THE SMALL INDEX PROPERTY FOR ω-STABLE ω-CATEGORICAL STRUCTURES AND FOR THE RANDOM GRAPH

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ABSTRACT

We give a criterion involving existence of many generic sequences of automorphisms for a countable structure to have the small index property. We use it to show that (i) any ω-stable ω-categorical structure, and (ii) the random graph have the small index property. We also show that the automorphism group of such a structure is not the union of a countable chain of proper subgroups.

1. *Introduction*

Let M be a countably infinite structure, and G be Aut(M). Following [27] we write automorphisms on the right; so if $g \in G$ and $a \in M$ we write ag for the image of a under g. If $\bar{a} = (a_1, \ldots, a_n) \in M$ is a (finite) tuple of elements of M, we write $\bar{a}g$ for $(a_1 g, \ldots, a_n g)$. We write $G_{\bar{a}}$ for the subgroup $\{g \in G : \bar{a}g = \bar{a}\}$ of G. We shall sometimes use the M^{eq} of [25]; note that essentially $G = \mathrm{Aut}(M^{\mathrm{eq}})$ also. If A is a subset of the domain of M^{eq}, we write Ag for $\{ag : a \in A\}$, and G_A for $\{g \in G : ag = a$ for all $a \in A\}$.

The group G is a topological group for which the basic open sets are the cosets of the $G_{\bar{a}}$ for $\bar{a} \in M$. Note that $G_{\bar{a}}g = gG_{\bar{a}g}$, so that in the definition we do not need to specify whether cosets are left or right. The open subgroups form a base of open neighbourhoods of 1. In fact, G is a *Polish space* (see §2).

A subgroup H of G is said to have *small index* in G if $|G:H| < 2^{\omega}$, and *large index* otherwise. If $\bar{a} \in M$, the right cosets of $G_{\bar{a}}$ in G are in bijection with $\{\bar{a}g : g \in G\}$. Hence $G_{\bar{a}}$, and so any open subgroup of G, has small (indeed countable) index in G. We say that M has the *small index property* if the converse holds: that is, every subgroup $H \leqslant G$ of small index is open in G.

If M has the small index property, the topological structure of G can be recovered from its abstract group structure. This has applications in reconstructing a structure from its automorphism group; [18] has more information. For related applications of the small index property see [11, 12].

EXAMPLE 1.1. We list some countable structures with the small index property.

1. The infinite set without structure: proved first by Semmes [24], and (later and independently) in [4].
2. The countable dense linear ordering $(\mathbb{Q}, <)$: proved first by Truss in [26]. Another proof of this result is given in [22].

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3. The countable atomless Boolean algebra [**26**].
4. A vector space of dimension ω over a finite or countable division ring (due to Evans [**7**]; one can add a non-degenerate bilinear form).
5. Any Boolean power of a finite simple group by the countable atomless Boolean algebra: Evans, unpublished.
6. Any countable 2-homogeneous tree [**5**].
7. Any ω-categorical abelian group: Evans, [**9**].

Any 2-homogeneous dense subset of $\mathbb{R}$ has the small index property [**6**]. In [**17**] it is shown that almost strongly minimal structures have a closely related property. An example (due to Cherlin and Hrushovski) of an ω-categorical structure without the small index property is given in [**16**]. For more on the small index property and its variants for uncountable structures see [**16, 18, 19**].

We shall prove the following theorem.

THEOREM 1.2. *If M is a countable ω-stable ω-categorical structure, or if M is the random graph, then M has the small index property. Also,* Aut(M) *is not the union of a countable chain of proper subgroups.*

Our method uses generic automorphisms of M. Recall (for example from [**23**] or [**27**]) that a subset C of a topological space X is *comeagre* if it contains a countable intersection of dense open subsets of X. In [**27**] an automorphism of M was said to be *generic* if its conjugacy class in G was comeagre. If $g \in G$, let us write (M, g) for the expansion of M obtained by adding a function symbol to the language of M and interpreting it as g. Since Aut(M) is a Polish space, comeagre sets are non-empty, and hence any two comeagre sets have non-empty intersection. Thus if g, h are generic then $(M, g) \cong (M, h)$.

In the cases we consider, G will contain a open subgroup K such that for each non-zero $n < \omega$ there is a comeagre subset S of K^n consisting of 'generic sequences' of automorphisms: for all $(g_1, \dots, g_n)$, $(h_1, \dots, h_n) \in S$, the expanded structures $(M, g_1, \dots, g_n)$ and $(M, h_1, \dots, h_n)$ are isomorphic. Moreover, they are *homogeneous* in that (roughly speaking) there are arbitrarily large finite subsets A of M such that A is g_i-closed for each $i \leqslant n$, and if $g_i \restriction A = h_i \restriction A$ for each i then the isomorphism can be chosen to fix A pointwise. We prove this using amalgamation as in [**27**], following Fraïssé.

If a subgroup H of G has small index but is not open, then for each $s \in {}^{<\omega}2$ we can find $g_s, \theta_s \in G$ so that for all s:

- $g_{s^\frown 0} \in H$ and $g_{s^\frown 1} \notin H$,
- $\theta_\varnothing = \theta_{s^\frown 0} = 1$, and $\theta_{s^\frown 1}: (M, g_{s\restriction 1}, \dots, g_s, g_{s^\frown 0}) \to (M, g_{s\restriction 1}, \dots, g_s, g_{s^\frown 1})$ is an isomorphism.

Using homogeneity we can choose the θ_s to fix increasing finite amounts of M as s increases. In this way we can arrange that the product $\theta_\sigma \overset{\text{def}}{=} \prod_{i<\omega} \theta_{\sigma \restriction i}$ exists for each $\sigma \in {}^{\omega}2$. Let σ, τ be distinct elements of ${}^{\omega}2$, and suppose that $i < \omega$ is such that $\sigma \restriction i = \tau \restriction i = s$ (say), $\sigma(i) = 0$ and $\tau(i) = 1$. Then by continuity of the group product operation, $\theta_\sigma^{-1} . \theta_\tau : (M, g_{s^\frown 0}) \to (M, g_{s^\frown 1})$ is an isomorphism, so that $\theta_\sigma^{-1} . \theta_\tau \in G \backslash H$. It follows that H has large index in G. This contradiction proves the small index property for M. The argument to show that G is not the union of a countable chain of proper subgroups is similar.

A similar technique can be applied to uncountable structures. Let M be an L-structure, let I be any set and let f_i $(i \in I)$ be function symbols not occurring in L. In [19], a sequence $(g_i : i \in I)$ of automorphisms of M is said to be *existentially closed* if the following holds. Assume that $M \preccurlyeq N$, h_i is an automorphism of N extending g_i (for each $i \in I$), and $\phi(\bar{x}, \bar{y})$ is a conjunction of (a) L-formulas, (b) formulas of the form $f_i(v_1) = v_2$ for $i \in I$ and v_1, v_2 in $\bar{x}^\frown\bar{y}$. Then for all $\bar{a} \in M$, if $(N, h_i : i \in I) \models \exists\bar{y}\phi(\bar{a}, \bar{y})$ then $(M, g_i : i \in I) \models \exists\bar{y}\phi(\bar{a}, \bar{y})$. Using amalgamation of existentially closed sequences of automorphisms, Lascar and Shelah prove that if M is a saturated model of a first order theory T of cardinality $\lambda = \lambda^{<\lambda} > |T|$, then whenever $H \leqslant \mathrm{Aut}(M)$ has index at most λ, there is $A \subseteq M$ with $|A| < \lambda$ and

$$\mathrm{Aut}_A(M) \stackrel{\text{def}}{=} \{g \in \mathrm{Aut}(M) : ag = a \text{ for all } a \in A\} \leqslant H.$$

The proofs will appear in [19] but are similar to the ones we present here. Shelah has announced that the result also holds when λ is singular (in which case T is stable).

We remark that the topological arguments we give can often be replaced by game-theoretic ones, by dint of the Banach–Mazur theorem [23, Theorem 6.1].

The layout of the paper is as follows. In §2 we define the notion of an amalgamation base for M, and show that if such a base exists then M has many homogeneous generic sequences of automorphisms of all finite lengths. In §3 we show that ω-stable ω-categorical structures and the random graph have amalgamation bases. In §4 we prove that a meagre subgroup (that is, one with comeagre complement) of a Polish group has large index. This will be needed in §5 when we show that the existence of many homogeneous generic sequences of all finite lengths implies the small index property. Finally in §6 we establish the result on ascending chains of subgroups. Theorem 1.2 follows from Theorems 2.9, 5.3, 6.1 and the results of §3.

Some history may be helpful. When Shelah visited Hodges in the summer of 1989, he sketched a strategy for proving that the random graph has the small index property, and Hodges took notes. Hodkinson later extracted versions of Theorems 2.9 and 5.3 from these notes, and proved Theorem 4.1. Lascar improved the argument, and used it and an earlier result of Hrushovski (Theorem 3.1 below) in showing that ω-stable ω-categorical structures have the small index property. Finally Hrushovski [14] proved Theorem 3.6, completing the argument for the random graph, and Lascar obtained Theorem 6.1.

NOTATION. Throughout the paper M will be a countably infinite structure, with further conditions where stated; A, B, etc. will generally denote sets of elements, and $A \subseteq M$ ($A \subseteq M^{\mathrm{eq}}$) will mean that A is a set of elements of the domain of M (respectively, of M^{eq}). If $A \subseteq M^{\mathrm{eq}}$, we write $\mathrm{Aut}(A)$ for the set of M^{eq}-elementary permutations of A. Except in §4, the symbol G will denote $\mathrm{Aut}(M)$; we shall identify $\mathrm{Aut}(M)$ with $\mathrm{Aut}(M^{\mathrm{eq}})$. We write $H \leqslant G$ to mean that H is a subgroup of G. If $g, h \in G$, we shall use g^h to denote $h^{-1}gh$. If α is an ordinal, ${}^{\alpha}2$ is the set $\{f : f : \alpha \to 2\}$ of sequences of zeros and ones of length α. If $f \in {}^{\alpha}2$, we let $f^\frown 0, f^\frown 1 \in {}^{\alpha+1}2$ denote the extensions f^* of f with $f^*(\alpha) = 0, 1$ respectively. If $\beta \leqslant \alpha$, $t \in {}^{\beta}2$ and $s \in {}^{\alpha}2$, we write $t \leqslant s$ if $s \restriction \beta = t$. We use ${}^{<\alpha}2$ to denote $\bigcup_{\beta<\alpha} {}^{\beta}2$.

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2. *Homogeneous generic sequences of automorphisms*

Here we consider generic sequences of automorphisms. We shall give a sufficient condition for a countably infinite ω-categorical structure M to have many homogeneous generic sequences of automorphisms of all finite lengths.

2.1. *Generic sequences of automorphisms*

DEFINITION 2.1. A *base* for M is a set $\mathscr{B}(M)$ of subsets of M^{eq} satisfying:
1. G_A is open in G for all $A \in \mathscr{B}(M)$,
2. if $A \in \mathscr{B}(M)$ and $g \in G$ then $Ag \in \mathscr{B}(M)$.

DEFINITION 2.2. Let $\mathscr{B}(M)$ be a base for M, and let $0 < n < \omega$. We say that $(g_1, \dots, g_n) \in G^n$ is $\mathscr{B}(M)$-*generic*, or just *generic*, if the following two conditions hold.
1. If $A \in \mathscr{B}(M)$ then $\{G_B : A \subseteq B \in \mathscr{B}(M),\ Bg_i = B \text{ for all } i \leqslant n\}$ is a base of open neighbourhoods of 1 in G.
2. Let $A \in \mathscr{B}(M)$ be such that $Ag_i = A$ $(1 \leqslant i \leqslant n)$. Let $B \in \mathscr{B}(M)$ extend A; and let $h_i \in \mathrm{Aut}(B)$ extend $g_i \upharpoonright A$ $(1 \leqslant i \leqslant n)$. Then there is $\alpha \in G_A$ such that $g_i^\alpha \overset{\mathrm{def}}{=} \alpha^{-1} g_i \alpha$ extends h_i for all i.

Recall that a *Polish space* is a separable topological space that can be made a complete metric space. If we let $M = \{a_n : n < \omega\}$ we can metrise G by: $d(g,h) = 0$ if $g = h$, and $1/2^n$ otherwise, where n is the least natural number such that $a_n g \neq a_n h$ or $a_n g^{-1} \neq a_n h^{-1}$. Then G is complete under this metric, which yields the topology defined in §1, so G is a Polish space. If $n > 1$ then G^n, endowed with the product topology, is also a Polish space. A sequence $(g_i : i < \omega)$ of elements of G is a Cauchy sequence if and only if for every open subgroup $K \leqslant G$ there is $n < \omega$ such that $g_i g_j^{-1} \in K$ and $g_i^{-1} g_j \in K$ for all $i,j > n$. Similarly, if $p_j \in G$ and $f_j = p_0 p_1 \cdots p_j$ for each $j < \omega$, the sequence $(f_j : j < \omega)$ is Cauchy if and only if for all open subgroups $K \leqslant G$ there is $n < \omega$ such that $p_j \in K \cap f_n^{-1}.K.f_n$ for all $j > n$.

We now show that once two generic sequences agree on an element of the base, they are conjugate over that element.

PROPOSITION 2.3 (Homogeneity). *Let $\mathscr{B}(M)$ be a base for M, let $n < \omega$ be non-zero, and let $(g_1, \dots, g_n)$, $(h_1, \dots, h_n)$ each be $\mathscr{B}(M)$-generic. Let $B \in \mathscr{B}(M)$ and suppose that $g_i \upharpoonright B = h_i \upharpoonright B \in \mathrm{Aut}(B)$ for each $i \leqslant n$. Then there is $f \in G_B$ such that $g_i^f = h_i$ for all $i \leqslant n$.*

(So $f : (M, g_1, \dots, g_n) \to (M, h_1, \dots, h_n)$ is an isomorphism.)

Proof. The proof is a standard back-and-forth argument, with the complication that the elements of $\mathscr{B}(M)$ are subsets of M^{eq}. Let us say that an M^{eq}-elementary map θ is *determined* on an element $a \in M$ if $G_{\mathrm{dom}(\theta)} \leqslant G_a$. Clearly, any two automorphisms extending θ will agree on a. We let Θ be the set $\{f \restriction A : f \in G_B,\ A \in \mathscr{B}(M),\ A \supseteq B,\ Ag_i = A$ for all $i \leqslant n, h_i$ extends $g_i^f \restriction (Af)$ for all $i \leqslant n\}$. Clearly Θ is non-empty, as it contains the identity map on B.

CLAIM. *Let $\theta \in \Theta$, and let $a \in M$. Then there is $\theta' \in \Theta$ extending θ and determined on a. Similarly, there is $\theta'' \in \Theta$ extending θ and such that θ''^{-1} is determined on a.*

Proof of Claim. Let $f \in G_B$, $A \in \mathscr{B}(M)$ be such that $f \restriction A = \theta$. As $(g_1, \ldots, g_n)$ is generic, we can choose $A' \supseteq A$ in $\mathscr{B}(M)$ such that $G_{A'} \leqslant G_a$ and $A'g_i = A'$ for all $i \leqslant n$. As $\theta \in \Theta$, the automorphisms g_i^f and h_i agree on Af for each $i \leqslant n$. Thus, $A'f \in \mathscr{B}(M)$ extends Af, and $g_i^f \restriction (A'f) \in \mathrm{Aut}(A'f)$ extends $h_i \restriction (Af)$ for all $i \leqslant n$. So as $(h_1, \ldots, h_n)$ is generic, there is $f^* \in G_{Af}$ such that h_i extends $g_i^{ff^*} \restriction (A'ff^*)$ for all $i \leqslant n$. Then the map $\theta' = (ff^*) \restriction A'$ is in Θ, extends θ and is determined on a, as required. The other half of the claim is proved similarly.

Enumerate the domain of M as $\{a_m : m < \omega\}$. By the claim, Θ is in effect a back-and-forth system, and we can define, by induction in the usual way, an increasing chain of elements $B_m \in \mathscr{B}(M)$ for $m < \omega$, all containing B, and automorphisms $f_m \in G_B$ for $m < \omega$, in such a way that $\theta_m \stackrel{\mathrm{def}}{=} f_m \restriction B_m \in \Theta$, both θ_m and its inverse are determined on a_m, and θ_{m+1} always extends θ_m. Now if $k, k' \geqslant m$ then f_k and $f_{k'}$ both extend θ_m, so they agree on a_m. Similarly, their inverses agree on a_m. It follows that $(f_m : m < \omega)$ is a Cauchy sequence. Let f be its limit. We check that f is as required. Certainly $f \in G_B$. Let $i \leqslant n$; we check that $g_i^f = h_i$. Let $m < \omega$. Then $f \restriction B_m = \theta_m \in \Theta$, so h_i extends $g_i^f \restriction (B_m f)$. Hence $g_i^f h_i^{-1} \in G_{B_m f} \leqslant G_{a_m}$, so g_i^f and h_i agree on a_m. As this holds for all m, we obtain $g_i^f = h_i$, as required.

2.2. *Extending M^{eq}-elementary maps*

We shall need the following 'folklore' lemma (appearing in early papers of Cameron), and corollaries on extending M^{eq}-elementary maps to automorphisms of M.

LEMMA 2.4. *Assume that M is ω-categorical. Then for each open subgroup K of G, there are only finitely many subgroups H of G that contain K.*

Proof. It suffices to prove the result for $K = G_{\bar{a}}$, for arbitrary $\bar{a} \in M$. Assume $\bar{a}$ has length n. By the Ryll-Nardzewski–Engeler–Svenonius theorem (see [2]), there are only finitely many orbits of G on M^{2n}. If $H \leqslant G$, let

$$\Phi = \{(\bar{a}g, \bar{a}hg) : g \in G, h \in H\}.$$

Then Φ is a union of orbits of G on $2n$-tuples, so it can take only finitely many values as H ranges over subgroups of G. The lemma will follow if we show that if $G_{\bar{a}} \leqslant H$ then $H = \{f \in G : (\bar{a}, \bar{a}f) \in \Phi\}$, so that Φ determines H in this case.

Clearly $(\bar{a}, \bar{a}h) \in \Phi$ for every $h \in H$. For the other direction, let $(\bar{a}, \bar{a}f) \in \Phi$, so that $(\bar{a}, \bar{a}f) = (\bar{a}g, \bar{a}hg)$ for some $g \in G$, $h \in H$. It is immediate that $g, hgf^{-1} \in G_{\bar{a}}$. But $G_{\bar{a}} \leqslant H$, so $g, hgf^{-1} \in H$, and it follows that $f \in H$, as required.

COROLLARY 2.5. *Assume that M is ω-categorical. Let f be an M^{eq}-elementary map with domain $D \subseteq M^{eq}$, and suppose that G_D is open. Then there is a $g \in G$ extending f.*

Proof. Clearly, $G_D = \bigcap_{\bar{a} \in D} G_{\bar{a}}$. So by Lemma 2.4 there is a finite $D_0 \subseteq D$ such that $G_D = G_{D_0}$. Now as M is saturated, it is easily seen that for all finite $X \subseteq D$, the map $f \restriction X$ extends to some $f_X \in G$. Let $g = f_{D_0}$. Then $f_X . g^{-1} \in G_{D_0} \leqslant G_X$ for all finite X with $D_0 \subseteq X \subseteq D$, so $g \restriction X = f_X \restriction X = f \restriction X$. As such an X can be found containing any chosen element of D, we see that g extends f.

COROLLARY 2.6. *Assume that M is ω-categorical. Let $A \subseteq M^{eq}$ be such that G_A is open. Then* $\mathrm{Aut}(A)$ *is (at most) countable.*

Proof. Choose finite $B \subseteq M$ with $G_B \leqslant G_A$. Using Corollary 2.5, for each $g \in \mathrm{Aut}(A)$ choose $g^+ \in G$ extending g. Then if $g \neq h$ in $\mathrm{Aut}(A)$, the right cosets $G_B g^+$ and $G_B h^+$ are distinct. Hence $|\mathrm{Aut}(A)| \leqslant |G:G_B| \leqslant \omega$.

2.3. *Existence of many generic sequences*

DEFINITION 2.7. Let $\mathscr{B}(M)$ be a base for M. We say that M has *ample $\mathscr{B}(M)$-generic automorphisms* if for all non-zero $n < \omega$, the set of $\mathscr{B}(M)$-generic elements of G^n is comeagre in G^n in the product topology.

We say that M has *ample homogeneous generic automorphisms* if there exists a base $\mathscr{B}(M)$ for M such that M has ample $\mathscr{B}(M)$-generic automorphisms.

We shall give a sufficient condition for M to have ample homogeneous generic automorphisms. For this we need another definition.

DEFINITION 2.8. A base $\mathscr{A}(M)$ for M is said to be an *amalgamation base* if (a) it is countable, and (b) the following conditions hold.

Cofinality. Let $e_1, \ldots, e_n$ be finite elementary maps from M into M. Let $A \in \mathscr{A}(M)$. Then there is $B \in \mathscr{A}(M)$ containing A, and $f_i \in \mathrm{Aut}(B)$ extending e_i for each $i \leqslant n$.

Amalgamation property. Let $A, B, C \in \mathscr{A}(M)$ with $A \subseteq B$, $A \subseteq C$. Then there is $\alpha \in G_A$ such that whenever $g \in \mathrm{Aut}(B\alpha)$, $h \in \mathrm{Aut}(C)$ satisfy $g \restriction A = h \restriction A \in \mathrm{Aut}(A)$, then $g \cup h$ is an elementary map in M^{eq}.

THEOREM 2.9. *Let M be a countable ω-categorical structure and let $\mathscr{A}(M)$ be an amalgamation base for M. Then M has ample $\mathscr{A}(M)$-generic automorphisms.*

Proof. Let $n \geqslant 1$; we show that $\{\bar{g} \in G^n : \bar{g}$ is $\mathscr{A}(M)$-generic$\}$ is comeagre in G^n in the product topology.

CLAIM 1. *Let $A \in \mathscr{A}(M)$ and $\bar{a} \in M$. Then the set*

$$X(A, \bar{a}) \overset{\text{def}}{=} \{(g_1, \ldots, g_n) \in G^n : \exists B \in \mathscr{A}(M)\,(B \supseteq A, B = Bg_i \text{ for all } i \leqslant n, G_B \leqslant G_{\bar{a}})\}$$

is an open dense subset of G^n.

Proof of Claim 1. Since $\mathscr{A}(M)$ is a base for M, we know that G_B is open for all $B \in \mathscr{A}(M)$, and it follows that $X(A,\bar{a})$ is open in G^n. To prove that it is dense, let $S \subseteq G^n$ be a non-empty open subset. We show $S \cap X(A,\bar{a}) \neq \varnothing$. We can replace S by a smaller set, so we can assume it has the form

$$\{(g_1,\ldots,g_n) \in G^n : g_i \text{ extends } e_i \text{ for all } i \leqslant n\}$$

for some finite elementary maps $e_1,\ldots,e_n$ of M into M, and by Corollary 2.5 we can assume that the e_i are defined on $\bar{a}$. As $\mathscr{A}(M)$ is an amalgamation base, by the cofinality condition there is $B \in \mathscr{A}(M)$ containing A and the domain and range of each e_i, such that each e_i can be extended to $f_i \in \mathrm{Aut}(B)$. By Corollary 2.5 there are $g_i \in G$ extending f_i (all i). Then $(g_1,\ldots,g_n) \in S \cap X(A,\bar{a})$. This proves the claim.

Now suppose that $A \subseteq B$ in $\mathscr{A}(M)$, $h_i \in \mathrm{Aut}(B)$ and $Ah_i = A$ for each $i \leqslant n$. Let $\bar{h} = (h_1,\ldots,h_n)$, and write:

$$Y(A,B,\bar{h}) \overset{\text{def}}{=} \{(g_1,\ldots,g_n) \in G^n : \text{if } g_i \restriction A = h_i \restriction A \text{ (all } i \leqslant n) \text{ then there is } \alpha \in G_A \text{ such that } (g_i^\alpha) \restriction B = h_i \text{ (all } i \leqslant n)\}.$$

CLAIM 2. *The set $Y(A,B,\bar{h})$ is open and dense in G^n.*

Proof of Claim 2. This is similar to the proof of Claim 1. As before, $Y(A,B,\bar{h})$ is open. To show density, let

$$S = \{(g_1,\ldots,g_n) \in G^n : g_i \text{ extends } e_i \text{ for each } i \leqslant n\},$$

where the e_i are finite elementary maps on M. We show that $S \cap Y(A,B,\bar{h}) \neq \varnothing$.

Again we can replace S by a smaller set, so as $\mathscr{A}(M)$ is an amalgamation base we can assume that $S = \{(g_1,\ldots,g_n) \in G^n : g_i \text{ extends } f_i \text{ for all } i \leqslant n\}$, where $f_i \in \mathrm{Aut}(C)$ extends e_i for all $i \leqslant n$, and $C \in \mathscr{A}(M)$ contains A. Note that G_C is open in G, so S is open in G^n, and by Corollary 2.5, $S \neq \varnothing$.

If $f_i \restriction A \neq h_i \restriction A$ for some $i \leqslant n$, then $S \subseteq Y(A,B,\bar{h})$, and we are done. Assume that we are in the other case. As $\mathscr{A}(M)$ is an amalgamation base, there is $\alpha \in G_A$ such that for each i, the map $f_i \cup h_i^\alpha$ is elementary, so by Corollary 2.5 extends to an automorphism g_i of M. Then $(g_1,\ldots,g_n) \in S \cap Y(A,B,\bar{h})$, proving the claim.

Now $\mathscr{A}(M)$ is countable, and by Corollary 2.6, so is $\mathrm{Aut}(B)$ for all $B \in \mathscr{A}(M)$. The theorem now follows, since

$$\{\bar{g} \in G^n : \bar{g} \text{ is } \mathscr{A}(M)\text{-generic}\} = \bigcap_{A,\bar{a}} X(A,\bar{a}) \cap \bigcap_{A,B,\bar{h}} Y(A,B,\bar{h}),$$

a countable intersection of dense open sets.

3. *Amalgamation bases*

We now show that ω-stable ω-categorical structures and the random graph have amalgamation bases.

Firstly let M be a countably infinite structure whose theory is ω-stable and ω-categorical. We work in the real M, but we shall use M^{eq} a little. In particular,

$\mathrm{acl}(A)$ for $A \subseteq M$ will always denote the algebraic closure of A in M^{eq}. But we will make no distinction between $G = \mathrm{Aut}(M)$, and $\mathrm{Aut}(M^{\mathrm{eq}})$. We begin with the following theorem, due to E. Hrushovski (unpublished).

THEOREM 3.1. *If $B \subseteq M$ is finite and $A = \mathrm{acl}(B)$ then G_A is open in G.*

Proof. Without loss of generality we may assume that $B = \emptyset$. We must find a finite set $C \subseteq M$ such that $G_C \leqslant G_A$.

For each $n < \omega$ and each 0-definable equivalence relation R on n-tuples of M with a finite number m_R of classes, add to the language of M new n-ary relation symbols $U_{R,1}, \ldots, U_{R,m_R}$, and interpret them as the R-equivalence classes. Let M' be the resulting expanded structure.

As M is ω-categorical, two n-tuples in M lie in the same orbit of $\mathrm{Aut}(M')$ if they lie in the same R-equivalence class for each finite equivalence relation R on n-tuples as above. But there are only finitely many M-inequivalent R of this kind. Hence M' is ω-categorical; and clearly it is ω-stable.

Now by [**13**, Theorem 2.1], the language of an ω-categorical ω-stable theory is essentially finite. Hence there is a finite subset $\{U_1, \ldots, U_s\}$ of the $U_{R,m}$ such that all the others are definable from them. Let C be a finite subset of M such that for each $i \leqslant s$ there is $\bar{c}_i \in C$ with $M' \models U_i(\bar{c}_i)$. Then $G_C \leqslant G_A$.

This argument also shows that M is G-finite (see [**16**] for the definition of G-finiteness).

DEFINITION 3.2. We say that $B \subseteq M$ is *homogeneous* if for all $\bar{a}, \bar{b} \in B$, if there is $g \in G$ such that $\bar{a}g = \bar{b}$, then there is $g \in G$ such that $Bg = B$ and $\bar{a}g = \bar{b}$. We write $\mathscr{H}(M)$ for $\{B \subseteq M : B$ is finite and homogeneous$\}$, and $\mathscr{K}(M)$ for $\{\mathrm{acl}(B) : B \in \mathscr{H}(M)\}$.

Clearly $\mathscr{K}(M)$ is closed under automorphisms. From this and Theorem 3.1, we deduce that $\mathscr{K}$ is a base for M.

FACT 3.3. *We have that $\mathscr{H}(M)$ is cofinal in the set $\wp_{<\omega}(M)$ of all finite subsets of M.* (See [**3**].)

PROPOSITION 3.4. *The set $\mathscr{K}(M)$ is an amalgamation base for M.*

Proof. Clearly $\mathscr{K}(M)$ is countable. Let $e_1, \ldots, e_n$ be finite elementary maps on M, and let $A \in \mathscr{K}(M)$. Choose $B \in \mathscr{H}(M)$ such that $A = \mathrm{acl}(B)$. Using Fact 3.3 let $C \in \mathscr{H}(M)$ contain B and the domains and ranges of the e_i. By saturation of M there are $g_1, \ldots, g_n \in G$ extending $e_1, \ldots, e_n$, and by homogeneity we can assume that $Cg_i = C$ for all i. Let $D = \mathrm{acl}(C) \in \mathscr{K}(M)$. Then $Dg_i = D$ for all i. So $g_i \restriction D \in \mathrm{Aut}(D)$ extends e_i for all i.

Now let $A, B, C \in \mathscr{K}(M)$ with $A \subseteq B, A \subseteq C$. There is $\alpha \in G_A$ such that $B\alpha$ and C are independent over A in M^{eq}. The proof is then completed by quoting the following fact.

FACT 3.5. *Let $A, B, C \in \mathscr{K}(M)$ with $A \subseteq B$, $A \subseteq C$ be such that B and C are independent over A. If $f \in \mathrm{Aut}(B)$, $g \in \mathrm{Aut}(C)$ and $f \restriction A = g \restriction A \in \mathrm{Aut}(A)$, then $f \cup g$ is elementary.*

Proof. The proof is an easy modification of [**16**, Theorem 3.3].

Secondly let M be the random graph. See [**1**] for information on the random graph. We do not use M^{eq} here. We shall prove that the base $\mathscr{P}_{<\omega}(M)$, the set of all finite subsets of M, is an amalgamation base for M.

As M has quantifier elimination, the elementary maps on M are just the isomorphisms of induced subgraphs of M. Let $A, B, C \in \mathscr{P}_{<\omega}(M)$ with $A \subseteq B$, $A \subseteq C$. There is $\alpha \in G_A$ such that $B\alpha \cap C = A$ and there are no graph edges between $B\alpha \setminus A$ and $C \setminus A$. Then whenever $g \in \mathrm{Aut}(B\alpha)$, $h \in \mathrm{Aut}(C)$ satisfy $g \restriction A = h \restriction A \in \mathrm{Aut}(A)$, the map $g \cup h$ is M-elementary. Hence the amalgamation condition of Definition 2.8 holds.

For cofinality it is enough to show that whenever $e_1, \ldots, e_n$ are isomorphisms of finite induced subgraphs of M, there is a finite subgraph A of M and $g_i \in \mathrm{Aut}(A)$ extending e_i for each $i \leqslant n$. As M is homogeneous and universal for finite graphs, the following theorem, recently proved by Hrushovski, establishes this.

THEOREM 3.6. *Let X be a finite graph. Then there exists a finite graph Z containing X as an induced subgraph, such that any isomorphism between induced subgraphs of X extends to an automorphism of Z.*

Proof. See [**14**].

J. Truss [**27**] proved earlier in a different way that a single isomorphism of subgraphs of X may be extended to an automorphism of a larger finite graph.

4. *Polish groups*

In this section we work in the slightly more general setting of a *Polish group*: a group G that is also a Polish space with a (countable) set of open subgroups forming a base of neighbourhoods of the identity. (The group $G = \mathrm{Aut}(M)$ is an example of such a group.) Then G is a complete metric space, and we write $d(g, h)$ for the metric on G.

A subset S of a topological space X is said to be *meagre* if $X \setminus S$ is comeagre. We shall prove the following result, needed in §5.

THEOREM 4.1. *Any meagre subgroup of G has index 2^ω in G.*

REMARKS 4.2. 1. By [**16**, Lemma 2.6], any subgroup H of G with the Baire property is either meagre (and so by Theorem 4.1 of large index), or open. As closed sets have the Baire property (see [**23**, Chapter 4]), Theorem 4.1 implies the result of Evans [**8**] that for any countable structure M, any *closed* subgroup of $\mathrm{Aut}(M)$ of small index is open. (Evans' result in turn generalises the definability theorem of Kueker [**15**].)

2. If H is meagre in G then so is each coset of H; it follows that $|G:H| > \omega$. So Theorem 4.1 needs no proof if one wishes to assume the continuum hypothesis. If $\kappa < 2^\omega$ then MA_κ implies that any union of κ meagre subsets of a Polish space is meagre, so Theorem 4.1 follows trivially from Martin's Axiom. See [**21**] for more information.

3. For the Polish space $\mathbb{R}$, Solovay [**21**, §4.2] proved by forcing that there is a model of $ZFC + \neg CH$ in which there exist meagre sets $X_i \subseteq \mathbb{R}$ $(i < \omega_1)$ such that $\bigcup_{i<\omega_1} X_i = \mathbb{R}$.

4. Theorem 4.1 is an easy consequence of the game-theoretic argument of Hodges [**10**, Theorem 4.1.5], which is an adaptation of [**25**, Theorem IV.5.16].

DEFINITION 4.3. A *coset system* is a pair (X, λ), where X is a non-empty set and λ is a map providing for each $x, y \in X$ a non-empty open subset $\lambda(x, y)$ of G. If there is no ambiguity we write G_{xy} for $\lambda(x, y)$, and G_x for $\lambda(x, x)$. We require that for all $x, y, z \in X$:

- $G_{xy} = (G_{yx})^{-1}$,
- $G_{xy} . G_{yz} = G_{xz}$.

Here, for subsets $S, T \subseteq G$, we write S^{-1} for $\{s^{-1} : s \in S\}$ and $S . T$ for $\{st : s \in S, t \in T\}$, as usual.

As an example, if D is an orbit of $G = \mathrm{Aut}(M)$ on M^n, then each non-empty set X and map $\sigma : X \to D$ yields a coset system (X, ν), where

$$\nu(x, y) = \{g \in G : (\sigma(x))g = \sigma(y)\}.$$

REMARK 4.4. Let (X, λ) be a coset system and let $x, y \in X$. The axioms yield $\lambda(x, x) = \lambda(x, x) . \lambda(x, x)^{-1}$. Hence G_x is an open subgroup of G. As $G_{xy} = G_x . G_{xy}$, it is clear that G_{xy} is a union of right cosets of G_x. But if $g, h \in G_{xy}$, then $gh^{-1} \in G_{xy} . G_{yx} = G_x$. Hence G_{xy} is a single right coset of G_x, and so a closed subset of G (for all x, y). Similarly, G_{xy} is a single left coset of G_y. If also $z \in X$, choose arbitrary $g_y \in G_{xy}$ and $g_z \in G_{xz}$; then $G_{yz} = g_y^{-1} . G_x . g_z$. As a special case, G_y is the conjugate of G_x by g_y.

DEFINITION 4.5. If (X, λ) and (X, μ) are coset systems, (X, μ) is said to *refine* (X, λ) if $\mu(x, y) \subseteq \lambda(x, y)$ for all $x, y \in X$.

LEMMA 4.6. *Let (X, λ) be a coset system, let $x, y \in X$ be distinct, and let $S \subseteq \lambda(x, y)$ be open and non-empty. Then there is a refinement (X, μ) of (X, λ) such that $\mu(x, y) \subseteq S$.*

Proof. Choose, for each $z \in X$, an element $g_z \in \lambda(x, z)$ in such a way that $g_x = 1$ and $g_y \in S$. Then $1 \in S(g_y^{-1}) \subseteq \lambda(x, x)$, and $S(g_y^{-1})$ is open. Choose an open subgroup $K \leqslant S(g_y^{-1})$, and if $z, z' \in X$ define $\mu(z, z') = g_z^{-1} . K . g_{z'}$. Then (X, μ) is a coset system with the required properties.

DEFINITION 4.7. If (X, λ) is a coset system, and $S \subseteq G$, we write $(X, \lambda) \subseteq S$ if $G_{xy} \subseteq S$ for all *distinct* $x, y \in X$.

COROLLARY 4.8. *Let $D \subseteq G$ be open and dense, and let (X, λ) be a coset system such that X is finite. Then there is a refinement (X, μ) of (X, λ) such that $(X, \mu) \subseteq D$.*

Proof. Let $x, y \in X$ be distinct. By definition, $\lambda(x, y)$ is open, so that $\lambda(x, y) \cap D$ is non-empty and open. By Lemma 4.6, there is a refinement (X, μ) of (X, λ) such that $\mu(x, y) \subseteq D$. As X is finite, the proposition follows by induction.

DEFINITION 4.9. If X is finite and (X, λ) is a coset system, we shall write $\mathrm{diam}(X, \lambda)$ for $\max\{\mathrm{diam}(\lambda(x, y)) : x, y \in X\}$, where if $S \subseteq G$ is non-empty, $\mathrm{diam}(S) \stackrel{\mathrm{def}}{=} \sup\{d(g, h) : g, h \in S\} \in \mathbb{R} \cup \{\infty\}$.

Using Lemma 4.6, we can also prove the following corollary.

COROLLARY 4.10. *Let X be finite, let (X, λ) be a coset system, and let $\varepsilon > 0$. Then there is a coset system (X, μ) refining (X, λ) and of diameter less than ε.*

Proof. If $x, y \in X$ are distinct, then choose $g \in \lambda(x, y)$ and let

$$S = \{h \in \lambda(x, y) : d(g, h) < \tfrac{1}{2}\varepsilon\}.$$

Clearly S is non-empty and open. By Lemma 4.6 we can choose a refinement (X, μ) of (X, λ) such that $\mu(x, y) \subseteq S$. Hence $\operatorname{diam}(\mu(x, y)) < \varepsilon$. Note that $\mu(x, x) = K$ in Lemma 4.6 can also be taken to have diameter less than ε. As X is finite, the corollary now follows by induction, as before.

DEFINITION 4.11. A *homomorphism* from a coset system (Y, μ) into a coset system (X, λ) is a map $\nu : Y \to X$ such that $\mu(y, y') \subseteq \lambda(\nu(y), \nu(y'))$ for all $y, y' \in Y$. A homomorphism ν as above is said to be *surjective* if it is so as a map from Y onto X.

LEMMA 4.12. *Let (X, λ) be a coset system, let Y be any non-empty set and let $\nu : Y \to X$ be given. Then there is a canonical coset system (Y, μ) such that ν is a homomorphism from (Y, μ) into (X, λ).*

Proof. Define $\mu(y, y') = \lambda(\nu(y), \nu(y'))$.

THEOREM 4.13. *Let C be a comeagre subset of G. Then for each $\sigma, \tau \in {}^{\omega}2$ there is an element $g_{\sigma\tau} \in G$, such that for all $\sigma, \tau, \upsilon \in {}^{\omega}2$,*

- $g_{\sigma\tau} \cdot g_{\tau\upsilon} = g_{\sigma\upsilon}$,
- *if $\sigma \neq \tau$ then $g_{\sigma\tau} \in C$.*

Proof. Let $D_i \subseteq G$ $(i < \omega)$ be dense open sets such that $\bigcap_{i<\omega} D_i \subseteq C$. We can assume that if $i < j < \omega$ then $D_i \supseteq D_j$. For each $n < \omega$ we shall define a coset system $T_n = ({}^{n}2, \lambda_n)$ by induction on n. We shall require that:

1. $T_n \subseteq D_n$ for all $n \geqslant 1$,
2. $\operatorname{diam}(T_n) < 1/n$ if $n \geqslant 1$,
3. if $n < m < \omega$, the map $(s \mapsto s \restriction n)$ for $s \in {}^{m}2$ is a (surjective) homomorphism from T_m onto T_n.

Define $T_0 = (\{\varnothing\}, \lambda_0)$ by $\lambda_0(\varnothing, \varnothing) = G$. Inductively assume that T_n has been defined. By Lemma 4.12 we can define a coset system $T^* = ({}^{n+1}2, \lambda^*)$ on ${}^{n+1}2$ by: $\lambda^*(s, t) = \lambda_n(s \restriction n, t \restriction n)$. By Corollaries 4.8 and 4.10 there is a refinement T_{n+1} of T^* of diameter less than $1/(n+1)$ such that $T_{n+1} \subseteq D_{n+1}$. Then $(s \mapsto s \restriction n)$ is a homomorphism from T_{n+1} onto T_n. This completes the definition of the T_n.

For each $\sigma, \tau \in {}^{\omega}2$ and each $n < \omega$, choose some $g_{\sigma\tau, n} \in \lambda_n(\sigma \restriction n, \tau \restriction n)$. It follows from (2) and (3) that $(\lambda_n(\sigma \restriction n, \tau \restriction n) : n < \omega)$ is a decreasing chain of subsets of G and $\operatorname{diam}(\lambda_n(\sigma \restriction n, \tau \restriction n)) < 1/n$. Hence $(g_{\sigma\tau, n} : n < \omega)$ is a Cauchy sequence; we define $g_{\sigma\tau}$ to be its limit.

Let $\sigma, \tau, \upsilon \in {}^{\omega}2$. We show that $g_{\sigma\tau} . g_{\tau\upsilon} = g_{\sigma\upsilon}$. If $K \leqslant G$ is open, then for all large enough n we have

$$K g_{\sigma\tau} = K g_{\sigma\tau, n} \quad \text{and} \quad (g_{\sigma\tau}^{-1} K g_{\sigma\tau}) g_{\tau\upsilon} = (g_{\sigma\tau}^{-1} K g_{\sigma\tau}) g_{\tau\upsilon, n}.$$

Hence

$$Kg_{\sigma\tau}g_{\tau\upsilon} = g_{\sigma\tau}(g_{\sigma\tau}^{-1}Kg_{\sigma\tau})g_{\tau\upsilon} = g_{\sigma\tau}(g_{\sigma\tau}^{-1}Kg_{\sigma\tau})g_{\tau\upsilon,n} = Kg_{\sigma\tau,n}\cdot g_{\tau\upsilon,n}$$

for all large enough n, so $(g_{\sigma\tau,n}\cdot g_{\tau\upsilon,n}:n<\omega)$ converges to $g_{\sigma\tau}\cdot g_{\tau\upsilon}$. But also, $g_{\sigma\tau,n}\cdot g_{\tau\upsilon,n}\in\lambda_n(\sigma\lceil n,\upsilon\lceil n)$. It now follows from (2) that $d(g_{\sigma\tau,n}\cdot g_{\tau\upsilon,n}, g_{\sigma\upsilon,n})<1/n$ if $n\geqslant 1$. So we obtain that $g_{\sigma\tau}\cdot g_{\tau\upsilon}=g_{\sigma\upsilon}$, as required.

Finally let $\sigma\neq\tau$ be in ${}^{\omega}2$. By Remark 4.4, each $\lambda_n(\sigma\lceil n,\tau\lceil n)$ is closed, so since $g_{\sigma\tau,m}\in\lambda_n(\sigma\lceil n,\tau\lceil n)$ if $m\geqslant n$, we have that $g_{\sigma\tau}\in\lambda_n(\sigma\lceil n,\tau\lceil n)$ for all n. By (1), for all n so large that $\sigma\lceil n\neq\tau\lceil n$, we have that $\lambda_n(\sigma\lceil n,\tau\lceil n)\subseteq D_n$. Since we assumed that the D_n form a decreasing chain, this is enough to ensure that $g_{\sigma\tau}\in\bigcap_{n<\omega}D_n\subseteq C$. The proof is complete.

Proof of Theorem 4.1. Suppose that $H\leqslant G$ is meagre. Choose elements $g_{\sigma\tau}\in G$ ($\sigma,\tau\in{}^{\omega}2$) as in Theorem 4.13 such that if $\sigma\neq\tau$ then $g_{\sigma\tau}\notin H$. Evidently $g_{\sigma\tau}^{-1}=g_{\tau\sigma}$ for all σ,τ. Let $\sigma\in{}^{\omega}2$ be arbitrary. If τ,υ are distinct, $g_{\tau\sigma}\cdot g_{\upsilon\sigma}^{-1}=g_{\tau\upsilon}\notin H$. Hence the right cosets $Hg_{\tau\sigma}$ ($\tau\in{}^{\omega}2$) are all distinct, and so $|G:H|=2^{\omega}$.

COROLLARY 4.14. *Assume that $H\leqslant G$ has small index but is not open. Let $C\subseteq G$ be any comeagre set. Then for all open subgroups $K\leqslant G$, we have that*
1. $(C\cap K)\cap H\neq\varnothing$,
2. $(C\cap K)\backslash H\neq\varnothing$.

Proof. Note that K is also a Polish group in which $C\cap K$ is comeagre.

(1) If $C\cap K\subseteq K\backslash H$ then $H\cap K$ is meagre in K. Hence by Theorem 4.1 it has large index in K, a contradiction.

(2) If $C\cap K\subseteq H$ then $H\cap K$ is comeagre in K. By translating, it follows that all cosets of $H\cap K$ in K are also comeagre in K. But any two comeagre sets intersect, so $K\leqslant H$ and H is open, also a contradiction.

5. *Generic automorphisms and the small index property*

We can now prove the first part of Theorem 1.2. We let $G=\mathrm{Aut}(M)$ again.

NOTATION 5.1. If $X\subseteq G^{n+1}$ (where $n<\omega$) and $\bar{g}=(g_1,\dots,g_n)\in G^n$ (we let $G^0=\{\varnothing\}$), we define:

$$X_{\bar{g}}=\{g\in G:(g_1,\dots,g_n,g)\in X\}.$$

We then define:

$$\partial X=\{\bar{g}\in G^n: X_{\bar{g}}\text{ is comeagre in }G\}.$$

FACT 5.2 (Kuratowski, Ulam). *If $n<\omega$ is non-zero, and $C\subseteq G^{n+1}$ is comeagre (in the product topology) then ∂C is comeagre in G^n.*

Proof. See [**23**, Theorem 15.1], for example.

THEOREM 5.3. *If M is a countable structure with ample homogeneous generic automorphisms, then M has the small index property.*

Proof. Let $\mathscr{B}(M)$ be a base for M such that $\{\bar{g}\in G^n:\bar{g}$ is $\mathscr{B}(M)$-generic$\}$ is comeagre in G^n in the product topology for all $n\geqslant 1$. We begin with the following claim.

CLAIM. *Let $n<\omega$ and let $\bar{g}=(g_1,\ldots,g_n)\in G^n$ be generic (or, if $n=0$, empty). Then $\{f\in G:(\bar{g},f)$ is generic$\}$ is comeagre in G.*

Proof of Claim. If $n=0$ the result is given. Suppose that $n>0$. By assumption, the set C of generic $(n+1)$-tuples is comeagre in G^{n+1}. By the Kuratowski–Ulam Theorem, ∂C is comeagre, and hence dense, in G^n. Choose $B\in\mathscr{B}(M)$ such that $Bg_i=B$ for $i=1,\ldots,n$, and then $\bar{h}=(h_1,\ldots,h_n)\in\partial C$ such that g_i and h_i agree on B for each i. By Proposition 2.3 there is $\theta\in G_B$ such that $h_i^\theta=g_i$ for each $i\leqslant n$. Then $\{f\in G:(\bar{g},f)$ is generic$\}=\theta^{-1}.C_{\bar{h}}.\theta$, which is comeagre in G. This proves the claim.

Now assume for contradiction that H is a subgroup of G of small index that is not open in G. Enumerate the domain of M as $\{a_n:n<\omega\}$. We shall define by induction on $s\in{}^{<\omega}2$:

- a set $B_s\in\mathscr{B}(M)$,
- elements $\gamma_s, g_{s\frown 0}, g_{s\frown 1}\in G$.

We shall require that $\gamma_\varnothing=1$, and that for each $s\in{}^{<\omega}2$:

1. if $t\in{}^{<\omega}2$, $t\leqslant s$ and $t\neq\varnothing$, then $B_s g_t=B_s$;
2. $g_{s\frown 0}\in G_{B_s}\cap H$, and $g_{s\frown 1}\in G_{B_s}\setminus H$;
3. if $s\in{}^n2$ for $n>0$ then the tuple $g_{\bar{s}}=(g_{s\restriction 1},\ldots,g_{s\restriction n})$ is generic;
4. $(g_t)^{\gamma_s}=(g_t)^{\gamma_t}$ for all t such that $\varnothing<t\leqslant s$;
5. if $s\in{}^n2$ then $\gamma_{s\frown 0}\gamma_s^{-1}$, $\gamma_{s\frown 1}\gamma_s^{-1}\in G_{a_i}\cap G_{a_i\gamma_s^{-1}}$ for all $i\leqslant n$.

Let $s\in{}^n2$ for some $n<\omega$, and assume that if $t<s$ then B_t has been defined, and that if $t\leqslant s$, $t\neq\varnothing$, then g_t and γ_t have been defined. We shall define B_s, $g_{s\frown 0}$, $g_{s\frown 1}$, $\gamma_{s\frown 0}$ and $\gamma_{s\frown 1}$.

Let $A=\{a_0,\ldots,a_n,a_0\gamma_s^{-1},\ldots,a_n\gamma_s^{-1}\}$. As $\bar{g}_s$ (if non-empty) is $\mathscr{B}(M)$-generic, we can choose $B_s\in\mathscr{B}(M)$ with $G_{B_s}\leqslant G_A$, and such that $B_s g_t=B_s$ for all $t\leqslant s$ with $t\neq\varnothing$. With this choice, (1) holds. By the claim, $C=\{g\in G:(\bar{g}_s,g)$ is generic$\}$ is comeagre in G. So given our assumption on H, by Corollary 4.14 we can choose $g_{s\frown 0}\in(C\cap G_{B_s})\cap H$ and $g_{s\frown 1}\in(C\cap G_{B_s})\setminus H$. Hence $g_{s\frown 0}$ and $g_{s\frown 1}$ satisfy (2) and (3).

Now $(\bar{g}_s,g_{s\frown 0})$ and $(\bar{g}_s,g_{s\frown 1})$ are generic and agree on $B_s\in\mathscr{B}(M)$. So by Proposition 2.3 there is $f_s\in G_{B_s}$ such that $(g_t)^{f_s}=g_t$ for all $t\leqslant s$, and $(g_{s\frown 1})^{f_s}=g_{s\frown 0}$. Define:

$$\gamma_{s\frown 0}=\gamma_s,\quad \gamma_{s\frown 1}=f_s.\gamma_s.$$

Clearly clauses (4) and (5) are satisfied. This completes the construction.

If $\sigma\in{}^\omega 2$, then by Clause 5 of the construction, $(\gamma_{\sigma\restriction n}:n<\omega)$ is a Cauchy sequence. Let γ_σ be its limit. Suppose that $\sigma\neq\tau$ and n,s are such that $\sigma\restriction n=s\frown 0$ and $\tau\restriction n=s\frown 1$. By (4) and continuity of the product, we have:

$$(g_{s\frown 0})^{\gamma_\sigma}=\lim_{n\to\infty}(g_{s\frown 0})^{\gamma_{\sigma\restriction n}}=(g_{s\frown 0})^{\gamma_{s\frown 0}}=(g_{s\frown 0})^{\gamma_s};$$

$$(g_{s\frown 1})^{\gamma_\tau}=\lim_{n\to\infty}(g_{s\frown 1})^{\gamma_{\tau\restriction n}}=(g_{s\frown 1})^{\gamma_{s\frown 1}}=(g_{s\frown 0})^{\gamma_s}.$$

Hence $(g_{s^\frown 0})^{\gamma_\sigma \gamma_\tau^{-1}} = g_{s^\frown 1}$. Since $g_{s^\frown 0} \in H$ and $g_{s^\frown 1} \notin H$, we see that $\gamma_\sigma . \gamma_\tau^{-1} \notin H$. Thus the right cosets $H\gamma_\sigma$ $(\sigma \in {}^\omega 2)$ are all distinct, and H has index 2^ω, a contradiction.

6. *Ascending chains of subgroups*

To complete the proof of Theorem 1.2, we use the techniques of §5 again to show the following.

THEOREM 6.1. *Assume that M is countably infinite and ω-categorical, and has ample homogeneous generic automorphisms. Then $G = \mathrm{Aut}(M)$ is not the union of a countable chain of proper subgroups.*

Proof. Assume for contradiction that G is the union of an increasing chain $(H_n : n < \omega)$ of proper subgroups. By Lemma 2.4, no H_n is open in G, so for all $\bar{a} \in M$ and $n < \omega$, the subgroup $G_{\bar{a}} \cap H_n$ is not comeagre in $G_{\bar{a}}$. Also, as the union of countably many meagre sets is meagre, by discarding finitely many of the H_n we can assume that no H_n is meagre in G. Hence $G_{\bar{a}} \cap H_n$ is not meagre in $G_{\bar{a}}$ for any $\bar{a}$ and n.

As M has ample homogeneous generic automorphisms, we can undertake the construction of Theorem 5.3 again. But this time we use the above to arrange that $g_{s^\frown 0} \in H_n$ and $g_{s^\frown 1} \notin H_n$ for each $n < \omega$ and $s \in {}^n 2$. Define γ_σ $(\sigma \in {}^\omega 2)$ as before. So if $s \in {}^n 2$, $\sigma > s^\frown 0$ and $\tau > s^\frown 1$ in ${}^\omega 2$, then $(g_{s^\frown 0})^{\gamma_\sigma \gamma_\tau^{-1}} = g_{s^\frown 1}$, so that $\gamma_\sigma \gamma_\tau^{-1} \notin H_n$.

Now as $G = \bigcup_{n<\omega} H_n$, there are $n < \omega$ and uncountable $\Sigma \subseteq {}^\omega 2$ such that $\gamma_\sigma \in H_n$ for all $\sigma \in \Sigma$. Choose $m \geqslant n$ and $\sigma, \tau \in \Sigma$ such that $\sigma \restriction m = \tau \restriction m$ and $\sigma \restriction m+1 \neq \tau \restriction m+1$. Then $\gamma_\sigma \gamma_\tau^{-1} \notin H_m \supseteq H_n$, a contradiction.

The results of Sections 2 and 3 show that any ω-stable ω-categorical structure satisfies the conclusion of the theorem, as does the random graph. Macpherson and Neumann **[20]** prove that the conclusion holds when M is an infinite set without further structure.

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