod I, contradiction to the choice of $g_{\xi(\alpha)+\omega+1}$. $\square_{1.2}$

Main Claim 1.3 Suppose λ is singular, for $i < \kappa$, λ_i is a regular cardinal and $\kappa < \lambda_i < \lambda$, I is an ideal on κ and for every $\mu < \lambda$ $\{i : \lambda_i \le \mu\} \in I$. Then:

(i) for some $B \in I^+$, (i.e. $B \notin I$) we have $(\prod_{i\in B} \lambda_i, <_{I\restriction B})$ has true cofinality λ^+

or

(ii) there are $\langle \lambda'_i : i < \kappa\rangle$, $\bar f = \langle f_\alpha : \alpha < \lambda^+\rangle$ such that:
- (α) $\lambda'_i = \mathrm{cf}\ \lambda'_i < \lambda_i$
- (β) for $\mu < \lambda$ we have $\{i : \lambda'_i \le \mu\} \in I$
- (γ) $\prod_{i<\kappa} \lambda'_i/I$ has true cofinality λ^+ as witnessed by $\bar f$; i.e. $f_\alpha \in \prod_{i<\kappa} \lambda'_i$ for $\alpha < \lambda^+$, $\bar f$ is $<_I$-increasing with $f = \langle \lambda'_i : i < \kappa\rangle$ the $<_I$ -eub of $\langle f_\alpha : \alpha < \lambda^+\rangle$.
- (δ) Moreover $\langle f_\alpha : \alpha < \lambda^+\rangle$ from (γ) satisfies: if $\delta < \lambda^+$ is limit, then for some club $C \subseteq \delta$ and $\langle s_\alpha : \alpha \in C\rangle$, $s_\alpha \in I$ we have: $[i < \kappa \Rightarrow \langle f_\alpha(i) : \alpha \in C, i \in \kappa\backslash s_\alpha\rangle$ is strictly increasing$]$.
- (ϵ) In (δ) if $\mathrm{cf}(\delta) > \mathrm{gen}(I)$, then without loss of generality $s_\alpha = s$; but we replace C by an unbounded subset (or a stationary subset).

Proof: Suppose that (i) fails.
Choose for each limit $\delta < \lambda^+$, a closed unbounded subset C_δ of δ of order type $\mathrm{cf}(\delta)$. In a sense we will be using the following "silly" weak square: $\langle\{C_\delta \cap \alpha : \alpha \in C_\delta, \delta < \lambda^+, \delta \text{ limit}\} : \alpha < \lambda\rangle$.
Now we choose by induction on $\alpha < \lambda^+$, $\langle g^\delta_\alpha : \delta < \lambda^+ \text{ limit}\rangle$ and f^0_α, f_α such that:

(1) $g^\delta_\alpha \in \prod_{i<\kappa} \lambda_i$
(2) $f^0_\alpha, f_\alpha \in \prod_{i<\kappa} \lambda_i$
(3) if $\beta < \alpha$, then $f_\beta <_I f^0_\alpha$
(4) if $\alpha \in C_\delta$, $\delta < \lambda^+$ is limit then $f^0_\alpha < g^\delta_\alpha$ (i.e. $f^0_\alpha(i) < g^\delta_\alpha(i)$ for every i)
(5) if $\beta \in C_\delta \cap \alpha$ (δ limit) and $\alpha \in C_\delta$ then:
$i < \kappa$ & $\lambda_i > |C_\delta \cap \alpha| \Rightarrow g^\delta_\beta(i) < g^\delta_\alpha(i)$.
(6) $f^0_\alpha < f_\alpha$
(7) if $\alpha \in C_\delta$, $\delta < \lambda^+$ is a limit ordinal then $g^\delta_\alpha <_I f_\alpha$.

For each α, first define f^0_α satisfying (3) (and (2)) (easy as $\prod \lambda_i/I$ is λ^+-directed or see below). Next define $\langle g^\delta_\alpha : \delta < \lambda^+ \text{ limit}\rangle$ such that (4) + (5) and (1) hold. This is trivial. Next define f_α such that (6)+(7) holds (and (2), of course), for this it suffices to show that $(\prod_{i<\kappa} \lambda_i, <_I)$ is λ^{++}-directed. By [Sh345a,1.5] $\left(\prod\{\lambda_i : i < \kappa\}, <_{J_{<\lambda^{++}}[\{\lambda_i : i<\kappa\}]}\right)$ is λ^{++}-directed; hence it suffices to show:

if $B \subseteq \kappa$ and $\{\lambda_i : i \in B\} \in J_{<\lambda^{++}}[\{\lambda_i : i < \kappa\}]$ then $B \in I$.

This holds by "not (i) (of 1.3)" + [Sh345a,1.8(1)]+[Sh345a,1.3(4)(ii)].
Now we apply claim 1.2 to $\langle f_\alpha : \alpha < \lambda^+\rangle$ so (i) or (ii) or (iii) there holds.

We next show that (i) of Claim 1.2 is impossible. So suppose that D is an ultrafilter on κ disjoint to I, and $\langle s_i : i < \kappa\rangle$, $\langle \alpha_\zeta : \zeta < \lambda^+ = \mathrm{cf}(\lambda^+)\rangle$, $\langle h_\zeta : \zeta < \lambda^+\rangle$ exemplify $(*)_D$ (of (i) of 1.2)).

Let $\delta < \lambda^+$ be a limit ordinal of cofinality $> \kappa$, such that

$$\delta = \sup\{\alpha_\zeta : \zeta < \delta\}.$$

So let

$$C^0 \subseteq C_\delta,\ C^0 = \{\beta_{\delta,\epsilon} : \epsilon < \mathrm{cf}\ \delta\},\ [\epsilon_1 < \epsilon_2 \Rightarrow \beta_{\delta,\epsilon_1} < \beta_{\delta,\epsilon_2}]$$

be such that: for each $\epsilon < \mathrm{cf}(\delta)$ for some $\zeta(\epsilon), \xi(\epsilon) < \delta$ we have

$$\beta_{\delta,\epsilon} < \alpha_{\zeta(\epsilon)} < \alpha_{\xi(\epsilon)} < \beta_{\delta,\epsilon+1}$$

so we have

$$f_{\alpha_{\zeta(\epsilon)}} <_D h_{\zeta(\epsilon)} <_D f_{\alpha_{\zeta(\epsilon)+1}} \leq_I f_{\alpha_{\xi(\epsilon)}}.$$

By (5),

$$|C_\delta| < \lambda_i \Rightarrow \langle g^\delta_{\beta_{\delta,\epsilon}}(i) : \epsilon < \mathrm{cf}(\delta)\rangle \text{ is strictly increasing.}$$

Also

$$g^\delta_{\beta_{\delta,\epsilon}} <_I f_{\beta_{\delta,\epsilon}} <_I f_{\alpha_{\zeta(\epsilon)}} <_D h_{\zeta(\epsilon)} <_D f_{\alpha_{\xi(\epsilon)}} <_I f^0_{\beta_{\delta,\epsilon+1}} <_I g^\delta_{\beta_{\delta,\epsilon+1}}$$

and $\{j : |C_\delta| \geq \lambda_j\} \in I$ (by an assumption on I in 1.3 for $\mu = |C_\delta| = \mathrm{cf}(\delta) < \lambda$ since λ is singular), so for some $t_\epsilon \in \mathcal{P}(\kappa)\backslash D$ (hence $\kappa\backslash t_\epsilon \notin I$) we have:

$$i \in \kappa\backslash t_\epsilon \Rightarrow g^\delta_{\beta_{\delta,\epsilon}}(i) < h_{\zeta(\epsilon)}(i) < g^\delta_{\beta_{\delta,\epsilon+1}}(i).$$

Choose for each $\epsilon < \mathrm{cf}(\delta)$ an ordinal $i_\epsilon \in \kappa\backslash t_\epsilon$ such that $\lambda_{i_\epsilon} > |C_\delta|$; so as $\mathrm{cf}(\delta) > \kappa$, for some $i(*)$, $\{\epsilon < \mathrm{cf}(\delta) : i_\epsilon = i(*)\}$ is unbounded. Now $\langle g^\delta_{\beta_{\delta,\epsilon}}(i_\epsilon) : \epsilon < \mathrm{cf}(\delta), i_\epsilon = i(*)\rangle$ is strictly increasing hence

$$\langle h_{\zeta(\epsilon)}(i_\epsilon) : \epsilon < \mathrm{cf}(\delta), i_\epsilon = i(*)\rangle$$

is strictly increasing, hence

$$\{h_{\zeta(\epsilon)}(i_\epsilon) : \epsilon < \mathrm{cf}(\delta), i_\epsilon = i(*)\}$$

is a subset of $s_{i(*)}$ of cardinality $\mathrm{cf}(\delta) > \kappa$, contradiction to the choice of $\langle s_i : i < \kappa\rangle$ above (see Claim 1.2(i)).

Next we show that (iii) of Claim 1.2 is impossible. So suppose that $g \in {}^\kappa\mathrm{Ord}$, $\langle t_\alpha : \alpha \in A\rangle$ and $A \subseteq \lambda^+$ exemplify $(***)_I$ of (iii) from 1.2. Let $\delta < \lambda^+$ be a limit ordinal of cofinality $> \kappa$ such that $\delta = \sup(A \cap \delta)$, and let $C_\delta = \{\beta_\epsilon : \epsilon < \mathrm{cf}\delta\}$($\beta_\epsilon$ increasing in ϵ).

So $g^\delta_{\beta_\epsilon} \leq_I f_{\beta_\epsilon} \leq_I f^0_{\beta_{\epsilon+1}} \leq_I g^\delta_{\beta_{\epsilon+1}}$, hence

$$s_\epsilon = \left\{i < \kappa :\ \text{not}\ \left[g^\delta_{\beta_\epsilon}(i) \leq f_{\beta_\epsilon}(i) < g^\delta_{\beta_{\epsilon+1}}(i)\right]\right\} \in I.$$

Let $a = \{i < \kappa : \lambda_i \leq |C_\delta|\}$, clearly $a \in I$.

Now as $\epsilon < \zeta < \mathrm{cf}\delta \Rightarrow g^\delta_{\beta_\epsilon}(i) < g^\delta_{\beta_\zeta}(i)$ for $i \notin a$, clearly the set

$$t^*_{\beta_\epsilon} =: \{i < \kappa : g^\delta_{\beta_\epsilon}(i) \leq g(i), i \notin a\}$$

decreases with ϵ, so as $\mathrm{cf}\delta > \kappa$ for some $\epsilon(*) < \mathrm{cf}\delta$ we have

$$\epsilon(*) \leq \epsilon < \mathrm{cf}\delta \Rightarrow t^*_{\beta_{\epsilon(*)}} = t^*_{\beta_\epsilon}.$$

However, by the choice of s_ϵ we have:

$$i \in \kappa\backslash s_\epsilon\backslash a \Rightarrow \left[g^\delta_{\beta_\epsilon}(i) \leq f_{\beta_\epsilon}(i) < g^\delta_{\beta_{\epsilon+1}}(i)\right]\ \&\ i \notin a$$
$$\Rightarrow \left[i \notin t^*_{\beta_\epsilon} \to i \notin t_{\beta_\epsilon}\ \&\ i \notin t_{\beta_\epsilon} \to i \notin t^*_{\beta_{\epsilon+1}}\right]$$

(on $i \notin t_{\beta_\epsilon}$ see (β) of $(***)_I$ of (iii) of 1.2) hence

$$\left[t^*_{\beta_\epsilon}\backslash s_\epsilon\backslash a \supseteq t_{\beta_\epsilon}\backslash s_\epsilon \setminus a \supseteq t^*_{\beta_{\epsilon+1}}\backslash s_\epsilon \setminus a\right].$$

Hence, if $\epsilon \geq \epsilon(*)$ then $t^*_{\beta_\epsilon}\backslash s_\epsilon\backslash a = t_{\beta_\epsilon}\backslash s_\epsilon \setminus a = t^*_{\beta_{\epsilon(*)}}\backslash s_\epsilon \setminus a$, but $s_\epsilon \cup a \in I$ hence $\langle t_{\beta_\epsilon} : \epsilon < \mathrm{cf}\delta\rangle$ is eventually constant modulo I. But $\langle t_\beta : \beta < \delta\rangle$ is monotonic mod I, hence it is eventually constant mod I, but this contradicts $\delta = \sup(A \cap \delta)$ (and clause (α) of $(***)_I$ of 1.2(iii)). So also (iii) of 1.2 is impossible in our case.

We conclude that when we have applied Claim 1.2 to $\langle f_\alpha : \alpha < \lambda^+\rangle$, possibility (ii) of 1.2 holds; say for f. Without loss of generality $f(i) \leq \lambda_i$ for every $i < \kappa$. Let $\lambda^*_i =: \mathrm{cf}[f(i)]$ (so $\lambda^*_i \leq \lambda_i$). Let E_i be a club of $f(i)$ of order type $\mathrm{cf}[f(i)]$. Let $f^*_\alpha \in \prod_{i<\kappa} \lambda^*_i$ be: $f^*_\alpha(i) = \mathrm{otp}(f_\alpha(i) \cap E_i)$. So clearly for some club C of λ^+, $[\alpha < \beta\ \&\ \beta \in C \Rightarrow f^*_\alpha <_I f^*_\beta]$ and $\langle f^*_\alpha : \alpha \in C\rangle$ is $<_I$-increasing and witnesses $\left(\prod_{i<\kappa} \lambda^*_i, <_I\right)$ has true cofinality λ^+. If $B =: \{i : \lambda^*_i = \lambda_i\} \notin I$, clearly $\{\lambda_i : i \in B\} \in J_{<\lambda^{++}}(\{\lambda_i : i < \kappa\})$ hence condition (i) of our Claim 1.3 holds so without loss of generality $\lambda^*_i < \lambda_i$.

If for some $\mu < \lambda$, $B = \{i : \mathrm{cf}[f(i)] \leq \mu\} \notin I$, choose an ultrafilter D on κ disjoint to I such that $B \in D$. Now we can repeat the previous argument (that condition (i) of 1.2 is impossible in our case) (using as $\langle \alpha_\zeta : \zeta < \lambda^+\rangle$ an enumeration of C, s_i a club of $f(i)$, otp $s_i = \mathrm{cf}(f(i))$ and choosing $\delta < \lambda^+$, $\mathrm{cf}(\delta) > \mu$ and $\delta = \sup\{\alpha_\zeta : \zeta < \delta\}$)). So $\langle \lambda^*_i : i < \kappa\rangle$

satisfies $(\alpha)+(\beta)$ of (ii) of Claim 1.3. Now (γ) is quite easy, witnessed by $\langle f^*_\alpha : \alpha \in C\rangle$, as well as (δ) [for a limit ordinal $\delta < \lambda^+$ of cofinality $> \kappa$, we know that for $i < \kappa$, such that $\lambda_i > |C_\delta|$ we have $\langle g^\delta_\alpha(i) : \alpha \in C_\delta\rangle$ is strictly increasing so, as above, we can find $s_\alpha \in I$ for $\alpha \in C_\delta$ such that for each $i < \kappa$, $\langle f_\alpha(i) : \alpha \in C_\delta, i \notin s_\alpha\rangle$ is strictly increasing hence $\langle f^*_\alpha(i) : \alpha \in C_\delta, i \notin s_\alpha\rangle$ is non-decreasing (see the definition of f^*_α). So if C is as above, $\delta = \sup(\delta \cap C)$, using the club $C_\delta \cap C$ and increasing the s_α's we get (δ)]. Lastly (ϵ) is trivial.
So assuming (i) of 1.3 fails, we have proved (ii) of 1.3 holds, thus finishing. $\square_{1.3}$

Claim 1.4 (1) Suppose in Claim 1.3 we assume also $\mu = \mathrm{cf}(\mu) > \lambda$ and $(\prod_{i<\kappa}\lambda_i, <_I)$ is μ-directed. Then in the conclusion we can replace λ^+ by μ (except that in (δ) of (ii) we should restrict ourselves to δ of cofinality $< \lambda$).

(2) In (γ) of (ii) of Claim 1.3 we can add: $\langle f_\alpha : \alpha < \lambda^+\rangle$ is $<_J$-increasing where $J = \{A \subseteq \kappa : \sup\{\lambda'_i : i \in A\} < \lambda\}$.

(3) In (1) (i.e. 1.4(1)) it follows that when possibility (ii) holds, if $A \subseteq \mu$, $|A| \le \lambda$ then we can find $s_\alpha \in I$ for $\alpha \in A$ such that: for each $i < \kappa$ and ζ we have: $|\{\alpha \in A : i \notin s_\alpha, f_\alpha(i) = \zeta\}| \le 1$; and $\langle f_\alpha(i) : i \notin s_\alpha, \alpha \in A\rangle$ is strictly increasing.

Proof: 1) Same proof as of Claim 1.3.
2) Easy. $\square_{1.4}$
3) By induction on $\mathrm{otp}(\mathrm{cl}(A))$.

We can conclude by 1.4:

Theorem 1.5 *If λ is singular,* cf $\lambda = \kappa$, *then for some strictly increasing sequence $\langle\lambda_i : i < \kappa\rangle$ of regular cardinals, $\lambda = \sum_{i<\kappa}\lambda_i$ and $(\prod_{i<\kappa}\lambda_i, <_{J^{bd}_\kappa})$ has true cofinality λ^+.*

Proof: Choose $\langle\lambda_i : i < \kappa\rangle$ a strictly increasing sequence of regular cardinals $> \mathrm{cf}\lambda$ with $\sum_{i<\kappa}\lambda_i = \lambda$, and $I = J^{bd}_\kappa$; now apply 1.3. If (i) of 1.3 holds, say for $B \in I^+$, (i.e. $B \subseteq \kappa$, $|B| = \kappa$), let $\{j(i) : i < \kappa\}$ enumerate B in increasing order and now $\lambda'_i =: \lambda_{j(i)}$ (for $i < \kappa$) are as required. If (ii) of 1.3 holds, say for $\langle\lambda'_i : i < \kappa\rangle$ then $\langle\lambda'_i : i < \kappa\rangle$ are almost as required. We know that $\prod_{i<\kappa}\lambda'_i/I$ has true cofinality λ^+, and

$$[\mu < \lambda \Rightarrow \{i : \lambda'_i \le \mu\} \in I]$$

hence for every $i < \kappa$ for some $j_i < \kappa$, $\lambda'_{j_i} > \lambda_i$ and without loss of generality $j_i > \cup\{j_{i_1} : i_1 < i\}$. As $I = J^{bd}_\kappa$, $\{j_i : i < \kappa\} \in I^+$, so

$$\prod_{i<\kappa}\lambda'_{j_i}/I \cong \prod_{i<\kappa}\lambda'_i/(I + (\kappa\backslash\{j_i : i < \kappa\})$$

which has true cofinality λ^+. $\square_{1.5}$

Claim 1.5A If λ is singular, $\mathrm{pp}^*_I(\lambda) > \mu$ and I an ideal on $\kappa < \lambda$, then we can find $f_\alpha : \kappa \to \lambda$ for $\alpha < \mu$ such that:
for every $A \subseteq \mu$, $|A| < \lambda$ there is a sequence $\langle s_\alpha : \alpha \in A\rangle$ such that:

(i) $s_\alpha \in I$

(ii) $\langle \mathrm{Rang}[f_\alpha\restriction(\kappa\backslash s_\alpha)] : \alpha \in A\rangle$ are pairwise disjoint.

Proof: If $\mu \le \lambda$ this is trivial: use f_α being constantly α, so they have pairwise disjoint ranges. If $\mu = \mathrm{cf}(\mu) > \lambda$ by the assumptions there are λ_i (for $i < \kappa$) such that $\lambda_i = \mathrm{cf}\lambda_i < \mu$, $\mathrm{tlim}_I\lambda_i = \lambda$ and $\prod_{i<\kappa}\lambda_i/I$ has true cofinality $> \mu$. Now we apply 1.4(1) (so read 1.3 again); now possibility (i) cannot hold ($\prod_{i<\kappa}\lambda_i/I$ is μ^+-directed), so we have

$$\langle\lambda'_i : i < \kappa\rangle, \quad \bar f = \langle f_\alpha : \alpha < \lambda^+\rangle$$

as in possibility (ii). Clearly $f_\alpha : \kappa \to \lambda$, the f_α's are as required by 1.4(3). Lastly if $\mu > \mathrm{cf}(\mu) + \lambda$ we can combine the results for the regular $\mu' \le \mu$ (and a pairing function on λ) to get the desired result.
I.e. let $\mu = \sum_{\zeta<\theta}\mu^+_\zeta$ with $\mu^+_\zeta < \mu$, $\theta = \mathrm{cf}\mu < \mu$; by what we have proved for each $\zeta < \theta$ there is a sequence $\langle f^\zeta_\alpha : \alpha < \mu^+_\zeta\rangle$ of functions as required in the claim, and similarly $\langle f_\zeta : \zeta < \theta\rangle$. Let $<,>$ be a pairing function on λ. Now we define $f_\alpha : \kappa \to \lambda$ as follows:

$$\text{if } \bigcup_{\xi<\zeta}\mu^+_\xi \le \alpha < \mu^+_\zeta \text{ we let for } i < \kappa : f_\alpha(i) = \langle f_\zeta(i), f^\zeta_\alpha(i)\rangle.$$

Check. $\square_{1.5A}$

Remark 1.5B (1) If for some $\lambda_i = \mathrm{cf}(\lambda_i) < \lambda$ for $i < \kappa$, $\kappa + \mathrm{cf}(\lambda) < \lambda_i$, I an ideal on κ, $\mathrm{tlim}_I\lambda_i = \lambda$ and $(\prod_{i<\kappa}\lambda_i, <_I)$ does not have true cofinality λ^+, then $\mathrm{pp}_\kappa(\lambda) > \lambda^+$ and, replacing I by some proper ideal $I' \supseteq I$, condition (ii) of 1.3 holds.

(2) If in addition $(\prod_{i<\kappa}\lambda_i, <_I)$ is μ-directed but does not have true cofinality μ then $\mathrm{pp}_\kappa(\lambda) > \mu$.

(3) In 1.4(1), if $(\prod_{i<\kappa}\lambda_i, <_I)$ is μ^+-directed, then the possibility (ii) holds.

(4) In 1.5A, we can add to *(ii)*:
$\alpha < \beta$ & $\alpha \in A$ & $\beta \in A$ & $i \in \kappa\backslash s_\alpha \cup s_\beta \Rightarrow f_\alpha(i) < f_\beta(i)$.

(5) If for $\alpha < \mu$, $f_\alpha \in \prod_{i<\kappa}\theta_i$, $\kappa < \theta_i = \mathrm{cf}\theta_i$, $\langle f_\alpha : \alpha < \mu\rangle$ is $<_I$-increasing with $<_I$ –eub $\langle\theta_i : i < \kappa\rangle$, $\mu = \mathrm{cf}\mu > \kappa^+$, then for every regular $\sigma < \mu$ for stationarily many $\delta < \mu$, $\mathrm{cf}\delta = \sigma$ and for some unbounded subset A of δ, $\langle f_\alpha : \alpha \in A\rangle$ is as in 1.4(1) clause (ii) (see 1.3).

Proof:
1) Without loss of generality $\lambda_i > \kappa$. Let $\mathfrak{a} =: \{\lambda_i : i < \kappa\}$, by [Sh345a,1.8] the assumption implies that
$J_{<\lambda^{++}}[\mathfrak{a}]$ is a proper ideal while
$(\prod \mathfrak{a}, <_{J_{<\lambda^{++}}}[\mathfrak{a}])$ is λ^{++}-directed.
Hence, letting J is the ideal generated by

$$I \cup \{a \subseteq \kappa : \{\lambda_i : i \in a\} \in J_{<\lambda^{++}}[\mathfrak{a}]\}$$

we have

$$(\prod_{i<\kappa} \lambda_i, <_J) \text{ is } \lambda^{++}\text{-directed.}$$

Now use Claim 1.3.
2) Same proof.
3) Left to the reader.
4) Easy.
5) Same proof as 1.3. $\square_{1.5B}$

Claim 1.6 (1) For $\langle f_\alpha : \alpha < \delta\rangle$, κ, I as in the hypothesis of Claim 1.2 such that $\mathrm{cf}(\delta) > \kappa^+$, the following are equivalent:
(a) there are $A \subseteq \delta$ unbounded, and $s_\alpha \in I$ for $\alpha \in A$ such that: $\langle f_\alpha(i) : \alpha \in A, i \in \kappa \backslash s_\alpha\rangle$ is strictly increasing in α for each $i < \kappa$,
(b) $(**)_I$ of Claim 1.2 holds for some f and

$$\{i < \kappa : \mathrm{cf}[f(i)] \neq \mathrm{cf}\ \delta\} \in I$$

(note: f/I is unique as a $<_I -$lub of $\langle f_\alpha : \alpha < \delta\rangle$).
(2) If $\mathrm{cf}\delta >$ gen I, in clause (a) above without loss of generality $s_\alpha = s$ for $\alpha \in A$; if $I = J^{\mathrm{bd}}_\kappa$ without loss of generality $s_\alpha = s = [i(*), \kappa)$ for some fixed $i(*)$.

Remark 1.6A Note that the assumptions of 1.6(1) imply

$$\{i < \kappa : \mathrm{cf}[f(i)] > \mathrm{cf}(\delta)\} \in I,$$

and if condition (b) of 1.6 holds then for every $\mu \neq \mathrm{cf}(\delta)$,

$$\{i < \kappa : \mathrm{cf}[f(i)] = \mu\} \in I.$$

Proof: <u>(a) ⇒ (b)</u>.
So let $s_\alpha \in I$, $A \subseteq \delta$ be as mentioned in (a). We shall show that (b) holds and in particular $(**)_I$ from Claim 1.2 holds.

We define $f \in {}^\kappa\mathrm{Ord}$:

$$f(i) = \sup\{f_\alpha(i) : \alpha \in A, i \in \kappa \backslash s_\alpha\}.$$

We shall show that f is as required there.

<u>Condition α :</u> Suppose $\beta < \delta$ then we can find $\alpha \in (\beta, \delta) \cap A$, now: $f_\beta <_I f_\alpha$ [by assumption] and $f_\alpha \leq_I f$ [as $f_\alpha \restriction (\kappa \backslash s_\alpha) \leq f \restriction (\kappa \backslash s_\alpha)$ by the definition of f, remembering $s_\alpha \in I$] hence together $f_\beta <_I f$ as required.

<u>Condition β :</u> Suppose $f' \in {}^\kappa\mathrm{Ord}$ and let $B = \{i \in \kappa : f'(i) < f(i)\}$; and assume $\kappa \setminus B \in I$. For each $i \in B$ let

$$\alpha_i = \min\{\alpha \in A : i \in \kappa \backslash s_\alpha, f_\alpha(i) > f'(i)\};$$

by the definition of f and B, α_i is well defined and belongs to A. As $\mathrm{cf}(\delta) > \kappa$, $\alpha(*) =: \sup_{i \in B} \alpha_i$ is well defined and is $< \delta$; choose $\alpha \in A$, $\alpha > \alpha(*)$ (α exists as $\delta = \sup A$), so by the assumptions on s_α, A we have

$$i \in \kappa \backslash s_\alpha \ \&\ i \in B \Rightarrow f'(i) < f_{\alpha_i}(i) < f_\alpha(i);$$

so s_α, $(\kappa \setminus B) \in I$ hence $f' <_I f_\alpha$ as required in (β).

<u>Condition γ :</u> We prove more: the additional condition from clause (b) of 1.6.
Let $B = \{i < \kappa : \{\alpha \in A : i \in \kappa \backslash s_\alpha\}$ is bounded in $A\}$. Clearly for $i \in (\kappa \backslash B)$, $\mathrm{cf}[f(i)] = \mathrm{cf}(\delta)$. Let $\alpha_i =: \sup\{\beta \in A : i \in \kappa \backslash s_\beta\}$ so $[i \in B \Leftrightarrow \alpha_i < \delta]$; and let $\alpha^* = \sup\{\alpha_i + 1 : i < \kappa, i \in B\}$ and let $\alpha = \min(A \backslash \alpha^*)$, so α is $< \delta$ (as $\mathrm{cf}(\delta) > \kappa$). Now $[i \in \kappa \backslash s_\alpha \Rightarrow \alpha \leq \alpha_i]$ hence, by α's definition, $B \subseteq s_\alpha$, hence $B \in I$. But $i \in (\kappa \backslash B) \Rightarrow \mathrm{cf}(f(i)) = \mathrm{cf}(\delta)$, (and "$\mathrm{cf}(\delta) > \kappa$" is an assumption of 1.6) so we finish.

<u>(b) ⇒ (a)</u>
Let $B = \{i < \kappa : \mathrm{cf}[f(i)] = \mathrm{cf}\ \delta\}$, so $B = \kappa \bmod I$, and for $i \in B$ let $\langle \gamma^i_\zeta : \zeta < \mathrm{cf}\ \delta\rangle$ be a strictly increasing continuous sequence of ordinals with limit $f(i)$; for $i \in \kappa \backslash B$ let $\gamma^i_\zeta = 0$. Define for $\zeta < \mathrm{cf}(\delta)$ a function $g_\zeta \in {}^\kappa\mathrm{Ord}$ by $g_\zeta(i) = \gamma^i_\zeta$. Let $\langle \alpha_\zeta : \zeta < \mathrm{cf}(\delta)\rangle$ be an increasing continuous sequence of ordinals with limit δ. Now clearly $f' <_I f \Leftrightarrow (\exists \zeta < \mathrm{cf}(\delta))[f' <_I g_\zeta]$ but we also know $f' <_I f \Leftrightarrow (\exists \zeta < \mathrm{cf}\ \delta)[f' <_I f_{\alpha_\zeta}]$. Hence for some club C of $\mathrm{cf}(\delta)$:

$$\text{for } \zeta < \xi \text{ from } C : g_\zeta <_I f_{\alpha_\xi} \ \&\ f_{\alpha_\zeta} <_I g_\xi.$$

Let us enumerate C in increasing order: $C = \{\zeta(\epsilon) : \epsilon < \mathrm{cf}\ \delta\}$; let

$$A = \{\alpha_{\zeta(3\epsilon+1)} : \epsilon < \mathrm{cf}(\delta)\}$$

and if $\alpha = \alpha_{\zeta(3\epsilon+1)}$

$$s_\alpha = \{i < \kappa : i \notin B \text{ or } g_{\zeta(3\epsilon)}(i) \geq f_{\alpha_{\zeta(3\epsilon+1)}}(i) \text{ or } f_{\alpha_{\zeta(3\epsilon+1)}}(i) \geq g_{\zeta(3\epsilon+2)}(i)\}.$$

We leave checking to the reader. $\square_{1.6}$

(2) Left to the reader.

We can unite 1.2 and part of the proof of 1.3 to

Claim 1.7 Assume $f_\alpha \in {}^\kappa\mathrm{Ord}$ for $\alpha < \delta$, I an ideal on κ, $\bar f = \langle f_\alpha : \alpha < \delta\rangle$ is $<_I$-increasing and $\mathrm{cf}\delta > \mu = \mathrm{cf}\mu > \kappa$.
Then

$$\bar f \text{ has a } <_I \text{-eub } f \text{ (see } (\alpha) + (\beta) \text{ of } (**)_I \text{ of (ii) of 1.3) and}$$

$$\{i : \mathrm{cf}[f(i)] < \mu\} \in I \text{ if}$$

$\otimes$ the set of $\alpha < \delta$ satisfying the following is stationary in $\delta : \mathrm{cf}\alpha \geq \mu$ and $\bar f \restriction \alpha$ satisfies condition (a) of 1.6,

i.e.

$\otimes_\alpha$ there is an unbounded $A \subseteq \alpha$ and $g_\beta \in {}^\kappa\mathrm{Ord}$ for $\beta \in A$ such that: for $\beta < \gamma$ in A we have:

$$g_\beta < g_\gamma \text{ and} f_\beta \leq_I g_\beta \leq_I f_\gamma$$

$$(\text{or } g_\beta \leq_I f_\beta \leq_I g_\gamma).$$

§2 On pp instead of the Singular Cardinal Problem

Here we shall get some conclusions of the basic advances in §1. The most direct is 2.3 where we get basic properties of pp, here we have explicitly (2.3(1),(3)), the characterization of PP in $\otimes_1$ from the foreward of §1, (for any family Γ of ideals). Conventional cardinal arithmetic has obvious monotonicity properties which are not shared by pp; in fact a kind of inverse monotonicity is gotten (2.3(2),(3)):

$\otimes_1$ if $\lambda < \mu \leq \mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda)$ and λ, μ are singulars with cofinality in the interval $[\sigma, \theta)$ and $\mathrm{cf}(\theta) = \theta \vee \mathrm{cf}\theta < \sigma$ then $\mathrm{pp}^+_{\Gamma(\theta,\sigma)}(\mu) \leq \mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda)$.

So in such a situation the minimal member has the largest pp! (see 2.3(6) for exact formulation).
We still have some continuity (2.3(4)):

$\otimes_2$ if $\sigma < \theta$, $\mathrm{cf}\theta = \theta \vee \mathrm{cf}\theta < \sigma \vee \mathrm{cf}\theta \neq \mathrm{cf}\lambda$, $\mathrm{cf}\lambda \in [\sigma, \theta)$, $\theta < \lambda < \chi$ and for arbitrarily large $\lambda' < \lambda$, $\mathrm{cf}\lambda' \in [\sigma, \theta)$, $\mathrm{pp}^+_{\Gamma(\theta,\sigma)}(\lambda') > \chi$ then $\mathrm{pp}^+_{\Gamma(\theta,\sigma)}(\lambda) > \chi$.

If we have been happy to get in the first section a representation of λ^+ (λ singular) as the true cofinality of $\prod_{i<\mathrm{cf}\lambda} \lambda_i / J^{\mathrm{bd}}_{\mathrm{cf}\lambda}$, we should be even more happy to get a better representation even at some price: λ of uncountable cofinality:

$\otimes_3$ if $\lambda > \mathrm{cf}\lambda > \aleph_0$, $\langle \lambda_i : i < \mathrm{cf}\lambda\rangle$ increasing continuous then for some club E of λ, $\prod_{i\in E} \lambda_i^+ / J^{\mathrm{bd}}_E$ has true cofinality λ^+.

We would be even more happy to add

$$\text{“and } \lambda > \max\mathrm{pcf}\{\lambda_i : i \in E \cap \alpha\} \text{ for } \alpha \in E\text{”},$$

but do not know how to do that. We shall return to "good" representations in [Sh371,§1].

Another direction is that we are used to proving things from instances of GCH, but rarely from their negation. By the result of §1 we can (in 2.2) show that $\mathrm{pp}(\lambda) > \lambda^+$ (our version of failure of an instance of GCH) implies the failure of corresponding instances of Chang's conjecture. This generalizes Solovay's theorem (on SCH above supercompact), we shall return to it in §5.

* * *

Claim 2.1 Suppose λ is singular of uncountable cofinality κ. Let $\langle \lambda_i : i < \kappa\rangle$ be strictly increasing continuous with limit λ. Then for some club C of κ, $\left(\prod_{i\in C} \lambda_i^+, <_{J^{\mathrm{bd}}_C}\right)$ has true cofinality λ^+. (So $\lambda^+ = \max\mathrm{pcf}\{\lambda_i^+ : i \in C\}$).

Question 2.1A Can we get $\langle f_\alpha : \alpha < \lambda^+\rangle$ witnessing the cofinality such that, for each $i \in C$ we have $|\{f_\alpha \restriction i : \alpha < \lambda\}| < \lambda_{i+1}$?

(this is equivalent to the formulation after $\otimes_3$ by later theorems)

Proof:
Without loss of generality for every i, $\lambda_i > \kappa^+$ and λ_i is singular; let $\mathfrak{a} = \{\lambda_i^+ : i < \kappa\}$; and apply [Sh345a, Def. 1.2(2)] so the sequence

$$\langle J_{<\mu}[\mathfrak{a}] : \mu \in \mathrm{pcf}(\mathfrak{a})\rangle$$

is well defined. Clearly, (see [Sh345a, 1.3(4)]) $J_{<\lambda^+}[\mathfrak{a}] \subseteq J^{\mathrm{bd}}_{\mathfrak{a}}$ (as if D an ultrafilter on $\mathfrak{a}$ such that $\mu < \lambda \Rightarrow \{\theta \in \mathfrak{a} : \theta > \mu\} \in D$, then $\mathrm{cf}[\prod \mathfrak{a}/D] > \lambda$). Now we know by [Sh345a,3.2(2)] that for some $\mathfrak{b}_\alpha \in J_{\leq\lambda^+}[\mathfrak{a}]$

(i) for $\alpha < \lambda$, $[\alpha < \beta \Rightarrow \mathfrak{b}_\alpha \subseteq \mathfrak{b}_\beta \mod J_{<\lambda^+}[\mathfrak{a}]]$

(ii) if $\mathfrak{c} \subseteq \mathfrak{a}\backslash\mathfrak{b}$, $\mathfrak{c} \in J_{\leq\lambda^+}[\mathfrak{a}]$ then for some $\alpha < \lambda$, $\mathfrak{c} \setminus \mathfrak{b}_\alpha$ is included in a countable union of members of $J_{<\lambda^+}[\mathfrak{a}]$, but this ideal is included in $J^{\mathrm{bd}}_{\mathfrak{a}}$ which is κ-complete, $\kappa > \aleph_0$, hence in our case this implies that $\mathfrak{c} \setminus \mathfrak{b}_\alpha$ belongs to $J^{\mathrm{bd}}_{\mathfrak{a}}$ (as $\mathrm{cf}(\lambda) = \kappa > \aleph_0$).

Let

$$I=\{A\subseteq\kappa: \text{ for some club } E \text{ of } \kappa \text{ and } \alpha<\lambda^+, i\in A\Rightarrow i\notin E \vee \lambda_i\in \mathfrak{b}_\alpha\}.$$

If $\kappa\in I$ we are done, so assume not. Let I_0 be the ideal of non stationary subsets of κ, $B_\alpha=\{i:\lambda_i^+\in\mathfrak{b}_\alpha\}$, so B_α is increasing mod I_0, hence $I=\{A:\bigvee_{\alpha<\kappa}A\setminus B_\alpha\in I_0\}$ is a proper normal ideal on κ; $\mathrm{tlim}_I\lambda_i=\lambda$; we apply Claim 1.3. If 1.3(i) holds, then for some $B\subseteq\kappa$, $\prod\{\lambda_i^+:i\in B\}/I\restriction B$ has true cofinality λ^+, which contradicts the choice of $\langle\mathfrak{b}_\alpha:\alpha<\lambda\rangle$. So (ii) of Claim 1.3 holds for some $\lambda_i'<\lambda_i^+$ for $i<\kappa$, hence $\lambda_i'<\lambda_i$ (as λ_i' is regular). So (by Fodor's lemma, i.e. I's normality) for some $i(*)$ we have $S'=:\{i<\kappa:\lambda_i'\le\lambda_{i(*)}\}$ is stationary; but this contradicts (β) of (ii) of 1.3. $\square_{2.1}$

Conclusion 2.2 Suppose λ is singular $>\kappa$ and

$(*)$ for some $\mu<\lambda$, $\mu\ge\kappa\ge\mathrm{cf}\lambda$ and $(\lambda^+,\lambda)\xrightarrow{\mathrm{cf}\lambda}(\mu^+,\mu)$

(if you do not know the notation, use $(*)'$),

or even

$(*)'$ if M is a model with universe λ^+ and $\kappa\ge\mathrm{cf}\lambda$ many functions, <u>then</u> for some submodel $N\subseteq M$, $\lambda\ge\|N\|>|N\cap\lambda|+\kappa$.

<u>Then</u>

(1) statement (ii) of Claim 1.3 holds for no I, $\langle\lambda_i:i<\kappa\rangle$, as there (so $\lambda=\mathrm{tlim}_I\lambda_i$);

(2) $\mathrm{pp}(\lambda)=\lambda^+$.

(3) More generally, if $\mu<\lambda<\lambda^*<\mathrm{pp}_\kappa(\lambda)$, $(\lambda^*,\lambda)\xrightarrow{\kappa}(\mu^+,\mu)$ (or the parallel of $(*)'$) then 2.2(1) holds.

Remark 2.2A (1) This generalizes Solovay's theorem on SCH above supercompact cardinals.

(2) So $\lambda^{\mathrm{cf}\lambda}=\lambda^+$ if $(\forall\mu<\lambda)[\mu^{\mathrm{cf}\lambda}<\lambda]$, $\mathrm{cf}\lambda>\aleph_0$ and $(*)'$ holds.

Proof: (1) Clearly $(*)\Rightarrow(*)'$, so assume $(*)'$. Suppose (ii) of 1.3 holds for I, $\langle\lambda_i:i<\kappa\rangle$, and let $\langle f_\alpha:\alpha<\lambda^+\rangle$ exemplify this. Let F be the following two place function from λ^+ to λ:

$$F(\alpha,i)=\begin{cases} f_\alpha(i) & \text{if } i<\kappa\\ 0 & \text{if } i\ge\kappa\end{cases}$$

Now $M=:(\lambda^+,F,i)_{i<\kappa}$ is as required in $(*)'$ (each $i<\kappa$ is an individual constant), so there is a submodel N of M satisfying $\lambda\ge\|N\|>|N\cap\lambda|$. As $\lambda\ge\|N\|$, by 1.4(3) we can find $\langle s_\alpha:\alpha\in N\rangle$ such that:

(i) $s_\alpha\in I$

(ii) $\mathrm{Rang}(f_\alpha\restriction(\kappa\setminus s_\alpha)$ for $\alpha\in N$ are pairwise disjoint (even more).

But $\alpha\in N$, $i<\kappa\Rightarrow\{\alpha,i\}\in N\Rightarrow f_\alpha(i)\in N$; hence

$$\alpha\in N\Rightarrow\mathrm{Rang}(f_\alpha)\subseteq N.$$

So $\langle\mathrm{Rang}((f_\alpha\restriction(\kappa\setminus s_\alpha)):\alpha\in N\rangle$ is a sequence of $\|N\|$ pairwise disjoint non-empty subsets of $N\cap\lambda$, contradiction to "$\|N\|>|N\cap\lambda|$".

(2) Easy (see 1.5B).

(3) Easy $\square_{2.2}$

The following is an application not used later, so if you have not heard on supercompact, just ignore this.

Conclusion 2.2B (1) Suppose in the universe V, μ is supercompact, $\lambda>\mu$, λ singular and $\mathrm{cf}\lambda<\mu$. If V' is an extension of V (so V is a transitive class of V', they have the same ordinals and both are models of ZFC) and $V'\models\mathrm{pp}\lambda>\lambda^+$, V'' an extension of V' and in V'' the ordinal λ^+ (i.e. $(\lambda^+)^V$) is still a cardinal then in V'', μ is not supercompact.

(2) This holds for "compact cardinal", too.

Remark 2.2C The condition "$\mathrm{cf}\lambda<\mu$" in 2.2B can be omitted by 2.4(1) below.

The No Hole Conclusion 2.3 (1) If $\mathrm{cf}\,\lambda\le\kappa<\lambda$, <u>then</u>

$\{\mu:\mu=\mathrm{cf}\,\mu$ and for some ideal I on κ and $\langle\lambda_i:i<\kappa\rangle$ we have: each λ_i is regular and $\mathrm{tlim}_I\lambda_i=\lambda$ and $\mu=\mathrm{tcf}\left(\prod_{i<\kappa}\lambda_i,<_I\right)\}$

is of the form $\{\mu:\lambda\le\mu\le^+\mathrm{pp}_\kappa(\lambda)$ and μ is regular$\}$, (see Definition 1.1(4)).

(2) If $\lambda<\mu$ are singulars of cofinality $\le\kappa$ (and $\kappa<\lambda$) and $\mathrm{pp}_\kappa(\lambda)\ge\mu$ <u>then</u> $\mathrm{pp}_\kappa(\mu)\le^+\mathrm{pp}_\kappa(\lambda)$.

(3) We can in (1) [in (2)] restrict ourselves to Γ (so pp_κ is replaced by pp_Γ), any set of ideals on κ [closed under sums; i.e.: assume $I\in\Gamma$ with domain without loss of generality κ, $I_i\in\Gamma$ for $i<\kappa$ with domain without loss of generality κ_i and $J^*=\sum_{i<\kappa}I_i$ (thus

$$\mathrm{Dom}\,J^*=\bigcup_i(\{i\}\times\kappa_i),$$

$$J^*=\{A\subseteq\mathrm{Dom}\,J^*:\{i<\kappa:\{j<\kappa_i:(i,j)\in A\}\notin I_i\}\in I\})$$

<u>then</u> for some $A\in J^+$, $J^*\restriction A$ is isomorphic to a member of Γ].

(3A) If $\sigma<\theta$ and $\mathrm{cf}\theta=\theta\vee\mathrm{cf}\theta<\sigma$ then $\Gamma(\theta,\sigma)$ is closed under sums.

(4) If $\mathrm{cf}(\lambda) \le \kappa < \lambda$ and

$$(\forall \mu < \lambda)(\exists \mu')[\mu < \mu' < \lambda \ \&\ \mathrm{cf}(\mu') \le \kappa \ \&\ \chi \le^+ \mathrm{pp}_\kappa(\mu')]$$

then $\chi \le^+ \mathrm{pp}_\kappa(\lambda)$. Also if $\sigma < \theta$, $\mathrm{cf}\theta = \theta \vee \mathrm{cf}\theta < \sigma$, $\mathrm{cf}\lambda \in [\sigma, \theta)$, $\chi > \lambda > \theta$ and for arbitrarily large $\mu' < \lambda$ we have $\chi \le^+ \mathrm{pp}_{\Gamma(\theta,\sigma)}(\mu')$ then $\chi \le^+ \mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda)$. [We can even use any Γ closed under sums as in (3)].

(5) If $\chi \in \mathrm{pcf}_{\Gamma(\theta,\sigma)}(\mathfrak{a})$, $\chi \notin \mathfrak{a}$, then for some μ,

$$\mu = \sup(\mathfrak{a} \cap \mu) \text{ and } \chi < \mathrm{pp}^+_{\Gamma(\theta,\sigma)}(\mu)$$

(hence $\sigma \le \mathrm{cf}\mu < \theta$). This is true for any property Γ closed under restriction (i.e. if $I \in \Gamma$, $A \in I^+$ then $I + (\mathrm{Dom}\, I \setminus A) \in \Gamma$), and even weakly closed under restriction (i.e. if $I \in \Gamma$, $A \in I^+$, then for some $B \in I^+$, $B \subseteq A$ and $I \restriction B$ is (isomorphic) to a member of Γ).

(6) Assume: $\sigma < \theta < \lambda_0 < \lambda_1 < \lambda$, $\mathrm{cf}\theta = \theta \vee \mathrm{cf}\theta < \sigma$, $S \subseteq [\lambda_0, \lambda_1]$ a set of cardinals, λ a limit cardinal, and for every $\chi < \lambda$ for some $\mu \in S$ we have $\mathrm{cf}\mu \in [\sigma, \theta)$ and $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\mu) \ge \chi$.
Then for some $\mu \in S$, $\mathrm{cf}\mu \in [\sigma, \theta)$ and $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\mu) \ge \lambda$. (In fact, the minimal $\mu \in S$ with $\mathrm{cf}(\mu) \in [\sigma, \theta)$, $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\mu) \ge \lambda_1$ is as required).
[We can use any Γ closed under sums].

Proof: (1) Easily every μ in the set is regular and $\ge \lambda$ and by definition 1.1(4) is $\le^+ \mathrm{pp}(\lambda)$. The absence of "holes" follows from 1.4(1).
(2) Follows by (3).
(3) The parallel to (1) has the same proof. So, it suffices to prove: if $\theta < \mathrm{pp}^+_\Gamma(\mu)$ then $\theta < \mathrm{pp}^+_\Gamma(\lambda)$. So assume $\mu < \theta = \mathrm{cf}\theta < \mathrm{pp}^+_\Gamma(\mu)$, so by the definition of pp^+ for some ideal $I \in \Gamma$ with domain say κ, for some sequence $\langle \lambda_i : i < \kappa \rangle$ of regular cardinals with $\mathrm{tlim}_I \lambda_i = \mu$ we have $\prod_{i<\kappa} \lambda_i / I$ has a true cofinality θ^* and $\theta^* \ge \theta$. As $\mathrm{tlim}_I \lambda_i = \mu$ and $\lambda < \mu$ without loss of generality $\bigwedge_i \lambda < \lambda_i < \mu$. As $\lambda_i < \mu$ necessarily $\lambda_i < \mathrm{pp}^+_\Gamma(\lambda)$, and as said above $\lambda_i > \lambda$, so by the parallel to 2.3(1) here, there is an ideal $I_i \in \Gamma$ over say κ_i and a sequence $\langle \lambda_{i,j} : j < \kappa_i \rangle$ of regular cardinals $< \mu$ and with tlim_{I_i} of it being λ and $\prod_{j<\kappa_i} \lambda_{i,j}/I_i$ having true cofinality λ_i. Now clearly if $(A_i, <_i)$ is a linear order of true cofinality λ_i then $\prod_{i<\kappa}(A_i, <_i)/I$ has true cofinality θ^*, see [Sh345a,1.3(6)], hence $\prod_{i<\kappa}\left(\prod_{j<\kappa_i} \lambda_{i,j}/I_i\right)/I$ has true cofinality θ^*. Let $I^* = \sum_{i<\kappa} I_i$ be the ideal on $A^* = \bigcup_{i<\kappa} \{i\} \times \kappa_i$, (i.e. this $\mathrm{Dom}\, I^*$) defined by

$$B \in I^* \text{ iff } B \subseteq A^* \text{ and: } \{i < \kappa : \{j < \kappa_i : (i,j) \in A^*\} \notin I_i\} \in I.$$

$$\prod_{(i,j)\in A^*} \lambda_{i,j}/I^* \text{ is isomorphic to } \prod_{i<\kappa}\left(\prod_{j<\kappa_i} \lambda_{i,j}/I_i\right)/I,$$

hence has true cofinality θ^*. But by assumptions for some $A \subseteq \mathrm{Dom}\, I^*$, $A \ne \emptyset \bmod I^*$ and $I^* \restriction A$ is isomorphic to some $I' \in \Gamma$. So we have shown $\mathrm{pp}^+_{\{I'\}}(\lambda) > \theta^* \ge \theta$, as required.
(3A) Check.
(4) Like (2),(3).
(5) So $\mathfrak{a}$ is a set of regular cardinals, $\sigma \le |\mathfrak{a}| < \theta$, $|\mathfrak{a}| < \min \mathfrak{a}$, and for some ideal I on $\mathfrak{a}$, $\chi = \mathrm{tcf} \prod \mathfrak{a}/I$, and I is σ-complete. Let μ be the minimal cardinality such that $\mathfrak{a} \cap \mu \notin I$. Now $\mathfrak{a}' = \mathfrak{a} \cap \mu$, $I' = I \restriction \mathfrak{a}'$ exemplify the desired conclusion.
(6) By (2), (3).

Claim 2.4 Suppose $\langle \lambda_i : i < \kappa \rangle$ is increasing continuously, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\lambda = \sum_{i<\kappa} \lambda_i$ and $\kappa < \lambda_0$.
(1) If $S = \{i < \kappa : \mathrm{pp}(\lambda_i) \le \lambda_i^+\}$ is stationary then $\mathrm{pp}(\lambda) \le \lambda^+$.
(2) If J is an ideal on κ, $\mathrm{pp}_J(\lambda_i) = \lambda_i^{+h(i)}$, then $\mathrm{pp}_J(\lambda) \le \lambda^{+\|h\|_J}$ (see 2.4A(2)).
(3) If $\lambda = \aleph_{\alpha+\delta} > \delta$ then $\mathrm{pp}(\lambda) < \aleph_{\alpha+(|\delta|^\kappa)^+}$ (remember $\mathrm{cf}(\delta) = \kappa > \aleph_0$).
(4) If $\lambda = \aleph_{\alpha+\delta}$, $\kappa = \mathrm{cf}(\delta) > \aleph_0$, $\mu > 2^\kappa + \sup\{\chi^\kappa : \mathrm{cf}\chi = \kappa, \chi \le \delta\}$ then $\mathrm{pp}(\lambda) < \aleph_{\alpha+\mu}$.

Remark 2.4A (1) Part (1) of 2.4 generalizes Silver's theorem.
(2) Part (2) of 2.4 generalizes the Galvin Hajnal Lemma. Remember that for a function h from κ to ordinals,

$$\|h\| = \sup\{\|h'\| + 1 : h' < h \bmod D_\kappa\},$$

where D_κ is the filter generated by the closed unbounded subsets of κ. Similarly $\|h\|_J$ is defined for any ideal J on κ; the rank is $< \infty$ if J is $\aleph_1$-complete.
(3) Part (3) generalizes the Galvin Hajnal Theorem [GH].
(4) Part (4) generalizes [Sh71] (which followed [GH]); see representation in Erdös Hajnal Maté Rado [EHMR 47.6, p.296].

On generalizing [Sh111], [Sh256], see [Sh371] and [Sh386].

Proof: Now 2.4(1) holds by 2.1. [Why? W.l.o.g. $\mathrm{cf}(\lambda_i) < \kappa$ for every i, and $\mathrm{tcf}(\prod \lambda_i^+/J^{\mathrm{bd}}_\kappa) = \lambda^+$. For simplicity first assume $\mathrm{pp}_{<\kappa}(\lambda_i) = \lambda_i^+$; assume $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda$, $\lambda = \sup(\mathfrak{a})$, $|\mathfrak{a}| \le \kappa$, let $\mathfrak{a}_\epsilon$ $(\epsilon < \kappa)$ be increasing $|\mathfrak{a}_\epsilon| > \kappa$, $\mathfrak{a} = \bigcup_{\epsilon<\kappa} \mathfrak{a}_\epsilon$. Now for $i \in S$ and $\epsilon < \kappa$, if $\lambda_i = \sup(\mathfrak{a}_\epsilon \cap \lambda_i)$ then $\prod(\lambda_i \cap \mathfrak{a}_\epsilon)/J^{\mathrm{bd}}_{\mathfrak{a}_\epsilon}$

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(4) If $\lambda = \aleph_{\alpha+\delta}$, $\kappa = \mathrm{cf}(\delta) > \aleph_0$, $\mu > 2^\kappa + \sup\{\chi^\kappa : \mathrm{cf}\chi = \kappa, \chi \le \delta\}$ <u>then</u> $\mathrm{pp}(\lambda) < \aleph_{\alpha+\mu}$.

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On generalizing [Sh111], [Sh256], see [Sh371] and [Sh386].

Proof: Now 2.4(1) holds by 2.1. [Why? W.l.o.g. $\mathrm{cf}(\lambda_i) < \kappa$ for every i, and $\mathrm{tcf}(\prod \lambda_i^+ / J^{\mathrm{bd}}_\kappa) = \lambda^+$. For simplicity first assume $\mathrm{pp}_{<\kappa}(\lambda_i) = \lambda_i^+$; assume $\mathfrak{a} \subseteq \mathrm{Reg} \cap \lambda$, $\lambda = \sup(\mathfrak{a})$, $|\mathfrak{a}| \le \kappa$, let $\mathfrak{a}_\epsilon$ ($\epsilon < \kappa$) be increasing $|\mathfrak{a}_\epsilon| > \kappa$, $\mathfrak{a} = \bigcup_{\epsilon<\kappa} \mathfrak{a}_\epsilon$. Now for $i \in S$ and $\epsilon < \kappa$, if $\lambda_i = \sup(\mathfrak{a}_\epsilon \cap \lambda_i)$ then $\prod(\lambda_i \cap \mathfrak{a}_\epsilon)/J^{\mathrm{bd}}_{\mathfrak{a}_\epsilon}$

has true cofinality λ_ϵ^+ so for some $\chi_{\epsilon,i} < \lambda_\epsilon$ we have $\mathfrak{a}_\epsilon \cap \lambda_i \setminus \chi_{\epsilon,i} \in J_{\le\lambda_i^+}[\mathfrak{a}]$, but $J_{\le\lambda_i^+}[\mathfrak{a}]/J_{<\lambda_i^+}[\mathfrak{a}]$ is κ^+-directed so there is $\mathfrak{b}_i \in J_{\le\lambda_i^+}[\mathfrak{a}]$ such that $\mathfrak{a}_\epsilon \cap \lambda_i \setminus \chi_{\epsilon,i} \setminus \mathfrak{b}_i \in J_{<\lambda_i^+}[\mathfrak{a}]$, but $J_{<\lambda_i^+}[\mathfrak{a}] \subseteq J^{\rm bd}_{\mathfrak{a}\cap\lambda_i}$, and $\mathrm{cf}(\lambda_i) < \kappa$, no neccessarily $\mathfrak{a} \cap \lambda_i \setminus b_i \subseteq \chi_i$ for some $\chi_i < \lambda_i$; now we can apply [Sh345a, 1.10] with $D_i = \{\mathfrak{b} \subseteq \mathfrak{a} : \sup(\mathfrak{a} \cap \lambda_i \setminus \mathfrak{b}) < \mu_i\}$, for $i \in S$, $E = \{a \subseteq \kappa : \sup(S' \setminus a) < \kappa\}$ (would be slightly shorter if $\mathrm{pp}_\kappa(\lambda_i) = \lambda_i$). Why can we assume $\mathrm{pp}_{<\kappa}(\lambda_i) = \lambda_i^+$? By [Sh371, 3.6].]

We can prove (2) by induction on $\|h\|_J$. Now (3) follows by 2.3 and also (4). For more details — see [Sh371,1.10]. $\square_{2.4}$

§3 The cofinality of $\prod \mathfrak{a}$

The reader may remember that promising clarity we have passed in [Sh345a] from $\mathrm{cf} \prod \mathfrak{a}$ to $\mathrm{cf}(\prod \mathfrak{a}/D)$ for D an ultrafilter (and various ideals), but it may seem the original cofinality, that of the partial order $\prod \mathfrak{a}$, was forgotten. Not so, in 3.1 we prove:

$\otimes_1$ $\mathrm{cf}(\prod \mathfrak{a}) = \max \mathrm{pcf}\ \mathfrak{a}$.

So, of course, we could have used $\mathrm{cf}(\prod \mathfrak{a})$ as the central notion.
Another way to say this is that: by "looking in a single direction"; i.e. dividing by an ultrafilter, we do not decrease cofinality in general, i.e.

$\otimes_1'$ there is an ultrafilter D on $\mathfrak{a}$ such that the linear order $\prod \mathfrak{a}/D$ has the same cofinality as $\prod \mathfrak{a}$ itself.

We can similarly characterize $\prod \mathfrak{a}/I$ (I an ideal on $\mathfrak{a}$, see 3.2). We then return to a recurrent theme: nice representation of λ^+ — (in 3.3) if λ singular:

$\otimes_2$ if λ singular, $\lambda_0 < \lambda$, $\theta = \lambda^+$ <u>then</u> there is a strictly increasing sequence $\langle \lambda_i : i < \delta \rangle$ of regular cardinals, δ limit $\le \mathrm{cf}\ \lambda$, such that:
  (a) $\theta = \max \mathrm{pcf}\{\lambda_i : i < \delta\}$
  (b) for $\alpha < \delta$, $\lambda_\alpha > \max \mathrm{pcf}\{\lambda_\beta : \beta < \alpha\}$

(the gain is (b) whereas the main price is that possibly $\delta < \mathrm{cf}\lambda$; of course, if $\mathrm{cf}\lambda \le \kappa \le \lambda_0 < \lambda < \theta = \mathrm{cf}\theta < \mathrm{pp}^+_\kappa(\lambda)$ we can get the same with $\delta < \kappa^+$).
In 3.5 we get something for such $\bar\lambda$, i.e.

$\otimes_3$ if $\bar\lambda = \langle \lambda_i : i < \delta \rangle$ is a sequence of regulars $> |\delta|$ satisfying (b) of $\otimes_2$ and $\theta = \max \mathrm{pcf}\mathfrak{a}$ then we can find a sequence $\langle f_\alpha : \alpha < \theta \rangle$ which is $<_{J_{<\theta}\{\lambda_i : i<\delta\}}$-increasing and cofinal in $\prod_{i<\delta} \lambda_i / J_{<\theta}[\{\lambda_i : i < \delta\}]$ satisfying
  (*) for $i < \delta$, $\{f_\alpha \restriction i : \alpha < \theta\}$ has cardinality $< \lambda_\alpha$
    [if we just demand $\max \mathrm{pcf}\{\lambda_\beta : \beta < \alpha\} < \bigcup_{\gamma<\delta} \lambda_\gamma$ then we get corresponding restriction]. [We can add "$\{f_\alpha \restriction i : \alpha < \theta, i < \delta\}$ is a tree"].

For this we need 3.4, the second central Lemma of this section, in which another recurrent theme appears: computing the characteristic function

$\langle \sup(N \cap \theta) : \theta \in \mathfrak{a} \rangle$ for suitably closed elementary submodel N of some fragment $H(\chi)$ of set theory.
Another use of 3.4 is computing $\mathrm{cf}(\mathcal{S}_{\le\lambda_0}(\lambda), \subseteq)$ when $\lambda = \lambda_0^{+\alpha}$, $\alpha < \lambda_0$ as $\max \mathrm{pcf}\{\lambda^{+(\beta+1)} : \beta < \alpha\}$. This is part of our program to compute the natural measures of variants of the power set by pp's.
We shall return to such problems in §5 and in [Sh400,§3,§5].

* * *

Lemma 3.1 *Suppose* $|\mathfrak{a}|^+ < \min \mathfrak{a}$. <u>*Then*</u>
$\mathrm{cf}(\prod \mathfrak{a}, <) = \max \mathrm{pcf}(\mathfrak{a})$.

Proof: We prove by induction on $\lambda \in \mathrm{pcf}(\mathfrak{a})$ that for every $\mathfrak{b} \in J_{\le\lambda}[\mathfrak{a}]$, $\mathrm{cf}(\prod \mathfrak{b}, <) = \max \mathrm{pcf}(\mathfrak{b})$. Suppose we have proved it for every $\lambda' < \lambda$ and let $\mathfrak{b} \in J_{\le\lambda}[\mathfrak{a}]$. Without loss of generality $\mathfrak{a} = \mathfrak{b}$. By the induction hypothesis without loss of generality $\mathfrak{b} \in J_{\le\lambda}[\mathfrak{a}] \setminus J_{<\lambda}[\mathfrak{a}]$. Let $\langle f_\alpha : \alpha < \lambda \rangle$ be such that:

(i) $f_\alpha \in \prod \mathfrak{b}$
(ii) $\alpha < \beta \Rightarrow f_\alpha < f_\beta \bmod J_{<\lambda}[\mathfrak{a}]$
(iii) for every $g \in \prod \mathfrak{b}$ for some α, $g <_{J_{<\lambda}[\mathfrak{a}]} f_\alpha$.

Using $\{f_\alpha : \alpha < \lambda\}$ easily $\lambda \le \mathrm{cf}(\prod \mathfrak{b}, <)$; i.e. $\max \mathrm{pcf}(\mathfrak{b}) \le \mathrm{cf}(\prod \mathfrak{b}, <)$. For each $\mathfrak{c} \in J_{<\lambda}[\mathfrak{a}]$, let $F_\mathfrak{c} \subseteq \prod \mathfrak{c}$ be cofinal (i.e. $(\forall g \in \prod \mathfrak{c})(\exists f \in F_\mathfrak{c})[g < f]$) and $|F_\mathfrak{c}| = \max \mathrm{pcf}(\mathfrak{c}) < \lambda$; it exists by the induction hypothesis. (If $2^{|\mathfrak{a}|}$ were $\le \lambda$, the proof would be easier). Let χ be large enough regular, and we now define by induction on $i < |\mathfrak{a}|^+$, N_i, g_i such that:

(A) (i) $N_i \prec (H(\chi), \in, <^*_\chi)$
  (ii) $\|N_i\| = \lambda$
  (iii) $\langle N_j : j \le i \rangle \in N_{i+1}$
  (iv) $\langle N_i : i < |\mathfrak{a}|^+ \rangle$ is increasing continuous
  (v) $\{i : i \le \lambda + 1\} \subseteq N_0$, $\mathfrak{b} \in N_0$, $\langle f_\alpha : \alpha < \lambda \rangle \in N_0$ and the function $\mathfrak{c} \mapsto F_\mathfrak{c}$ belongs to N_0.

(B) (i) $g_i \in \prod \mathfrak{b}$ and $g_i \in N_{i+1}$
  (ii) for no $f \in N_i \cap \prod \mathfrak{b}$ does $g_i < f$ (equivalently, for no $f \in N_i \cap \prod \mathfrak{b}$, we have $g_i \le f$)
  (iii) $j < i \Rightarrow \bigwedge_{\theta\in\mathfrak{a}} g_j(\theta) < g_i(\theta)$.

There is no problem to define N_i, and if we cannot choose g_i this means that $N_i \cap \prod \mathfrak{b}$ exemplifies $\mathrm{cf}(\prod \mathfrak{b}, <) \le \lambda$ as required. So assume $\langle N_i, g_i : i < |\mathfrak{a}|^+ \rangle$ is defined. For each $i < |\mathfrak{a}|^+$ for some $\alpha(i) < \lambda$, $g_i < f_{\alpha(i)} \bmod J_{<\lambda}[\mathfrak{a}]$ hence $\alpha(i) \le \alpha < \lambda \Rightarrow g_i <_{J_{<\lambda}[\mathfrak{a}]} f_\alpha$. Choose $\alpha < \lambda$ such that $\alpha > \bigcup_{i<|\mathfrak{a}|^+} \alpha(i)$. Let $\mathfrak{c}_i = \{\theta \in \mathfrak{b} : g_i(\theta) \ge f_\alpha(\theta)\}$; so $\langle \mathfrak{c}_i : i < |\mathfrak{a}|^+ \rangle$ is increasing with i (by (B)(iii)), hence for some $i(*) < |\mathfrak{a}|^+$, $\mathfrak{c}_i = \mathfrak{c}_{i(*)}$ for every $i \in [i(*), |\mathfrak{a}|^+)$. Note that $\mathfrak{c}_{i(*)} \in J_{<\lambda}[\mathfrak{a}]$ (as $g_i < f_\alpha \bmod J_{<\lambda}[\mathfrak{a}]$

hence $F_{\mathfrak{c}_{i(*)}}$ is well defined. Now $\mathfrak{c}_{i(*)} \in N_{i(*)+1}$ (as $f_\alpha, g_{i(*)} \in N_{i(*)+1}$) hence $F_{\mathfrak{c}_{i(*)}} \subseteq N_{i(*)+1}$. Now

$$g_{i(*)+1}\restriction(\mathfrak{b}\setminus\mathfrak{c}_{i(*)}) = g_{i(*)+1}\restriction(\mathfrak{b}\setminus\mathfrak{c}_{i(*)+1}) < f_\alpha\restriction(\mathfrak{b}\setminus\mathfrak{c}_{i(*)+1}) = f_\alpha\restriction(\mathfrak{b}\setminus\mathfrak{c}_{i(*)})$$

(the $<$: by the definition of $\mathfrak{c}_{i(*)+1}$) and (as $\mathfrak{c}_{i(*)} \in J_{<\lambda}[\mathfrak{a}]$ and $F_{\mathfrak{c}_{i(*)}} \subseteq N_{i(*)+1}$) we know $g_{i(*)+1}\restriction\mathfrak{c}_{i(*)}$ is $< f$ for some $f \in F_{\mathfrak{c}_{i(*)}} \subseteq N_{i(*)+1}$. So $g_{i(*)+1} \leq \max\{f_\alpha, f\} \in N_{i(*)+1}$, contradiction to the choice of $g_{i(*)+1}$. $\square_{3.1}$

Conclusion 3.2 For any ideal I on κ and sequence $\langle\lambda_i : i < \kappa\rangle$ of regular cardinals such that $\lambda_i > \kappa^+$, we have

$$\text{cf}(\textstyle\prod_{i<\kappa}\lambda_i, <_I) = \sup\{\text{cf}(\prod\lambda_i, <_D) : D \text{ an ultrafilter on } \kappa \text{ disjoint from } I\}.$$

Proof: By [Sh345a,1.8], for some $A \in I$, the right hand side is

$$\max\text{pcf}(\{\lambda_i : i \in \kappa\setminus A\}).$$

Now

$$\text{cf}\Big(\prod_{i<\kappa}\lambda_i, <_I\Big) = \text{cf}\Big[\prod_{i<\kappa, i\notin A}\lambda_i, <_{I\restriction(\kappa\setminus A)}\Big]$$

and by 3.1 the latter is $\leq \max\text{pcf}\{\lambda_i : i \in \kappa\setminus A\}$. The other inequality is even easier. (We have quietly used [Sh345a, 1.3(8)]). $\square_{3.2}$

Claim 3.3 (1) Suppose λ is singular, $\lambda_0 < \lambda$. Then we can find a limit $\zeta(*) \leq \text{cf}\lambda$ and an increasing sequence of regular cardinals $\langle\lambda_i : i < \zeta(*)\rangle$ such that:
(i) $\max\text{pcf}\{\lambda_i : i < \zeta(*)\} = \lambda^+$
(ii) for $j < \zeta(*)$, $\text{cf}(\prod_{i<j}\lambda_i, <) < \lambda_j$.

(2) If $\langle\lambda_i : i < \delta\rangle$ is a strictly increasing sequence of regulars $> |\delta|$ with limit λ, $\text{pcf}\{\lambda_i : i < \delta\} \cap \theta \subseteq \bigcup_{i<\delta}\lambda_i$ and $\theta = \text{tcf}(\prod_{i<\delta}\lambda_i/J^{\text{bd}}_\delta)$, then for some $\mathfrak{a} \subseteq \{\lambda_i : i < \delta\}$ with no last element, $\theta = \max\text{pcf}(\mathfrak{a})$ and $\sigma \in \mathfrak{a} \Rightarrow \max\text{pcf}(\mathfrak{a}\cap\sigma) < \sigma$.

Proof:
1) By 1.5 there is $\langle\lambda^i : i < \text{cf}(\lambda)\rangle$, an increasing sequence of cardinals, $\lambda_0 < \lambda^0$, $\lambda = \sum_{i<\text{cf}\lambda}\lambda^i$ such that $\text{tcf}\big(\prod\lambda^i, <_{J^{\text{bd}}_{\text{cf}\lambda}}\big) = \lambda^+$ and each λ^i is regular. By [Sh345a,1.8] without loss of generality $\lambda^+ = \max\text{pcf}\{\lambda^i : i < \text{cf}(\lambda)\}$. Try to choose by induction on $\zeta < \text{cf}\,\lambda$, $i(\zeta) < \text{cf}\,\lambda$ such that:

$$\lambda^{i(\zeta)} > \max\text{pcf}\{\lambda^{i(\xi)} : \xi < \zeta\}.$$

Suppose $i(\zeta)$ is defined iff $\zeta < \zeta(*)$. Necessarily, $\zeta(*)$ is a limit ordinal $\leq \text{cf}\lambda$, also $\lambda^{i(\epsilon)} \in \text{pcf}\{\lambda^{i(\xi)} : \xi < \zeta\}$ if $\epsilon < \zeta < \zeta(*)$, hence $\langle\lambda^{i(\zeta)} : \zeta < \zeta(*)\rangle$ is strictly increasing.
We have still to get that $\lambda^+ = \max\text{pcf}\{\lambda^{i(\zeta)} : \zeta < \zeta(*)\}$. If $\zeta(*) = \text{cf}\lambda$ this is clear, otherwise $\max\text{pcf}\{\lambda^{i(\zeta)} : \zeta < \zeta(*)\}$ is $\geq \bigcup_{i<cf\lambda}\lambda^i$ (as we cannot choose $\zeta(*)$) hence (as $\{\lambda^{i(\zeta)} : \zeta < \zeta(*)\} \subseteq \{\lambda^i : i < \text{cf}\lambda\}$, $\lambda^+ = \max\text{pcf}\{\lambda^i : i < \text{cf}\lambda\}$) we have $\lambda^+ = \max\text{pcf}\{\lambda^{i(\zeta)} : \zeta < \zeta(*)\}$.
2) A similar proof. $\square_{3.3}$

Lemma 3.4 *Suppose $|\mathfrak{a}|^+ < \min(\mathfrak{a})$, where $\mathfrak{a}$ is a set of regular cardinals. Then we can find a family F, $F \subseteq \prod\mathfrak{a}$, of cardinality $\max\text{pcf}(\mathfrak{a})$ such that:*

(*) *If*

(a) χ is regular $> 2^{2^{\sup\mathfrak{a}}}$, $\langle N_i : i \leq \delta\rangle$ is an increasing continuous sequence of elementary submodels of $(H(\chi), \in, <^*_\chi)$, $[j < i \Rightarrow N_j \in N_i]$, $\{F, \mathfrak{a}\} \in N_0$, $\mathfrak{a} \subseteq N_0$, $\|N_i\| < \min\mathfrak{a}$, and $|\mathfrak{a}| < \text{cf}(\delta) < \min(\mathfrak{a})$,

then

(b) for some $f \in F$, $(\forall\theta \in \mathfrak{a})\,[f(\theta) = \sup(\theta \cap \bigcup_{i<\delta}N_i)]$.

Proof: For every $\mathfrak{b} \subseteq \mathfrak{a}$ let $\lambda(\mathfrak{b}) = \max\text{pcf}(\mathfrak{b})$; so by 3.1 there is $F^0_\mathfrak{b} \subseteq \prod\mathfrak{b}$ of cardinality $\lambda(\mathfrak{b})$ such that $(\forall f \in \prod\mathfrak{b})(\exists g \in F^0_\mathfrak{b})[f < g]$ (i.e. $\bigwedge_{\theta\in\mathfrak{b}} f(\theta) < g(\theta)$).

Subfact 3.4A For every $\mathfrak{b} \subseteq \mathfrak{a}$ there is $\langle f^\mathfrak{b}_\alpha : \alpha < \lambda(\mathfrak{b})\rangle$ such that:
(i) $f^\mathfrak{b}_\alpha \in \prod\mathfrak{b}$
(ii) $\alpha < \beta \Rightarrow f^\mathfrak{b}_\alpha < f^\mathfrak{b}_\beta \bmod J_{<\lambda(\mathfrak{b})}[\mathfrak{b}]$
(iii) if $|\mathfrak{a}| < \text{cf}(\alpha) < \min(\mathfrak{a})$, then for $\theta \in \mathfrak{b}$
$f^\mathfrak{b}_\alpha(\theta) = \min\{\bigcup_{\beta\in C} f^\mathfrak{b}_\beta(\theta) : C \text{ a club of } \alpha\}$
(iv) for every $f \in \prod\mathfrak{b}$, $\alpha < \lambda(\mathfrak{b})$ for some β, $\alpha < \beta < \lambda(\mathfrak{b})$ and $f < f^\mathfrak{b}_\beta$ (and not just $f <_{J_{<\lambda}[\mathfrak{b}]} g$!)

Proof: Immediate by 3.1 [for (iii) note that if $\langle f^\mathfrak{b}_\beta : \beta < \alpha\rangle$ has been defined, and we define $f^\mathfrak{b}_\alpha$ as there, then for each $\theta \in \mathfrak{b}$, for some club C_θ of α we get the minimum, hence $C = \bigcap_{\theta\in\mathfrak{a}} C_\theta$ is a club of α (as $\text{cf}(\alpha) > |\mathfrak{a}|$); so $f^\mathfrak{b}_\alpha \in \prod\mathfrak{b}$ (as $|C| = \text{cf}(\alpha) < \min\mathfrak{a}$) and $f^\mathfrak{b}_\beta \leq f^\mathfrak{b}_\alpha$ for $\beta \in C$, hence (ii) holds (remember α is limit)]. $\square_{3.4A}$

Continuation of the Proof of 3.4: Now, for $\mathfrak{b} \subseteq \mathfrak{a}$, we define $F_\mathfrak{b}$ by induction on $\max\text{pcf}(\mathfrak{b})$, (using $\langle f^\mathfrak{b}_\alpha : \alpha < \lambda(\mathfrak{b})\rangle$ from 3.4A).

$$F_\mathfrak{b} = \cup\{(f^\mathfrak{b}_\gamma\restriction(\mathfrak{b}\setminus\mathfrak{c})) \cup g\restriction\mathfrak{c} : \gamma < \lambda(\mathfrak{b}) \text{ and for some } \alpha < \beta < \lambda(\mathfrak{b}),\ \mathfrak{c} = \mathfrak{c}(f^\mathfrak{b}_\alpha, f^\mathfrak{b}_\beta) \text{ and } g \in F_\mathfrak{c}\}$$

where $\mathfrak{c}(f^\mathfrak{b}_\alpha, f^\mathfrak{b}_\beta) = \{\theta \in \mathfrak{b} : f^\mathfrak{b}_\alpha(\theta) > f^\mathfrak{b}_\beta(\theta)\}$.

Note: $F_{\mathfrak{b}}$ is well defined as for $\alpha < \beta < \max\mathrm{pcf}(\mathfrak{b})$, $\mathfrak{c}(f^{\mathfrak{b}}_\alpha, f^{\mathfrak{b}}_\beta) \in J_{<\lambda(\mathfrak{b})}[\mathfrak{b}]$ (by (ii) of 3.4A) hence $\lambda\left(\mathfrak{c}(f^{\mathfrak{b}}_\alpha, f^{\mathfrak{b}}_\beta)\right) < \lambda(\mathfrak{b})$, so no vicious circle arises.

In particular, $F_{\mathfrak{a}}$ is defined and we shall prove that $F =: F_{\mathfrak{a}}$ is as required in 3.4. We rather prove that $F_{\mathfrak{b}}$ is as required there with $\mathfrak{b}$ instead of $\mathfrak{a}$, by induction on $\max\mathrm{pcf}(\mathfrak{b})$. So suppose $\langle N_i : i < \delta\rangle$ is as mentioned there. Now

$(*)_1$ for each $\theta \in \mathfrak{b}$, $\langle \sup(N_i \cap \theta) : i \le \delta\rangle$ is a strictly increasing continuous sequence of ordinals $< \theta$.

[Why? $\sup(N_i \cap \theta) < \theta$ as $\|N_i\| < \min\mathfrak{a}$ and for $i < j$,

$$\sup(N_i \cap \theta) < \sup(N_j \cap \theta) \text{ as } \{\theta, N_i\} \in N_j$$

hence $\sup(N_i \cap \theta)$ belongs to N_j].

Clearly

$(*)_2$ letting $g_i \in \prod\mathfrak{b}$ be $g_i(\theta) = \sup(N_i \cap \theta)$; then $i < j \Rightarrow g_i < g_j$

$(*)_3$ $g_i \in N_{i+1}$

$(*)_4$ $\langle \sup(N_i \cap \lambda(\mathfrak{b})) : i \le \delta\rangle$ is strictly increasing continuous.

Let $\gamma(i) = \sup(N_i \cap \lambda(\mathfrak{b}))$.

$(*)_5$ There is $\beta(i) \in N_{i+1}$ such that $\gamma(i) < \beta(i) < \gamma(i+1)$ and $g_i < f^{\mathfrak{b}}_{\beta(i)}$.

[Why? As $N_i \in N_{i+1}$ hence γ_i, $g_i \in N_{i+1}$ and use 3.4A(iv)].

$(*)_6$ if $f \in N_i \cap \prod\mathfrak{b}$ <u>then</u> $f < g_i$.

Now for some club C_1 of $\gamma(\delta)$, for every $\theta \in \mathfrak{b}$, $f^{\mathfrak{b}}_{\gamma(\delta)}(\theta) = \bigcup_{\beta\in C_1} f^{\mathfrak{b}}_\beta(\theta)$ [by 3.4A(iii); and see its proof]. As we can decrease C_1, without loss of generality $C_1 = \{\gamma(i) : i \in C\}$ where C is a club of $\mathrm{cf}(\delta)$.

By $(*)_5 + (*)_6$,

$(*)_7$ for $i_1 < i_2 \le i_3 < \delta$ we have $f^{\mathfrak{b}}_{\beta(i_1)} < g_{i_2} < f^{\mathfrak{b}}_{\beta(i_3)}$

hence (by $(*)_2$) we have $\langle f^{\mathfrak{b}}_{\beta(i)}(\theta) : i < \delta\rangle$ is strictly increasing for each $\theta \in \mathfrak{b}$; also $f^{\mathfrak{b}}_{\beta(i)} <_{J_{<\lambda(\mathfrak{b})}[\mathfrak{b}]} f^{\mathfrak{b}}_{\gamma(\delta)}$ (by 3.4A (iii) and the choice of C_1 and C) hence

$$\mathfrak{c}_i =: \{\theta \in \mathfrak{b} : f^{\mathfrak{b}}_{\beta(i)}(\theta) \ge f^{\mathfrak{b}}_{\gamma(\delta)}(\theta)\}$$

is a member of $J_{<\lambda(\mathfrak{b})}[\mathfrak{b}]$ and $\langle \mathfrak{c}_i : i < \delta\rangle$ is increasing, and $\mathfrak{c}_i \subseteq \mathfrak{b} \subseteq \mathfrak{a}$. As $\mathrm{cf}(\delta) > |\mathfrak{a}|$, for some $i(*) < \delta$ we have

$$[i(*) \le i < \delta \Rightarrow \mathfrak{c}_i = \mathfrak{c}_{i(*)}],$$

and for the same reasons (see $(*)_7$)

$(*)_8$ $i(*) < i < \delta \Rightarrow \mathfrak{c}_{i(*)} = \{\theta : g_i(\theta) \ge f^{\mathfrak{b}}_{\gamma(\delta)}\theta\}$.

It is also clear that $[\theta \in \mathfrak{b}\backslash\mathfrak{c}_{i(*)} \Rightarrow f^{\mathfrak{b}}_{\gamma(\delta)}(\theta) = g_\delta(\theta)]$.

[The inequality $\ge$ follows from $\theta \in \mathfrak{b}\backslash\mathfrak{c}_{i(*)}$ and $(*)_8$ (and $g_\delta(\theta) = \bigcup_{\beta<\delta} g_\beta(\theta)$).

The inequality $\le$ follows from $f^{\mathfrak{b}}_{\gamma(\delta)}(\theta) = \bigcup_{\beta\in C_1} f^{\mathfrak{b}}_\beta(\theta)$, $f^{\mathfrak{b}}_\beta(\theta) \in N_\delta$].

Next let $j(*) \in C$ (see above) be such that $\beta(i(*)) < \gamma(j(*))$ hence $f^{\mathfrak{b}}_{\beta(i(*))} < f^{\mathfrak{b}}_{\gamma(j(*))} \bmod J_{<\lambda(\mathfrak{b})}[\mathfrak{b}]$, hence

$$\mathfrak{c} = \mathfrak{c}(f^{\mathfrak{b}}_{\beta(i(*))}, f^{\mathfrak{b}}_{\gamma(j(*))}) =: \{\theta \in \mathfrak{b} : f^{\mathfrak{b}}_{\beta(i(*))}(\theta) \ge f^{\mathfrak{b}}_{\gamma(j(*))}(\theta)\}$$

belongs to $J_{<\lambda(\mathfrak{b})}[\mathfrak{b}]$. As ($j(*) \in C$ and) $f^{\mathfrak{b}}_{\gamma(j(*))} \le f^{\mathfrak{b}}_{\gamma(\delta)}$, necessarily $\mathfrak{c}$ includes $\mathfrak{c}_{i(*)}$.

Now $\mathfrak{c} \in N_i$ for $i < \delta$ large enough, hence without loss of generality $F_{\mathfrak{c}} \in N_i$ (that is, there is always some function $\mathfrak{e} \mapsto F'_{\mathfrak{e}}$ satisfying the same requirements, including $F'_{\mathfrak{b}} = F_{\mathfrak{b}}$, which belongs to N_i). Now apply the induction hypothesis and the definition of $F_{\mathfrak{b}}$. $\square_{3.4}$

Conclusion 3.5 Suppose δ is a limit ordinal, $\langle \lambda_i : i < \delta\rangle$ is a strictly increasing sequence of regular cardinals $> |\delta|^+$, and for $\alpha < \delta$,

$$\lambda_\alpha > \max\mathrm{pcf}\{\lambda_i : i < \alpha\}.$$

<u>Then</u> we can find $f_\alpha \in \prod_{i<\delta} \lambda_i$ for $\alpha < \max\mathrm{pcf}(\{\lambda_i : i < \delta\}$ such that:

(a) for $\zeta < \xi < \max\mathrm{pcf}\{\lambda_i : i < \delta\}$ we have

$$f_\zeta < f_\xi \bmod J_{<\max\ \mathrm{pcf}\{\lambda_i : i<\delta\}}[\{\lambda_i : i < \delta\}]$$

(b) for $\alpha < \delta$ we have $|\{f_\zeta{\restriction}i : \zeta\}| < \lambda_i$ (in fact $\le \max\mathrm{pcf}\{\lambda_j : j < i\}$)

(b)$^+$ $f_\zeta(\alpha) = f_\xi(\beta) \Rightarrow f_\zeta{\restriction}\alpha = f_\xi{\restriction}\beta$ (and so $\alpha = \beta$; i.e. we have a tree).

Proof: For each $\alpha < \delta$ let $F_\alpha \subseteq \prod_{i<\alpha} \lambda_i$ be as guaranteed by 3.4. Let

$$F^* = \{f : f \text{ belongs to } \Pi_{i<\delta}\lambda_i \text{ and for every } \ \alpha < \delta \ \ f{\restriction}\alpha \in F_\alpha\}.$$

By $(*)$ of 3.4, applied simultaneously to $\mathfrak{a}_j = \{\lambda_i : i < j\}$ (for $j < \delta$) for every $g \in \prod_{i<\delta} \lambda_i$ we can find $f \in F^*$, $g \le f$. [Why? Let $\chi = (2^{2^{\sup\mathfrak{a}}})^+$ and for $i < |\delta|^+$ we choose $N_i \prec (H(\chi), \in, <^*_\chi)$, $\|N_i\| = |\delta|$, $g \in N_0$, $\delta + 1 \subseteq N_0$, $[j < i \Rightarrow N_j \prec N_j]$, $[j < i \Rightarrow N_j \in N_i]$, and $\{F^*, F_\alpha : \alpha < \delta\} \subseteq N_0$. Let $N^* = \bigcup_{i<|\delta|^+} N_i$. So for each $\alpha < \delta$ for some $f_\alpha \in F_\alpha$ we have $(\forall\beta < \alpha)[f_\alpha(\lambda_\beta) = \sup(N^* \cap \lambda_\beta)]$. So $f =: \bigcup_{\alpha<\delta} f_\alpha$ is as required.]

Now it is easy to get $\langle f_\zeta : \zeta < \max\mathrm{pcf}\{\lambda_i : i < \delta\}\rangle$ as required in (a) + (b), choosing f_ζ by induction on ζ.

As for $(b)^+$, let

$$F_\alpha = \{f^\alpha_\xi : \xi < \mu_\alpha < \lambda_\alpha\}$$

and let us define

$$f'_\zeta \in \prod_{i<\delta} \lambda_i : f'_\zeta(i) = \mu_i \times f_\zeta(i) + \xi_\zeta(i) \text{ where } f_\zeta \restriction i = f^i_{\xi_\zeta(i)} (\in F_i).$$

$\square_{3.5}$

Question 3.5A Suppose $|\mathfrak{a}|^+ < \min \mathfrak{a}$, $\lambda = \max \mathrm{pcf}(\mathfrak{a})$; then is the ideal $J_{<\lambda}[\mathfrak{a}]$ generated by $< \lambda$ sets? (See [Sh371,§2]).

Theorem 3.6 *If* $\lambda = \lambda_0^{+\alpha}$, $\alpha < \lambda_0$, *<u>then</u>:*

$$\mathrm{cf}\,(\mathcal{S}_{\leq\lambda_0}(\lambda), \subseteq) = \max \mathrm{pcf} \left\{ \lambda^{+(\beta+1)} : \beta < \alpha \right\}.$$

Remark 3.6A Remember $\mathcal{S}_{\leq\lambda_0}(\lambda) = \{a \subseteq \lambda : |a| \leq \lambda_0\}$, this is partially ordered by $\subseteq$ (in fact is λ_0^+-directed), hence its cofinality is well defined.

Proof: By 3.4. [Without loss of generality, $\lambda_0 > \aleph_0$, let F be as in 3.4 for $\mathfrak{a} = \{\lambda_0^{+\beta+1} : \beta < \alpha\}$, let M be the model with universe λ, and functions: f, g, such that $\{f(\alpha, i) : i < \alpha\} = \{j : j < |\alpha|\}$, $g(\alpha, f(\alpha, i)) = i$; now for $h \in F$ let

$$N_h = \bigcap \left\{ \text{Skolem Hull } \langle \lambda_0 \cup \bigcup_{\beta<\alpha} C_\beta \rangle_\mu : C_\beta \text{ a club of } \ f(\lambda^{+(\beta+1)}) \right\}.$$

Now $\{N_h : h \in F\}$ exemplify the inequality $\leq$; the other is easier]. $\square_{3.6}$

§4 Applications

We have claimed to have applications of the theory, and this section is an implementation of this theme. First, we deal with colouring theorems (in other words, negative partition relation): such theorems mainly say there is a (two place symmetric) function c from λ to κ such that any subset of λ of cardinality λ is "complicated", [4.1,4.7,4.8]. As a conclusion we get the non-productivity of λ^+-c.c. for Boolean algebras [see 4.2].

In another direction we prove that for regular cardinals λ, if on λ there is no Jonsson algebra (i.e. is a Jonsson cardinal) then it is quite large (say inaccessible or successor of a singular which is a limit of inaccessible Jonsson cardinals). Note, however, that there may be Jonsson cardinals $< 2^{\aleph_0}$.

In 4.10, we define an (λ)-entangled linear order (4.10(5)) and an (λ)-entangled sequence of linear orders; so our aim is to prove in ZFC that there are such objects. So if $\lambda > \mathrm{cf}\lambda$ there is an entangled sequence of linear orders of cardinality λ^+, of length $\mathrm{cf}\lambda$ (by 4.9, and more in 4.11). We then give sufficient conditions for the existence of such a sequence (4.12) which gives, for example, there is one in λ^+ if λ is singular $\leq 2^{\aleph_0}$.

* * *

We first get some strong negative partition relations.

Conclusion 4.1 If λ is singular, <u>then</u>:

(a) $\mathrm{Pr}_1(\lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$

(b) $\mathrm{Pr}_2(\lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$ if $(\forall \alpha < \lambda)(|\alpha|^{<\mathrm{cf}\lambda} < \lambda)$

(c) $2^{\mathrm{cf}\lambda} \geq \lambda$ <u>implies</u> $\mathrm{Pr}_0(\lambda^+, \aleph_0, \aleph_0)$

Remark 4.1A (1) $\mathrm{Pr}_\ell(\lambda, \kappa, \theta)$ is defined in the Appendix, Definitions 1.1, 1.2, 1.3 for $\ell = 0, 1, 2$ respectively but will be clear in the proofs.
(2) Historically note: by [Sh282, Lemma 40] we have that part (b) follows from 1.5 for almost the result and part (c) follows by part (a) + [Sh282] (or [Sh327,Lemma 1]).

Proof: a) By 4.1B below, we have that $\mathrm{Pr}_1(\lambda^+, \mathrm{cf}(\lambda), \mathrm{cf}(\lambda))$ follows from 1.5.
b) Follows from 1.5 by 4.1D below (and 2.1).
c) This follows from part (a) by [Sh365,4.5(3)], so the reader is allowed to ignore it.
(In details, let λ^+, λ^+, $\mathrm{cf}(\lambda)$, $\aleph_0$, $\mathrm{cf}(\lambda)$ here stand for λ, μ, σ, θ, χ there, so the assumptions there hold, as:

(i) $\mathrm{Pr}_1(\lambda, \mu, \sigma, \theta)$ there means $\mathrm{Pr}_1(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \aleph_0)$ and it holds because of part (a) and monotonicity of Pr_1,

(ii) $\chi^{<\theta} \leq \sigma$ means $\mathrm{cf}(\lambda)^{\aleph_0} = \mathrm{cf}(\lambda)$, which obviously holds,

(iii) $\sigma = \sigma^{<\theta}$ means $\mathrm{cf}(\lambda) = \mathrm{cf}(\lambda)^{<\aleph_0}$ which holds,

(iv) $\mathrm{cf}(\mu) > \chi^{<\theta}$ means $\mathrm{cf}(\lambda^+) > \mathrm{cf}(\lambda)^{<\aleph_0}$ i.e. $\lambda^+ > \mathrm{cf}(\lambda)$ which obviously holds.

So the conclusion of [Sh365, 4.5(3)] holds; it says $\mathrm{Pr}_0(\lambda, \mu, \sigma, \theta)$ which means $\mathrm{Pr}_0(\lambda^+, \lambda^+, \mathrm{cf}(\lambda), \aleph_0)$ as required. Note: if $\mathrm{cf}(\lambda)^{<\theta} = \mathrm{cf}(\lambda)$ then we can replace $\aleph_0$ by θ.)

$\square_{4.1}$

Lemma 4.1B *Suppose:*

$\mathfrak{c}$ is a set of regular cardinals, $|\mathfrak{c}| < \min(\mathfrak{c})$,
$\mathrm{tcf}\,(\prod \mathfrak{c}, \leq_{J^{bd}_{\mathfrak{c}}}) = \lambda$,
$\mathfrak{c}$ has no last element so $(\forall \mu \in \mathfrak{c})(\mu < \lambda)$ *and*
let $\chi = \min\{|\mathfrak{c} \setminus \mu| : \mu \in \mathfrak{c}\} < \sup \mathfrak{c}$.

<u>Then</u> $\mathrm{Pr}_1(\lambda, \chi, \mathrm{cf}(\mathfrak{c}))$.

Remark 4.1C 0) Remember: for a set A of ordinals with no last element $J^{bd}_A = \{B \subseteq A : \sup B < \sup A\}$.
1) Every unbounded $\mathfrak{c}' \subseteq \mathfrak{c}$ satisfies all the assumptions (though: maybe with a smaller χ). So without loss of generality $|\mathfrak{c}| = \chi$.
2) $\mathrm{cf}(\mathfrak{c})$ is the cofinality of the order type of $\mathfrak{c}$.

Proof: Let $\langle f_\alpha : \alpha < \lambda\rangle$ exemplify $\mathrm{tcf}\,(\prod \mathfrak{c}, \le_{J^{bd}_{\mathfrak{c}}}) = \lambda$, $\langle \mathfrak{c}_i : i < \chi\rangle$ a partition of $\mathfrak{c}$ to χ sets, each $\mathfrak{c}_i$ is an unbounded subset of $\mathfrak{c}$. Let $h : \mathfrak{c} \to \chi$ be such that $\theta \in \mathfrak{c}_{h(\theta)}$.

Let us define two symmetric two-place functions θ, e from λ with range of cardinality χ : for $\alpha < \beta < \lambda$, let $\theta(\alpha, \beta) = \sup\{\theta : f_\alpha(\theta) \ge f_\beta(\theta)\}$ (so if there is a maximal θ for which $f_\alpha(\theta) \ge f_\beta(\theta)$, it is $\theta(\alpha,\beta)$) and let:

$$e(\alpha,\beta) = h(\theta(\alpha,\beta)).$$

Suppose $\xi < \mathrm{cf}(\mathfrak{c})$, $\langle \alpha_{\beta,\zeta} : \zeta < \xi\rangle$ is a strictly increasing sequence of ordinals $< \lambda$ for each $\beta < \lambda$ and the $\alpha_{\beta,\zeta}$ pairwise distinct. Now for any given $i(*) < \chi$ we should find $\beta < \gamma < \lambda$ such that $\bigcup_\zeta \alpha_{\beta,\zeta} < \alpha_{\gamma,0}$ and for every $\zeta_1, \zeta_2 < \xi$, $\theta\,(\alpha_{\beta,\zeta_1}, \alpha_{\gamma,\zeta_2})$ belongs to $\mathfrak{c}_{i(*)}$.

Let χ^* be a regular large enough cardinal. Let M_0 be an elementary submodel of $(H(\chi^*), \in, <^*)$ where $<^*$ is a well ordering of $H(\chi^*)$, to which λ, $\mathfrak{c}$, ξ, $\langle < \alpha_{\beta,\zeta} : \zeta < \xi >: \beta < \lambda\rangle$, $\langle f_\alpha : \alpha < \lambda\rangle$ belongs, $\xi \cup \mathfrak{c} \subseteq M_0$, and $\|M_0\| < \sup \mathfrak{c}$. As the $\alpha_{i,\zeta}$ are distinct, by renaming without loss of generality $i \le \alpha_{i,\zeta}$ for every i.

Let $\mathfrak{c}' = \{\theta \in \mathfrak{c} : \sup(M_0 \cap \theta) < \theta\}$, so $\mathfrak{c}' \equiv \mathfrak{c} \bmod J^{bd}_{\mathfrak{c}}$. Define a function $g \in \prod \mathfrak{c}$ by: $g(\theta)$ is $\sup(M_0 \cap \theta)$ for $\theta \in \mathfrak{c}'$ and zero otherwise. As $\langle f_\beta : \beta < \lambda\rangle$ exemplify $\mathrm{tcf}\,(\prod \mathfrak{c}, <_{J^{bd}})= \lambda$, clearly for some $\beta(0) < \lambda$, $g < f_{\beta(0)} \bmod J^{bd}_{\mathfrak{c}}$, and $\beta(0) > \sup(M_0 \cap \lambda)$.

As $\alpha_{\beta(0),\zeta} \ge \beta(0)$ for each $\zeta < \xi$, clearly for some $\theta^0_\zeta \in \mathfrak{c}$,

$$[\theta^0_\zeta < \theta \in \mathfrak{c} \Rightarrow g(\theta) < f_{\alpha_{\beta(0),\zeta}}(\theta)].$$

Let $\theta(0) = \sup\{\theta^0_\zeta : \zeta < \xi\}$, so as $\mathrm{cf}(\mathfrak{c}) > \xi$ clearly $\theta(0) < \sup(\mathfrak{c})$. Let $\theta(1) \in \mathfrak{c}_{i(*)}$ be such that $\mathfrak{c} \setminus \theta(1) \subseteq \mathfrak{c}'$ and $\theta(1) > \theta(0)$. Let for $\beta < \lambda$, $f^*_\beta \in \prod \mathfrak{c}$ be defined by

$$f^*_\beta(\theta) = \min\{f_{\alpha_{\beta,\zeta}}(\theta) : \zeta < \xi\}.$$

Easily $f_\beta \le f^*_\beta \bmod J^{bd}_{\mathfrak{c}}$ (as $\beta \le \alpha_{\beta,\zeta}$ and $\mathrm{cf}(\mathrm{otp}(\mathfrak{c}) > \xi)$. Let

$$\mathfrak{c}^* = \{\theta \in \mathfrak{c} : \sup\{f^*_\beta(\theta) : \beta < \lambda\} = \theta\}.$$

So $\mathfrak{c}^* = \mathfrak{c} \bmod J^{bd}_{\mathfrak{c}}$ [why? otherwise define $g^* \in \prod \mathfrak{c}$:

$$g^*(\theta) = \begin{cases} \sup\{f^*_\beta(\theta) : \beta < \lambda\} & \text{if } \theta \in \mathfrak{c} \setminus \mathfrak{c}^* \\ 0 & \text{otherwise.}\end{cases}$$

So for some $\beta < \lambda$, $g^* < f_\beta \bmod J^{bd}_{\mathfrak{c}}$, and we get a contradiction easily].

So we could have chosen $\theta(1)$ such that it belongs to $\mathfrak{c}^*$, ($\theta(1) > \theta(0)$, $\theta(1) \in \mathfrak{c}_{i(*)}$, $\mathfrak{c}\setminus\theta(1) \subseteq \mathfrak{c}'$ and) $\theta(1) > \|M_0\|$ and then we can choose $\beta(1) < \lambda$ such that:

$$\gamma =: f^*_{\beta(1)}(\theta(1)) > \sup\{f_{\alpha_{\beta(0),\zeta}}(\theta(1)) : \zeta < \xi\}.$$

Let M_1 be the Skolem Hull of $M_0 \cup \{\gamma\}$ (in $(H(\chi^*), \in, <^*)$). Now clearly $(H(\chi^*), \in, <^*) \models (\exists \beta < \lambda) f^*_\beta(\theta(1)) = \gamma$; as λ, $\langle f^*_\beta, \beta < \lambda\rangle$, $\theta(1)$, γ are in M_1 there is $\beta(2) \in M_1 \cap \lambda$ such that $f^*_{\beta(2)}(\theta(1)) = \gamma$. So

$$\Big[\zeta_1, \zeta_2 < \xi \Rightarrow f_{\alpha_{\beta(0),\zeta_1}}(\theta(1)) < \gamma = f^*_{\beta(1)}(\theta(1)) = f^*_{\beta(2)}(\theta(1)) \le f_{\alpha_{\beta(2),\zeta_2}}(\theta(1))\Big].$$

Easily, for every regular cardinal $\sigma \in M_0$, if $\sigma > \theta(1)$, then

$$\sup(M_0 \cap \sigma) = \sup(M_1 \cap \sigma).$$

As $\beta(0) > \sup(M_0 \cap \lambda)$, also $\beta(0) > \sup(M_1 \cap \lambda)$, but $\beta(2) \in M_1$, so $\beta(0) > \beta(2)$, and similarly $\alpha_{\beta(2),\zeta_1} < \beta(0) \le \alpha_{\beta(0),\zeta_2}$ (for $\zeta_1, \zeta_2 < \xi$). Also for every $\theta \in \mathfrak{c}$, if $\theta > \theta(1)$, $\zeta_1, \zeta_2 < \xi$, then $f_{\alpha_{\beta(2),\zeta_1}}(\theta) \in M_1$ hence $f_{\alpha_{\beta(2),\zeta_1}}(\theta) < \sup(M_0 \cap \theta)$ hence $f_{\alpha_{\beta(2),\zeta_1}}(\theta) < g(\theta)$, but $g(\theta) < f_{\alpha_{\beta(0),\zeta_2}}(\theta)$. So $\theta\,(\alpha_{\beta(2),\zeta_1}, \alpha_{\beta(2),\zeta_2}) = \theta(1)$, but $h(\theta(1)) = i(*)$, so we finish. $\square_{4.1B}$

Claim 4.1D Assume

(i) $\lambda = \max \mathrm{pcf}\, \mathfrak{a}$, $\lambda \notin \mathfrak{a}$
(ii) for $\mu \in \mathfrak{a}$, $\mu > [\max \mathrm{pcf}(\mathfrak{a} \cap \mu)]^{<\theta_\mu}$
(iii) $\mathrm{Pr}_2(\mu, \sigma_\mu, \theta_\mu)$ for $\mu \in \mathfrak{a}$.
(iv) $\theta \le \mathrm{cf}(\mathfrak{a})$, $\theta = \bigcup_{\mu\in\mathfrak{a}} \theta_\mu = \mathrm{tlim}_J \langle \theta_\mu : \mu \in \mathfrak{a}\rangle = \theta$
(v) J a θ-complete proper ideal on $\mathfrak{a}$ extending $J_{<\lambda}[\mathfrak{a}]$
(vi) $\sigma = \bigcup_{\mu\in\mathfrak{a}} a_\mu$, where $|a_\mu| = \sigma_\mu$ and for each $\epsilon < \sigma$ and $\xi < \theta$, we have

$$\{\mu \in \mathfrak{a} : \epsilon \in a_\mu \text{ and } \xi < \theta_\mu\} \notin J.$$

<u>Then</u> $\mathrm{Pr}_2(\lambda, \sigma, \theta)$.

Proof: Let $\langle f_\alpha : \alpha < \lambda\rangle$ be a $<_{J_{<\lambda}[\mathfrak{a}]}$-increasing cofinal sequence of members of $\prod \mathfrak{a}$ such that for every $\mu \in \mathfrak{a}$ we have $|\{f_\alpha \restriction (\mathfrak{a} \cap \mu) : \alpha < \lambda\}|^{<\theta_\mu} < \mu$

(see 3.5, using assumption (ii)). Let c_μ be a two place function from μ to a_μ, exemplifying $\mathrm{Pr}_2(\mu,\sigma_\mu,\theta_\mu)$. Define

$$e(\beta,\alpha)=\min\{\mu\in\mathfrak{a}: f_\alpha(\mu)\neq f_\beta(\mu)\}$$

$$c(\beta,\alpha)=c_{e(\beta,\alpha)}\left(f_\beta[e(\beta,\alpha)], f_\alpha[e(\beta,\alpha)]\right).$$

Given $\varepsilon<\sigma$ and $\langle\langle\alpha(\beta,\zeta):\zeta<\xi\rangle:\beta<\lambda\rangle$, $\alpha_{\beta,\zeta}<\lambda$ with no repetition where $\xi<\theta$, then for each $\beta<\lambda$ for some $\mu_\beta\in\mathfrak{a}$, $\langle f_{\alpha(\beta,\zeta)}\restriction\mu_\beta:\zeta<\xi\rangle$ are pairwise distinct (note $\theta\leq\mathrm{cf}\mathfrak{a}$) and without loss of generality for every $\beta<\lambda$, $\mu_\beta=\mu^*$ (as $|\mathfrak{a}|<\lambda=\mathrm{cf}(\lambda)$) and $f_{\alpha(\beta,\zeta)}\restriction(\mathfrak{a}\cap\mu^*)$ does not depend on β (use condition (ii) and the choice of $\langle f_\alpha :< \lambda\rangle$). As λ is regular, without loss of generality $\beta<\gamma<\lambda\Rightarrow\bigwedge_{\zeta_1,\zeta_2<\xi}\alpha(\beta,\zeta_1)<\alpha(\gamma,\zeta_2)$. Define $f^*_\beta\in\prod\mathfrak{a}$ as in the proof of 4.2A: $f_\beta(\mu)=\min\{f_{\alpha(\beta,\zeta)}(\mu):\zeta<\xi\}$, and define

$$\mathfrak{b}=\left\{\mu\in\mathfrak{a}:\mu=\sup_{\beta<\gamma<\lambda}f^*_\gamma(\mu)\ \text{ for every }\beta<\lambda\right\}.$$

Again, clearly $\mathfrak{a}\backslash\mathfrak{b}\in J$, choose $\mu\in\mathfrak{b}$, $\mu>\mu^*$ such that: $\theta_\mu>\xi\wedge\epsilon\in a_\mu$ and by induction on $\gamma<\mu$ choose $\beta(\gamma)>\cup\{\beta(\gamma'):\gamma'<\gamma\}$ such that $f^*_\gamma(\mu)>\gamma$. By condition (ii) and the choice of $\langle f_\alpha:\alpha<\lambda\rangle$, without loss of generality $f_{\alpha(\beta_\gamma,\zeta)}\restriction\mu$ does not depend on γ.
Now use (iii) (i.e. the choice of c_μ). $\square_{4.1D}$

Claim 4.1E Assume (i), (iv), (v), (vi) of 4.1D and
(ii)′ $J=J^{\mathrm{bd}}_{\mathfrak{a}}$
(iii)′ $\mathrm{Pr}_\ell(\mu,\sigma_\mu,\theta_\mu)$ where $\ell\in\{0,1\}$ is constant
Then $\mathrm{Pr}_\ell(\lambda,\sigma,\theta)$.

Proof: like 4.1A.

Conclusion 4.2 If λ is singular then for some Boolean algebra B, B satisfies the λ^+-c.c. but $B\times B$ does not (also λ^+-L spaces and λ^+-S spaces).

Proof: By the Appendix 1.6A(7).

Definition 4.3 (1) M is a Jonsson algebra if every proper subalgebra has smaller cardinality than the cardinality of M. Usually the universe of M is the cardinality of M and unless stated otherwise $L(M)$ (the language of M) is countable.
(2) λ is a Jonsson cardinal if there is no Jonsson algebra on λ.

We present the known:

Theorem 4.4 *(1) If on λ there is a Jonsson algebra then on λ^+ there is a Jonsson algebra (and on $\aleph_0$ there is a Jonsson algebra).*

(2) If D is a filter on κ, $\kappa\leq\lambda$ for $i<\kappa$, λ_i is a regular cardinal on which there is a Jonsson algebra, $(\forall\mu<\lambda)[\{i:\mu\leq\lambda_i\}\in D]$ and $(\prod_{i<\kappa}\lambda_i/D)$ has true cofinality λ^+, then on λ^+ there is a Jonsson algebra (really the "true" is not necessary).

(3) If λ is the successor of a regular cardinal, then on λ there is a Jonsson algebra.

(4) If on λ there is a Jonsson algebra M with $|L(M)|^+<\lambda$, then on λ there is Jonsson algebra.

Proof: 1) Let M be a Jonsson algebra on λ, and F_0, F_1 be two place functions from λ^+ to λ^+ such that for $\alpha\in[\lambda,\lambda^+)$ let $F_0(\alpha,-)$, $[F_1(\alpha,-)]$ be a one to one function from λ onto α [from α onto λ]. Let N be the model with universe λ^+ with: the functions of M and F_1,F_0 (in the places those are not defined — give zero as value). The rest should be clear and a similar one is done in the proof of 4.5 below.
2) By the proof 4.5 below only: use the ideal dual to D instead of J^{bd}_κ, defining M^+ we expand it also by individual constant λ_j for $j<\kappa$ (no harm done by 4.4(4) below).
3) Let $\lambda=\mu^+$.
Let $\chi>2^\lambda$, M be the model with universe λ and all functions from λ to λ definable in $(H(\chi),\in,<^*_\chi)$. Suppose $N\subseteq M$, $M\neq N$, $\|N\|=\lambda$, so $E=\{\delta<\lambda:\delta=\sup(\delta\cap A)\}$ is a club of λ. In $(H(\chi),\in,<^*_\chi)$ there is a sequence $\bar{C}=\langle C_\delta:\delta<\lambda,\delta\ \text{limit}\rangle$ where C_δ is a club of δ of order type $\mathrm{cf}(\delta)$, such that $[\alpha\in C_\delta\Rightarrow\mathrm{cf}\alpha<\mathrm{cf}\delta]$, $\bar{C}$ definable hence C_δ is definable from δ. So the function $F(\delta,i)=\min(C_\delta\backslash i)$ if $i<\delta$, δ limit, is a function of N. If $\delta\in E$, $\mathrm{cf}\delta=\mu$ then $\min(N\backslash\delta)=\delta$ (otherwise $C_{\min(N\backslash\delta)}\cap\delta$ has cardinality $<\mu$, hence is bounded in δ hence there is $i\in(\sup(C_\delta\cap\delta),\delta)\cap N$, so $F(\min(N\backslash\delta),i)$ is necessarily in $[\min(N\backslash\delta),\delta)$ but this interval is disjoint to N). So it is enough to find a function $h:\lambda\to\lambda$ such that $\{\alpha<\lambda:\mathrm{cf}(\alpha)=\lambda$ and $h(\alpha)=\zeta\}$ is stationary for every $\zeta<\lambda$ (as then there is such h definable in $(H(\chi),\in,<^*_\chi)$). There is such h by a theorem of Solovay; really, as we can restrict ourselves to one cofinality, known earlier from Ulam's proof, i.e.

Observation 4.4B If $\lambda>\mu$ are regular cardinals then for some $h:\lambda\to\lambda$ for every $\zeta<\lambda$ we have $\{\delta<\lambda:h(\delta)=\zeta$ and $\mathrm{cf}(\delta)=\mu\}$ is stationary.

Proof: Why does such h exist? For $\langle C_\delta:\delta<\lambda\rangle$ as above define for each $i<\mu$, $\alpha<\lambda$ the set

$$A^i_\alpha=\{\delta:\ \text{the } i\text{th member of } C_\delta \text{ is } \alpha\}.$$

Clearly $\alpha<\beta<\lambda\Rightarrow A^i_\alpha\cap A^i_\beta=\emptyset$ hence it suffice to find i such that

$$B_i = \{\alpha < \lambda : A^i_\alpha \text{ is a stationary subset of } \lambda\}$$

has λ members. If this fails, then $i < \mu \Rightarrow |B_i| < \lambda$ hence

$$|\bigcup_{i<\mu} B_i| \le \mu \times \mu < \lambda$$

so $\bigcup_{i<\mu} B_i$ is bounded by say α^*; so for $i < \mu$, $\alpha \in (\alpha^*, \lambda)$ there is a club $E_{i,\alpha}$ of λ disjoint to A^i_α; for $i < \mu$, $\alpha \le \alpha^*$, let $E_{i,\alpha} = \lambda$. Hence

$$E = \{\delta : \delta \text{ a limit ordinal } < \lambda,\ \delta > \alpha^* \text{ and } i < \mu \ \&\ \alpha < \delta \Rightarrow \delta \in E_{i,\alpha}\}$$

is a club of λ; let $\delta \in E$, $\mathrm{cf}(\delta) = \mu$, now $\delta = \sup C_\delta$ so there is $\alpha \in C_\delta$, $\alpha > \alpha^*$, let $i = \mathrm{otp}(\alpha \cap C_\delta)$, so $\delta \in A^i_\alpha$, but $\delta \in E_{i,\alpha}$, contradiction. $\square_{4.4B}$

Continuation of the Proof of 4.4: (4) Define a model N, with universe λ, $|L(N)| = \aleph_0$, $<^N$ the usual order, N has Skolem functions and (letting $\kappa = \aleph_0 + |L(M)|$, $\langle f^n_i : i < \alpha_n \le \kappa\rangle$ be the n-place functions of M), and F^N_n is the $(n+1)$-place function defined by:

$$F^N_{n+1}(i, \alpha_1, \ldots, \alpha_n) = \begin{cases} f^n_i(\alpha_1, \ldots, \alpha_n) & \text{if } i < \kappa \\ \kappa^+ & \text{if } i \ge \kappa \end{cases}.$$

Without loss of generality $N{\restriction}\kappa^{++}$ is a Jonsson algebra (by part (3), note that $\lambda \ge \kappa^{++}$) and κ, κ^+, κ^{++} individual constants. If $N' \subseteq N$, $N' \ne N$ and $\|N'\| = \lambda$ then necessarily $\kappa \not\subseteq N'$ (otherwise we contradict "M is a Jonsson algebra"), hence $\kappa^{++} \not\subseteq N'$, hence $\sup(N' \cap \kappa^{++}) < \kappa^{++}$. Let N'' be the Skolem Hull of $N' \cup \{i : i < \kappa\}$; so easily $\sup(N' \cap \kappa^{++}) = \sup(N'' \cap \kappa^{++})$ so N' contradicts "M is a Jonsson algebra". $\square_{4.4}$

Conclusion 4.5 The first regular Jonsson cardinal is a limit cardinal.

Proof: Suppose not, so the first regular Jonsson cardinal is a successor, say λ^+, by 4.4(3) λ is singular. Let $\kappa = \mathrm{cf}\lambda$. We can easily find a model M such that:

(a) M has universe λ^+ and countable vocabulary.

(b) if $\mu < \lambda^+$ and on μ there is a Jonsson algebra, N a submodel of M, $\mu \in N$ & $|N \cap \mu| = \mu$, <u>then</u> $\mu \subseteq M$.

(c) for some function symbols f, g, for every $\alpha < \lambda^+$, $f(\alpha, -)$ is a one to one function from $|\alpha|$ onto α and $g(\alpha, f(\alpha, i)) = i$ for $i < |\alpha|$.

By 1.5 there is a strictly increasing sequence $\langle \lambda_i : i < \kappa = \mathrm{cf}(\lambda)\rangle$ of regular cardinals with limit λ such that $\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa$ has true cofinality λ^+ and let $\langle f_\alpha : \alpha < \lambda^+\rangle$ exemplify this. Let $J = J^{\mathrm{bd}}_\kappa$.
Let M^+ be M expanded by two functions:

(i) H, two place such that $H(\alpha, i) = f_\alpha(i)$ for $\alpha < \lambda^+$, $i < \kappa$.

(ii) h, one place such that $h(\alpha) = \min\{\lambda_i : i < \kappa, \lambda_i > \alpha\}$ if the minimum exists.

Suppose N is a submodel of M^+ of cardinality λ^+. Necessarily $|N|$ is unbounded in λ^+ hence for some $\alpha < \lambda^+$, $|N \cap \alpha| = \lambda$, so without loss of generality $\alpha \in N$. Now $f(\alpha, -)$ is a one to one function from α to λ, so

$$|N \cap \lambda| \ge |\{f(\alpha, i) : i \in N \cap \alpha\}| = \lambda.$$

Hence N is unbounded in λ and using h we see $A = \{i < \kappa : \lambda_i \in N\}$ is unbounded in κ. Let $B = \{i \in A : \lambda_i = \sup(N \cap \lambda_i)\}$.

So using H, h and the choice of $\langle f_\alpha : \alpha < \lambda^+\rangle$ we have: $B \ne \emptyset \bmod J$ hence is unbounded in κ (i.e. let $f \in \prod_{i<\kappa} \lambda_i$ be: $f(i)$ is $\sup(N \cap \lambda_i)$ if $\sup(N \cap \lambda_i) < \lambda_i$, $f(i)$ is zero otherwise. Let $\beta < \lambda^+$ be such that $f < f_\beta \bmod J$ and without loss of generality $\beta \in N$, hence $\bigwedge_{i \in A} f_\beta(i) \in N$ and the contradiction is easy).

For every $i \in B$, $\lambda_i = \sup(N \cap \lambda_i)$ so as λ_i is regular $\lambda_i = |N \cap \lambda_i|$. As $\lambda_i \in N$ (as $B \subseteq A$) by (b) above, $\lambda_i \subseteq N$. So $\bigcup_{i \in B} \lambda_i \subseteq N$; but B is an unbounded subset of κ, $\langle \lambda_i : i < \kappa\rangle$ increasing with limit λ, hence $\lambda = \bigcup_{i \in B} \lambda_i$, so $\lambda \subseteq N$. Now for every $\alpha < \lambda^+$ for some $\beta \in N$, $\alpha < \beta$ & $\lambda < \beta$; so $f(\beta, \alpha)$ is well defined and $< \lambda$ hence $f(\beta, \alpha) \in N$. As β, $f(\beta, \alpha)$ are in N, so is $\alpha = g(\beta, f(\beta, \alpha))$. As $\alpha < \lambda^+$ was arbitrary we have $\lambda^+ \subseteq N$, as required. $\square_{4.5}$

Conclusion 4.6 Suppose that λ is singular and

(α) $\lambda <$ first inaccessible

or

(β) $\{\mu : \mu < \lambda$ is a (weakly) inaccessible Jonsson cardinal$\}$ is bounded in λ

or

(γ) for some regular non-Jonsson cardinals $\lambda_i < \lambda$ (for $i < \mathrm{cf}(\lambda) = \kappa$),

$$\lambda = \bigcup_{i<\kappa} \lambda_i \text{ and } \mathrm{tcf}\Big(\prod_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa}\Big) = \lambda^+$$

or

(δ) for no $\mu(*) < \lambda$, $\lambda = \sum_{i<\mathrm{cf}\lambda} \lambda_i$, $\lambda_i < \lambda$, λ_i is inaccessible do we have:

$$(\lambda_i, \mu^+) \to (\lambda_i, \mu) \text{ when } i < \mathrm{cf}(\lambda) \text{ and } \mu \in (\mu(*), \lambda_i) \text{ is regular}$$

or

(ϵ) $\{\mu < \lambda : \mu^+$ satisfies one of (α) – (δ) (or just has a Jonsson algebra)$\}$ is stationary in λ.

Then on λ^+ there is a Jonsson algebra.

Proof: Note that for each λ we have $(\alpha) \Rightarrow (\beta) \Rightarrow (\gamma)$: Why? $(\alpha) \Rightarrow (\beta)$ trivially; $(\beta) \Rightarrow (\gamma)$ we prove by induction on λ, in the induction step use 1.5 + $[\mathfrak{a} \subseteq \mathrm{pcf}(\mathfrak{b})$ & $|\mathfrak{b}| < \min \mathfrak{b}$ & $|\mathfrak{a}| < \min \mathfrak{a} \Rightarrow \mathrm{pcf}(\mathfrak{a}) \subseteq \mathrm{pcf}(\mathfrak{b})]$ — (see [Sh345a,1.12] and 4.4(3).

We prove the statements by induction on λ. Now use 4.4(2) (for (γ) this is all, and for (ϵ) use 2.1, for (δ) repeat the proof of 4.5). $\square_{4.6}$

True book readers should ignore meanwhile the following conclusion as it depends on later chapters (or even worse: on outside work).

Conclusion 4.7 $\mathrm{Pr}_0(\lambda^+, \lambda^+, \aleph_0)$ (see the Appendix, Definition 1.1, 1.5) provided that λ regular $> \aleph_1$ or is singular and $(\alpha) \vee (\beta)$ of 4.6 holds or λ is singular and at least one of the following hold:

(α)′ $\lambda <$ first inaccessible

(β)′ $\{\mu : \mu < \lambda$ is weakly inaccessible and $\neg\mathrm{Pr}_0(\mu, \mu, \aleph_0)\}$ is bounded below λ.

(γ)′ for some regular λ_i ($i < \mathrm{cf}\lambda = \kappa$)

$$\lambda = \bigcup_{i<\kappa} \lambda_i,$$

$$\lambda^+ = \mathrm{tcf}(\prod_{i<\kappa} \lambda_i / J^{\mathrm{bd}}_\kappa) \text{ and}$$

$$\mathrm{Pr}_0(\lambda_i, \lambda_i, \aleph_0)$$

(δ)′ $\{\mu < \lambda : \mathrm{Pr}_0(\mu^+, \mu^+, \aleph_0)\}$ is stationary.

In case (γ)′ we can replace λ^+ by any regular. We can also replace Pr_0 by Pr_1.

Proof: We prove this by induction on λ.
If $\lambda > \aleph_1$ is regular use [Sh365,4.8(1)]. Let λ be singular. In order to have $\mathrm{Pr}_0(\lambda^+, \lambda^+, \aleph_0)$ use the following: our Conclusion 1.5 and 4.8 below. $\square_{4.7}$

Fact 4.8 Suppose $\tau \leq \mathrm{cf}\delta$, $\langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals, $\delta < \lambda_0$ for $i < \delta$ $\mathrm{Pr}_0(\lambda_i, \sigma_i, \tau_i)$, the sequences $\langle \sigma_i : i < \delta \rangle$, $\langle \tau_i : i < \delta \rangle$ are non-decreasing, $\tau_i < \delta < \sigma_i \leq \lambda_i$, $\sigma = \bigcup_{i<\delta} \sigma_i$, $\tau = \bigcup_{i<\delta} \tau_i$ and $\mathrm{tcf}\left(\prod_{i<\delta} \lambda_i, <_{J^{\mathrm{bd}}_\delta}\right) = \lambda$. Then

(i) $\mathrm{Pr}_0(\lambda, \sigma, \tau)$ where $\sigma = \bigcup_{i<\delta} \sigma_i$, $\tau = \bigcup_{i<\delta} \tau_i$

(ii) if $\lambda = \mu^+$, $\mu = \mu^{<\tau} = \bigcup_{i<\delta} \sigma_i = \bigcup_{i<\delta} \lambda_i$ and $\tau = \bigcup_{i<\delta} \tau_i$ then $\mathrm{Pr}_0(\lambda, \lambda, \tau)$.

Remark 4.8A (1) If θ is a limit cardinal $\leq \kappa$, $\langle \theta_i : i < \delta \rangle$ increasing with limit $\theta \leq \mathrm{cf}\delta$, and we assume $\mathrm{Pr}_0(\lambda_i, \sigma, \theta_i)$ we can get $\mathrm{Pr}_0(\lambda^+, \sigma, \theta)$.

(2) We can in 4.8 replace Pr_0 by Pr_1 (in the assumption and the conclusion). The proof is the same, except the natural change in the choice of e_i.

(3) We can use σ_i an ordinal, and note: $\mathrm{Pr}_\ell(\lambda, \sigma, \tau)$ iff $\mathrm{Pr}_\ell(\lambda, |\sigma|, \tau)$.

Proof: Let $\mathfrak{c} = \{\lambda_i : i < \delta\}$.

Let $\langle f_\alpha : \alpha < \lambda \rangle$ exemplify $\mathrm{tcf}(\prod_{i<\delta} \lambda_i, <_{J^{\mathrm{bd}}_\delta}) = \lambda$ and let us define for $\alpha < \beta < \lambda$ an ordinal $\theta(\alpha, \beta)$: it is the maximal cardinal θ in $\mathfrak{c}$ such that $f_\alpha(\theta) \geq f_\beta(\theta)$ if there is such θ and undefined otherwise. Next let e_i (for $i < \delta$) be a two place symmetric function from λ_i to σ_i exemplifying $\mathrm{Pr}_0(\lambda_i, \sigma_i, \tau_i)$. For proving (i) define a two place function e from λ^+ to σ by: for $\alpha < \beta < \lambda^+$:

$$e(\alpha, \beta) = \begin{cases} e_i(f_\alpha(i), f_\beta(i)) & \text{if } \lambda_i = \theta(\alpha, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

For proving (ii) let for $\beta < \lambda$, g_β be a function from μ onto β; and we let for $\alpha < \beta < \lambda$:

$$e(\alpha, \beta) = \begin{cases} g_\beta[e_i(f_\beta(i), f_\alpha(i))] & \text{if } \lambda_i = \theta(\alpha, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

For proving (i) repeat the proof of 4.1D.

For proving (ii) we need a short preliminary argument.

Now suppose for $\beta < \lambda$ the sequence $\langle \alpha_{\beta,\zeta} : \zeta < \zeta^* \rangle$ is increasing (where $\zeta^* < \tau$), and for any $\beta_1 \neq \beta_2$ the sequences are disjoint. Let h be a two place function from $\{\zeta : \zeta < \zeta^*\}$ to λ. Without loss of generality $\alpha_{\beta,\zeta} > \max \mathrm{Rang}\, h$ and $\alpha_{\beta,\zeta} > \mu$. For each $\beta < \lambda, \xi < \zeta^*$ and $\zeta < \zeta^*$ let $\gamma_{\beta,\xi,\zeta} < \mu$ be such that $g_{\alpha_{\beta,\zeta}}(\gamma_{\beta,\xi,\zeta}) = h(\xi, \zeta)$. So, as $\mu = \mu^{<\tau} \geq \mu^{|\zeta^*|}$ without loss of generality for some $\gamma_{\xi,\zeta}$ for every β we have $\gamma_{\beta,\xi,\zeta} = \gamma_{\xi,\zeta}$, and let $i(*) < \delta$ be such that $\bigwedge_{\xi,\zeta} \gamma_{\xi,\zeta} < \lambda_{i(*)}$ and $\zeta^* < \tau_{i(*)}$. The rest is as previously by the proof of 4.1D. $\square_{4.8}$

Lemma 4.9 *Suppose λ is singular then* $\mathrm{Ens}(\lambda^+, \mathrm{cf}\,\lambda)$ *where:*

Definition 4.10 (1) $\mathrm{Ens}(\lambda, \mu, \kappa)$ means: there are linear orderings $\langle \mathcal{I}_\alpha : \alpha < \kappa \rangle$ witnessing it, which means:

(a) $\mathcal{I}_\alpha$ is a linear order of power λ

(b) if $n < \omega$ $\alpha_1 < \cdots < \alpha_n < \kappa$, $w \subseteq \{1, ..., n\}$, $t^\ell_\zeta \in \mathcal{I}_{\alpha_\ell}$ for $\zeta < \mu$, $\ell = 1, ..., n$ and $[\zeta_1 \neq \zeta_2 \Rightarrow t^\ell_{\zeta_1} \neq t^\ell_{\zeta_2}]$ then for some $\zeta < \xi < \mu$,

$$[\ell \in w \Rightarrow \mathcal{I}_{\alpha_\ell} \models t^\ell_\zeta < t^\ell_\xi]$$

$$[1 \le \ell \le n \wedge \ell \notin w \Rightarrow \mathcal{I}_{\alpha_\ell} \models t^\ell_\zeta > t^\ell_\xi].$$

(1A) In this case we call $\langle \mathcal{I}_\alpha : \alpha < \kappa \rangle$ a (λ, μ)-entangled sequence (of linear orders).

(2) $\mathrm{Ens}(\lambda, \mu, k)$ is defined similarly but $n \le k$. $\mathrm{Ens}_\kappa(\lambda, \mu))$ means $\mathrm{Ens}(\lambda, \mu, \kappa)$.

(3) If we omit μ this means $\lambda = \mu$.

(4) A linear order $\mathcal{I}$ is (μ, n)-entangled if: for every pairwise distinct $t^\ell_\zeta \in \mathcal{I}$ $(1 \le \ell \le n, \zeta < \mu)$ such that $t^1_\zeta < t^2_\zeta < \cdots < t^n_\zeta$ and $w \subseteq \{1, ..., n\}$ <u>there are</u> $\zeta < \xi < \mu$ such that:

(*) $1 \le \ell \le n \Rightarrow [\ell \in w \Leftrightarrow \mathcal{I} \models t^\ell_\zeta < t^\ell_\xi]$.

(5) We omit μ if $|\mathcal{I}| = \mu$, we omit n if $\mathcal{I}$ is (μ, n)-entangled for all $n < \omega$.

(6) The linear orders $\mathcal{I}_0, \mathcal{I}_1$ are κ-far if they have no isomorphic subsets of cardinality κ.

(7) $\mathcal{I}^*$ is the inverse of $\mathcal{I}$.

We shall prove 4.9 later.

Fact 4.10A Suppose $\langle \lambda_i : i < \delta \rangle$ is strictly increasing sequence of regular cardinals, $\bigwedge_{i<\delta} \lambda_i < \lambda = \mathrm{cf}(\lambda)$, $\lambda_i > |\delta|$, D a filter on δ containing all cobounded subsets, $\mathrm{tcf}(\prod_{i<\delta} \lambda_i / D) = \lambda$ i.e. there is $\{f_\alpha : \alpha < \lambda\} \subseteq \prod_{I<\delta} \lambda_i$ such that:

(i) $\alpha < \beta < \lambda \Rightarrow f_\alpha <_D f_\beta$

(ii) $(\forall f \in \prod_{i<\delta} \lambda_i)\,(\exists \alpha < \lambda)(f <_D f_\alpha)$.

Suppose $A_i \subseteq \delta (i < \kappa)$ are such that: in $\mathcal{P}(\delta)/D$, $\{A_i : i < \kappa\}$ is independent and for $i < \delta$, $|\{f_\alpha \restriction i : \alpha < \lambda\}| < \lambda_i$. <u>Then</u> $\mathrm{Ens}(\lambda, \kappa)$.

Proof: Let $\mathcal{I} = \{f_\alpha : \alpha < \lambda\}$. For each $\zeta < \kappa$ we define a linear order $<^*_\zeta$ of $\mathcal{I}$:

$f_\alpha <^*_\zeta f_\beta$ <u>iff</u> for some $i < \delta$:

$$f_\alpha(i) \neq f_\beta(i) \ \&\ f_\alpha \restriction i = f_\beta \restriction i \ \&\ [f_\alpha(i) < f_\beta(i) \iff i \in A_\zeta].$$

Let $n < \omega$, $\zeta_1 < \cdots < \zeta_n < \kappa$, $t^\ell_\gamma = f_{\alpha(\ell,\gamma)}$ be pairwise distinct for $\ell = 1, ..., n$ and $\gamma < \lambda$; and let $w \subseteq \{1, ..., n\}$; we should find $\zeta < \xi < \lambda$ as in 4.10(1)(b). Let

$$g_\gamma(i) =: \min\{f_{\alpha(\ell,\gamma)}(i) : \ell \in \{1, ..., n\}\},$$

$$i_\gamma =: \min\{i : \langle f_{\alpha(\ell,\gamma)} \restriction i : \ell \in \{1, ..., n\}\rangle \text{ are pairwise distinct }\}.$$

Without loss of generality $i_\gamma = i^*$ for every $\gamma < \lambda$.

Let
$B = \{i < \delta :$ for every $\xi < \lambda_i$, there are λ ordinals $\gamma < \lambda$ such that $g_\gamma(i) > \xi\}$.

We shall prove

Claim 4.10B $B \in D$.

Proof of Claim 4.10B: Suppose that $B \notin D$. Then, since

$$D = \cap\{F : D \subseteq F \text{ and } F \text{ is an ultrafilter on } \delta\},$$

there is an ultrafilter F on δ such that $B \notin F$ and $D \subseteq F$. So $C =: \delta \backslash B \in F$. From the definition of B,

$$(\forall i \in C)(\exists \xi_i < \lambda_i)(\exists \gamma_i < \lambda)(\forall \gamma)(\gamma_i \le \gamma < \lambda \Rightarrow g_\gamma(i) \le \xi_i).$$

Let, for $i \in C$, ξ_i, γ_i be as stated. Define $h \in \prod_{i<\delta} \lambda_i$ by:

$$h(i) =: \begin{cases} \xi_i + 1 & \text{if } i \in C \\ 0 & \text{if } i \notin C \end{cases}$$

$\langle f_\alpha / D : \alpha < \lambda \rangle$ is cofinal in $\prod_{i<\delta} \lambda_i / D$, hence $\langle f_\alpha / F : \alpha < \lambda \rangle$ is cofinal in $\prod_{i<\delta} \lambda_i / F$, so there exists $\beta < \lambda$ such that:

$$h < f_\beta \bmod F.$$

Without loss of generality $\bigcup_{i\in C} \gamma_i < \beta$(since $C \subseteq \delta, |\delta| < \lambda = \mathrm{cf}(\lambda)$ and $\bigwedge_{i\in C}(\gamma_i < \lambda)$). Since $\alpha(\ell, \zeta)$, $(1 \le \ell \le n, \zeta < \lambda)$ are pairwise distinct, and $\beta < \lambda$, there exists $\zeta < \lambda$ such that $\bigwedge^n_{\ell=1}[\alpha(\ell, \zeta) > \beta]$. Without loss of generality $\bigcup_{i\in C} \gamma_i < \zeta$.
So $\bigwedge^n_{\ell=1} (f_\beta < f_{\alpha(\ell,\zeta)} \bmod F)$. That means

$$E =: \left\{ i < \delta : \bigwedge_{\ell=1}^{n} f_\beta(i) < f_{\alpha(\ell,\zeta)}(i) \right\} \in F.$$

So

$$E = \{i < \delta : f_\beta(i) < g_\zeta(i)\} \in F,$$

using the definition of g_ζ. Since $h < f_\beta \bmod F$, it now follows that

$$\{i < \delta : h(i) < g_\zeta(i)\} \in F$$

and so

$$C \cap \{i < \delta : h(i) < g_\zeta(i)\} \in F.$$

Choosing i in this (non-empty) intersection, one obtains

$$g_\zeta(i) \le \xi_i < \xi_i + 1 = h(i) < g_\zeta(i)$$

— a contradiction. So $B \in D$, proving the claim. $\square_{4.10B}$

Continuation of the Proof of 4.10A: Now choose $i < \delta$ as follows. First note that since $|\{f_\alpha \restriction i : \alpha < \lambda\}| < \lambda_i$ for each $i < \delta$, and $\mathrm{cf}(\prod_{i<\delta} \lambda_i / D) = \lambda$, D cannot contain any bounded subsets of δ. By a hypothesis,

$$A =: \bigcap_{\ell \in w} A_{\zeta_\ell} \cap \bigcap_{\ell \notin w} (\delta \backslash A_{\zeta_\ell}) \neq \emptyset \bmod D,$$

so $\delta \backslash A \notin D$ and there exists an ultrafilter F on δ such that $D \subseteq F$ and $A \in F$. Hence

$$C =: \{i < \delta : i^* < i\} \cap A \cap B \in F$$

and one can choose $i \in C$.
So we have chosen i:

$$i^* < i \in B \cap \bigcap_{\ell \in w} A_{\zeta_\ell} \cap \bigcap_{1 \le \ell \le n, \ell \notin w} (\delta \backslash A_{\zeta_\ell}).$$

For each $\xi < \lambda_i$ choose γ_ξ such that $g_{\gamma_\xi}(i) > \xi$. For some unbounded $S \subseteq \lambda_i$ we have: $\xi_1 < \xi_2 \in S \Rightarrow \bigwedge_{\ell,m} f_{\alpha(\ell,\gamma_{\xi_1})}(i) < f_{\alpha(m,\gamma_{\xi_2})}(i)$. Without loss of generality $\langle f_{\alpha(\ell,\gamma_\xi)} \restriction i : \xi \in S \rangle$ is constant (by a hypothesis). The conclusion should be clear now. $\square_{4.10}$

Fact 4.10C If $\langle \lambda_i : i < \delta \rangle$ is a strictly increasing sequence of regular cardinals, $\bigwedge_{i<\delta} \lambda_i < \lambda = \mathrm{cf}\,\lambda$, $\lambda_i > |\delta|$, D an ultrafilter on δ containing the cobounded subsets of δ, $\mathrm{tcf}(\prod_{i<\delta} \lambda_i / D) = \lambda$, and there is $\langle f_\alpha / D : \alpha < \lambda \rangle$ $<_D$-increasing cofinal in $\prod_{i<\delta} \lambda_i / D$ such that for $i < \delta$ we have $\mu_i =: |\{f_\alpha \restriction i : \alpha < \lambda\}| < \lambda_i$ and $\mathrm{Ens}(\lambda_i, \mu_i)$, <u>then</u> there is an entangled linear order of power λ.

Proof: Let $\langle f_\alpha : \alpha < \lambda \rangle$ be as mentioned above. Let $\langle \mathcal{I}^i_\eta : \eta \in \prod_i \rangle$ where $\prod_i = \{f_\alpha \restriction i : \alpha < \lambda\}$, witness $\mathrm{Ens}(\lambda_i, \mu_i)$; without loss of generality $\mathcal{I}^i_\eta$ has universe λ_i.
Define $<^*$ on $\mathcal{I} =: \{f_\alpha : \alpha < \lambda\}$:
$f_\alpha <^* f_\beta$ <u>iff</u> there is $i < \delta$ such that:

$$f_\alpha \restriction i = f_\beta \restriction i$$

$$\mathcal{I}^i_{f_\alpha \restriction i} \models f_\alpha(i) < f_\beta(i).$$

Clearly $<^*$ linearly orders $\mathcal{I}$, and $\mathcal{I}$ has cardinality λ.
Proving $\mathcal{I}$ is as required, is easy, choosing $i \in \{i < \delta : i^* < i\} \cap B$ and $S \subseteq \lambda_i$ in the notation of the proof of 4.10A. $\square_{4.10C}$

Remark 4.10D So we have another way to get:
if $\lambda = \beth_\lambda > \mathrm{cf}\,\lambda$, then for some regular $\kappa \in (\lambda, 2^\lambda]$ there is an entangled order of cardinality κ.

Fact 4.10E Suppose $\langle \lambda_i : i < \delta \rangle$ is strictly increasing, D a filter on δ containing the cobounded filter on δ, $\mathrm{tcf}(\prod \lambda_i / D) = \lambda$, $\mu < |\delta| = \delta < \lambda_0$, $\mu < \lambda_0 < \bigcup_{i<\delta} \lambda_i < \mathrm{Ded}\,\mu$, $2^\mu < \lambda$. <u>Then</u> $\mathrm{Ens}_2(|\delta|, \lambda)$ (remember that $\mathrm{Ded}\,\mu = \bigcup\{|\mathcal{I}|^+ : \mathcal{I}$ a linear order with a dense subset of cardinality $\le \mu\}$).

Proof: Let $\mathcal{J}$ be a dense linear order of power $\bigcup_{i<\delta} \lambda_i$ with a dense subset $\mathcal{I}$ of power μ. Let $t^i_\zeta (i < \delta, \zeta < \lambda_i)$ be distinct members of $\mathcal{J}$. Let $\langle f_\alpha : \alpha < \lambda \rangle$ witness $\mathrm{tcf}(\prod_{i<\delta} \lambda_i / D) = \lambda$. For each α let $\mathcal{I}_\alpha = \{t^i_{f_\alpha(i)} : i < \delta\}$. For $\alpha < \lambda$ let $A_\alpha =: \{\beta : \mathcal{I}_\alpha, \mathcal{I}_\beta$ are not $|\delta|$-far or $\mathcal{I}_\alpha, \mathcal{I}^*_\beta$ are not $|\delta|$-far$\}$. We shall prove that $|A_\alpha| < \lambda$, suppose not. Now for each $\beta \in A_\alpha$ there are $K_{\alpha,\beta} \subseteq \mathcal{I}_\alpha$, $L_{\alpha,\beta} \subseteq \mathcal{I}_\beta$ each of power δ and $h_{\alpha,\beta}$ an isomorphism or anti-isomorphism from $K_{\alpha,\beta}$ onto $L_{\alpha,\beta}$; let $M_{\alpha,\beta}$ be a dense subset of $K_{\alpha,\beta}$ of power $\le \mu$ such that if $x \in \mathcal{I}$ and $\min\{y \in K_{\alpha,\beta} : y > x\}$ is well defined then it is in $M_{\alpha,\beta}$; similarly with $\max\{y \in K_{\alpha,\beta} : y < x\}$; similarly for $h''_{\alpha,\beta}(M_{\alpha,\beta})$, $L_{\alpha,\beta}$ [possible as $|\mathcal{I}| \le \mu$].
Assume $|A_\alpha| = \lambda$. As $2^\mu < \lambda$ for some $A'_\alpha \subseteq A_\alpha$, $|A'_\alpha| = \lambda$ and for some M^*_α, h_α we have: $[\beta \in A'_\alpha \Rightarrow M_{\alpha,\beta} = M^*_\alpha \;\&\; h_{\alpha,\beta} \restriction M^*_\alpha = h_\alpha]$. Essentially h_α defines uniquely $h_{\alpha,\beta}(x)$ where $x \in \mathrm{Dom}\,h_{\alpha,\beta}$. More fully, let
$\mathcal{I}^\alpha =: \{x \in \mathcal{I}_\alpha :$ there is $y \in \mathcal{J}$, x, y are single in the Dedekind cut each realizes over M^*_α, $h''_\alpha(M^*_\alpha)$ respectively, and $(\forall z \in M^*_\alpha)[z < y \equiv h_\alpha(z) < x]\}$.
Now $[\beta \in A'_\alpha \Rightarrow \mathrm{Dom}\,h_{\alpha,\beta} \subseteq \mathcal{I}^\alpha \subseteq \mathcal{I}_\alpha]$ and $h^\alpha =: \bigcup_{\beta \in A'_\alpha} h_{\alpha,\beta}$ is a function from $\mathcal{I}^\alpha$ into $\mathcal{J}$.
Now define $g^\alpha \in \prod_{i<\delta} \lambda_i : g^\alpha(i) = \sup\{\zeta < \lambda_i : t^i_\zeta \in \mathrm{Rang}(h^\alpha)\}$, $g^\alpha(i) < \lambda_i$ as $|\mathrm{Rang}\,h^\alpha| = \mathrm{Dom}\,h^\alpha = |\mathcal{I}^\alpha| \le |\mathcal{I}_\alpha| \le |\delta| < \lambda_0 \le \lambda_i$ so $[\beta \in A'_\alpha \Rightarrow f_\beta \le g^\alpha]$. But $|A'_\beta| = \lambda$; contradiction.
Hence for each $\alpha < \lambda$ we have $|A_\alpha| < \lambda$, so we can find an $A^* \subseteq \lambda$ such that A^* unbounded in λ and:

$$\alpha < \beta \;\&\; \alpha \in A^* \;\&\; \beta \in A^* \Rightarrow \beta \notin A_\alpha.$$

I.e. we have λ linear orders, each of power $\delta > \mu$, any two are $|\delta|$-far and even one is $|\delta|$-far to the inverse of the other. By 4.10(2) we finish. $\square_{4.10E}$

Claim 4.10F In Fact 4.10A suppose in addition μ is a limit cardinal, $\prod_{i<\delta} \lambda_i \ge \mu \ge \mathrm{cf}\mu = \lambda$. <u>Then</u>

(1) $\mathrm{Ens}(\mu, \kappa)$.
(2) Moreover, there are $\langle \mathcal{I}_\zeta : 1 + \zeta < \kappa \rangle$ exemplifying $\mathrm{Ens}(\mu, \kappa)$ such that:
  (a) for each $\theta < \mu$ there is a linear order of power θ embeddable in every $\mathcal{I}_\zeta$.

(b) each $\mathcal{I}_\zeta$ has dense subset of cardinality $\sum_{i<\delta} \lambda_i < \mu$.

Proof: 1) Let $\mu = \bigcup_{\alpha<\lambda} \mu_\alpha$, $\mu_\alpha < \mu$, $[\alpha < \beta \Rightarrow \mu_\alpha < \mu_\beta]$ and $\langle f_\alpha/D : \alpha < \lambda\rangle$ be $<_D$-increasing and cofinal in $\prod \lambda_i/D$ and be as in 4.10A. So for each α, as $\prod_{i<\delta}\{\zeta : f_\alpha(i) < \zeta < \lambda_i\}$ has cardinality $\prod_{i<\delta} \lambda_i \geq \mu$, it has a subset F_α of cardinality μ_α^+; as $\langle f_\alpha/D : \alpha < \lambda\rangle$ is cofinal in $\prod_{i<\delta} \lambda_i/D$, for some $\gamma_\alpha < \lambda$,

$$F'_\alpha =: \{g \in F_\alpha : g/D < f_{\gamma_\alpha}/D\} \text{ has power } \geq \mu_\alpha,$$

and without loss of generality $\gamma_\alpha = \alpha + 1$. Let $\mathcal{I} = \bigcup_{\alpha<\lambda} F'_\alpha$ and proceed as before (in 4.10A).

2) Without loss of generality $A =: \bigcap_{\zeta<\kappa} A_\zeta$ is such that $\prod_{i\in A} \lambda_i \geq \mu$. [Why? Let us use $\langle A'_\zeta : \zeta < \kappa\rangle$ where: $A'_\zeta =: A_0 \cup A_{1+\zeta}$ if $\prod_{i\in A_0} \lambda_i \geq \mu$ and $A'_\zeta = (\delta\backslash A_0) \cup A_{1+\zeta}$ if $\prod_{i\in A_0} \lambda_i < \mu$]. Now, as above, we can choose $F_\alpha \subseteq \prod \lambda_i$ such that:

(i) $|F_\alpha| = \mu_\alpha$

(ii) for some $\gamma_\alpha < \lambda$, $g \in F_\alpha \Rightarrow f_\alpha \leq g \leq_D f_{\gamma_\alpha}$

(iii) $g, h \in F_\alpha \Rightarrow g\restriction(\delta\backslash A) = h\restriction(\delta\backslash A)$.

So on F_α all orders $<^*_\zeta$ are the same, and so $\langle(\bigcup_{\alpha<\lambda} F_\alpha, <^*_\zeta) : \zeta < \kappa\rangle$ are as required. $\square_{4.10F}$

Fact 4.10G In 4.10E, suppose in addition $\mathrm{cf}\chi = \mathrm{cf}\delta < \chi \leq \bigcup_{i<\delta} \lambda_i$. <u>Then</u> we can find $\langle \mathcal{I}_\zeta : \zeta < \lambda\rangle$ such that:

(a) $\mathcal{I}_\zeta$ is a linear order of power χ with a dense subset of power μ.

(b) The linear orders $\{\mathcal{I}_\zeta : \zeta < \lambda\}$ are pairwise far (and $\mathcal{I}_\zeta, \mathcal{I}^*_\xi$ are).

Proof: Use 4.10E, $D = \{A \subseteq \delta : \delta\backslash A \text{ is bounded}\}$, $\chi = \sum_{i<\delta} \chi_i$, $\chi_i > \sum_{j<i} \chi_j$; replace t^i_ζ by χ_i elements. $\square_{4.10G}$

Proof of 4.9: By 1.5 there is $\bar\lambda = \langle \lambda_i : i < \mathrm{cf}\ \lambda\rangle$ a strictly increasing sequence of regular cardinals each $> \mathrm{cf}(\lambda)$, $\lambda = \sum_{i<\mathrm{cf}\lambda} \lambda_i$, and

$$\lambda^+ = \mathrm{tcf}\Big(\prod_{i<\mathrm{cf}\lambda} \lambda_i, <_{J^{\mathrm{bd}}_{\mathrm{cf}\lambda}}\Big).$$

<u>Case I:</u> For some unbounded $A \subseteq \mathrm{cf}(\lambda)$ for every $i < \mathrm{cf}\ \lambda$,

$$\lambda_i > \max \mathrm{pcf}\{\lambda_j : j \in i \cap A\}.$$

We have $\langle f_\alpha : \alpha < \lambda^+\rangle$ with $f_\alpha \in \prod_{i\in A} \lambda_i$ and $\alpha \neq \beta \Rightarrow f_\alpha \neq f_\beta$, but $|\{f_\alpha\restriction i : \alpha < \lambda^+\}| < \lambda_i$, by Conclusion 3.5. By Appendix 1.7, there is a sequence $\langle A_i : i < 2^{\mathrm{cf}(\lambda)}\rangle$ of subsets of $\mathrm{cf}(\lambda)$ independent modulo $J^{\mathrm{bd}}_{\mathrm{cf}(\lambda)}$. By 4.10A we know that $\mathrm{Ens}(\lambda^+, 2^{\mathrm{cf}(\lambda)})$ which is more than enough.

<u>Case II:</u> Not Case I. Let $\mu = \lambda^+$, $\kappa = \mathrm{cf}\ \lambda$.

Then, by Claim 3.3(2), we can choose by induction on $\alpha < \kappa = \mathrm{cf}(\lambda)$, $\mathfrak{a}_\alpha \subseteq \{\lambda_i : i < \mathrm{cf}\ \lambda\}$ such that:

(i) $\mathfrak{a}_\alpha$ is a set of regular cardinals,

(ii) $\sup\left(\bigcup_{\beta<\alpha} \mathfrak{a}_\beta\right) < \min\ \mathfrak{a}_\alpha$

(iii) $\sup \mathfrak{a}_\alpha < \lambda$

(iv) $\lambda^+ = \max \mathrm{pcf}(\mathfrak{a}_\alpha)$ and $J^{\mathrm{bd}}_{\mathfrak{a}_\alpha} \subseteq J_{<\lambda^+}[\mathfrak{a}_\alpha]$

(v) for $\theta \in \mathfrak{a}_\alpha$, $\max \mathrm{pcf}(\mathfrak{a}_\alpha \cap \theta) < \theta$

By 3.5 for each $\alpha < \kappa$ we can find $\langle f^\alpha_i : i < \lambda^+\rangle$ which is increasing mod $J_{<\lambda^+}[\mathfrak{a}_\alpha]$, cofinal in $\left(\prod \mathfrak{a}_\alpha, <_{J_{<\lambda^+}[\mathfrak{a}_\alpha]}\right)$ and for $\theta \in \mathfrak{a}_\alpha$ we have

$$|\{f_i\restriction\theta : i < \lambda^+\}| < \theta.$$

Now use Lemma 4.11 below. $\square_{4.9}$

Lemma 4.11 *Suppose λ is regular and $\langle \mathfrak{a}_\epsilon, I_\epsilon : \epsilon < \kappa\rangle$ are such that:*

(i) $\mathfrak{a}_\epsilon$ a set of regular cardinals (such that $|\mathfrak{a}_\epsilon|^+ < \min\ \mathfrak{a}_\epsilon$) with no last element.

(ii) $I_\epsilon = J_{<\lambda}[\mathfrak{a}_\epsilon]$ include $J^{\mathrm{bd}}_{\mathfrak{a}_\epsilon}$, $\lambda = \max \mathrm{pcf}(\mathfrak{a}_\epsilon)$

(iii) $\mathrm{tcf}(\prod \mathfrak{a}_\epsilon, <_{I_\epsilon}) = \lambda$, moreover there is a $<_{I_\epsilon}$-increasing and cofinal sequence $\bar f^\epsilon = \langle f^\epsilon_\alpha : \alpha < \lambda\rangle$ such that for $\theta \in \mathfrak{a}_\epsilon$ we have

$$\{f^\epsilon_\alpha\restriction\theta : \alpha < \lambda\} \text{ has power } < \theta$$

(iv) if $\epsilon_1 < \epsilon_2 < \kappa$, then $\sup \mathfrak{a}_{\epsilon_1} < \sup \mathfrak{a}_{\epsilon_2}$.

<u>*Then*</u> $\mathrm{Ens}(\lambda, \kappa)$.

Remark: For the existence of such λ, $\mathfrak{a}_\epsilon$'s see the proof of 4.9 above or 3.3(2)+3.5. If $\lambda = \mu^+$, $\mathrm{pp}(\mu) = \mu^+$ then there are necessarily many suitable $\mathfrak{a}$'s.

Proof: For $\epsilon < \kappa$, let $\mathcal{I}_\epsilon$ be the set $\{f^\epsilon_\alpha : \alpha < \lambda\}$ ordered by $<_{\ell x}$ (i.e. $f < g$ iff for some γ, $f\restriction\gamma = g\restriction\gamma$ and $f(\gamma) < g(\gamma)$). To prove our conclusion, let $k < \omega$, $\epsilon_k < \ldots < \epsilon_1 < \kappa$, let w be a subset of $\{1, \ldots, k\}$ and $t^\ell_i \in \mathcal{I}_{\epsilon_\ell}$ ($\ell = 1, \ldots, n$, and $i < \lambda$) be pairwise distinct. We have to find $\zeta < \xi < \lambda$ as demanded in 4.10(1)(b).

Of course, we can look for $\zeta < \xi$ in any subset S of λ of cardinality λ. Let $\mathfrak{a}_{\epsilon_\ell} = \{\lambda^\ell_i : i < \delta_\ell\}$, λ^ℓ_i increasing with i and let $\lambda^\ell = \sup_{i<\delta_\ell} \lambda^\ell_i$. Now choose $\mu_\ell < \lambda^\ell$ for ℓ, $1 \leq \ell \leq k$ such that $\mu_\ell > \lambda^{\ell+1}$ when $\ell < k$ and $\mu_\ell > \delta_\ell$. We can choose by induction on $\ell \leq k$, S_ℓ and $i(\ell) < \delta_\ell$ and g^ℓ such that:

$S_0 = \lambda$

$S_{\ell+1} \subseteq S_\ell$
$|S_\ell| = \lambda$
for $\ell > 0$, $\zeta \in S_\ell$, $f^\ell_\zeta \restriction \lambda^\ell_{i(\ell)} = g^\ell$
$\lambda^\ell_{i(\ell)} > \mu_\ell$

Next now define for $\ell \le k$, T_ℓ and $\langle S_\nu : \nu \in T_\ell \rangle$ such that:

$T_0 = \{<>\}$;
T_ℓ is a set of sequences of length 2ℓ ;
if $\rho \in T_{\ell+1}$, then $\rho \restriction 2\ell \in T_\ell$;
for $\nu \in T_\ell$, S_ν is a subset of S_k of cardinality λ;
for $\rho \in T_{\ell+1}$, $S_\rho \subseteq S_{\rho \restriction 2\ell}$;
if $\ell > 0$, $\nu \in T_{\ell-1}$, then $(\exists \rho \in T_\ell)[\rho \restriction 2\ell = \nu]$ and
for some $j_\nu < \delta_\ell$ and $h_\nu \in \{f^\ell_\zeta \restriction \lambda^\ell_{j_\nu} : \zeta \in S_\nu\}$, we have:

$$(\forall \rho \in T_\ell)(\forall \zeta)\Big[\nu = \rho \restriction 2\ell \;\&\; \zeta \in S_\rho \Rightarrow h_\nu = f^\ell_\zeta \restriction \lambda^\ell_{j_\nu} \;\&\; \rho(2\ell) = h_\nu \;\&\; \rho(2\ell+1) = f^\ell_\zeta(\lambda^\ell_{j_\nu})\Big];$$

if $\ell > 0$, $\rho \in T_\ell$ then $\{\rho'(2\ell-1) : \rho' \in T_\ell, \rho' \restriction 2\ell = \rho \restriction 2\ell\}$
is an unbounded subset of $\lambda^\ell_{j_\nu}$ (where $\nu = \rho \restriction (2\ell - 2)$).

We do this by defining T_ℓ, $\langle S_\nu : \nu \in T_\ell \rangle$ for $\ell = 0, ..., k$ by induction on ℓ.

For $\ell = 0$ — no problem.

For $\ell + 1$ — also easy.

By the pigeon hole principle for trees of finite height (see [RSh117]), remembering $\epsilon_k < ... < \epsilon_1$, we can assume that j_ν, h_ν for all $\nu \in T_\ell$ are the same.

Then we can get the conclusion. $\square_{4.11}$

Lemma 4.12 *(1) Suppose $\mathfrak{a}$ is a set of regular cardinals satisfying $|\mathfrak{a}|^+ <$ min $\mathfrak{a}$, $\lambda = \max \mathrm{pcf}(\mathfrak{a})$ and $[\theta \in \mathfrak{a} \Rightarrow \theta > \max \mathrm{pcf}(\theta \cap \mathfrak{a})]$. Suppose $\kappa = |\mathfrak{a}|$ and for $\epsilon < \kappa$ $\mathfrak{a}_\epsilon \subseteq \mathfrak{a}$ are pairwise disjoint, not in $J_{<\lambda}[\mathfrak{a}]$. If $2^\kappa \ge \lambda$ or just $2^\kappa \ge \sup \mathfrak{a}$, <u>then</u> there is an entangled linear order of cardinality λ; equivalently a λ-narrow Boolean algebra of a linear order.*

(2) We can replace "$\kappa = |\mathfrak{a}|$" by "$\mathrm{cf}(\sup \mathfrak{a}) \le \kappa$ and $\mathfrak{a}$ has no last element". Clearly $\mathfrak{a}$ has no last element.

Proof: 1) Let $\langle f_\alpha : \alpha < \lambda \rangle$ be as in 4.11(iii).

We can find for each $\theta \in \mathfrak{a}$, sets $F_{\theta,\zeta} (\zeta < \kappa)$ such that:

$$F_{\theta,\zeta} \subseteq \{f_\alpha \restriction \theta : \alpha < \lambda\},$$

and for any finite disjoint subsets X, Y of $\{f_\alpha \restriction \theta : \alpha < \lambda\}$, for some $\zeta < \kappa$,

(possible as $2^\kappa \ge |\{f_\alpha \restriction \theta : \alpha < \lambda\}|$-see Appendix 1.7). As $\mathfrak{a}$ can be partition to κ pairwise disjoint sets each not in $J_{<\lambda}[\mathfrak{a}]$, and as $J^{\mathrm{bd}}_{\mathfrak{a}} \subseteq J_{<\lambda}[\mathfrak{a}]$, clearly we can find $\langle (\theta_\sigma, \zeta_\sigma) : \sigma \in \mathfrak{a} \rangle$ such that: ($\theta_\sigma \in \mathfrak{a}, \sigma \ge \theta_\sigma, \zeta_\sigma < \kappa$ and)

(*) for each $\theta \in \mathfrak{a}$, $\zeta < \kappa$ the set

$$\{\sigma \in \mathfrak{a} : \theta_\sigma = \theta, \zeta_\sigma = \zeta\}$$

is $\neq \emptyset$ mod $J_{<\lambda}[\mathfrak{a}]$.

Now we define a linear order $<_{et}$ on $\{f_\alpha : \alpha < \lambda\}$
$f_\alpha <_{et} f_\beta$ <u>iff</u> for some $\sigma \in \mathfrak{a}$, we have

$$f_\alpha \restriction (\mathfrak{a} \cap \sigma) = f_\beta \restriction (\mathfrak{a} \cap \sigma),$$

$$f_\alpha(\sigma) \neq f_\beta(\sigma) \text{ and}$$

$$f_\alpha(\sigma) < f_\beta(\sigma) \Leftrightarrow f_\alpha \restriction \theta_\sigma \in F_{\theta_\sigma, \zeta_\sigma}.$$

Now the set $\mathcal{I} = \{f_\alpha : \alpha < \lambda\}$ linearly ordered *by* $<_{et}$ is as required: note that in the definition of $f_\alpha <_{et} f_\beta$ we have $f_\alpha \restriction \theta_\sigma = f_\beta \restriction \theta_\sigma$ as $\theta_\sigma \le \sigma$ and check by cases. The proof is similar to that of 4.11. As for the Boolean algebra — see Appendix 2.8.

2) Same proof only in (*) we replace "for each $\theta \in \mathfrak{a}$" by "for arbitrarily large $\theta \in \mathfrak{a}$". $\square_{4.12}$

Claim 4.13 (1) If $\mathrm{cf}\lambda < \lambda < 2^{\aleph_0}$ <u>then</u> there is an entangled linear order of cardinality λ^+, equivalently — a λ^+-narrow Boolean algebra of a linear order.

(2) If λ is singular, $\kappa = \mathrm{cf}\lambda < \lambda < 2^\kappa$, $\lambda^{<\kappa} = \lambda$ <u>then</u> there is an entangled order of power λ^+, equivalently — a λ^+-narrow Boolean algebra of a linear order.

Proof: 1) The equivalence is by Appendix 2.3. The conclusion follows by 4.12 above for $\kappa = \aleph_0$ (and 1.5) when $\mathrm{cf}\lambda = \aleph_0$. So assume $\mathrm{cf}\lambda > \aleph_0$, and let $\langle \lambda_i : i < \mathrm{cf}\lambda \rangle$ be an increasing continuous sequence of singular cardinals with limit λ. By 2.1 without loss of generality $\lambda^+ = \max \mathrm{pcf}\{\lambda_i^+ : i < \mathrm{cf}\lambda\}$. Of course, for some $A \subseteq \{i < \mathrm{cf}\lambda : \mathrm{cf} i = \aleph_0\}$ with no last element, we have

$$i \in A \Rightarrow \lambda_i^+ > \max \mathrm{pcf}\{\lambda_j^+ : j \in i \cap A\}$$

$$\lambda^+ = \max \mathrm{pcf}\{\lambda_i^+ : i \in A\}.$$

By the case we already proved ($\mathrm{cf}\lambda = \aleph_0$) for every $i \in A$ there is an entangled linear order of cardinality λ_i^+ hence by Appendix 2.2(5) $\mathrm{Ens}(\lambda_i^+, \lambda_i^+)$. So apply 4.10C.

2) By 4.12 (and 1.5). [Note: $\lambda = \lambda^{<\kappa}$ is in order that for $\mathfrak{a} \subseteq \lambda$ of power $< \kappa$, max pcf $\mathfrak{a} \le \lambda$ (hence $< \lambda$)]. $\square_{4.13}$

Remark 4.14 We can:

(1) Generalize 4.11: $\sup \mathfrak{a}_{\epsilon_1} < \sup \mathfrak{a}_{\epsilon_1}$ is replaced by

$$\sup \mathfrak{a}_{\epsilon_1} = \sup \mathfrak{a}_{\epsilon_2} \Rightarrow \mathfrak{a}_{\epsilon_1} \cap \mathfrak{a}_{\epsilon_2} \in I_{\epsilon_1} \cap I_{\epsilon_1}.$$

(2) On generalizing 4.13 to the theorem saying: for many $\lambda's$; see [Sh371].

(3) In 4.12, if $\kappa^{<\sigma} = \kappa$, $\sigma \le \mathrm{cf}(\mathrm{otp}\mathfrak{a})$ <u>then</u> we can demand on the entangled linear order $\mathcal{I}$ we get that Definition 4.10(4), (5) we have "$n < \sigma$" instead "$n < \omega$".

§5 Covering numbers, pp

Another major player makes it appearance now - $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$, covering number. One of our themes is trying to measure $\mathcal{S}_{\le\kappa}(\lambda)$ $(= [\lambda]^{\le\kappa})$ in various ways (looking at its cardinality, λ^κ, as the most rough, and at $\mathrm{pp}_{\Gamma(\mathrm{cf}\mu)}(\mu)$ as the most fine). So probably the most natural of these is just the cofinality of $(\mathcal{S}_{\le\kappa}(\lambda), \subseteq)$, now $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ is a finer dissection of this to finer parts, see Def. 5.1 below.

Note for example that, as everyone knows:

$$\lambda^\kappa = 2^\kappa + \mathrm{cf}\,(\mathcal{S}_{\le\kappa}(\lambda), \subseteq)$$

You may think this cofinality is in some sense the roughest one below the actual cardinal exponentiation, but $\min\{|S| : S \subseteq \mathcal{S}_{\le\kappa}(\lambda)$ is stationary$\}$ is rougher (where $S \subseteq \mathcal{S}_{\le\kappa}(\lambda)$ is stationary if for every model with universe λ and $\le \kappa$ relations and functions, there is $N \prec M$ with universe $\in S$).

In 5.2, 5.3 we give some basic properties, including monotonicity, computation in degenerated cases, various interactions including for example (5.3(10)):

$\otimes_1$ if $\lambda \ge \mu > \theta = \mathrm{cf}\theta > \sigma \ge \aleph_0$, $\mathrm{cf}\mu \in [\sigma, \theta)$ then for some $\mu_1 < \mu$,

$$\mathrm{cov}(\lambda, \mu, \theta, \sigma) = \mathrm{cov}(\lambda, \mu_1, \theta, \sigma)$$

(in a sense, the cases $\lambda = \mu > \theta = \sigma^+$ are sufficient, see 5.3(7)).
Then comes the section's main theorem which says that indeed $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ can be expressed by pp's (when σ is uncountable)

$\otimes_2$ if $\lambda \ge \mu > \theta > \sigma > \aleph_0$ then $\mathrm{cov}(\lambda, \mu, \theta, \sigma)$ is the supremum of $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\lambda^*)$, $\lambda^* \in [\mu, \lambda]$, $\mathrm{cf}\lambda^* \in [\sigma, \theta)$ (or both are "degenerated", i.e. $\le \lambda$).

You can ask about more strict equality considering attainment of the sup's, if $\mu = \theta$ we get this. Of course, the case $\sigma = \aleph_0$ is missing, mainly for the case of fix points (i.e. there is $\aleph_\delta = \delta \in [\mu, \lambda]$) as otherwise 3.6 tells us something. (We shall return later in the book to those points).

We then draw some conclusions; mainly expressing $\mathrm{pp}_{\Gamma(\theta,\sigma)}$ by $\mathrm{pp}_{\Gamma(\tau)}$'s (5.8(1)) and so saying more on the parallel to cov (5.8(2) improving 5.3(7)), expressing $\sup_{I\in\Gamma(\theta,\sigma)} T_I(\lambda)$ (supremum of the cardinalities of families of almost disjoint functions modulo an ideal, prominent in Galvin Hajnal's work). In 5.12 we investigate another problem: is there a tree with λ nodes and $\ge \chi$ λ-branches (for $\chi \le 2^\lambda$ regular) when $2^{<\lambda} < 2^\lambda$.

* * *

Definition 5.1 $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is the first cardinal μ such that:
there is a family $\mathcal{P}$ of μ subsets of λ, each of cardinality $< \kappa$, such that

$$t \subseteq \lambda \;\&\; |t| < \theta \Rightarrow (\exists \mathcal{P}')\Big[\mathcal{P}' \subseteq \mathcal{P} \;\&\; |\mathcal{P}'| < \sigma \;\&\; t \subseteq \bigcup_{A\in\mathcal{P}'} A\Big].$$

We always assume $\lambda \ge \theta \ge \sigma$, $\kappa \ge \aleph_0$, $\theta > 1$, $\sigma > 1$,

$$\kappa \ge \theta \vee [\kappa^+ = \theta \;\&\; \mathrm{cf}\theta < \sigma],$$

so $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is well defined.

The covering numbers, pp and cardinal arithmetic are very closely related. In fact the covering numbers and pp carry almost the same information; cardinal arithmetic may deviate because (2^κ for κ regular are quite arbitrary and) exponentiation by $\aleph_0$ is less clear to us, and this unclarity spreads.

Observation 5.2 (1) $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is monotonically increasing in λ and θ and monotonically decreasing in κ, σ.

(2) $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is 1 <u>iff</u> $\lambda < \kappa$.

(3) For singular λ, $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is $\mathrm{cf}\lambda$ <u>iff</u> $\lambda = \kappa$, $\mathrm{cf}\lambda \notin [\sigma, \theta)$, (i.e. $\mathrm{cf}(\lambda) < \sigma$ or $\mathrm{cf}(\lambda) \ge \theta$).

(4) $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is $\le \lambda$ <u>if</u>: $\lambda = \kappa = \mathrm{cf}\lambda \ge \theta$ or $\lambda < \kappa$ or $\theta \le \sigma$ or $\lambda = \kappa > \mathrm{cf}\lambda \notin [\sigma, \theta)$.

(5) $\mathrm{cov}(\lambda, \kappa, \theta, \sigma)$ is $\ge \lambda$ <u>iff</u> λ is regular $\ge \kappa$ or $\lambda > \kappa$ or

$$\lambda = \kappa > \mathrm{cf}(\lambda) \;\&\; \mathrm{cf}(\lambda) \in [\sigma, \theta).$$

(6) $\mathrm{cov}(\lambda, \kappa, \kappa, 2) = \mathrm{cf}\,(\mathcal{S}_{<\kappa}(\lambda), \subseteq)$.

(7) $\mathrm{cov}(\lambda, \kappa, \theta, \sigma) < \lambda$ <u>iff</u> $\lambda < \kappa$ or $\lambda = \kappa > \mathrm{cf}\lambda \notin [\sigma, \theta)$
<u>iff</u> $\mathrm{cov}(\lambda, \kappa, \theta, \sigma) \in \{1, \mathrm{cf}\lambda\} \setminus \{\lambda\}$

(8) $\mathrm{cov}(\lambda_1, \kappa_1, \theta_1, \sigma_1) \le \mathrm{cov}(\lambda_2, \kappa_2, \theta_2, \sigma_2)$ <u>if</u> $\lambda_1 \le \lambda_2$, $\kappa_1 \ge \kappa_2$, $\theta_1 \le \theta_2$, $\sigma_1 \ge \sigma_2$.

Convention 5.2A Dealing with $\mathrm{cov}(\lambda,\kappa,\theta,\sigma)$ we usually assume $\lambda \geq \kappa \geq \theta > \sigma \geq 2$ (for example in 5.3(3)). Note that otherwise by 5.2(2)+(4), we know $\mathrm{cov}(\lambda,\kappa,\theta,\sigma) \leq \lambda$ for "uninteresting" reasons.

Observation 5.3 (1) If $\mathrm{cf}\lambda \geq \theta$ <u>then</u>

$$\mathrm{cov}(\lambda,\kappa,\theta,\sigma) \leq \sum_{\alpha<\lambda} \mathrm{cov}(|\alpha|,\kappa,\theta,\sigma) \leq \lambda + \sup_{\lambda_1<\lambda} \mathrm{cov}(\lambda_1,\kappa,\theta,\sigma).$$

(2) $\mathrm{cov}(\lambda,\kappa,\theta,\sigma) \leq \mathrm{cov}(\lambda^+,\kappa,\theta,\sigma) \leq \lambda^+ + \mathrm{cov}(\lambda,\kappa,\theta,\sigma)$.
(3) $\mathrm{cov}(\lambda,\kappa,\theta,\sigma) \leq \mathrm{cov}(\lambda,\kappa^+,\theta,\sigma)+\kappa^+ \leq \mathrm{cov}(\lambda,\kappa,\theta,\sigma)+\kappa^+$ and if $\kappa < \lambda$, equality holds.
(4) If $\lambda > \kappa(\geq \theta > \sigma)$, σ regular <u>then</u>

$$\mathrm{cov}(\lambda,\kappa,\theta,\sigma) = \sum_{\mu\in[\kappa,\lambda]} \mathrm{cov}(\mu,\mu,\theta,\sigma).$$

(5) $\theta + \mathrm{cov}(\lambda,\kappa,\theta,\sigma) = \theta + \sup\{\mathrm{cov}(\lambda,\kappa,\theta_1^+,\sigma) : \theta_1 < \theta\}$.
(6) If $\lambda \geq \kappa \geq \theta_1 \geq \theta_2 \geq \theta_3$, $\mathrm{cf}\kappa \notin [\theta_3,\theta_2)$ and $\lambda^* = \mathrm{cov}(\lambda,\kappa,\theta_1,\theta_2)$ <u>then</u> $\mathrm{cov}(\lambda,\kappa,\theta_1,\theta_3) \leq \mathrm{cov}(\lambda^*,\kappa,\theta_2,\theta_3)$.
(7) If $\lambda \geq \kappa \geq \theta > \sigma = \mathrm{cf}(\sigma)$, $\mathrm{cf}\kappa \geq \theta$, $\lambda_0 = \lambda$,

$$\lambda_{n+1} = \sup\{\mathrm{cov}(\mu,\mu,\tau^+,\tau) : \kappa \leq \mu \leq \lambda_n,\ \mathrm{cf}\mu = \tau \in [\sigma,\theta)\}$$

<u>then</u>

$$\mathrm{cov}(\lambda,\kappa,\theta,\sigma) \leq \bigcup_{n<\omega} \lambda_n.$$

(For more on this see 5.8(2)).
(8) Suppose $\lambda \geq \kappa \geq \theta > \sigma$, σ a limit uncountable cardinal and $\mathrm{cf}\theta \neq \mathrm{cf}\sigma$. <u>Then</u>

$$\mathrm{cov}(\lambda,\kappa,\theta,\sigma) = \min_{\sigma_1<\sigma} \mathrm{cov}(\lambda,\kappa,\theta,\sigma_1)$$

[check first if θ successor].
(9) $\mathrm{cov}(\lambda,\kappa,\theta,\aleph_0) = \mathrm{cov}(\lambda,\kappa,\theta,2)$ when $\mathrm{cov}(\lambda,\kappa,\theta,\aleph_0) \geq \aleph_0$.
(10) If $\lambda \geq \mu > \theta > \sigma$, $\mathrm{cf}\mu \in [\sigma,\theta)$ and $\mathrm{cf}(\mu) \neq \mathrm{cf}(\theta)$ <u>then</u> for some $\mu_1 < \mu$ we have

$$\mathrm{cov}(\lambda,\mu,\theta,\sigma) = \mathrm{cov}(\lambda,\mu_1,\theta,\sigma).$$

Proof: Check. For example:
6) Let $\mathcal{P}^*$ be a family of subsets of λ each of cardinality $< \kappa$, $|\mathcal{P}^*| = \lambda^*$, $\mathcal{P}^*$ exemplifying the definition of $\lambda^* = \mathrm{cov}(\lambda,\kappa,\theta_1,\theta_2)$. Let

$$\mathcal{P}^* = \{A_i : i < \lambda^*\}.$$

Let $\mathcal{P}$ be a family of subsets of λ^* each of cardinality $< \kappa$ exemplifying the definition of $\mathrm{cov}(\lambda^*,\kappa,\theta_2,\theta_3)$. If κ is regular then $\{\bigcup_{i\in x} A_i : x \in \mathcal{P}\}$ exemplify $\mathrm{cov}(\lambda,\kappa,\theta_1,\theta_3) \leq |\mathcal{P}|$ giving the conclusion. Otherwise we let $\kappa = \sum_{i<\mathrm{cf}\kappa} \kappa_i$, $\kappa_i < \kappa$, and use

$$\{\cup\{A_i : i \in x, |A_i| < \kappa_j\} : x \in \mathcal{P}, j < \mathrm{cf}\kappa\}.$$

7) Let χ be regular large enough, by induction on n choose $N_n \prec (H(\chi),\in)$ of cardinality λ_n such that

$$\{N_0,\dots,N_{n-1},\lambda,\kappa,\theta,\sigma\} \cup (\lambda_n+1) \subseteq N_n,$$

and

$$\mathcal{P}_n = \{A \in N_n : |A| < \kappa, A \subseteq \lambda\}$$

and $\mathcal{P}_\omega = \bigcup_{n<\omega}\mathcal{P}_n$. Suppose $X \subseteq \lambda$, $|X| < \theta$ and for no $\mathcal{P} \subseteq \mathcal{P}_\omega$, $|\mathcal{P}| < \sigma$ is $X \subseteq \bigcup_{A\in\mathcal{P}} A$; let I be the σ-complete ideal on X generated by $\{X \cap A : A \in \mathcal{P}_\omega\}$, so $X \notin I$. Let

$$\theta_n = \min\Big\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{P}_n,\ \bigcup_{A\in\mathcal{P}} A \cap X \notin I\Big\};$$

now $\theta_n \leq |X| < \theta$ and $\mathrm{cf}\theta_n \geq \sigma$ and $\theta_{n+1} < \theta_n$ (use 5.3(4) applied to $\mathrm{cov}(\lambda_n,\kappa,\theta_n,\theta_n)$), contradiction.
10) Let $\mu = \sum\{\mu_\alpha : \alpha < \mathrm{cf}\mu\}$, $[\alpha < \beta \Rightarrow \theta < \mu_\alpha < \mu_\beta < \mu]$. By 5.2(8) we know $\mathrm{cov}(\lambda,\mu,\theta,\sigma) \leq \mathrm{cov}(\lambda,\mu_\alpha,\theta,\sigma)$ for every $\alpha < \mathrm{cf}(\mu)$. Let $\mathcal{P}$ exemplify $\mathrm{cov}(\lambda,\mu,\theta,\sigma)$, so the bad case is

$$\bigwedge_{\alpha<\mathrm{cf}\mu} \mathrm{cov}(\lambda,\mu,\theta,\sigma) < \mathrm{cov}(\lambda,\mu_\alpha,\theta,\sigma)$$

so for each $\alpha < \mathrm{cf}\mu$,

$$\mathcal{P}_\alpha = \{A \in \mathcal{P} : |A| < \mu_\alpha\} \text{ cannot exemplify } \mathrm{cov}(\lambda,\mu_\alpha,\theta,\sigma)$$

so some $A_\alpha \in [\lambda]^{<\theta}$ exemplifies this. As $|A_\alpha| < \mu$, $\mathrm{cf}(\mu) \neq \mathrm{cf}(\theta)$ for some $\theta_1 < \theta$, $B =: \{\alpha < \mathrm{cf}\mu : |A_\alpha| \leq \theta_1\}$ is unbounded in $\mathrm{cf}\alpha$. So $A =: \bigcup_{\alpha\in B} A_\alpha$ contradicts the choice of $\mathcal{P}$ because $\mathrm{cf}(\mu) \in [\sigma,\theta)$. $\square_{5.3}$

Remark: Concerning 5.3(7), on the other direction see 5.8(2).

The cov vs pp Theorem 5.4 Remember
$\Gamma = \Gamma(\theta,\sigma) =: \{I :$ for some cardinal $\theta_I < \theta$, I is a σ-complete ideal on θ_I (proper of course) $\}$
(and $\Gamma(\sigma) = \Gamma(\sigma^+,\sigma)$).
Suppose σ is regular $> \aleph_0$ and $\lambda \geq \kappa \geq \theta > \sigma$, <u>then</u>:

(1) $\sup\{pp_\Gamma(\lambda^*) : \lambda^* \in [\kappa,\lambda]$ (and $\sigma \le cf(\lambda^*) < \theta)\} + \lambda = cov(\lambda,\kappa,\theta,\sigma) + \lambda$.
(2) Moreover, if $\mu =: cov(\lambda,\kappa,\theta,\sigma)$ is a regular cardinal $> \lambda$ then for some $I \in \Gamma(\theta,\sigma)$ and $\langle \lambda_\alpha : \alpha \in \mathrm{Dom}\, I\rangle$ we have: $\mu \le tcf(\prod \lambda_\alpha, <_I)$ (hence we can also get equality) and $\theta < \lambda_\alpha \le \lambda$.[4]
(3) In (1) for the inequality $\le$, "$\sigma > \aleph_0$" is not needed.
(4) In (1), if both sides of the equation are $> \lambda$ then we can replace sup by max.

Proof of 5.4(1):
The inequality $\le$.

If $\mu < \sup\{pp_\Gamma(\lambda_1) : \kappa \le \lambda_1 \le \lambda\}$ then for some $\lambda^* \in [\kappa,\lambda]$ we have $\mu < pp_\Gamma(\lambda^*)$ so for some $\theta^* < \theta$ and σ-complete ideal I on θ^*, and $\langle \lambda_i : i < \theta^*\rangle$ we have:

$$\lambda_i = cf\lambda_i,\ \lambda^* = tlim_I\langle \lambda_i : i < \theta^*\rangle,\ \text{and}\ \mu < \mu^* = tcf\Big(\prod_{i<\theta^*} \lambda_i, <_I\Big).$$

So let $\langle f_\alpha : \alpha < \mu^*\rangle$ be a $<_I$-increasing sequence from $\prod_{i<\theta^*} \lambda_i$ cofinal in it. Suppose $\mathcal{P}$ exemplifies $\mu' = cov(\lambda,\kappa,\theta,\sigma)$, so for every $\alpha < \mu^*$ for some $A_\alpha \in \mathcal{P}$, $\{i < \theta^* : f_\alpha(i) \in A_\alpha\} \ne \emptyset \bmod I$.

If $\mu' < \mu^*$, without loss of generality $A_\alpha = A^*$ for every $\alpha < \mu^*$ (remember that μ^* is necessarily regular), but $|A^*| < \kappa \le \lambda^*$, so for some $t \in I$ we have $|A^*| < \min\{\lambda_i : i < \theta^* \text{ and } i \notin t\}$. Now for every $i < \theta^*$

$$g(i) =: \sup[\{f_\alpha(i) : f_\alpha(i) \in A^*,\ \alpha < \theta^* \ \text{and}\ i \notin t\} \cup \{0\}]$$

is $< \lambda_i$, hence $g \in \prod_{i<\theta^*} \lambda_i$, but for each $\alpha < \mu^*$ we have $\neg(g \le_I f_\alpha)$ (by the choice of $A_\alpha = A^*$ as $t \in I$); so we get a contradiction to $\mu^* = tcf(\prod_{i<\theta^*} \lambda_i, <_I)$. So $\mu^* \le \mu'$, i.e. $\mu^* \le cov(\lambda,\kappa,\theta,\sigma)$.
As μ was an arbitrary cardinal $< \sup\{pp_\Gamma(\lambda^*) : \kappa \le \lambda^* \le \lambda\}$ and $\mu^* > \mu$ we conclude

$$cov(\lambda,\kappa,\theta,\sigma) \ge \sup\{pp_\Gamma(\lambda^*) : \lambda \ge \lambda^* \ge \kappa\}.$$

The inequality $\ge$

Without loss of generality $\lambda = \kappa$ (by 5.3(4)), λ singular (by 5.2(4)) and $cf\kappa \in [\sigma,\theta)$ (by 5.2(3)).

Suppose μ is regular, $\lambda < \mu \le cov(\lambda,\kappa,\theta,\sigma)$. We shall prove that for some $I \in \Gamma(\theta,\sigma)$ and $\langle \lambda_\alpha : \alpha \in \mathrm{Dom}\, I\rangle$ we have: $tlim_I\langle \lambda_\alpha : \alpha \in \mathrm{Dom}\, I\rangle$ is in $[\kappa,\lambda]$, i.e. is λ (remember $\kappa = \lambda$) and $tcf(\prod \lambda_\alpha, <_I)$ is $\ge \mu$ or μ is (weakly) inaccessible and for every regular $\mu_1 < \mu$ we have this. We assume that this fails.

[4]We have not required here "$\sup_\alpha \lambda_\alpha \in [\kappa,\lambda]$"

Let $\chi = (2^{2^\lambda})^+$, and choose an elementary submodel N of $(H(\chi), \in, <^*_\chi)$ of power $< \mu$, such that $N \cap \mu$ is an ordinal and $\langle \mu,\lambda,\kappa,\theta,\sigma\rangle \in N$. Now by the assumption on μ, $\mathcal{P} =: \{A : A \subseteq \lambda, A \in N, |A| < \kappa\}$ does not satisfy the requirements in Definition 5.1, hence there are $\theta^* \in [\sigma,\theta)$ and a function $f^* : \theta^* \to \lambda$ such that for no $\zeta^* < \sigma$, $A_\zeta \in \mathcal{P}$ (for $\zeta < \zeta^*$) is (Rang $f) \subseteq \bigcup_{\zeta<\zeta^*} A_\zeta$. We let

$$I = \big\{B : B \subseteq \theta^* \text{ and } \{f(\alpha) : \alpha < \theta^*, \alpha \in B\} \text{ is included in the union of some } < \sigma \text{ members of } \mathcal{P}\big\}.$$

So I is a σ-complete ideal on θ^*, $\theta^* \notin I$ (but singletons belong to it). Let

$$H = \big\{h : h \text{ is a function with domain } \theta^*,\ h(i) \text{ a subset of } \lambda \text{ which belongs to } N,\ \{i : f^*(i) \in h(i)\} \equiv \theta^* \bmod I \text{ and for some } \zeta^* < \sigma, \text{ and } \langle X_\zeta : \zeta < \zeta^*\rangle \text{ we have: } X_\zeta \in N,\ X_\zeta \text{ a subset of } \mathcal{P}(\lambda),\ |X_\zeta| < \kappa \text{ and } \mathrm{Rang}(h) \subseteq \bigcup_{\zeta<\zeta^*} X_\zeta\big\}.$$

$$G = \big\{g : \text{for some } h \in H,\ g \text{ is a function with domain } \theta^* \text{ and } g(i) = |h(i)|\big\}.$$

Now $G \ne \emptyset$ (by the choice of H and G the constant function with domain θ^* and value λ belongs to H and to G, for H with witness $\langle X_\zeta : \zeta < \zeta^*\rangle$, $\zeta^* = 1$, $X_\zeta = \{\lambda\}$). Clearly $g(i)$ is a cardinal $\le \lambda$ for each $i < \theta^*$, $g \in G$. As I is σ-complete, $\sigma > \aleph_0$, there is $g^* \in G$ such that for no $g \in G$, do we have $g < g^* \bmod I$. So we can find $h^* \in H$ such that $(\forall i < \theta^*)[g^*(i) = |h^*(i)|]$; as $h^* \in H$ we have Rang $h^* \subseteq N$, Dom $h^* = \theta^*$, $h^*(i)$ a subset of λ and $\{i : f(i) \in h^*(i)\} = \theta^* \bmod I$. Let $\lambda_i = cf(g^*(i))$, so λ_i is regular $\le \lambda$.[5]
Let $\{X_j : j < j(*) < \sigma\}$ exemplify $h^* \in H$.
Now for each $y \in \mathcal{P}(\lambda)$, let $\langle y^{[\epsilon]} : \epsilon < cf|y|\rangle$ be an increasing continuous sequence of subsets of y of power $< |y|$ with $y = \bigcup_\epsilon y^{[\epsilon]}$. Without loss of generality the function $y \mapsto \langle y^{[\epsilon]} : \epsilon < cf|y|\rangle$ for $y \in \mathcal{P}(\lambda)$ belongs to N, hence for $X \in N$, $X \subseteq \mathcal{P}(\lambda)$ we have $\langle\langle y^{[\epsilon]} : \epsilon < cf|y|\rangle : y \in X\rangle \in N$. Let for $\tau < \kappa$ and $X \subseteq \mathcal{P}(\lambda)$:

$$X^\tau = X \cup \{y^{[\epsilon]} : y \in X, cf(|y|) \le \tau, \text{ and } \quad \epsilon < cf(|y|)\}.$$

$$X^{\tau,0} = X,\ X^{\tau,n+1} = (X^{\tau,n})^\tau,\ X^{\tau,\omega} = \bigcup_{n<\omega} X^{\tau,n}.$$

Clearly for $\zeta \le \omega$: $|X_j^{\tau,\zeta}| \le |X_j| + \tau < \kappa$ and $X_j^{\tau,\omega} \in N$ (as $X_j \in N$, $\tau \in \lambda \subseteq N$).

Let $\tau^* = \sup_j |X_j|$. As $\lambda = \kappa$ has cofinality $\ge \sigma > j(*)$, and each X_j has cardinality $< \kappa$, clearly $\tau^* < \kappa$. Now $\{X_j^{\tau^*,\omega} : j < j(*)\}$ satisfies all the

[5]We could have noted:
$(*)_0$ $\theta^* \notin I^* = \{A \subseteq \theta^* : \text{for some } g \in G,\ g < g^* \bmod (I + (\theta \backslash A))\}$,
I^* is σ-complete, and for no $g \in G$, $\{i < \theta^* : g(i) < g^*(i)\} \ne \emptyset \bmod I^*$.

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requirements on $\{X_j : j < j(*)\}$, so without loss of generality $X_j = X_j^{\tau^*,\omega}$ for each $j < j(*)$.
Now if for some $\tau < \kappa$, $A_\tau =: \{i < \theta^* : \lambda_i \le \tau\}$ is $\equiv \theta^*$ mod I (remember $\lambda_i = \mathrm{cf}[g^*(i)]$) we shall easily contradict the choice of g^* as follows. Define h^{**}, g^{**} :
Dom $h^{**} = \theta^* =$ dom g^{**}, and for each $i < \theta^*$, if $\mathrm{cf}(g^*(i)) > \tau$, $h^{**}(i) = h^*(i)$; if $\mathrm{cf}(g^*(i)) \le \tau$ let $h^{**}(i) = h^*(i)^{[\epsilon]}$ for the minimal $\epsilon < \mathrm{cf}(g^*(i))$ such that $f^*(i) \in h^*(i)^{[\epsilon]}$; and let $g^{**}(i) = |h^{**}(i)|$. Now $g^{**} \in G$ (as exemplified by h^{**} which belongs to H, which in turn is exemplified by $\langle X_j^\tau : j < j(*)\rangle$), but $g^{**} < g^*$ mod I contradicting the choice of g^*. We conclude:

$A_\tau = \{i < \theta^* : \lambda_i = \mathrm{cf}(g^*(i)) \le \tau\} \not\equiv \theta^*$ mod I for every $\tau < \kappa$.

Let J be the ideal which $I \cup \{A_\tau : \tau < \kappa\}$ generates. As $[\kappa_1 < \kappa_2 \Rightarrow A_{\kappa_1} \subseteq A_{\kappa_2}]$ and as $\mathrm{cf}\kappa \ge \sigma$ (we have assumed $\lambda = \kappa$ hence this holds because of 5.2(3)) clearly J is σ-complete and $\theta^* \notin J$ and $\mathrm{tlim}_J \lambda_i = \lambda = \kappa$.

We can suppose

$(*)_1$ if J' is a σ-complete (proper) ideal on θ^* extending J with

$$\mu' =: \mathrm{tcf}(\prod \lambda_i, <_{J'})$$

well defined, then $\mu' < \mu$

(otherwise such a J' is as required).

Now $\{\lambda_i : i < \theta^*\}$ does not necessarily belong to N, but

$$\mathfrak{b}_j =: \{\mathrm{cf}|y| : y \in X_j \text{ and } \mathrm{cf}|y| > |X_j|\}$$

belongs to N for each $j < j(*)$ and letting

$$t = \{i : \lambda_i = 1 = g^*(i)\}$$

we have

$$t \in I, \quad \{\lambda_i : i \in \theta^* \backslash t\} \subseteq \bigcup_j \mathfrak{b}_j \text{ and } |\mathfrak{b}_j| < \kappa.$$

[Why? The last clause is totally trivial: $|\mathfrak{b}_j| < \kappa$ as $|\mathfrak{b}_j| \le |X_j|$ and we have assumed $|X_j| < \kappa$. For the one before last, first remember $X_j = X_j^{\tau^*,\omega}$, hence $\aleph_0 \le \mathrm{cf}[g^*(i)] \le \tau^*$ is impossible, and even

$$\mathrm{cf}[g^*(i)] \le \tau^* \Rightarrow g^*(i) = 1.$$

Now $t = \{i < \theta^* : g^*(i) = 1\}$ belongs to I (as $\mathrm{Rang}(f \restriction t) \subseteq \bigcup_{j<j(*)} Y_j$ where $Y_j = \{\alpha < \lambda : \{\alpha\} \in X_j\}$ is a subset of λ from N of cardinality $< \lambda$; and see definition of I). So together

$$t = \{i : \mathrm{cf}[g^*(i)] \le \tau^*\} \in I$$

as required].

Let $J^j =: \{\mathfrak{b} : \mathfrak{b} \subseteq \mathfrak{b}_j, \sup \mathfrak{b} < \kappa\}$.
By [Sh345a,2.12(2)] there is a (σ-generating) sequence

$$\langle \mathfrak{b}_{j,\tau,\epsilon} : \epsilon < \tau \in \mathfrak{b}_j^* \subseteq \mathrm{pcf}(\mathfrak{b}_j)\rangle, \ \mathfrak{b}_{j,\tau,\epsilon} \subseteq \mathfrak{b}_j$$

such that $\mathfrak{b}_{j,\tau,\epsilon} \in J_{\le\tau}[\mathfrak{b}_j]$ and:

$(*)_2$ (a) for each $\epsilon < \tau \in \mathfrak{b}_j^*$, the σ-complete ideal $J^j_{\tau,\epsilon}$ on $\mathfrak{b}_j$ generated by $J^j \cup \{\mathfrak{b}_{j,\tau_1,\zeta} : \zeta < \tau_1 < \tau \ \& \ \tau_1 \in \mathfrak{b}_j^*\} \cup \{\mathfrak{b}_j \backslash \mathfrak{b}_{j,\tau,\epsilon}\}$ is proper and satisfies $\left(\prod \mathfrak{b}_j, <_{J^j_{\tau,\epsilon}}\right)$ has true cofinality τ.

(b) if J' is a σ-complete ideal on $\mathfrak{b}_j$ extending J^j and $\tau' = \mathrm{tcf}(\prod \mathfrak{b}_j, <_{J'})$ is well defined then:
(i) $\tau' \in \mathfrak{b}_j^*$
(ii) $\tau \in \mathfrak{b}_j^* \ \& \ \epsilon < \tau < \tau' \Rightarrow \mathfrak{b}_{j,\tau,\epsilon} \in J'$
(iii) $\mathfrak{b}_j \backslash \mathfrak{b}_{j,\tau',\epsilon} \in J'$ for some $\epsilon < \tau'$

and without loss of generality $\langle \mathfrak{b}_{j,\tau,\zeta} : \zeta < \tau \in \mathfrak{b}_j^*\rangle \in N$ for each $j < j(*)$.

Let $\mathfrak{b}_j^{**} = \{\tau \in \mathfrak{b}_j^* :$ for some $\zeta < \tau$ and $\mathfrak{d} \subseteq \mathfrak{b}_{j,\tau,\zeta}$, $|\mathfrak{d}| \le \theta^*$ and $\mathfrak{d}$ is not in the σ-complete ideal generated by $J^j \cup \{\mathfrak{b}_{j,\tau_1,\epsilon} : \epsilon < \tau_1 \in \mathfrak{b}_j^*, \tau_1 < \tau\}\}$.
Suppose the σ-complete ideal J_j'' generated by

$$J^j \cup \{\mathfrak{b}_{j,\tau,\zeta} : \zeta < \tau \in \mathfrak{b}_j^{**}, \tau < \mu\}$$

does not include $\{\mathfrak{d} \subseteq \mathfrak{b}_j : |\mathfrak{d}| \le \theta^*\}$.
So by [Sh345a,1.8] for some $\mathfrak{c} \subseteq \mathfrak{b}_j$, we have

$$|\mathfrak{b}_j \backslash \mathfrak{c}| \le \theta^* \text{ and } (\prod \mathfrak{b}_j, <_{J_j''+\mathfrak{c}}) \text{ has true cofinality (and } J_j'' + \mathfrak{c} \text{ is proper)}$$

hence this true cofinality is $\ge \mu$ (as $\mathfrak{b}_{j,\tau,\zeta} \in J_j'' \subseteq J_j'' + \mathfrak{c}$); as $\mathrm{tlim}_{J_j''+\mathfrak{c}}(\mathfrak{b}_j) = \kappa = \lambda$ (as $J^j \subseteq J_j'' + \mathfrak{c}$) we have (as $|\mathfrak{b}_j \backslash \mathfrak{c}| \le \theta^*$) $\mathrm{pp}^+_{\Gamma(\theta,\sigma)}(\kappa) > \mu$, as desired.

So we assume that

$(*)_3$ every $\mathfrak{c} \subseteq \mathfrak{b}_j$ of cardinality $\le \theta^*$ belongs to the σ-complete ideal generated by $J^j \cup \{\mathfrak{b}_{j,\tau,\zeta} : \zeta < \tau \in \mathfrak{b}_j^{**}, \tau < \mu\}$.

Let $\tau(j)$ be minimal such that for some $\kappa_j < \kappa$ we have $\{\lambda_i : i < \theta^*\} \cap \mathfrak{b}_j \backslash \kappa_j$ is included in a union of $< \sigma$ sets from $\{\mathfrak{b}_{j,\tau,\zeta} : \zeta < \tau \in \mathfrak{b}_j^*$ and $\tau \le \tau(j)\}$, say $\langle \mathfrak{b}_{j,\tau_{j,\epsilon},\zeta_{j,\epsilon}} : \epsilon < \epsilon_j(*)\rangle$, $\epsilon_j(*) < \sigma$. Now $\tau(j)$ exists and is $\le \mu$ by the previous paragraph. Note that $\tau(j)$ has cofinality $< \sigma$ or $\tau(j) \in \mathfrak{b}_j^* \subseteq \mathrm{pcf}(\mathfrak{b}_j)$ hence by $(*)_3$ we have $\tau(j) < \mu$.

So we have $\tau(j) < \mu$, note that $\{\mathfrak{b}_{j,\tau_{j,\epsilon},\zeta_{j,\epsilon}} : \epsilon < \epsilon_j(*)\}$ is not necessarily in N. But by 3.1[6] for each $\zeta < \tau \in \mathfrak{b}^*_j$ there is $F_{j,\tau,\zeta} \subseteq \prod \mathfrak{b}_{j,\tau,\zeta}$ of cardinality $\le \tau$ such that: for every $f \in \prod \mathfrak{b}_{j,\tau,\zeta}$ there is a partition $\langle \mathfrak{c}_\alpha : \alpha < \alpha(*) < \sigma\rangle$ of $\mathfrak{b}_{j,\tau,\zeta}$ and $f_\alpha \in F_{j,\tau,\zeta}$ for $\alpha < \alpha(*)$ such that $f < \bigcup_{\alpha<\alpha(*)} f_\alpha{\restriction}\mathfrak{c}_\alpha$. Without loss of generality $\langle F_{j,\tau,\zeta} : \zeta < \tau \in \mathfrak{b}^*_j\rangle$ belongs to N. Hence:

$(*)_4$ $\zeta < \tau \in \mathfrak{b}^*_j \;\&\; \tau < \sup(N \cap \mu) \Rightarrow F_{j,\tau,\zeta} \subseteq N$.

Define a function $f_j \in \prod \mathfrak{b}_j$:

$$f_j(\tau) = \min\{\gamma : \text{if for any } i < \theta^*,\ f^*(i) \in h^*(i) \text{ and } \tau = \mathrm{cf}[|h^*(i)|] \text{ then } f^*(i) \in h^*(i)^{[\gamma]}\}.$$

So clearly there is $F^*_j \subseteq \bigcup_{\epsilon<\epsilon_{j(*)}} F_{j,\tau_{j,\epsilon},\zeta_{j,\epsilon}}$, $|F^*_j| < \sigma$ such that

$$\bigwedge_{\tau\in\mathfrak{b}_j\setminus\kappa_j}\ \bigvee_{f\in F^*_j} f_j(\tau) < f(\tau).$$

If $F^*_j \subseteq N$ for each j, we let (for $j < j(*)$, $f \in F^*_j$) :

$$X^f_j = X^{\kappa_j}_j \cup \left\{ y^{[f(\mathrm{cf}|y|)]} : y \in X_j \text{ and } \mathrm{cf}(|y|) \in \mathfrak{b}_j \right\}.$$

Now we can easily get a contradiction to the minimality of g^*. Does $F^*_j \subseteq N$? By $(*)_4$ above and the assumption $\tau(j) < \mu$ above, if $\sup(\mathfrak{b}^{**}_j \cap \mu) < \mu$ then yes; this occurs if μ is a successor cardinal, but this case is enough for 5.4(1) (check definition of $\mathfrak{b}^{**}_j$ and see $(*)_2(a)$).

Proof of 5.4(2): We have almost proved it in the proof of the " $\ge$ " in 5.4(1). The remaining case there, is that for some j, ϵ we have $\tau_{j,\epsilon} \notin N$, so not only $\sup(\mathfrak{b}^{**}_j) = \sup(\mathfrak{b}^*_j \cap \mu) = \mu$; i.e.

$$\sup\left\{\tau \in \mathfrak{b}^*_j : \text{for some } \mathfrak{c} \subseteq \mathfrak{b}_j, |\mathfrak{c}| \le \theta^* \text{ and } \mathfrak{c} \text{ is not in the } \sigma\text{-complete ideal generated by } \{\mathfrak{b}_{j,\tau_1,\zeta} : \zeta < \tau_1 \in \mathfrak{b}^*_j \cap \tau\}\right\}$$

is μ, but μ is not in the set. It is quite unclear whether this case is consistent with ZFC. However, $\mathrm{tlim}_I\langle\lambda_\alpha : \alpha < \theta^*\rangle = \lambda$ is not required for 5.4(2): just $\le \lambda$. Now for proving 5.4(2) we will repeat the proof of 5.4(1) with a change that makes it easier: redefining H we make it:

$H = \{h : \mathrm{Dom}\ h = \theta^*, h(i)$ is a subset of λ of cardinality $< \kappa$
and $\{i : f^*(i) \notin h(i)\} \in I$ and there are $j(*) < \sigma$
and $X_j \in \mathcal{P}(\mathcal{P}(\lambda)) \cap N$ for $j < j(*)$ such that: $|X_j| \le \theta^*$ and
$\{i < \theta^* : h(i) \in \bigcup_j X_j\} \equiv \theta^* \bmod I\}$.

After defining λ_i, $y^{[\epsilon]}$, instead of proving $A_\tau \ne \theta^* \bmod I$, for $\tau < \kappa$, we shall just let $A = \{i < \theta^* : \lambda_i > \theta^*\}$. For $i \in \theta^*\setminus A$ let $\gamma(i) = \min\{\gamma :$

[6] Of course, more holds by 3.1

$f^*(i) \in h^*(i)^{[\gamma]}\}$ (so $\gamma(i) \le \theta^*$ as for $i \in \theta^*\setminus A$ we have $\mathrm{cf}[h^*(i)] \le \theta^*$). Let $\mathfrak{c}_j = \{\mathrm{cf}|y| : y \in X_j, \mathrm{cf}|y| > \theta^*\}$, so $\mathfrak{c}_j$ is a set of regular cardinals $> \theta^*$ of cardinality $\le \theta^*$, $\mathfrak{c}_j \in N$. Let the function $y \mapsto \langle y^{[\epsilon]} : \epsilon < \mathrm{cf}(|y|)\rangle$ (for $y \subseteq \lambda$) be defined as there and for $\mathcal{C}$ a subset of $\bigcup_j J_{<\mu}[\mathfrak{c}_j]$ of cardinality $< \sigma$, $\bar{t} = \langle t_\mathfrak{c} : \mathfrak{c} \in \mathcal{C}\rangle$, $t_\mathfrak{c} \in N \cap \prod \mathfrak{c}$, let $h^*_{\bar{t}}$ be

$$h^*_{\bar{t}}(i) = \begin{cases} h^*(i)^{[t_\mathfrak{c}(\mathrm{cf} g^*(i))]} & \text{if } i \in A, \mathfrak{c} \in \mathcal{C} \text{ is } <^*_\chi\text{-minimal such that: } \mathrm{cf}(g^*(i) \in \mathfrak{c}, \\ h^*(i)^{[\gamma(i)]} & \text{otherwise} \end{cases}$$

$$g^*_{\bar{t}}(i) = |h^*_{\bar{t}}(i)|.$$

Let $X^*_j = X^{\theta^*}_j \cup \{y^{[t_\mathfrak{c}(\mathrm{cf}|y|)]} : \mathfrak{c} \in \mathcal{C}, \mathrm{cf}(|y|) \in \mathfrak{c}, y \in X_j\}$.

If for some $j < j(*)$ $\mathrm{pcf}_{\Gamma(\theta,\sigma)}(\mathfrak{c}_j)$ has a member $\ge \mu$, we finish; if not, then this set has a sup with cofinality $< \sigma$ hence is bounded below μ. Hence for each $j < j(*)$, we can find $\langle \mathfrak{c}_{\zeta,\epsilon} : \epsilon < \epsilon_j(*)\rangle$, $\epsilon_j(*) < \sigma$, $\mathfrak{c}_{j,\epsilon} \subseteq \mathfrak{c}$, $\max\mathrm{pcf}(\mathfrak{c}_{j,\epsilon}) < \mu$ and $\mathfrak{c}_j = \bigcup_\epsilon \mathfrak{c}_{\zeta,\epsilon}$. We let

$$\mathcal{C} = \bigcup\{\mathfrak{c}_{\zeta,\epsilon} : \epsilon < \epsilon_j(*) \text{ and } j < j(*)\}$$

and for each $\mathfrak{c} \in \mathcal{C}$ choose $t_\mathfrak{c}$ large enough and continue as in the proof of 5.4(1).

3) Read the proof of " $\le$ " of 5.4(1).

4) By 2.3(6) — use $\lambda^* \in [\kappa, \lambda]$ minimal with $\mathrm{pp}_\Gamma(\lambda^*) > \lambda$. $\square_{5.4}$

Conclusion 5.5 $\mathrm{cov}(\lambda, \lambda, (\mathrm{cf}\lambda)^+, \mathrm{cf}\lambda) = \mathrm{pp}_{\Gamma(\mathrm{cf}\lambda)}(\lambda)$ for λ singular of uncountable cofinality (remember $\Gamma(\sigma) = \Gamma(\sigma^+, \sigma)$).

Lemma 5.6 *(Cardinal arithmetic vs* cov*). Suppose $\lambda \ge \kappa \ge \theta > \sigma$, $\mathrm{cf}\kappa \ge \sigma$ and $[\mathrm{cf}\theta \ge \sigma \vee 2^{<\theta} < \lambda]$.*
<u>*Then*</u>

$$\lambda^{<\theta} = \mathrm{cov}(\lambda, \kappa, \theta, \sigma)^{<\sigma} + \sum_{\alpha<\kappa} |\alpha|^{<\theta}.$$

Proof: The proof of " $\le$ " is straightforward by the definition of cov. For the other direction, $\lambda \ge \kappa$ implies $\lambda^{<\theta} \ge \kappa^{<\theta} \ge \sum_{\alpha<\kappa} |\alpha|^{<\theta}$, and $\mathcal{P} = \{a \subseteq \lambda : |a| < \theta\}$ exemplifies $\lambda^{<\theta} \ge \mathrm{cov}(\lambda, \kappa, \theta, \sigma)$; hence it suffices to have $(\lambda^{<\theta})^{<\sigma} = \lambda^{<\theta}$. This holds if $\mathrm{cf}(\theta) \ge \sigma$; and also if $(\exists\theta(1) < \theta)[\lambda^{<\theta} = \lambda^{\theta(1)}]$ which holds if $2^{<\theta} < \lambda$ (by Hajnal [H] and later and independently [Sh233,2.12]). $\square_{5.6}$

Conclusion 5.7 Assume $\mathrm{cf}(\lambda) > \aleph_0$, then:

$$\lambda^\theta = \begin{cases} 2^\theta & \text{if } \lambda \le 2^\theta \\ \sum_{\lambda_1<\lambda} \lambda_1^\theta & \text{if } \mathrm{cf}\lambda > \theta \text{ or } (\exists\lambda_1)(\lambda_1<\lambda\le\lambda_1^\theta) \\ \mathrm{cov}(\lambda,\lambda,(\mathrm{cf}\lambda)^+,\mathrm{cf}\lambda) = \mathrm{pp}_{\Gamma(\mathrm{cf}\lambda)}(\lambda) & \text{otherwise} \end{cases}$$

Lemma 5.8 *(1) Suppose $\aleph_0 < \sigma = \mathrm{cf}\sigma < \theta$, $\Gamma = \Gamma(\theta,\sigma)$ (see 5.4).*
If $\mathrm{cf}(\lambda) \in [\sigma,\theta)$, $\lambda \ge \theta$ then $\mathrm{pp}_\Gamma(\lambda)$ is
$\sup\{\mu$ *: there are $\langle<\mu_\ell,\theta_\ell>: \ell \le n\rangle$ such that (for $\ell \le n$)*
$\theta_\ell = \mathrm{cf}\theta_\ell = \mathrm{cf}\mu_\ell \in [\sigma,\theta)$, $\mu_0 = \lambda$, and for $\ell < n$ we have $\mu_{\ell+1} \le \mathrm{pp}_{\Gamma(\theta_\ell)}(\mu_\ell)$ and $\mu \le^+ \mathrm{pp}_{\Gamma(\theta_n)}(\mu_n)$
(meaning that, if equality holds, the sup is obtained, Def.1.1)$\}$.
(2) In 5.3(7) equality holds when $\sigma = \mathrm{cf}\sigma > \aleph_0$.

Proof: (1) Easy by now: $\mathrm{pp}_\Gamma(\lambda)$ is at least as large as the sup by repeated use of 2.3(2),(3) (for θ singular, it is enough to do it for every $\theta_1 = \mathrm{cf}\theta_1 \in [\sigma,\theta)$). For the other inequality use 5.4(1) (to translate it to a problem on cov) and 5.3(7).
(2) Easy too. $\square_{5.8}$

Remark 5.8A Also for $\sigma = \aleph_0$ Lemma 5.8(1) is true (by [Sh371, §1]).

Conclusion 5.9 For $\lambda > 2^{<\theta}$, $\theta \ge \sigma$, $\sigma = \mathrm{cf}\sigma > \aleph_0$, $\Gamma = \Gamma(\theta,\sigma)$ we have

$$\mathrm{cov}(\lambda,\theta,\theta,\sigma) = \sup\{\mathrm{pp}_\Gamma(\lambda^*) : \theta \le \lambda^* \le \lambda \text{ (and } \sigma \le \mathrm{cf}\lambda^* < \theta)\}$$
$$= T_\Gamma(\lambda) = T^+_\Gamma(\lambda)$$

where, remember:

Definition 5.10 (1) For an ideal I,
$T_I(\lambda) = \sup\{|F| : F$ is a set of functions from Dom I to λ such that $f_1 \ne f_2 \in F \Rightarrow \{i : f_1(i) = f_2(i)\} \in I\}$.
(2) For a family Γ of ideals $T_\Gamma(\lambda) = \sup\{T_I(\lambda) : I \in \Gamma\}$.
(3) For a family Γ of ideals,
$T^+_\Gamma(\lambda) = \sup\{\mu$: there are (θ_i, f_i, I_i) for $i < \mu$ such that I_i an ideal on θ_i, $I_i \in \Gamma$ and f_i a function from θ_i to λ such that for $i \ne j$,
$\{\alpha < \theta_i : f_i(\alpha) \in \mathrm{Rang}\, f_j\} \in I_i\}$.

Proof of 5.9: First equality by 5.4, second term is $\le$ third term easily, third term $\le$ fourth term by Definition 5.10 and fourth term $\le \mathrm{cov}(\lambda,\theta,\theta,\sigma)$ as $2^{<\theta} < \lambda$, so we finish. $\square_{5.9}$

Lemma 5.11 *For every regular λ at least one of the following holds:*
(a) $2^\lambda = 2^{<\lambda}$
(b) for some μ, $\lambda = \mathrm{cf}(\mu) < \mu \le 2^{<\lambda}$, and $\mathrm{pp}_{\Gamma(\lambda)}(\mu) =^+ 2^\lambda$ (of course, for any such μ, $\mathrm{pp}_{\Gamma(\lambda)}(\mu) \le \mu^\lambda = 2^\lambda$).
(c) (i) there are $f_i : \lambda \to \lambda$ for $i < 2^\lambda$ such that for $i \ne j$ f_i, f_j are different on a cobounded subset of λ and
(ii) for each regular $\chi \le 2^\lambda$ there is a dense linear order $\mathcal{I}$ of power λ with χ Dedekind cuts with cofinality λ both sides (equivalently, a tree of cardinality λ with $\ge \chi$ λ-branches).

Remark 5.11A (1) It is known that part (c) here implies: for every $\chi < 2^\lambda$ and normal filter D on λ in $\mathcal{P}(\lambda)/D$ there are χ pairwise disjoint elements.

Proof: Without loss of generality $\lambda > \aleph_0$ (otherwise (c) holds). We assume not (a) nor (b). Let χ be a cardinal, $2^{<\lambda} < \chi \le 2^\lambda, \mathrm{cf}(\chi) > 2^{<\lambda}$. Let μ be the minimal cardinal such that there is F, $|F| \ge \chi$ satisfying $(*)^\mu_\lambda(F)$ below where μ stands for the constant function with domain λ and value μ, and for $g : \lambda \to \mathrm{Ord}$ we let:

$(*)^g_\lambda(F)$ F is a family of functions from λ to ordinals, such that

$$f \ne h \in F \Rightarrow \exists\alpha<\lambda, \forall\beta,\gamma<\lambda[\alpha\le\beta \ \&\ \alpha\le\gamma \Rightarrow g(\beta) > f(\beta) \ne h(\gamma)].$$

Now $\mu = 2^{<\lambda}$ is *O.K.* [Let $H : {}^{\lambda>}2 \to 2^{<\lambda}$ be one to one, so if we let for $\eta \in {}^{\lambda>}2$, f_η be the function from λ to $2^{<\lambda}$ defined by $f_\eta(\alpha) = H(\eta{\restriction}\alpha)$, then $F =: \{f_\eta : \eta \in {}^\lambda 2\}$ has power $2^\lambda \ge \chi$ and is as required].
Let $\Gamma = \{I : I$ a λ-complete ideal on $\lambda\}$.
By 5.4(2), applied to $\mathrm{cov}(2^{<\lambda},\lambda^+,\lambda^+,\lambda)$, it is $< 2^\lambda$ (remember we are assuming that (b) fails because as in 5.4(4) (i.e. using 2.3) the sup is max). So for some $A \subseteq \mu \subseteq 2^{<\lambda}$, $|A| = \lambda$, $\{\eta \in F : |\mathrm{Rang}(f_\eta) \cap A| = \lambda\}$ has power $\ge \chi$. This exemplifies (c)(i) (well if 2^λ is singular, we can glue the various examples using a pairing function on λ).
As χ was an arbitrary cardinal $\le 2^\lambda$ with cofinality $> 2^{<\lambda}$, we have finished the proof that (a) or (b) or (c)(i) holds.

We want to get (c)(ii) in the case that (c)(i) is proved. Let us consider $({}^\lambda 2, <_{\ell x})$, where $<_{\ell x}$ is lexicographic order; it is a dense linear order if we omit the eventually zero sequences; let $\eta_i \in {}^\lambda 2$ for $i < 2^\lambda$ be distinct not eventually constant; and let $f_i : \lambda \to {}^{\lambda>}2$ be $f_i(\alpha) = \eta_i{\restriction}\alpha$. By the proof above if (a), (b) fail then: for each χ, $2^{<\lambda} < \chi \le 2^\lambda$, χ regular there is $T \subseteq {}^{\lambda>}2$, such that $|T| = \lambda$ and:

$$Y =: \{i < 2^\lambda : |T \cap \mathrm{Rang}(f_{\eta_i})| = \lambda\} \text{ has cardinality } \ge \chi;$$

now let

$$T^+ = \{\nu \in {}^{\lambda}2\colon \text{for some } \eta \in T, \text{ and } \alpha < \lambda \text{ we have: } \nu\restriction\alpha = \eta\restriction\alpha,\ (\forall\beta \in [\alpha,\lambda))\,\nu(\beta) = 0\}.$$

So T^+ is a linear order, and for each $i \in Y$ we have a Dedekind cut as required, and the cuts are distinct. $\square_{5.11}$

Claim 5.12 Suppose $\mu = \text{cov}(\mu, \theta_1, \theta_1, \theta_2)$, $\theta_1 > \theta_2$ are regular uncountable. Then we can find a family $\mathcal{P}$ of μ subsets of μ, each of power $< \theta_1$ such that: <u>for any</u> model M of universe μ, with vocabulary of power $< \theta_1$, <u>there is</u> $N \prec M$ (of cardinality $< \theta_1$) whose universe is the union of $< \theta_2$ member of $\mathcal{P}$.

Remark 5.12A If $\mu_1 = \text{cov}(\mu_0, \theta_1, \theta_1, \theta_2)$ <u>then</u> for $\mu = \mu_0$, we can find such $\mathcal{P}$ of power μ_1 see 5.3(7) + 5.8.

Proof: Easy. Choose by induction on n, $N_n \prec M$, $\mathcal{P}_n \subseteq \mathcal{P}$ such that $N_n \subseteq \bigcup_{A\in\mathcal{P}_n} A \subseteq N_{n+1}$, $\|N_n\| < \theta_1$, $|\mathcal{P}_n| < \theta_2$. $\square_{5.12}$

§6 λ-Freeness

We give here other applications: for example, to the problem of the existence of a family of λ sets, each of cardinality $\leq \kappa$, which is λ-free not free (this property is called NPT(λ,κ)). Here, free means having a one-to-one choice function, and λ-free means having all subsets of cardinality $< \lambda$, free. (We also can have versions with more parameters, which we ignore for simplicity.) remember that for $\kappa = \aleph_0$ this has an equivalent algebraic form — there is an λ-free not free abelian group. It is also known that for λ singular this cannot occur (none of the results, except as elucidation, are used in this book).
Note that again failure of remnants of GCH gives us information.
Now by 6.5 we get

$\otimes_1$ if $\lambda > \text{cf}\lambda = \kappa$, $\text{pp}(\lambda) > \lambda^+$ then NPT(λ,κ) holds

more than this:

$\otimes_2$ if $\lambda > \kappa \geq \text{cf}\lambda$, $\text{pp}_\kappa(\lambda) > \mu > \lambda$, we get a family of μ subsets of λ, which is λ^+-free, trivially not free, and more.

Still there was a gap in 5.4 for the case $\sigma = \aleph_0$, so exactly for the case $\lambda > \kappa =\colon \text{cf}\lambda = \aleph_0$, $\text{cov}(\lambda,\lambda,\aleph_1,2) > \mu > \lambda$ we get nothing. But by 6.3, for $\mu = \lambda^+$ we get the conclusion of $\otimes_2$ and this is generalized in 6.8. In 6.2 we collect some basic facts and in 6.7 investigate the relations between the cases of the NPT.

In 6.9 (and 6.9A — 6.9E) we deal with the following problem: let for simplicity λ be singular strong limit of cofinality $\aleph_0$, $\theta^* < \lambda$. Is there $T \subseteq {}^{\omega}\lambda$ (i.e. T is a set of ω-sequences of ordinals $< \lambda$) which has cardinality $\lambda^{\aleph_0}(= 2^\lambda)$ but has no perfect subset of density character θ^*? Of course, it would be better to partition ${}^{\omega}\lambda$ to two sets with this property and to have $\theta^* = \aleph_0$; it then comes close to a well known topological question: can you divide a topological space to two, each part with no compactum. What we get is that

$\otimes_3$ for some θ^*, for every strong limit $\lambda > \theta^*$ of cofinality $\aleph_0$ there is $T \subseteq {}^{\omega}\lambda$ of cardinality $\lambda^{\aleph_0} = 2^\lambda$ with no perfect subset of density character $\geq \theta^*$.

Note that the results involving an ideal I gotten from an assumption of the form $\text{pp}(\lambda) > \mu$, give better conclusions when (see more cases in [Sh371, §1]) we can represent cardinals $\mu = \text{cf}\mu \in [\lambda, \text{pp}^+_\kappa(\lambda))$ in the form $\mu = \prod \mathfrak{a}/J^{\text{bd}}_{\mathfrak{a}}$, $\lambda = \sup \mathfrak{a}$, by results like $\text{pp}_{J^{\text{bd}}_\kappa}(\lambda) = \text{pp}(\lambda)$ for suitable $\lambda > \text{cf}\lambda = \kappa > \aleph_0$.

* * *

Definition 6.1 (1) NPT(λ,κ) means: there is a family $\mathcal{P}$ of λ subsets of λ each of which has cardinality $\leq \kappa$, $\mathcal{P}$ has no transversal (= one to one choice function) but every $\mathcal{P}' \subseteq \mathcal{P}$ of cardinality $< \lambda$ has a transversal.

(2) NPT$_J(\lambda,\kappa)$, where J is an ideal on κ, means: there is a family $\mathcal{P}$ of λ functions from κ to λ, such that:
 (i) if $\mathcal{P}' \subseteq \mathcal{P}$, $|\mathcal{P}'| < \lambda$ then $\mathcal{P}'$ is J-free;
 i.e. there are $\langle s_f : f \in \mathcal{P}'\rangle$, $s_f \in J$, and:

$$f \neq g \in \mathcal{P}' \ \&\ i \in \kappa\backslash s_f \setminus s_g \Rightarrow f(i) \neq g(i)$$

 (ii) $\mathcal{P}$ is not J-free.

(3) NPT$_J(\mu,\lambda,\theta,\kappa)$ where J is an ideal on κ, $\mu \geq \lambda \geq \theta \geq \kappa$, means: there is a family $\mathcal{P}$ of μ functions from κ to λ, $\mathcal{P}$ not J-free, but every $\mathcal{P}' \subseteq \mathcal{P}$ of cardinality $< \theta$ is J-free.

(4) In (3) if $J = J^{\text{bd}}_\kappa$, we omit it; if $\theta = \lambda$ we omit it.

(5) NPT$_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$ where J is an ideal on κ, $\mu \geq \lambda \geq \theta_1 \geq \theta_2 + \kappa$, means: there is a family $\mathcal{P}$ of μ functions from κ to λ such that:
 (a) $\mathcal{P}$ is (θ_1,θ_2)-free which means: for $\mathcal{P}' \subseteq \mathcal{P}$, if $|\mathcal{P}| < \theta_1$, then there are $\langle s_f : f \in \mathcal{P}'\rangle$, $s_f \in J$ and for each $f \in \mathcal{P}'$,

$$|\{g \in \mathcal{P}' : (\exists i < \kappa)[i \notin s_f \cup s_g \ \&\ g(i) = f(i)]\}| < \theta_2.$$

 (b) for $\mathcal{P}' = \mathcal{P}$ the condition above fails, i.e. $\mathcal{P}$ is not (μ^+,θ_2)-free.

(6) $\mathrm{NPT}^-_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$ is defined similarly replacing (a) by:
(a$^-$) $\mathcal{P}$ is weakly (θ_1,θ_2)-free which means: for every regular $\theta\in[\theta_2,\theta_1]$ and $\mathcal{P}'\subseteq\mathcal{P}$ of cardinality θ there is a J-free $\mathcal{P}''\subseteq\mathcal{P}'$ of cardinality θ.

Fact 6.2 (1) In the definition of $\mathrm{NPT}_J(\mu,\lambda,\theta,\kappa)$ if $\mu>\lambda$ then "$\mathcal{P}$ not J-free" follows, assuming J is proper.
(1a) Similarly for $\mathrm{NPT}_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$ and $\mathrm{NPT}^-_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$.
(2) If $\mu=\lambda$, $\mathrm{NPT}_J(\mu,\lambda,\theta,\kappa)$ is equivalent to

$$(\exists\lambda^*)[\theta\le\lambda^*\le\lambda\ \&\ \mathrm{NPT}_J(\lambda^*,\lambda^*,\lambda^*,\kappa)].$$

(3) If $\lambda>\mathrm{cf}(\lambda)+\kappa$, J an ideal on κ, then $\mathrm{NPT}_J(\lambda,\kappa)$ fails [by [Sh52] or see [Sh161,§0] but we shall have no essential use of it].
(4) $\mathrm{NPT}(\lambda,\aleph_0)$ iff there is a λ-free, non-free abelian group of power λ [holds by [Sh161] but we shall have no essential use of it].
(5) $\mathrm{NPT}_J(\mu,\lambda,\theta,\kappa)$ iff $\mathrm{NPT}_J(\mu,\lambda,\theta,2,\kappa)$.
(6) If $\theta_1<\theta_2<\theta_3$, $\mathrm{cf}(\theta_2)=\theta_2$, $\mu>\lambda$ and for $\ell=1,2$, $\mathrm{NPT}_J(\mu,\lambda,\theta_{\ell+1},\theta_\ell,\kappa)$ then $\mathrm{NPT}_J(\mu,\lambda,\theta_3,\theta_1,\kappa)$
[if for $\ell=1,2$ $\langle f^\ell_\alpha:\alpha<\mu\rangle$ is a witness for the corresponding assumption use $\langle f_\alpha:\alpha<\mu\rangle$, $f_\alpha(i)=\langle f^1_\alpha(i),f^2_\alpha(i)\rangle$ and an injection from $\lambda\times\lambda$ into λ].
(7) If $\sigma<\theta$ are regular, $\kappa_\ell<\sigma<\theta<\lambda_1<\lambda_2<\lambda_3$ and $\mathrm{NPT}_{J_\ell}(\lambda_{\ell+1},\lambda_\ell,\theta,\sigma,\kappa_\ell)$ for $\ell=1,2$ and $J=J_1\times J_2$ then $\mathrm{NPT}_J(\lambda_3,\lambda_1,\theta,\sigma,\kappa_1+\kappa_2)$.
(8) If $\mu>\lambda\ge\theta\ge\kappa$, and there is a family $\mathcal{P}$ of μ subsets of λ, each of power $\le\kappa$ with no transversal but such that every $\mathcal{P}'\subseteq\mathcal{P}$ of cardinality $<\theta$ has a transversal then for some regular ideal J on κ $\mathrm{NPT}_J(\mu,\lambda,\theta,\kappa^+,\kappa)$.
(9) We can in (8) allow $\mu=\lambda$ if we weaken somewhat the definition of NPT_J (calling it $\mathrm{NPT}'_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$: instead of $\mathcal{P}$ being a family of μ functions f from κ to λ, it is a family of μ sequences $\bar f=\langle f_\ell:\ell<n\rangle$, f_ℓ a function from κ to λ, $\mathcal{P}$ is (θ_1,θ_2)-free (which means: for $\mathcal{P}'\subseteq\mathcal{P}$ of cardinality $<\theta_1$ we can find $(\ell_{\bar f},s_{\bar f})$ for $\bar f\in\bar{\mathcal{P}}'$, $\ell_{\bar f}<\ell g(\bar f)$, $s_f\in J$ such that for every z we have

$$|\{\bar f\in\mathcal{P}':z\in\mathrm{Rang}(\bar f[\ell_{\bar f}]\restriction(\kappa\backslash s_{\bar f})\}|<\theta_1 \text{ and } \bar f\ne\bar g\Rightarrow\bar f[\ell_{\bar f}]\ne\bar g[\ell_{\bar g}],$$

but $\mathcal{P}$ is not (μ^+,θ_2)-free).
(10) Also the inverse implication holds in (8) (even if $\mu=\lambda$) and (9).
(11) If $\lambda>\mu\ge\mathrm{cf}\lambda+\theta$ and $\mathrm{NPT}(\lambda',\mu,\theta,\sigma,\kappa)$ holds for every $\lambda'<\lambda$ then $\mathrm{NPT}(\lambda,\mu,\theta,\sigma,\kappa)$ holds.

Proof: (7) Let $F_\ell=\{f^\ell_\alpha:\alpha<\lambda_{\ell+1}\}$ exemplify $\mathrm{NPT}_{J_\ell}(\lambda_{\ell+1},\lambda_\ell,\theta,\sigma,\kappa_\ell)$ for $\ell=1,2$ so $\kappa_\ell=\mathrm{Dom}f^\ell_\alpha$. Let for $\alpha<\lambda_3$, f_α be the following function; its domain is $\kappa_1\times\kappa_2$, $f_\alpha(\langle i,j\rangle)=f^1_{f^2_\alpha(j)}(i)$.
(8) Let $\{w_i:i<\kappa\}$ list the finite subsets of κ,

$$J_0=\{A\subseteq\kappa:\ \text{for some } i,\ \ (\forall j\in A)[w_i\not\subseteq w_j]\}$$

for $x=\{\alpha^x_\zeta:\zeta<\zeta_x\le\kappa\}\in\mathcal{P}$ let

$$f_x(i)=\{\alpha^x_j:j\in w_i\cap\zeta_x\}$$

(so $f_x(i)$ is a finite subset of $x\subseteq\lambda$ rather than a member of λ, this is minor), and let $F_0=\{f_x:x\in\mathcal{P}\}$.

We shall finish by proving that F_0 exemplifies $\mathrm{NPT}_{J_0}(\mu,\lambda,\theta,\kappa^+,\kappa)$. First we prove (θ,κ^+)-freeness. Suppose

$$F_1\subseteq F_0,\ |F_1|<\theta,\ F_1=\{f_{x_\zeta}:\zeta<\zeta^*<\theta\},$$

so $\{x_\zeta:\zeta<\zeta^*\}$ has a transversal h; now we shall define $s_f\in J_0$ for $f\in F_1$ such that: for each z the set

$$\{f\in F_1:z\in\mathrm{Rang}(f\restriction(\kappa\backslash s_f))\}$$

is finite: for $x=x_\zeta$, we let $s_{f_x}=\{j<\kappa:h(x)\notin\{\alpha^x_\zeta:\zeta\in w_j\}\}$.

So we have proved the (θ,κ^+)-freeness. As for the non-freeness part, as here $\mu>\lambda$ use 6.2(1).
(9) Use $\mathcal{P}$ whose existence is proved in [Sh161,§3] (so we shall not use it) (or see [EM]). $\square_{6.2}$

Remark 6.2A By the analysis in [Sh161,§3], if $\mu=\lambda$, part (8) of 6.2 may fail.

Theorem 6.3 *Suppose* $\lambda>\mathrm{cf}(\lambda)=\aleph_0$, $\mathrm{cov}(\lambda,\lambda,\aleph_1,2)>\lambda^+$ *(for example* $\lambda^{\aleph_0}>\lambda^+$ & $(\forall\mu<\lambda)$ $(\mu^{\aleph_0}<\lambda)$*. Then* $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda^+,\aleph_0)$.

Remark 6.3A In fact we prove $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda^+,\lambda,\lambda^+,\aleph_0)$.

Proof: Let $\lambda=\sum_{n<\omega}\lambda_n$, $\lambda_n<\lambda_{n+1}<\lambda$. For each $\alpha<\lambda^+$ let $\alpha=\bigcup_{n<\omega}A^\alpha_n$ where $|A^\alpha_n|\le\lambda_n$, $A^\alpha_0=\emptyset$, $A^\alpha_n\subseteq A^\alpha_{n+1}$. Now we choose by induction on $\alpha<\lambda^+$, x_α such that:
(a) x_α is a countable (infinite) subset of λ
(b) for no $\beta<\lambda^+$, $n<\omega$ is x_α a subset of $\cup\{x_\gamma:\gamma\in A^\beta_n\cap\alpha\}$.

There is no problem to carry the induction: at stage α,

$$\{\cup\{x_\gamma:\gamma\in A^\beta_n\cap\alpha\}:n<\omega,\beta<\lambda^+\}$$

is a family of λ^+ subsets of λ, each of power $<\lambda$, so as $\lambda^+<\mathrm{cov}(\lambda,\lambda,\aleph_1,2)$, there is x_α as required.

Now

(*) for each $\beta < \lambda^+, n < \omega$, there is a transversal f^β_n of $\{x_\alpha : \alpha \in A^\beta_n\}$. [Simply define $f^\beta_n \restriction (A^\beta_n \cap \alpha)$ by induction on α; for $\alpha = 0$, and α limit we have no problem; for α successor use (b) above].

It is also clear that (again by (b)):

(**) for $\alpha_1 \neq \alpha_2$, $x_{\alpha_1} \neq x_{\alpha_2}$.

Let η_α be an ω-sequence enumerating x_α (for $\alpha < \lambda^+$). Now for each $\beta < \lambda^+$, for $\alpha \in A^\beta_{n+1} \backslash A^\beta_n$ let $k = k_\beta(\alpha)$ be such that $f^\beta_n(\alpha) = \eta_\alpha(k)$. It is easy to check that for every $\nu \in {}^{\omega>}\lambda$, $\{\alpha < \beta : \nu = \eta_\alpha \restriction (k_\beta(\alpha)+1)\}$ is countable (has at most one member in each $A^\beta_{n+1} \backslash A^\beta_n$). We define a graph on $\{\alpha : \alpha < \beta\}$: α_1, α_2 are connected iff $\{\eta_{\alpha_1} \restriction \ell : k_\beta(\alpha_1) < \ell < \omega\}$, $\{\eta_{\alpha_2} \restriction \ell : k_\beta(\alpha_2) < \ell < \omega\}$ are not disjoint. So every node has valency $\leq \aleph_0$, so the connected components of the graph are countable (as by (**) $\alpha_1 \neq \alpha_2 \Rightarrow \eta_{\alpha_1} \neq \eta_{\alpha_2}$), so we can find a function $h_\beta : \beta \to \omega$ such that $\{\eta_\alpha \restriction \ell : h_\beta(\alpha) \leq \ell < \omega\}$, for $\alpha < \beta$, are pairwise disjoint. $\square_{6.3}$

Remark 6.4 1) Note: if $\langle x_\alpha : \alpha < \mu \rangle$ is a sequence of countable subsets of λ, $\lambda < \mu$ and $[A \subseteq \mu \ \& \ |A| < \theta \Rightarrow \langle x_\alpha : \alpha \in A \rangle$ has a transversal] (for example $\mathrm{NPT}_J(\mu, \lambda, \theta, \aleph_0)$) then $\mathrm{NPT}(\mu, \lambda, \theta, \aleph_0)$.
2) We can replace in 6.3, 6.4(1) ω by $\theta > \omega$ but use $1.5A + 5.4$ to get (*) below concerning 6.3, and add $(\forall \chi < \lambda)[\chi^{<\theta} < \lambda]$ for 6.4(1).

(*) if $\lambda < \mu = \mathrm{cf}\mu < \mathrm{cov}(\lambda, \lambda, (\mathrm{cf}\lambda)^+, \mathrm{cf}\lambda)$ and $\mathrm{cf}\lambda > \aleph_0$ then $\mathrm{NPT}(\mu, \lambda, \lambda, \mathrm{cf}\lambda)$.

3) So if $\mathrm{cf}(\lambda) > \aleph_0$ we can get a stronger conclusion in 6.3; however, if $\mathrm{cf}(\lambda) = \aleph_0$ (and $\aleph_\lambda = \lambda$) we do not know whether:

$$[\mathrm{cov}(\lambda, \lambda, \aleph_2, 2) > \lambda^+ \Leftrightarrow \mathrm{pp}(\lambda) > \lambda^+].$$

4) So generally, we pay less attention to the case $\mathrm{NPT}(\ldots, \sigma)$, $\sigma > \aleph_0$.

Claim 6.5 (1) If $\mu \leq^+ \mathrm{pp}^*_J(\lambda)$, μ regular and $\theta_2 = (2^{|\mathrm{Dom}\ J|})^+$, $\theta_2 < \theta_1 < \theta_2^{+\omega} \leq \lambda$, then $\mathrm{NPT}^-_J(\mu, \lambda, \theta_1, \theta_2, |\mathrm{Dom}\ J|)$.
(2) If $\mu < \mathrm{pp}^*_J(\lambda)$, μ regular $> \lambda$ and $\lambda^+ \geq \theta_1$ then $\mathrm{NPT}_J(\mu, \lambda, \theta_1, 2, |\mathrm{Dom}\ J|)$.
(3) If $\lambda < \mu \leq^+ \mathrm{pp}^*_J(\lambda)$, μ regular, $\theta \leq \lambda^+$ and $\{\delta < \mu : \mathrm{cf}\delta < \theta\} \in I[\mu]$ (see [Sh345a,2.3(5)]) then $\mathrm{NPT}_J(\mu, \lambda, \theta, |\mathrm{Dom}\ J|)$.
(4) In part (2), "$\mu \leq \mathrm{pp}^*_J(\lambda)$, μ singular" suffice.
(5) If $\mu \leq \mathrm{pp}^*_J(\lambda)$, μ singular $> \lambda$ then $\mathrm{NPT}_J(\mu, \lambda, \lambda, |\mathrm{Dom}\ J|)$.

Remark 6.5A Any (θ_1, θ_2) such that (*) below holds will do in 6.5(1).

(*) $\theta_2 < \theta_1 < \lambda$, and for $\sigma = \mathrm{cf}\sigma \in [\theta_2, \theta_1]$, if $f'_\alpha \in {}^{\mathrm{Dom}(J)}(\lambda)$ for $\alpha < \sigma$, and $\langle f'_\alpha : \alpha < \sigma \rangle$ is $<_J$-increasing then in 1.6(1) statement (a) holds.

Proof: (1) Straightforward (use (*) of 6.5A which holds by 1.2A(3)+1.6).
(2) Remember 1.5A.
(3) Easy, too. [Let $\langle C_\alpha : \alpha < \mu \rangle$ be such that $C_\alpha \subseteq \alpha$, $\beta \in C_\alpha \Rightarrow C_\beta = \alpha \cap C_\beta$, $\mathrm{otp}(C_\alpha) < \lambda$, E a club of μ, $[\delta \in E \ \& \ \mathrm{cf}(\delta) < \lambda \Rightarrow \delta = \sup C_\delta]$. By assumption there is $\bar\lambda = \langle \lambda_i : i < \kappa \rangle$, where $\lambda_i < \lambda = \mathrm{tlim}_J \lambda_i$, and $\prod_{i<\kappa} \lambda_i / J$ is μ-directed. Now choose by induction on $\alpha < \mu$, $f_\alpha \in \prod_{i<\kappa} \lambda_i$ such that:

(a) $f_\beta <_J f_\alpha$ for $\beta < \alpha$,
(b) $f_\beta(i) < f_\alpha(i)$ if $\beta \in C_\alpha$, $|C_\alpha| < \lambda_i$.

Now $\{f_\alpha : \alpha < \mu\}$ is as required: check as in the proof of clause (δ) of 1.3(ii).]
(4) See the proof of 6.9B.
(5) Use part (2) on many regulars $< \mu$ and combine. $\square_{6.5}$

Conclusion 6.6 If every λ-free abelian group is free, then:

$$\mu \text{ singular } \ \& \ \mu \geq \lambda \Rightarrow \mathrm{pp}(\mu) = \mu^+ \ \& \ \mathrm{cov}(\mu, \mu, (\mathrm{cf}\mu)^+, \mathrm{cf}\mu) \leq \mu^+.$$

Remark 6.6A Note that for μ singular,

$$\mathrm{cov}(\mu, \mu, (\mathrm{cf}\mu)^+, \mathrm{cf}\mu) \leq \mu^+ \Rightarrow \mathrm{cov}(\mu, \mu, (\mathrm{cf}\mu)^+, 2) \leq \mu^+$$

[as if $\mathcal{P} \subseteq [\mu]^{<\mu}$ exemplifies $\mathrm{cov}(\mu, \mu, (\mathrm{cf}\mu)^+, \mathrm{cf}\mu) \leq \mu^+$, let $\mathcal{P} = \{A_\alpha : \alpha < \mu^+\}$, let for $\alpha < \mu^+$, g_α be a one to one function from μ onto $1 + \alpha$; and now

$$\mathcal{P}' = \left\{ \cup \{A_{g_\alpha(i)} : i < i^*, |g_\beta(i)| < \mu^*\} : \alpha < \mu^+, i^* < \mu \text{ and } \mu^* < \mu \right\}$$

exemplifies $\mathrm{cov}(\mu, \mu, (\mathrm{cf}\mu)^+, 2) \leq \mu^+$].

Proof: By 6.2(4) $[\chi \geq \lambda \Rightarrow \neg\mathrm{NPT}(\chi, \aleph_0)]$, hence by 6.3

$$[\mu \geq \lambda \ \& \ \mathrm{cf}\mu = \aleph_0 \Rightarrow \mathrm{cov}(\mu, \mu, \aleph_1, 2) \leq \mu^+].$$

For the first conjunct (in the conclusion), by the above and 5.4(3)

$$[\mu \geq \lambda \ \& \ \mathrm{cf}\mu = \aleph_0 \Rightarrow \mathrm{pp}(\mu) = \mu^+],$$

now by 2.1 (see more 2.4(1) and [Sh371, 1.10]) the first conjunct in the conclusion holds also when $\mathrm{cf}\mu > \aleph_0$. Lastly the second conjunct holds also when $\mathrm{cf}\mu > \aleph_0$ by 5.4 (and 6.6A). $\square_{6.6}$

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Claim 6.7

(1) Suppose θ_2 is regular $> \aleph_0$ (or θ_2 is 2), $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda_2,\lambda_1,\theta_1,\theta_2,\omega)$ and $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda_3,\lambda_2,\theta_1,\theta_2,\omega)$ <u>then</u> $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda_3,\lambda_1,\theta_1,\theta_2,\omega)$.

(2) $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\mu,\lambda,\theta,\aleph_1,\aleph_0)$ is equivalent to $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\mu,\lambda,\theta,2,\aleph_0)$ is equivalent to $\mathrm{NPT}(\mu,\lambda,\theta,\aleph_0)$ (even by the same families).

(3) $\mathrm{NPT}_J(\mu,\lambda,\theta_1,\theta_2,\sigma)$, $\mu > \lambda$, $\mathrm{cf}(\lambda) \geq \sigma$, $(\forall \alpha < \lambda)[|\alpha|^{<\sigma} \leq \lambda]$ <u>implies</u> $\mathrm{NPT}_{J^{\mathrm{bd}}_\sigma}(\mu,\lambda,\theta_1,\theta_2,\sigma)$.

(4) If $\mathrm{NPT}_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$, J is a σ-complete ideal on κ, $\kappa < \theta^+ < \theta_1$, $\mu > \lambda \geq \theta_1 > \theta_2$ <u>then</u> $\mu \leq \mathrm{cov}(\lambda,\theta^+,\kappa^+,\sigma)$.

Proof: 1) Note that without loss of generality $\lambda_3 > \lambda_2 > \lambda_1$, the case $\theta_1 < \aleph_1$ is trivial so we shall ignore it. So for $\ell = 1,2$ there is $\langle f^\ell_\alpha : \alpha < \lambda_{\ell+1}\rangle$ which exemplify $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda_{\ell+1},\lambda_\ell,\theta_1,\theta_2,\omega)$. Now we define for $\alpha < \lambda_3$ a function f_α from ω to ${}^{\omega>}(\lambda_1)$:

$$f_\alpha(n) = \langle f^1_{f^2_\alpha(\ell)}(n) : \ell = 0,1,\dots,n\rangle.$$

Now $\langle f_\alpha : \alpha < \lambda_2\rangle$ exemplify $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda_3,\lambda_1,\theta_1,\theta_2,\omega)$ (if you are bothered by the f_α's having "wrong" range, rename it).

[Why? As for "not J^{bd}_ω-free" use 6.2(1). As for "(θ_1,θ_2)-free", let $A \subseteq \lambda_3$, $|A| < \theta_1$ and we shall find finite $s_\alpha \subseteq \omega$ for $\alpha \in A$ such that for every $\beta \in A$

$$\theta_2 > |\{\gamma \in A : (\exists \ell)[\ell < \omega \ \&\ \ell \notin s_\beta \ \&\ \ell \notin s_\gamma \ \&\ f_\beta(\ell) = f_\gamma(\ell)]\}|.$$

By the choice of $\langle f^2_\alpha : \alpha < \lambda_3\rangle$ there are finite $s^2_\alpha \subseteq \omega$ for $\alpha \in A$ such that for every $\beta \in A$, $|A^2_\beta| < \theta_2$ where

$$A^2_\beta =: \{\gamma \in A : (\exists \ell)[\ell < \omega \ \&\ \ell \notin s^2_\beta \ \&\ \ell \notin s^2_\gamma \ \&\ f^2_\beta(\ell) = f^2_\gamma(\ell)]\}.$$

Let $B = \cup\{\mathrm{Rang}\ f^2_\alpha : \alpha \in A\}$, this is a subset of λ_2 of cardinality $\leq |A| \times \aleph_0 < \theta_1$, hence by the choice of $\langle f^1_\alpha : \alpha < \lambda_2\rangle$ there are finite $s^1_\alpha \subseteq \omega$ for $\alpha \in B$ such that for every $\beta \in B$, $|A^1_\beta| < \theta_2$ where

$$A^1_\beta =: \{\gamma \in B : (\exists \ell)[\ell < \omega \ \&\ \ell \notin s^1_\beta \ \&\ \ell \notin s^1_\gamma \ \&\ f^1_\beta(\ell) = f^1_\gamma(\ell)]\}.$$

For each $\alpha \in A$ choose $n(\alpha)$ as the minimal $n > \sup s^2_\alpha$, and let

$$s_\alpha =: s^2_\alpha \cup s^1_{f^2_\alpha(n(\alpha))} \cup \{0,\dots,n(\alpha)\}.$$

We assume $\theta_2 = 2$. Now suppose

(*) $\ell < \omega, \ell \notin s_\beta, \ell \notin s_\gamma, f_\beta(\ell) = f_\gamma(\ell), n(\beta) = n(\gamma)$ but $\beta \neq \gamma \in A$.

Then $\ell \geq n(\beta)$, so $f_\beta(\ell)$ is a sequence of length $\geq n(\beta)+1$, with the $n(\beta)$-th member being $f^1_{f^2_\beta(n(\beta))}(\ell)$. Similarly $\ell \geq n(\gamma)$ so $f_\gamma(\ell)$ is a sequence of length $\geq n(\gamma)+1$, with $n(\gamma)$-th member being $f^1_{f^2_\gamma(n(\gamma))}(\ell)$. As $n(\beta) \notin s^2_\beta$, $n(\gamma) \notin s^2_\gamma$ (and $n(\beta) = n(\gamma)$) we have $f^2_\beta(n(\beta)) \neq f^2_\gamma(n(\gamma))$, and as $\ell \notin s^1_{f^2_\beta(n(\beta))}$ (remember $s^1_{f^2_\beta(n(\beta))} \subseteq s_\beta$ by its definition) and $\ell \notin s^1_{f^2_\gamma(n(\gamma))}$ (similarly) we have $f^1_{f^2_\beta(n(\beta))}(\ell) \neq f^1_{f^2_\gamma(n(\gamma))}(\ell)$. But we have assumed in (*) that $n(\beta) = n(\gamma)$, and the two ordinals above are the $n(\beta) = n(\gamma)$-th members of $f_\beta(\ell)$, $f_\gamma(\ell)$ respectively; contradiction. Now (*) is enough just for $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\lambda_3,\lambda_1,\theta_1,\theta_2,\omega)$ but by 6.7(2) we get the desired conclusion (the instance of (θ_1,θ_2)-freeness).

So next assume $\theta_2 \neq 2$, hence it is regular uncountable. Let us define for $\beta \in A$

$$A_\beta =: \bigcup_{\ell<\omega} A_{\beta,\ell}$$

where

$$A_{\beta,\ell} =: \{\gamma \in A : \ell \notin s_\beta, \ell \notin s_\gamma, f_\beta(\ell) = f_\gamma(\ell) \text{ and } n(\beta) = n(\gamma)\}.$$

It is enough to prove $|A_\beta| < \theta_2$ hence it is enough to prove $|A_{\beta,\ell}| < \theta_2$. For each $\zeta \in A^1_{f^2_\beta(n(\beta))}$ and $m < \omega$ choose $\gamma(\zeta,m) \in A$ such that $f^2_{\gamma(\zeta,m)}(m) = \zeta$ and $m \notin s^1_{\gamma(\zeta,m)}$, if there is such a $\gamma(\zeta,m)$. So assume $\gamma \in A_{\beta,\ell}$, then we can deduce $f^1_{f^2_\beta(n(\beta))}(\ell) = f^1_{f^2_\gamma(n(\gamma))}(\ell)$, $\ell \notin s^1_{f^2_\beta(n(\beta))} \cup s^1_{f^2_\gamma(n(\gamma))}$ hence $f^2_\gamma(n(\gamma)) \in A^1_{f^2_\beta(n(\beta))}$. Therefore

$$\gamma \in \bigcup\{A^2_{\gamma(\zeta,m)} : \zeta \in A^1_{f^2_\beta(n(\beta))}, m < \omega, \gamma(\zeta,m) \text{ is well defined}\}.$$

So

$$|A_{\beta,\ell}| \leq \sum\{|A^2_{\gamma(\zeta,m)}| : \zeta \in A^1_{f^2_\beta(n(\beta))}, m < \omega, \gamma(\zeta,m) \text{ is well defined}\} < \theta_2$$

(as θ_2 is regular).]

2) Easy (see in the proof of 6.3 starting with the choice of the η_α's).

3) Similarly: if $\{f_\alpha : \alpha < \mu\}$ exemplifies $\mathrm{NPT}_J(\mu,\lambda,\theta_1,\theta_2,\sigma)$, f^*_α is defined by: $f^*_\alpha(\gamma) = f_\alpha \restriction \gamma$ for $\gamma < \sigma$, then

$$\{f^*_\alpha : \alpha < \mu\} \text{ exemplify } \mathrm{NPT}_{J^{\mathrm{bd}}_\sigma}(\mu,\lambda,\theta_1,\theta_2,\sigma).$$

4) Suppose $\mu^* =: \mathrm{cov}(\lambda,\theta^+,\kappa^+,\sigma)$ and $\mathcal{P}$ exemplifies this and let $\{f_\alpha : \alpha < \mu\}$ exemplify $\mathrm{NPT}_J(\mu,\lambda,\theta_1,\theta_2,\kappa)$. For each $\alpha < \mu$, $\mathrm{Rang}(f_\alpha)$ is a subset

of λ of cardinality $\leq \kappa < \theta^+$, hence there is $t(\alpha) \subseteq \mathcal{P}$ such that $|t(\alpha)| < \sigma$ and $\mathrm{Rang}(f_\alpha) \subseteq \cup\{A : A \in t(\alpha)\}$. As J is σ-complete, for some $A_\alpha \in t(\alpha)$, $Y_\alpha =: \{i < \kappa : f_\alpha(i) \in A_\alpha\} \neq \emptyset \mod J$.

If $\mu^* < \mu$, for some $A \in \mathcal{P}$

$$|\{\alpha < \mu : A_\alpha = A\}| > \theta_1$$

(remember $\theta_1 \leq \lambda < \mu$).
This contradicts the (θ_1, θ_2)-freeness of $\{f_\alpha : \alpha < \mu\}$; so $\mu^* \geq \mu$ as desired. $\square_{6.7}$

Lemma 6.8 *Suppose*
(a) $\mathrm{cf}(\lambda) = \aleph_0 < \lambda$
(b) $\mu = \mathrm{cf}(\mu) > \lambda \geq \theta_1 > \theta_2$, *$\theta_2$ regular*
(c) $\mathrm{cov}(\mu, \lambda, \theta_1, \theta_2) < \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$
Then $\mathrm{NPT}_{J^{\mathrm{bd}}_\omega}(\mu, \lambda, \theta_1, \theta_2, \aleph_0)$.

Remark 6.8A (1) If λ is strong limit (or just $(\forall \lambda_1 < \lambda)[\lambda_1^{\aleph_0} < \lambda]$) then clause (c) is equivalent to $\mathrm{cov}(\mu, \lambda, \theta_1, \theta_2) < \lambda^{\aleph_0}$.
(2) If $\theta_2 = \aleph_1$ we can change it in the conclusion to 2 (by 6.7(2)).

Proof: Let $\mathcal{P}$ exemplifies $\chi =: \mathrm{cov}(\mu, \lambda, \theta_1, \theta_2)$. We now define by induction on $\alpha < \mu$ a set x_α such that:
(a) x_α is a countable subset λ,
(b) for no $B \in \mathcal{P}$ is x_α a subset *of* $\cup\{x_\gamma : \gamma \in B \cap \alpha\}$.
There is no problem to carry the construction as $|\mathcal{P}| = \chi < \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$ and $[B \in \mathcal{P} \Rightarrow |B| < \lambda]$. Also as in the proof of 6.3 for $B \in \mathcal{P}$, $\langle x_\alpha : \alpha \in B\rangle$ has a transversal which we call f_B. Now if $A \subseteq \mu$, $|A| < \theta_1$ then by the choice of $\mathcal{P}$ there are $i(*) < \theta_2$ and $B_i \in \mathcal{P}$ for $i < i(*)$ such that $A \subseteq \bigcup_{i<i(*)} B_i$. Now define $f : \mathrm{Dom}(f) = A$, and if $\alpha \in A \cap B_i \setminus \bigcup_{j<i} B_j$ we let $f(\alpha) = f_{B_i}(\alpha)$. So f is a $(< \theta_2)$ to 1 function. The rest is easy, too. $\square_{6.8}$

Lemma 6.9 *Suppose λ is strong limit,* $\mathrm{cf}(\lambda) = \aleph_0$. *Then there is* $T \subseteq {}^\omega\lambda$, $|T| = \lambda^{\aleph_0}$ *with no perfect subsets with density character θ^*, where Case a or Case b holds:*
Case a: $\theta_n^+ < \theta_n^{\aleph_0} < \theta_{n+1}$, $\theta^* = \bigcup_n \theta_n \leq \lambda$, *$\theta^*$ strong limit.*
Case b: $\lambda^{\aleph_0} = \lambda^+$, $\theta^* = (2^{\aleph_0})^+$.

Remark: The conclusion fails (for lack of $\theta^* \leq \lambda$) for at most ω many λ's (which are strong limits of cofinality ω).

Proof: First we show 6.9A, 6.9B.

Fact 6.9A If $\theta^+ < \theta^{\aleph_0} < \lambda$, $\mathrm{cf}(\lambda) = \aleph_0$, $\lambda^{\aleph_0}$ regular and for every $\mu < \lambda^{\aleph_0}$ we have $\mathrm{cov}(\mu, \theta^{++}, \theta^{++}, \theta^+) < \lambda^{\aleph_0}$ then there is $T \subseteq {}^\omega\lambda$, $|T| = \lambda^{\aleph_0}$, T contains no perfect subsets of power $> \theta^+$ and density $\leq \theta$.

Proof: By the hypothesis for each $\alpha < \lambda^{\aleph_0}$ there is a family $\mathcal{P}_\alpha$ of subsets of α, each of cardinality θ^+, $|\mathcal{P}_\alpha| < \lambda^{\aleph_0}$ such that: for every $A \subseteq \alpha$, $|A| = \theta^+$ for some $B \in \mathcal{P}_\alpha$, $|B \cap A| = \theta^+$. Now we choose by induction on $\alpha < \lambda^{\aleph_0}$, $\eta_\alpha \in {}^\omega\lambda$ such that:

(*) if $\beta \leq \alpha$, $A \in \mathcal{P}_\beta$ then η_α is not in the closure of $\{\eta_\gamma : \gamma \in A\}$.

This is easily possible as $\lambda^{\aleph_0}$ is regular and for $\beta \leq \alpha$:

$$\left|\bigcup\{\mathrm{closure}\{\eta_\gamma : \gamma \in A\} : A \in \mathcal{P}_\beta\}\right| \leq |\beta| \cdot (\theta^+)^{\aleph_0} < \lambda^{\aleph_0}.$$

Now $\{\eta_\alpha : \alpha < \lambda^{\aleph_0}\}$ is as required because $\theta^+ < \theta^{\aleph_0}$. $\square_{6.9A}$

Fact 6.9B If $\lambda^{\aleph_0}$ is singular, $(\forall \lambda_1 < \lambda)[\lambda_1^{\aleph_0} < \lambda < \lambda^{\aleph_0}]$, (hence $\mathrm{cf}(\lambda) = \aleph_0$) and $\theta_2 < \theta_1 \leq \lambda$, θ_2 regular and $\mathrm{cov}(\mu, \lambda, \theta_1, \theta_2) < \lambda^{\aleph_0}$ for every $\mu < \lambda^{\aleph_0}$ then we can find $T \subseteq {}^\omega\lambda$, $|T| = \lambda^{\aleph_0}$ which is (θ_1, θ_2)-free.

Proof: Let $\chi = \mathrm{cf}(\lambda^{\aleph_0})$, $\lambda^{\aleph_0} = \sum_{i<\chi} \mu_i$, each μ_i regular $> \lambda + \chi$. For

$$\mu \in \{\mu_i : i < \chi\},$$

by 6.8 (and see 6.8A(1)), we can find $\{\eta^\mu_\alpha : \alpha < \mu\} \subseteq {}^\omega\lambda$ which is (θ_1, θ_2)-free. Combining we get a (θ_1, θ_2)-free $T \subseteq {}^\omega\lambda$, $|T| = \lambda^{\aleph_0}$; i.e. we let for $\alpha < \lambda$, η_α be as follows:
let $i(\alpha)$ be minimal such that $\mu_i > \alpha$, and $\eta_\alpha(\ell) = \langle \eta^{\mu_0}_{i(\alpha)}(\ell), \eta^{\mu_{i(\alpha)}}_\alpha(\ell)\rangle$. $\square_{6.9B}$

Proof of 6.9: If $\lambda^{\aleph_0} = \lambda^+$, then, by 1.5, for some strictly increasing sequence $\langle \lambda_n : n < \omega\rangle$ of regular cardinals $< \lambda$, $\lambda = \sum_{n<\omega} \lambda_n$ and $\lambda^+ = \mathrm{tcf}(\prod \lambda_i, <_{J^\omega_{bd}})$; let $\langle f_\alpha : \alpha < \lambda^+\rangle$ exemplify it. Now for any $\theta_1 > \theta_2$ as in (*) of 6.5A, $\{f_\alpha : \alpha < \lambda^+\}$ is weakly (θ_1, θ_2)-free which easily implies that it contains no subset of power θ_2 with density $< \theta_2$. Clearly $\theta_1 = (2^{\aleph_0})^{++}$, $\theta_2 = (2^{\aleph_0})^+$ satisfies (*) of 6.5A so we have proved 6.9 under this assumption (more exactly, when Case b of 6.9 holds) (not using "λ strong limit"). Really we use just $\mathrm{pp}(\lambda) =^+ \lambda^{\aleph_0}$ (and $2^{\aleph_0} \leq \lambda$, but note that the case $2^{\aleph_0} = \lambda^+$ is trivial).
So assume $\lambda^{\aleph_0} > \lambda^+$, hence we are in Case a of 6.9.
First assume that there is $n < \omega$ such that:
$(*)_n$ $\left[\mu < \lambda^{\aleph_0} \Rightarrow \mathrm{cov}(\mu, \lambda, \theta_n^{++}, \theta_n^+) < \lambda^{\aleph_0}\right]$.
Then, if $\lambda^{\aleph_0}$ is regular by 6.9A our conclusion holds and if $\lambda^{\aleph_0}$ is singular by 6.9B our conclusion holds. So assume $(*)_n$ fails for each n; choose the

minimal $\mu < \lambda^{\aleph_0}$ such that: $\mu \geq \lambda$ and for some $\sigma < \theta^*$, $\mathrm{pp}_{\Gamma(\sigma^+,\aleph_1)}(\mu) \geq \lambda^{\aleph_0}$ (exists by 5.4(4)). Choose the minimal such σ (for the μ already chosen) hence for any $\tau < \theta^*$ we have $[\lambda \leq \mu' < \mu \Rightarrow \mathrm{pp}_{\Gamma(\tau,\aleph_1)}(\mu') < \lambda^{\aleph_0}]$ (Γ as in 5.4) hence by 5.4 $[\lambda \leq \mu' < \mu \Rightarrow \mathrm{cov}(\mu', \lambda, \tau, \aleph_1) < \lambda^{\aleph_0}]$.
For simplicity note that $\mathrm{cf}(\lambda^{\aleph_0}) > \lambda$ (as λ is a strong limit) hence for some $\aleph_1$-complete ideal J on σ, $\mathrm{pp}^*_J(\mu) \geq \lambda^{\aleph_0}$. If $\lambda_1 \in (\mu, \lambda^{\aleph_0}]$ and $\lambda_1 \leq^+ \mathrm{pp}^*_J(\mu)$ then by 6.5(1) (more exactly 6.5A) and 6.5(5), $\mathrm{NPT}^-_J(\lambda_1, \mu, \theta_1, \theta_2, \sigma)$ for any regular uncountable $\theta_1 > \theta_2$ in the interval $[\aleph_1, \theta^*)$ satisfying $(*)$ of 6.5A, for example $(2^\sigma)^{++}$, $(2^\sigma)^+$. We next prove that $\mathrm{NPT}^-_J(\lambda_1, \mu, \theta_1, \theta_2, \sigma)$ holds for $\lambda_1 = \lambda^{\aleph_0}$.
case (i): $\lambda^{\aleph_0}$ is singular.
Combining the result on λ_1, for $\mathrm{cf}(\lambda^{\aleph_0}) + \mu^+$ and for a sequence of regular cardinals converging to $\lambda^{\aleph_0}$, we get $\mathrm{NPT}^-_J(\lambda^{\aleph_0}, \mu, \theta_1, \theta_2, \sigma)$, like in 6.9B.
case (ii): $\lambda^{\aleph_0}$ successor.
We can apply the above directly for $\lambda_1 = \lambda^{\aleph_0}$,
case (iii): $\lambda^{\aleph_0}$ inaccessible.
By 5.4(2) $\lambda^{\aleph_0} \leq^+ \mathrm{pp}^*_J(\mu)$ and we act as in case (ii).
So, now we have $\mathrm{NPT}^-_J(\lambda^{\aleph_0}, \mu, \theta_1, \theta_2, \sigma)$.
Also necessarily $\mathrm{cf}\mu \leq \sigma$ (as $\mu < \lambda^{\aleph_0}$, $\mathrm{pp}_\sigma(\mu) \geq \lambda^{\aleph_0}$) so let

$$\mu = \sum_{\alpha<\sigma} \mu_\alpha,\ \mu_\alpha < \mu.$$

So $\mathrm{cov}(\mu_\alpha, \theta_1, \theta_2, \sigma) < \lambda^{\aleph_0}$ so by 6.8 $\mathrm{NPT}(\mu_\alpha, \lambda, \theta_1, \theta_2, \aleph_0)$ (see 6.8A(1)); as $\sigma < \lambda$ we easily get $\mathrm{NPT}(\mu, \lambda, \theta_1, \theta_2, \aleph_0)$ hence $\mathrm{NPT}_{J^{bd}_\omega}(\mu, \lambda, \theta_1, \theta_2, \aleph_0)$ (as in the proof of 6.7(2)). By (monotonicity of NPT and) 6.9C below (with $\lambda^{\aleph_0}, \mu, \lambda, \theta_1, \theta_2, \sigma, \aleph_0$ here standing for $\chi, \mu, \lambda, \theta_1, \theta_2, \sigma, \tau$ there) we can put together $\mathrm{NPT}_{J^{bd}_\omega}(\mu, \lambda, \theta_1, \theta_2, \aleph_0)$ and $\mathrm{NPT}^-_J(\lambda^{\aleph_0}, \mu, \theta_1, \theta_2, \sigma)$ to get the desired conclusion. $\square_{6.9}$

Fact 6.9C If $\chi \geq \mu \geq \lambda \geq \theta_1 > \theta_2 > \sigma \geq \tau$, $\mathrm{NPT}^{(-)}_J(\chi, \mu, \theta_1, \theta_2, \sigma)$, $\mathrm{NPT}_I(\mu, \lambda, \theta_1, \theta_2, \tau)$ and $(\forall \alpha < \lambda)[|\alpha|^{|\mathrm{Dom}\ J|} = |\alpha|^\sigma < \lambda]$, $\tau = \mathrm{cf}(\lambda)$, $\mathrm{cf}(\mu) \neq \tau$, θ_2 regular, ($> \aleph_0$ for simplicity) <u>then</u> $\mathrm{NPT}^{(-)}_{J^{bd}_\tau}(\chi, \lambda, \theta_1, \theta_2, \tau)$.

Proof: We concentrate on NPT_J.
Let $\langle f_\alpha : \alpha < \chi \rangle$ exemplify $\mathrm{NPT}^{(-)}_J(\chi, \mu, \theta_1, \theta_2, \sigma)$; without loss of generality $\langle \{f_\alpha(i) : \alpha < \chi\} : i < \sigma \rangle$ are pairwise disjoint and let $\langle g_\beta : \beta < \mu \rangle$ exemplify $\mathrm{NPT}_I(\mu, \lambda, \theta_1, \theta_2, \tau)$. Without loss of generality for each $i < \tau$ $\sup\{g_\beta(i) : \beta < \mu\} < \lambda$ (Why? as $2^\tau \leq 2^\sigma < \lambda$: let $\langle \lambda_i : i < \tau \rangle$ be an increasing sequence of regular cardinals $> \tau$ with limit λ, if $\mathrm{cf}\mu > 2^\sigma$ letting $h_\beta : \tau \to \tau$ be $h_\beta(i) = \min\{j : \lambda_j > g_\beta(i)\}$, without loss of generality $h_\beta = h$ for all β, then easy; if $\mathrm{cf}\mu \leq 2^\sigma$ decompose the problem).

For each $\alpha < \chi$ we define a function h_α : its domain is τ (like for the functions g_β) and for $i < \tau$ we let $h_\alpha(i) = \langle g_{f_\alpha(\zeta)} \restriction (i+1) : \zeta \in \mathrm{Dom}\ J \rangle$; so

$$|\{h_\alpha(i) : \alpha < \chi\}| \leq |\bigcup_{j \leq i} \{g_\zeta(j) : \zeta < \mu\}|^{|\mathrm{Dom}\ J|} < \lambda,$$

so except for a need to rename, $\langle h_\alpha : \alpha < \chi \rangle$ are of the right kind.

Now suppose $A \subseteq \chi$, $|A| < \theta_1$, so there are $\langle s_\alpha : \alpha \in A \rangle$ (for NPT^-_J, we have to shrink A) and an equivalence relation E^1 on A, with each class of cardinality $< \theta_2$ such that: $s_\alpha \in J$ and

$$[i \in (\mathrm{Dom}\ J) \backslash s_\alpha \setminus s_\beta \ \&\ \alpha \in A \ \&\ \beta \in A \ \&\ \neg \alpha E^1 \beta \Rightarrow f_\alpha(i) \neq f_\beta(i)].$$

Let $B = \bigcup_{\alpha \in A} \mathrm{Rang}\ f_\alpha$, it is a subset of μ of cardinality $< \theta_1$, hence (as by an assumption $\mathrm{NPT}_I(\mu, \lambda, \theta_1, \theta_2, \tau)$) there are $\langle t_\zeta : \zeta \in B \rangle$, and an equivalence relation E^2 on B with each equivalence class of cardinality $< \theta_2$ such that: $t_\zeta \in I$ and

$$[i \in (\mathrm{Dom}\ I) \backslash t_\zeta \setminus t_\xi \ \&\ \zeta \in B \ \&\ \xi \in B \ \&\ \neg \zeta\ E^2 \xi \Rightarrow g_\alpha(i) \neq g_\xi(i)].$$

We define an equivalence relation E^* on A: it is the minimal equivalence relation, such that for each $\alpha, \beta \in A$:
$[\alpha\ E^1 \beta \Rightarrow \alpha\ E^* \beta]$;
$[\zeta \in \mathrm{Rang}(f_\alpha \restriction (\mathrm{Dom}\ J \backslash s_\alpha)) \ \&\ \xi \in \mathrm{Rang}(f_\beta \restriction (\mathrm{Dom}\ J \backslash s_\beta)) \ \&\ \zeta E^2 \xi \Rightarrow \alpha E^* \beta]$.

As $|\mathrm{Dom}\ J| = \sigma < \theta_2$, θ_2 regular, also the E^*-equivalence classes have each cardinality $< \theta_2$ (remember that without loss of generality $\langle \{f_\alpha(i) : \alpha < \chi\} : i \in \mathrm{Dom}\ J \rangle$ are pairwise disjoint).
Let for $\alpha \in A$:

$$j(\alpha) =: \min\{j+1 : j < \tau, \text{ and for some } \ i \in \sigma \backslash s_\alpha, j \notin t_{f_\alpha(i)}\}.$$

Now $\{h_\alpha \restriction [j(\alpha), \sigma) : \alpha \in A\}$ are as required to prove that $\{h_\alpha : \alpha < \chi\}$ exemplify $\mathrm{NPT}_{J^{bd}_\sigma}(\chi, \lambda, \theta_1, \theta_2, \sigma)$ for the case we chose A: for each $\alpha \in A$:

$$\{\beta \in A : \mathrm{Rang}[h_\alpha \restriction [j(\alpha), \sigma)] \cap \mathrm{Rang}[h_\beta \cap [j(\beta), \sigma)] \neq \emptyset\}$$

has cardinality $< \theta_1$. $\square_{6.9C}$

Claim 6.9D (1) In 6.9C, instead of assuming $(\forall \alpha < \lambda)[|\alpha|^{|\mathrm{Dom}\ J|} < \lambda)]$, we can weaken the conclusion to $\mathrm{NPT}_{J \times I}(\chi, \mu, \theta_1, \theta_2, \sigma \times \tau)$.
(2) We also can note that if $\theta \leq \lambda^* \leq \lambda$, θ a limit cardinal, $\theta > \tau > \aleph_0$, $\theta < \mu < \mathrm{cov}(\lambda, \lambda^*, \theta, \tau)$ <u>then</u> for some $\lambda^{**} \in [\theta, \lambda]$, $\sigma \in [\tau, \theta)$ and τ-complete ideal J on σ (containing the singletons) $\mathrm{NPT}_J(\mu, \lambda^{**}, \theta, \sigma)$.

Proof of 6.9D(2): As θ is limit, $\lambda > \theta$, clearly

$$\mathrm{cov}(\lambda, \lambda^*, \theta, \tau) = \sum_{\substack{\sigma<\theta \\ \sigma\geq\tau}} \mathrm{cov}(\lambda, \lambda^*, \sigma^+, \tau),$$

so for some $\sigma \in [\tau, \theta)$ we have

$$\mu < \mathrm{cov}(\lambda, \lambda^*, \sigma^+, \tau).$$

By 5.4+5.8 for some regular $\tau^* \in [\tau, \sigma^+]$ and τ-complete filter J on τ^* and $\lambda^{**} \in [\theta, \lambda]$, $\mathrm{pp}^*_J(\lambda^{**}) > \mu$.
Now use 1.5A. $\square_{6.9D}$

Claim 6.9E If $\lambda < \mu < \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$, $\lambda > \mathrm{cf}\lambda = \aleph_0$ then $\mathrm{NPT}_I(\mu, \lambda, \lambda^+, 2, \sigma)$ for some $\sigma < \lambda$ and ideal I.

Remark 6.9F (1) If $\mathrm{cf}\lambda > \aleph_0$, 6.4(2)(*) gives more.
(2) As λ is singular, $\mathrm{NPT}(\mu, \lambda, \lambda^+, 2, \sigma)$ implies $\mathrm{NPT}(\mu, \lambda, \lambda, 2, \sigma)$ (like in 6.2(3) — compactness for singular cardinals).

Proof: If $\mathrm{cov}(\mu, \lambda, \lambda, \aleph_1) < \mathrm{cov}(\lambda, \lambda, \aleph_1, 2)$ then by 6.8 (with $\theta_1 = \lambda$, $\theta_2 = \aleph_1$) we have $\mathrm{NPT}(\mu, \lambda, \lambda, \aleph_1, \aleph_0)$ hence $\mathrm{NPT}(\mu, \lambda, \lambda, 2, \aleph_0)$ by 6.7(2). This works when μ is regular. when μ is singular, we obtain the same result by combining $\mathrm{NPT}(\mu_1, \lambda, \lambda, 2, \aleph_0)$ for many regular $\mu_1 < \mu$, as we did before in this section.

So assume not, so $\mu < \mathrm{cov}(\mu, \lambda, \lambda, \aleph_1)$, hence by 5.3(5) (as $\mathrm{cf}\lambda < \lambda < \mu$) for some $\theta_1 < \lambda$, $\mu < \mathrm{cov}(\mu, \lambda, \theta_1, \aleph_1)$. Let μ^* be the minimal cardinal satisfying: there is $\theta_1 < \lambda$ such that $\lambda \leq \mu^* \leq \mu < \mathrm{cov}(\mu^*, \lambda, \theta_1, \aleph_1)$; choose such θ_1, hence by 5.4 for some $\sigma \in [\aleph_1, \theta_1)$ and $\mu^{**} \in [\lambda, \mu^*]$ and $\aleph_1$-complete ideal J on σ, $\mathrm{pp}^*_J(\mu^{**}) > \mu$; by the minimality of μ^* (again by 5.4) necessarily $\mu^{**} = \mu^*$. Now for $\mu' < \mu^*$ we have $\mathrm{cov}(\mu', \lambda, \lambda, \aleph_1) \leq \mu$ (by the choice of μ^*) hence by 6.8 and 6.7(2) we have $\mathrm{NPT}(\mu', \lambda, \lambda, 2, \aleph_0)$; as necessarily $\mathrm{cf}(\mu^*) \leq \sigma$, clearly by 6.2(10) we have $\mathrm{NPT}(\mu^*, \lambda, \lambda, 2, \aleph_0)$. By 1.5A (as $\mathrm{pp}^*_J(\mu^{**}) > \mu$) (if μ is singular — its proof) we have $\mathrm{NPT}_J(\mu, \mu^*, \lambda, 2, \sigma)$. Assuming 6.9E is true for μ^*, and $\mu^* < \mu$ by 6.9C, $\mathrm{NPT}_I(\mu, \lambda, \lambda, 2, \sigma)$ for some ideal I.
If $\mu = \mu^*$, the conclusion trivially follows by $\mathrm{NPT}_J(\mu, \mu^*, \lambda, 2, \sigma)$. $\square_{6.9E}$

§7 Existence of $L_{\infty,\lambda}$-equivalent non-isomorphic models of singular cardinality λ

We give here an application to model theory (not used later). The reader is not supposed to know anything on the logic $L_{\infty,\lambda}$ (though he is expected to know what is isomorphism), as the following can serve as definition (it is usually a Theorem).

Definition 7.0 Models $\mathfrak{A}$, $\mathfrak{B}$ with the same vocabulary are called $L_{\infty,\lambda}$-equivalent if there is a non empty family $\mathcal{L}$ of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ (i.e. each $L \in \mathcal{L}$ is a one to one function, $\mathrm{Dom}\, L \subseteq \mathfrak{A}$, $\mathrm{Rang}\, L \subseteq \mathfrak{B}$, and for any n-place predicate R from the vocabulary and $a_0, \ldots, a_{n-1} \in \mathrm{Dom}\, L$ we have $\mathfrak{A} \models R[a_0, \ldots, a_{n-1}] \Leftrightarrow \mathfrak{B} \models R[L(a_0), \ldots, L(a_{n-1})]$, similarly for function symbols F and satisfaction of $F(a_0, \ldots, a_{n-1}) = b$), which satisfies

(*) if $L \in \mathcal{L}$, $A \subseteq \mathfrak{A}$ has cardinality $< \lambda$, $B \subseteq \mathfrak{B}$ has cardinality $< \lambda$, then for some L', $L \subseteq L' \in \mathcal{L}$, and $A \subseteq \mathrm{Dom}\, L'$, $B \subseteq \mathrm{Rang}\, L'$.

So our aim is to prove that there are non-isomorphic but very similar (i.e. $L_{\infty,\lambda}$-equivalent) models in the singular case; why not the regular? Too late. So why is the singular case harder? Trying to build the models and family $\mathcal{L}$ of partial isomorphism, without loss of generality their universe will be λ, so we do not have to consider all A, $B \subseteq \lambda$, a cofinal family is enough. So, if λ is regular, we can consider only λ such sets. But we have λ singular and so we cannot use such a list of a cofinal set of such subsets. Now if λ is strong limit (singular, $\mathrm{cf}\lambda > \aleph_0$) we can still for each $\lambda' < \lambda$ count the $A \cap \lambda'$ + the "type" of A for $A \subseteq \lambda$, $|A| < \lambda$; but it is too late for this, too. What we do is, if $\lambda > \mathrm{cf}\lambda > \aleph_1$, using mainly 2.1, to get a sequence of regular cardinals $\langle \lambda_i : i < \kappa \rangle$ increasing to λ such that for "many $\delta < \kappa$ of cofinality $\aleph_0$" for some unbounded $a = a_\delta \subseteq \delta$, $\max \mathrm{pcf}\{\lambda_i : i \in a\} < \lambda$ (see.7.6A). Probably even if $\mathrm{cf}\lambda = \aleph_1$ we can get such a sequence, we know how to prove it in many cases but not generally; for $\mathrm{cf}\lambda = \aleph_0$ there are no such models. The information on such sequence of cardinals $\langle \lambda_i : i < \mathrm{cf}\lambda \rangle$ together with "nice" families $F_\delta \subseteq \prod_{i\in a_\delta} \lambda_i$ (in particular cofinal and of cardinality $\mathrm{cf}(\prod_{i\in a_\delta} \lambda_i) < \lambda$) is then summed up in a "parameter" (all this is 7.1 — 7.4). All this prepares the ground to build such models from the parameters (7.5). We build a model $M = \cup\{P^M_f : f \in \cup F_\delta\}$, on each P_f we put an abelian group, but do not make + as a function of the model, we just give each function $x \mapsto x + c$ ($c \in P^M_i$), i.e. we do not "say" who is the zero (this is a widespread trick) and we essentially can compute automorphism groups so far and they are quite large. But then via projections we restrict it till it becomes trivial, but still for $a \neq b \in P^M_\emptyset$, (M, a), (M, b) are similar enough, i.e. $L_{\infty,\lambda}$-equivalent.
So the idea is in the exact fitting of the families F_δ with the partial isomorphisms.

* * *

From 2.1 we easily conclude:

Claim 7.1 Suppose λ is singular, $\aleph_1 < \kappa =: \mathrm{cf}(\lambda)$, $\langle \lambda_i : i < \kappa \rangle$ is a strictly increasing continuous sequence of cardinals with limit λ.
(1) For $\theta < \kappa$ regular the set

$$S_\theta =: \{i < \kappa : \mathrm{cf}(i) = \theta, \text{ and there is a sequence } \langle j^i_\alpha : \alpha < \theta\rangle \text{ such that: } \bigwedge_{\alpha<\beta} j^i_\alpha < j^i_\beta < i,\ i = \bigcup_{\alpha<\theta} j^i_\alpha \text{ and } \mathrm{tcf}(\prod_{\alpha<\theta} \lambda^+_{j^i_\alpha}/J^{\mathrm{bd}}_\theta) \text{ is well defined and } \le \lambda\}$$

is stationary.

(2) Moreover, $(\kappa \setminus \cup\{S_\theta : \theta < \kappa, \theta \text{ regular}\}) \cap \delta$ is not stationary in δ for any $\delta < \kappa$ of cofinality $> \aleph_0$.

(3) Moreover, if $\theta > \aleph_0$ then $\{i < \kappa : \mathrm{cf}(i) = \theta, i \notin S_\theta\}$ is empty.

Conclusion 7.2 If λ is singular, $\aleph_1 < \kappa = \mathrm{cf}\ \lambda$ and $\theta = \mathrm{cf}\theta < \kappa$, <u>then</u> we can find $\langle \lambda_i : i < \kappa\rangle$ strictly increasing sequence of regular cardinals with limit λ and $S \subseteq \kappa$ such that:

$$S'_\theta = \{i < \kappa : \mathrm{cf}(i) = \theta,\ i \notin S, \text{ and there are } \langle j^i_\alpha : \alpha < \theta\rangle \text{ such that } j^i_\alpha \in S,\ \bigwedge_{\alpha<\beta} j^i_\alpha < j^i_\beta < i,\ i = \bigcup_{\alpha<\theta} j^i_\theta \text{ and } \mathrm{tcf}(\prod_{\alpha<\theta} \lambda_{j^i_\alpha}/J^{\mathrm{bd}}_\theta) \text{ is } \lambda_i\}$$

is stationary, moreover: if $\theta > \aleph_0$ then $[i < \kappa\ \&\ \mathrm{cf}(i) = \theta \Rightarrow i \in S'_\theta]$ and if $\theta = \aleph_0$ then for every $\delta < \kappa$, if $\mathrm{cf}(\delta) > \aleph_0$ then $S'_\theta \cap \delta$ is stationary in δ.

Proof: Let $\langle \lambda^0_i : i < \kappa\rangle$ be as in 7.1 with $\lambda^0_i > \kappa^+$.
Choose by induction on i, λ_i, ℓ_i such that:

(a) λ_i is regular

(b) $\lambda^0_i < \lambda_i < \lambda$

(c) $\ell_i \in \{0, 1\}$

(d) if i is limit, $\mathrm{cf}(i) = \theta$ and there is $\langle j_\alpha : \alpha < \theta\rangle$ strictly increasing, $i = \bigcup_\alpha j_\alpha, \mathrm{tcf}(\prod_{\alpha<\theta} \lambda_{j_\alpha}/J^{\mathrm{bd}}_\theta) < \lambda$ (and is well defined) and $\ell_{j_\alpha} = 1$ for $\alpha < \mathrm{cf}(i)$ <u>then</u> let $\lambda_i = \mathrm{tcf}(\prod_{\alpha<\theta} \lambda_{j_\alpha}/J^{\mathrm{bd}}_\theta)$ and $\ell_i = 0$ (not necessarily for the same $\langle j_\alpha : \alpha < \mathrm{cf}(i)\rangle$)

(e) if i is limit and $\mathrm{cf}(i) \neq \theta$ or there is no $\langle j_\alpha : \alpha < \mathrm{cf}(i)\rangle$ as above <u>then</u> $\lambda_i = (\bigcup_{j<i} \lambda_j)^+$ and $\ell_i = 1$.

(f) if i is non-limit, λ_i is $\left(\lambda^0_{j(i)}\right)^+$ where $j(i) = \min\{j : \lambda^0_j > \lambda^0_i + \sum_{\zeta<i} \lambda_\zeta\}$.

Now use 2.1. $\square_{7.2}$

Definition 7.3 Call $\bar{p} = \langle \lambda_i, a_i, F_i : i < \delta(*)\rangle$ a suitable parameter if:

(i) $\langle \lambda_i : i < \delta(*)\rangle$ is non-decreasing sequence of regular cardinals $> \delta(*)$ which is not eventually constant

(ii) $i \in a_i \subseteq i + 1$

(iii) $j \in a_i \Rightarrow a_j = a_i \cap (j + 1)$

(iv) $F_i \subseteq \prod_{j\in a_i} \lambda_j$, $|F_i| \le \sum_{j\le i} \lambda_j = \lambda_i$

(v) for each $f \in \prod_{j\in a_i} \lambda_j$ for some $f^* \in F_i$, $f \le f^*$

(vi) if $f \in F_i$, $j \in a_i$ then $f{\restriction}a_j \in F_j$

(vii) for every $f \in \prod_{i<\delta(*)} \lambda_i$ for some $f^* \in \prod_{i<\delta(*)} \lambda_i$, $f < f^*$ and $\bigwedge_i (f^*{\restriction}a_i \in F_i)$

(viii) $S =: \{\delta < \delta(*) : \text{for some } i \ge \delta, \sup(a_i \cap \delta) = \delta\}$ is stationary.

(ix) for each $\alpha < \delta(*)$, the set $\{\delta < \delta(*) : \text{for some } \beta \text{ we have } \delta \in a_\beta \text{ and } \delta = \sup(\delta \cap a_\beta)\}$ is stationary.

We write $\lambda^{\bar{p}}_i$, $a^{\bar{p}}_i$, $F^{\bar{p}}_i$, $\delta(*)^{\bar{p}}$ for the λ_i, a_i, F_i, $\delta(*)$ respectively and

$$F^{\bar{p}} = \cup\{F^{\bar{p}}_i : i < \delta(*)^{\bar{p}}\}, \quad \lambda^{\bar{p}} = \sum\{\lambda^{\bar{p}}_i : i < \delta(*)^{\bar{p}}\};$$

lastly $S^{\bar{p}}$ is as in (viii).

Fact 7.4 1) If

$(*)_\lambda$ $\lambda > \mathrm{cf}\lambda > \aleph_0$, $\langle \lambda_i : i < \mathrm{cf}\lambda\rangle$ is a strictly increasing sequence of regular cardinals, $(\mathrm{cf}\lambda)^+ < \lambda_0$, $S \subseteq \{i < \mathrm{cf}(\lambda) : \mathrm{cf}(i) = \aleph_0\}$ is stationary and for each $i \in S$ there is a strictly increasing sequence of successor ordinals $\langle j^i_n : n < \omega\rangle$, $\bigcup_n j^i_n = i$, $\lambda_i = \mathrm{tcf}(\prod \lambda_{j^i_n}, <_{J^{\mathrm{bd}}_\omega})$,

<u>then</u> there is a suitable parameter $\bar{p}$, $\lambda^{\bar{p}} = \lambda$, $\delta(*)^{\bar{p}} = \mathrm{cf}(\lambda)$.

2) If $\lambda > \mathrm{cf}\lambda > \aleph_1$, then $(*)$ above holds for some $\langle \lambda_i : i < \mathrm{cf}(\lambda)\rangle$ and S as above.

Proof: 1) By easy manipulations (as in Definition 7.3 we require $\langle \lambda_i : i < \delta(*)\rangle$ only to be non decreasing). For finding the F_i's use 3.4 as in the proof of 3.5.

2) By 7.2. $\square_{7.4}$

Main Lemma 7.5 Suppose $\bar{p} = \langle \lambda_i, a_i, F_i : i < \delta(*)\rangle$ is a suitable parameter, $\lambda = \lambda^{\bar{p}}$, $\delta(*) = \kappa = \mathrm{cf}\ \kappa > \aleph_0$. <u>Then</u> there are $L_{\infty,\lambda}$-equivalent non-isomorphic models (with vocabulary — just one binary relation).

Proof: Stipulate $a_{\delta(*)} = \{i : i < \delta(*)\}$,

$$F_{\delta(*)} = \{f : f \in \prod_{i<\delta(*)} \lambda_i \text{ and } (\forall i < \delta(*))[f{\restriction}a_i \in F_i]\}.$$

We let, for $i \le \delta(*)$:

$$\Gamma_i = \{(j, f) : j < i \text{ and for some } g \in F_i,\ f = g{\restriction}[j, i]\}$$

$$S^a_i = \{\langle (j_0, f_0), (j_1, f_1), \ldots, (j_{n-1}, f_{n-1})\rangle : n < \omega,\ (j_\ell, f_\ell) \in \Gamma_{i_\ell},\ i_\ell \le i_{\ell+1} \le i,\ a_{i_\ell} \subseteq a_{i_{\ell+1}},\ j_\ell < j_{\ell+1} \text{ and } [j_{\ell+1} < \alpha \in a_{i_\ell} \Rightarrow f_\ell(\alpha) < f_{\ell+1}(\alpha)]\}$$

and if $i < \delta(*)$

$S^b_i = \{\langle (j_0, f_0), \ldots, (j_{n-1}, f_{n-1}), j_n\rangle : \langle (j_0, f_0), \ldots, (j_{n-1}, f_{n-1})\rangle \in S^a_i$
and $\delta(*) > j_n \geq i\}$

$S_i = S^a_i \cup S^b_i$

Note that S^b_i, S_i are defined only for $i < \delta(*)$.

We define a partial order $<_i$ on S_i:

$\eta \leq_i \nu$ if η is an initial segment of ν or

$$\eta = \langle (j_0, f_0), \ldots, (j_{n-1}, f_{n-1}), j_n\rangle \text{ and}$$

$$\nu = \langle (j_0, f_0), \ldots, (j_{n-1}, f_{n-1}), (j_n, f_n)\rangle.$$

Let $F = \bigcup_{i<\delta(*)} F_i$ and $\Gamma = \bigcup_{i<\delta(*)} \Gamma_i$. For $(j, f) \in \Gamma$ let $i(j, f)$ be the unique i such that $(j, f) \in \Gamma_i$ and for $f \in F$, $i(f)$ is the unique i such that $f \in F_i$.

Let $g \leq (j, f)$ mean $g\restriction[(j, \delta(*)) \cap \mathrm{Dom}\, f] = g\restriction\mathrm{Dom}(f) \leq f$.

Let for $f \in F_i$

$S_f =: \{\eta : \eta = \langle (j_0, f_0), ..., (j_{n-1}, f_{n-1}), x\rangle \in S_i$ such that:
$x = (j_n, f_n) \Rightarrow f \leq (j_n, f_n)\}$.

Let for $f \in F_i, G_f$ be the abelian group of order 2, generated freely by

$$\{x^f_\eta : \eta \in S_f\}$$

We can assume that the G_f's are pairwise disjoint.

If $g_1 \leq g_2$ are from F (so Dom g_1 is a subset, maybe proper, of Dom g_2) we define a homomorphism $h = h_{g_1,g_2}$ from G_{g_2} to G_{g_1}, by defining its value on the free generators of G_{g_2} :

<u>if</u> $\eta = \langle (j_0, f_0), ..., (j_{n-1}, f_{n-1}), x\rangle \in S_{g_2}$, x is (j_n, f_n) or j_n, let:
$\ell_{g_1,g_2} = \ell_{g_1,g_2}(\eta) = \min\{\ell : g_1 \leq (j_\ell, f_\ell) \text{ or } \ell = \ell g(\eta) - 1 = n\}$
and $g_1 \in F_{i_1}$,
and $g_2 \in F_{i_2}$; (so $i_1 \leq i_2$ as $a_{i_1} \subseteq a_{i_2}$, equivalently, $i_1 \in a_{i_2}$)
<u>then</u> $h(x^{g_2}_\eta) = x^{g_1}_\nu$ where

<u>Case 1:</u> <u>$i_1 = i_2$:</u> $\nu = \langle (j_\ell, f_\ell) : \ell \leq \ell_{g_1,g_2}(\eta)\rangle$ and if $\ell_{g_1,g_2}(\eta) = n$ then we mean $\nu = \eta$ (even if $x = j_n$).

<u>Case 2:</u> <u>$i_1 < i_2$:</u> let
$m_{g_1,g_2} = m_{g_1,g_2}(\eta) = \min\{m : j_m \geq i_1 \text{ or } m = \ell g(\eta) - 1\}$.
Note that $\ell_{g_1,g_2} \leq n$ and $m_{g_1,g_2} \leq n$. If $m_{g_1,g_2} \leq \ell_{g_1,g_2}$ we let

$$\nu = \langle (j_\ell, f_\ell\restriction a_{i_1}) : \ell < m_{g_1,g_2}\rangle\hat{}\langle j_{m_{g_1,g_2}}\rangle,$$

and if $m_{g_1,g_2} > \ell_{g_1,g_2}$ (so $\ell_{g_1,g_2} < n$) we let

$$\nu = \langle (j_\ell, f_\ell\restriction a_{i_1})) : \ell \leq \ell_{g_1,g_2}\rangle.$$

So we have finished defining h_{g_1,g_2}.

It is easy to check that

$(*)_0$ if $g_1 \leq g_2 \leq g_3$ are all in $\bigcup_{i<\delta(*)} F_i$ <u>then</u> $h_{g_1,g_2} \circ h_{g_2,g_3} = h_{g_1,g_3}$.

Now we define a model M :

(i) its universe is $\bigcup_{f\in F} G_f$

(ii) it has: the relations $P_f = G_f$, the partial functions h_{f_1,f_2} from P_{f_2} to P_{f_1} for $f_1 \leq f_2$ from F, the partial functions F^f_y ($f \in F, y \in G_f$) with domain P_f and $F^f_y(x) = x + y$ (the addition in G_f) (all one place).

Fact 7.5A M has no non-trivial automorphisms.

Proof: Suppose L is an automorphism of M. For each $f \in F$, L maps P^M_f onto P^M_f, let $y^L_f =: L(0_{G_f})$; as L commutes with F^f_y for each $y \in G_f$, we can show:

$(*)_1$ for $y \in P_f$, $L(y) = y + y^L_f$
(remember: $+_{G_f}$ is not a function of M).

For $f_1 \leq f_2$ from F, as L commutes with h_{f_1,f_2} we can show:

$(*)_2$ if $f_1 \leq f_2$ are from F <u>then</u> $h_{f_1,f_2}(y^L_{f_2}) = y^L_{f_1}$.

As $y^L_f \in G_f$, it is just a sum of a finite set of generators, let $n^L(f)$ be their number. For each $i < \delta(*)$, F_i is λ_0-directed (as each λ_i is regular $\geq \lambda_0$; see 7.3(i),(v)). Now $h_{f_1,f_2}(y^L_{f_2}) = y^L_{f_1}$ implies $n^L(f_1) \leq n^L(f_2)$; so by $(*)_2$

$$f_1 \leq f_2 \Rightarrow n^L(f_1) \leq n^L(f_2).$$

So for each $i < \delta(*)$ for some $f^*_i \in F_i$, $f^*_i \leq f \in F_i \Rightarrow n^L(f) = n^L(f^*_i)$. Let

$$F'_i = \{f \in F_i : f^*_i \leq f\}.$$

So we can let for $f \in F'_i$, $y^L_f = x^f_{\nu_1(f)} + \ldots + x^f_{\nu_n(f)}$ (no repetition) where $n = n^L(f^*_i)$ and

$$f_1 \leq f_2 \in F'_i \Rightarrow h_{f_1,f_2}(x^{f_2}_{\nu_\ell(f_2)}) = x^{f_1}_{\nu_\ell(f_1)}.$$

Hence, without loss of generality:

$$f_1 \leq f_2 \in F'_i \;\&\; 1 \leq \ell \leq n^L(f^*_i) \Rightarrow \nu_\ell(f_1) \leq_i \nu_\ell(f_2)$$

(see after the definition of S_i). By a similar argument, increasing f^*_i we can have:

$$f \in F'_i \Rightarrow \bigwedge_{\ell=1,\ldots,n^L(f)} \ell g(\nu_\ell(f)) = \ell g(\nu_\ell(f^*_i)) \;\&\; \nu_\ell(f) = \nu_\ell(f^*_i).$$

We next show that $\nu_\ell(f^*_i) \in S^b_i$; if not choose $f \in F'_i$ large enough and get a contradiction to the definition of S_f.

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Let

$$\alpha_i = \max\{\nu_\ell(f_i^*)[\ell g(\nu_\ell(f_i^*)) - 1] : 1 \le \ell \le n^L(f_i^*)\}$$

which is $< \delta(*)$ (note: the max is on a finite set of ordinals, ($< \delta(*)$) as $\nu_\ell(f_i^*) \in S_i^b$; hence it is well defined).
As $\delta(*)$ is an uncountable regular cardinal, the set

$$C = \{\delta < \delta(*) : i < \delta \Rightarrow \alpha_i < \delta\}$$

is a club of $\delta(*)$ hence there is a limit ordinal $\delta \in C \cap S^{\bar{p}}$, so (see 7.3(viii)) for some $\beta \in [\delta, \delta(*))$, $\delta = \sup(a_\beta \cap \delta)$. We can find $f \in F_\beta$ such that for every $i \in a_\beta \cap \delta$,

$$(\forall \alpha)[\alpha \in a_i \Rightarrow f_i^*(\alpha) < f(\alpha)]$$

(remember, F_β is λ_0- directed and $\lambda_0 > \delta(*)$). Look at y_f^L, it has the form $x_{\rho_1}^f + \cdots + x_{\rho_m}^f$. The set w of j's appearing in some ρ_ℓ for $\ell \in \{1, ..., m\}$; (i.e.

$$w = \{j : \text{for some } \ell \in \{1, ..., m\} \text{ and } f \text{ we have } (\exists k)[\rho_\ell(k) = (j, f) \vee \rho_\ell(k) = j]\})$$

is finite, hence there is $i \in a_\beta \cap \delta$ above $\max(w \cap \delta)$; now

$$h_{f\restriction a_i, f}(y_f^L) = y_{f\restriction a_i}^L = x_{\nu_1(f\restriction a_i)}^{f\restriction a_i} + \cdots + x_{\nu_n(f\restriction a_i)}^{f\restriction a_i},$$

where $n = n^L(f_i^*)$; but each $\nu_\ell(f\restriction a_i)$ is a sequence with last element an ordinal in $[i, \delta(*))$, (because $\nu_\ell(f\restriction a_i) \in S_i^b$ because $f\restriction a_i \in F_i'$) but $< \delta$ (as $\delta \in C$), so we get an easy contradiction, unless $n = 0$. So $y_{f\restriction a_i}^L = 0$ for large enough $i \in \delta \cap a_\beta$, hence (by the choice of f): $g \in F_i' \Rightarrow y_g^L = 0$. But for every $g_1 \in F_i$ there is $g_2 \in F_i'$, such that $g_1 < g_2$ hence $h_{g_1, g_2}(y_{g_2}^L) = y_{g_1}^L$ so: $g \in F_i \Rightarrow y_g^L = 0$. If $i \in a_\beta \cap \delta$ we can find j, $i < j \in a_\beta \cap \delta$, j large enough, so by the previous sentence $[f \in F_j \Rightarrow y_f^L = 0_{G_f}]$, now for any $f \in F_i$ we can find $g \in F_j$, $f \le g$, hence using $h_{f,g}$ we know $y_f^L = 0_{G_f}$. By 7.3(ix) this holds for every $i < \delta(*)$ (using some $\delta \in C \cap S^{\bar{p}}$, and β) so every $y_f^L = 0_{G_f}$ ($f \in F$) thus we have proved 7.5A. $\square_{7.5A}$

We define a family $\mathcal{L}$ of partial automorphisms of M:
$\mathcal{L}$ is the family of functions L_ν, $\nu \in S_{\delta(*)}^a$, where, for $\nu = \langle (j_0, f_0), ..., (j_{n-1}, f_{n-1}) \rangle$ we define L_ν by induction on n.

<u>if $n = 0$:</u> L_ν is the empty function
<u>if $n > 0$:</u> Dom $L_\nu = \cup\{P_f : f \le (j_{n-1}, f_{n-1})$ and $f \in F\}$;
(note: if $f \in F_i$, $i < j_{n-1}$ then $P_f \subset$ Dom L_ν).

Now if $i < \delta(*)$, $f \in F_i$, we let

$$n_{\nu,i,f} = \min\{m \le \ell g(\nu) : f \le (j_m, f_m)\}$$

and define:

$$\rho_{\nu,i,f} = \begin{cases} \langle (j_\ell, f_\ell\restriction a_i) : \ell \le n_{\nu,i,f} \rangle & \text{if } j_{n_{\nu,i,f}} < i \\ \langle (j_\ell, f_\ell) : \ell < n_{\nu,i,f} \rangle \char`^ \langle j_{n_{\nu,i,f}} \rangle & \text{if } j_{\nu,i,f} \ge i \end{cases}.$$

Lastly for $y \in P_f$, $f \in F_i$, $f \le (j_{n-1}, f_{n-1})$ we let $L_\nu(y) = y + x_{\rho_{\nu,i,f}}^f$.
Now

<u>Observation α:</u> L_ν is a partial automorphism of M
[preserving P_f — clear; commuting with h_{f_1,f_2} — see the definition of h_{f_1,f_2} and of L_ν remember that $j_\ell < j_{\ell+1}$ in the definition of S_i^a; commuting with F_y^f-check the definitions].

<u>Observation β:</u> if $\nu_2 \in S_{\delta(*)}^a$, $\nu_1 = \nu_2\restriction m$ then $L_{\nu_1} \subseteq L_{\nu_2}$ [check definition].

<u>Observation γ:</u> for every $A \subseteq M$, $|A| < \lambda$ and $\nu_1 \in S_{\delta(*)}^a$ there are $j < \delta(*)$ and $f \in F_{\delta(*)}$ such that $\nu_2 = \nu_1 \char`^ \langle (j, f\restriction(a_i \backslash (j+1))) \rangle$ belongs to $S_{\delta(*)}^a$ and $A \subseteq$ Dom L_{ν_α}.
[Choose j bigger than the first coordinate of each $\nu_1(\ell)$, ($\ell < \ell g(\nu_1)$) and such that $\lambda_j > |A|$; choose $f \in F_{\delta(*)}$ (i.e. for each $i < \delta(*)$ we have $f\restriction a_i \in F_i$ and) such that:
(a) $\alpha \in (j, \delta(*))$ & $g \in F_\alpha$ & $P_g \cap A \ne \emptyset \Rightarrow g\restriction(j, \alpha] < f\restriction a_\alpha$,
(b) $\ell < \ell g(\nu_1)$ & $\nu_1(\ell) = (j', f') \Rightarrow f' < f$.]

<u>Observation δ:</u> There are $\nu \in S_{\delta(*)}^a$ such that L_ν is not the identity. [Easy].

So for some $b \ne c \in P_f^M$ (for some f), (M, b), (M, c) are as required, except for a too large vocabulary, which can be corrected by coding (for example see [Sh189]). $\square_{7.5}$

Conclusion 7.6 If $\lambda > \text{cf}(\lambda) > \aleph_1$ then there are $M_1 \equiv_{L_{\infty,\lambda}} M_2$,

$$\|M_1\| = \|M_2\| = \lambda, \quad M_1 \not\cong M_2,$$

(and $L(M_2)$ is just one binary relation).

Remark 7.6A The remaining case is cf $\lambda = \aleph_1$ and if $\langle \lambda_i : i < \aleph_1 \rangle$ is increasing continuous with limit λ, then for some club C of ω_1,

$$\prod_{i \in C} \lambda_i^+ / \{w : w \subseteq C, |w| < \aleph_0\} \text{ has true cofinality } \lambda^+$$

[why? let

$$S = \{\delta < \omega_1 : \delta \text{ limit, and some } \theta \in (\lambda_\delta, \lambda) \text{ is in } \mathrm{pcf}\{\lambda_i^+ : \alpha < i < \delta\} \text{ for every } \alpha < \delta\}.$$

If S is stationary then as in the proof of 7.2 we can find a parameter and apply 7.5.

If S is not stationary for some club C of λ, $C \cap S = \emptyset$. We next assume $\prod_{i \in C} \lambda_i^+ / \{w \subseteq C : |w| < \aleph_0\}$ is not λ-directed then for some countable $a \subseteq C$, $\prod_{i \in a} \lambda_i^+ / \{w \subseteq a : |w| < \aleph_0\}$ has true cofinality $\theta \in \mathrm{Reg} \cap \lambda$ (just choose a minimal $\theta \in \mathrm{pcf}\{\lambda_i^+ : i \in a\}$ such that $J_{<\theta}[\{\lambda_i^+ : i \in a\}]$ is not included in $\{w \subseteq a : |w| < \aleph_0\}$). Let $j_n = n$th member of a, so $\bigcup_n j_n \in S$ but $\bigcup_n j_n \in C$, contradiction]

Note that by 7.1 the class of counterexamples reflects in no λ (and more). So probably counterexamples are very rare. It is an open question whether the existence of such λ is consistent with ZFC (assuming, of course, suitable large cardinals; it is clear that the consistency strength of this is high).

In fact, in the remaining case, if $\lambda_i = \mathrm{cf}\lambda_i < \lambda$ increasing for $i < \omega_1$ and $\lambda = \sup\{\lambda_i : i < \omega_1\}$ then for some $\alpha < \omega_1$

$$\{\delta < \omega_1 : \mathrm{pcf}\{\lambda_i : \alpha < i < \delta\} \text{ is disjoint to } (\bigcup_{i<\delta} \lambda_i, \lambda)\}$$

contains an end segment of ω_1 (as in 7.3(ix)).

Question 7.7 Does 7.6 hold for $\lambda > \mathrm{cf}\lambda = \aleph_1$ too?

III [Sh 365]

THERE ARE JONSSON ALGEBRAS IN MANY INACCESSIBLE CARDINALS

§0 Introduction

Here we prove (in ZFC) that there is a Jonsson algebra on λ if: λ is an inaccessible not ω-Mahlo or just λ is an inaccessible (not necessarily strong limit) cardinal which has a stationary subset not reflecting in any inaccessible cardinals. We also prove this for many successor of singulars. The method is "guessing clubs". We prove stronger theorems (strong colouring relations) in almost all those cases. In particular if $\lambda > \aleph_2$ is regular not Mahlo (or just has a stationary set which does not reflect in inaccessibles) then for some Boolean algebra B, B satisfies the λ-c.c. but $B \times B$ does not (on the quite long history of this problem see Appendix §1). So for every $\lambda > \aleph_1$ there is a topological space (in fact, coming from a Boolean algebra) having cellularity λ but its square has cellularity $> \lambda$. Note: on successors of regulars we know more so we usually ignore them.

On the history of Jonsson algebras, see introduction to [Sh355,§0], on colouring theorems see Appendix §1 and on guessing clubs see [Sh-e, 7.8A-...] but the presentation here is self contained. We use an indecomposable filter, see for example Kanamori Magidor [KM]. In the proof of the colouring theorems we use Todorcevic walks ([To2]). We use also Claim 3.2A which is a variant of Kanamori [Kn], Ketonen [Ke] (here: for filters which are not necessarily ultrafilters see 3.2A).

The structure of this chapter is as follows: in the first section we define ideals of guessing clubs and show their connections to the existence of Jonsson algebras. In the second section we prove the existence of various club guessing $\bar{C}$'s, we also repeat a theorem from [Sh-e, III 6.4], [Sh351]: for λ regular, $\{\delta < \lambda^+ : \mathrm{cf}\delta < \lambda\} \in I[\lambda^+]$. In the third section we prove the existence of the promised Jonsson algebras and in the fourth section— the colouring theorems.

For continuation, saying more on higher inaccessibles and successors of singulars, see [Sh380], [Sh413] and [Sh535].