

On Saturation for a Predicate

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Introduction We generalize the definition of saturated models by demanding that the types contain a fixed predicate. This leads to a natural extension of the notion of stability in which only those types which contain this predicate are counted. Later, we translate some basic theorems from stability to the new concept. In the same style, we also change the definition of “simple” theory from [2], where we claimed that T_{ind}^* is simple and unstable. We prove this fact here. In [2] we were interested in the following property. Given a theory T , and two cardinal numbers λ, κ such that $\lambda = \lambda^{|\mathcal{T}|} \geq \kappa$: Does any model M of T of cardinality λ have an elementary extension of the same cardinality which is κ -saturated? We denoted this property by $SP_T(\lambda, \kappa)$, and proved there that if T is nonsimple then $SP_T(\lambda, \kappa)$ is equivalent to $\lambda = \lambda^{<\kappa}$. Let $SP_T = \{(\lambda, \kappa): SP_T(\lambda, \kappa)\}$. If T is stable, the answer is yes, so we can concentrate on simple unstable T . If λ is strong limit, $SP_T(\lambda, \kappa) \Leftrightarrow \lambda = \lambda^{<\kappa}$. Hence, assuming *GCH*, $SP_T = SP_{T_{\text{ord}}}$ (T_{ord} = theory of linear order). It was also clear that $SP_{T_{\text{ind}}^*}$ is minimal, i.e., for any (simple unstable) T , $SP_T \supseteq SP_{T_{\text{ind}}^*}(\lambda, \kappa)$. In [2] we also prove the consistency of “for every (simple unstable) T , $SP_T = SP_{T_{\text{ind}}^*} \neq SP_{T_{\text{ord}}}$ ”. We ask whether this follows from *ZFC*. We shall prove that if we replace “ κ -saturated” by “ κ -saturated for a predicate P ”, the answer is negative.

We assume familiarity with the introduction of [2] and with a number of basic theorems from there (not with their proofs), and also with Sections II.1 and II.2 of [1].

1 General theory

Conventions Let T be a fixed theory, and let P be a one-place predicate. Assume that every formula is equivalent to an atomic formula.

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Definition 1.1 Let $M \models T$. Then M is (λ, P) -saturated (compact) if whenever $A \subseteq M$, $|A| < \lambda$, $p \in S(A)$, $(|p| < \lambda)$, $P(x) \in p$, then p is realized in M (cf. the proof of Chang's 2-cardinal theorem).

Definition 1.2 The notation $Q_T^P(\lambda, \kappa)$ stands for the following assertion: Every model M of T of power λ has an elementary extension N such that $|N| = \lambda$ and N is (κ, P) -saturated. We omit P when $P(x)$ is equivalent to $(x = x)$. In [2] this was denoted by $SP_T(\lambda, \kappa)$.

Remark 1.2A By [1], Chapter VIII, Theorem 4.7, for $\lambda \geq 2^{|T|}$, $Q_T(\lambda, \lambda)$ holds iff $\lambda = \lambda^{<\lambda}$ or T is stable in λ (which is equivalent to the statement $\lambda^{<\kappa(T)} = \lambda$).

Definition 1.3 The restriction of the theory T to P is $T^P = \{\psi: \psi^P \in T\}$.

Definition 1.4 The theory T is stable for P if there is some cardinal λ such that if $|A| \leq \lambda$, then $|\{q \in S(A): P(x) \in q\}| \leq \lambda$.

Claim 1.5 The theory T is P -unstable iff there is a model M of T with $\vec{a}_n \in M$, for $n \in \omega$, such that for some ϕ , the set $p_k = \{\phi(x, a): k \leq n < \omega\} \cup \{\sim\phi(x, \vec{a}_n): n < k\} \cup \{P(x)\}$ is consistent for every $k < \omega$.

The proof is the same as the proof that an unstable theory has the order property (see [1], Chapter II, Theorem 2.2, pp. 30-31).

Example 1.6 There exist T and P such that T is unstable for P but T^P is stable.

Let B be the set of all two-place relations on ω . Let $M = (\omega \cup B, P^M, R^M)$, where $P^M = \omega$, and $R^M = \{\langle a, b, c \rangle: a, b \in \omega, c \in B, \langle a, b \rangle \in c\}$. Let $T = Th(M)$. To conform to our convention we must add extra relations.

To see that T is P -unstable it is easiest to use Claim 1.5 rather than argue directly from the definition. It is easy to see that a sequence of functions from ω into ω of B , $a_i i \in \omega$ can be chosen so that if $\phi(a_i, a_j, v)$ is the formula that says a_i and a_j agree on v , then for any $n \in \omega$, the type $\{\phi(a_i, a_j, v): i \leq n\} \cup \{\sim\phi(a_i, a_j, v): i > n\} \cup \{P(v)\}$ is consistent. For example, let a_i be the identity on $\{0, \dots, i-1\}$ and send j to $j+i$ for $j \geq i$.

On the other hand, in T^P every formula is equivalent to a Boolean combination of equalities. Otherwise, there would be sequences \vec{x}, \vec{y} , and a formula ϕ such that $\left[\bigwedge_{i,j} [(x_i = x_j) \equiv (y_i = y_j)] \right] \& \phi(\vec{x}) \& \sim\phi(\vec{y})$. But, for such \vec{x}, \vec{y} in P^M , there is a permutation σ of ω taking \vec{x} to \vec{y} ; it extends naturally to B , and we get an automorphism of M .

Lemma 1.7 If T is unstable for P and $\lambda \geq |D(T)|$, then $Q_T^P(\lambda, \lambda)$ iff $\lambda^{<\lambda} = \lambda$.

Proof: If $\lambda^{<\lambda} = \lambda$, then by [1], Chapter I, Theorem 1.7, $Q_T(\lambda, \lambda)$, and, in particular, $Q_T^P(\lambda, \lambda)$. So, suppose $\lambda^{<\lambda} > \lambda$. Choose the least $\kappa < \lambda$ such that $2^\kappa > \lambda$, if it exists. Order $I = 2^{<\kappa}$, $|I| \leq \lambda$, lexicographically. Now by instability, find a model M of T , a formula ϕ , and $\vec{a}_t \in M (t \in I)$ such that for each $s \in I$, the set $\{\phi(x, \vec{a}_t): t \geq s, t \in I\} \cup \{\sim\phi(x, \vec{a}_t): t < s, t \in I\} \cup \{P(x)\}$ is consistent, and $|M| = \lambda$.

Suppose $N > M$, $|N| = \lambda$, and N is (λ, P) -saturated. For every $\eta \in {}^\kappa 2$, let $p_\eta = \{\phi(x, \vec{a}_{\eta|\alpha}) : \alpha < \kappa \ \& \ \eta(\alpha) = 0\} \cup \{\sim\phi(x, \vec{a}_{\eta|\alpha}) : \alpha < \kappa \ \& \ \eta(\alpha) = 1\} \cup \{P(x)\}$. Using the lexicographic order each p_η is consistent, and every pair is contradictory. Each one is realized in N . Therefore, $|N| \geq 2^\kappa > \lambda$, a contradiction. This completes the proof in case λ is not a strong limit cardinal.

Now suppose λ is a strong limit cardinal. Let $I = 2^{<\lambda}$, \vec{a}_η , M , N be as before. Let $\lambda = \sum_{i < cf(\lambda)} \lambda_i$, where $\lambda_{i+1} > 2^{\lambda_i}$, and express N as $\bigcup_{i < cf(\lambda)} N_i$, where $|N_i| \leq \lambda_i$. We now define, by induction on $i < cf(\lambda)$, functions $\eta_i \in {}^{\lambda_i} 2$ such that for $j < i$, η_j is an initial segment of η_i , and such that the type $p_{\eta_{i+1}}$ has a subset of two elements that is not realized in N_i . In the limit, we get a type p_η , with $P(x) \in p_\eta$, consistent and having a subset of power $cf(\lambda)$ that is not realized in N .

Let η_i be defined, and let $\vec{0}_\alpha$ be a sequence of α zeroes. Each of $\phi(x, \vec{a}_{\eta_i} \widehat{\vec{0}_\alpha})$ ($\alpha < \lambda_{i+1}$) defines a subset of N_i . There exist α, β such that $0 < \alpha < \beta < \lambda_{i+1}$ and $\phi(x, \vec{a}_{\eta_i} \widehat{\vec{0}_\alpha})$ and $\phi(x, \vec{a}_{\eta_i} \widehat{\vec{0}_\beta})$ are realized by exactly the same elements of N_i . Let η_{i+1} extend $\eta_i \widehat{\vec{0}_\beta} \langle 1 \rangle$ to an element of $2^{\lambda_{i+1}}$. Then $\{\phi(x, \vec{a}_{\eta_i} \widehat{\vec{0}_\alpha}), \sim\phi(x, \vec{a}_{\eta_i} \widehat{\vec{0}_\beta})\}$ is the required subset.

In fact, we have proved the following:

Conclusion 1.8 If λ is a strong limit cardinal and T is P -unstable, then $Q_T^P(\lambda, (cf(\lambda))^+)$ fails. This proof is essentially in [1].

Claim 1.9 Suppose T is P -stable (so T^P is stable). Then for any model M and finite sequence \vec{a} , $tp(\vec{a}, P^M)$ is “definable,” i.e., for each formula ϕ , there is a formula ψ with parameters $\vec{c} \in P^M$ such that for all $\vec{b} \in P^M$, $\phi(\vec{x}, \vec{b}) \in tp(\vec{a}, P^M)$ iff $M \models \psi(\vec{b}, \vec{c})$.

Proof: Otherwise, by [1], Chapter II, Theorem 2.2, there is some $A \subseteq P^M$ such that $|S(A)| > \lambda = \lambda^{|T|} > |A|$, so there exist $\phi, p_m \in S(A)$, $m \in \omega$ and $\vec{a}_n \in A$, $n \in \omega$ such that $\phi(\vec{x}, \vec{a}_n) \in p_k$ iff $n < k$. Choose \vec{b}_k realizing p_k , and let $q_n = \{\bigwedge P(x_i)\} \cup \{\phi(\vec{b}_k, \vec{x}) : n \leq k < \omega\} \cup \{\sim\phi(\vec{b}_k, \vec{x}) : k < n\}$. Clearly, the q_n show the P -instability except for the bar on \vec{x} , which is easy to remove.

Alternatively, note that $R^m(P(x), \Delta, 2) < \omega$ for every finite Δ (cf., [1]).

Claim 1.10 Suppose T is P -stable, and suppose M is a model such that the restriction of M to P^M is κ -saturated, where $\kappa > |T|$. Then M is (κ, P) -saturated.

Proof: Let $p \in S(A)$, where p includes $P(x)$, $|A| < \kappa$. For each $\phi = \phi(x, \vec{a}) \in p$, there is a $\psi_\phi = \psi_\phi(x, \vec{c})$, $\vec{c} \in P^M$, such that $M \models (\forall x)[P(x) \rightarrow [\phi(x, \vec{a}) \leftrightarrow \psi_\phi(x, \vec{c})]]$. Then $\{\psi_\phi : \phi \in p\}$ is consistent and has power less than κ , and, hence, is realized. Therefore, so is p .

Remark 1.11 We cannot omit the condition “ $\kappa > |T|$ ” although we may deal with compactness instead of saturation. This is shown by the following example.

Let $|M| = \omega_1 \cup {}^\omega \omega_1$. For each $i < \omega_1$, let F_i^M be the unary operation with values $F_i^M(\alpha) = \alpha$ for $\alpha < \omega_1$, and $F_i^M(\eta) = \eta(i)$ for $\eta \in {}^\omega \omega_1$. Let $P^M = \omega_1$. Let $M = (|M|, P^M, F_0^M, \dots, F_i^M, \dots)$. (To conform to our convention, we have to add names for every definable formula.) Clearly, M is not (\aleph_1, P) -saturated

(not even (\aleph_0, P) -saturated), (just consider the type $\{v \neq F_i(id) : i < \omega_1\} \cup \{P(v)\}$, where id is the identity function), but M^P is \aleph_1 -saturated (because every formula is equivalent to a Boolean combination of equalities).

Claim 1.12 If $M^P < N^*$, then there exists a model N such that $M < N$ and $N^* < N^P$.

The proof is like the proof of the Robinson Consistency Lemma.

Conclusion 1.13 For regular $\kappa > |T|$, $Q_T^P(\lambda, \kappa)$ iff $Q_{T^P}(\lambda, \kappa)$. (The assumption of regularity is not really needed.) We are assuming that T is P -stable.

Proof: Clearly, $Q_T^P(\lambda, \kappa)$ implies $Q_{T^P}(\lambda, \kappa)$. Assume $Q_{T^P}(\lambda, \kappa)$. Let M be a model of T of power λ . We must produce an elementary extension that is (P, κ) -saturated and has power λ . Inductively define a continuous elementary chain $M_i (i \leq \kappa)$, starting with $M_0 = M$. Given M_i of power λ , there are N_i^* and M_{i+1} such that $M_i^P < N_i^* < M_{i+1}^P$ and N_i^* is κ -saturated. Now $M_\kappa^P = \bigcup_i M_i^P = \bigcup_i N_i^*$ is κ -saturated (see [1], Chapter III, 3.11). Hence, M_κ is (P, κ) -saturated by Claim 1.10.

(When T is not P -stable, we do not have our conclusion for λ, κ but only for λ, λ .)

Claim 1.14 If $\lambda^{<\kappa} = \lambda \geq 2^{|\mathcal{T}|}$, then $Q_T^P(\lambda, \kappa)$. For example, if T is the theory of linear order, we get equivalence. For T stable, we know that $Q_T(\lambda, \kappa)$ iff either $\lambda^{<\kappa} = \lambda$ or else T is λ -stable.

Question 1.15 Is there an unstable theory for which $Q_T(\lambda, \kappa)$ is not equivalent to the condition $\lambda^{<\kappa} = \lambda$?

Under *GCH*, the answer is “No”, also the tree property for P (cf. [2]) implies the answer is “No”.

Definition 1.16 Let T_{ind}^* be the following theory, saying of a binary relation symbol $R(x, y)$ that it is symmetric, irreflexive, and except for these conditions, everything else possible occurs. The axioms are $R(x, y) \equiv R(y, x)$, $\sim R(x, x)$, and

$$(\forall y_0 \dots y_{2n-1}) \left[\bigwedge_{i,j < n} (y_i \neq y_{n+j}) \rightarrow (\exists x) \left[\bigwedge_{i < n} R(x, y_i) \wedge \bigwedge_{i < n} \sim R(x, y_{n+i}) \right] \right].$$

Claim 1.17 T_{ind}^* is \aleph_0 -categorical, has elimination of quantifiers, and is unstable.

Theorem 1.18 If $\mu = \mu^{<\kappa}$, $\mu \leq \lambda \leq 2^\mu$, then $Q_{T_{\text{ind}}^*}(\lambda, \kappa)$.

Remark 1.18A The theorem is interesting when $\lambda \neq \lambda^{<\kappa}$. For example, assume $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$, $2^{\aleph_2} = \aleph_{\aleph_7}$. If $\lambda = \aleph_{\aleph_1}$, $\mu = \aleph_2$, $\kappa = \aleph_2$, then $\lambda < \lambda^{<\kappa}$, and the same conclusion follows from $\lambda = \aleph_\omega$, $\mu = \aleph_2$, $\kappa = \aleph_1$.

Proof of Theorem 1.18: We shall give the proof in a number of stages. It is sufficient to prove the following:

*₁ If $|M| = \lambda$, there is a model N such that $M < N$, $|N| = \lambda$, and for every type $p(x)$ over M , if $|p| < \kappa$, then p is realized in N .

(Remember that T_{ind}^* is countable, so we can work with compact models instead of saturated ones.)

To prove $*_1$, it is sufficient to prove:

- $*_2$ If $|M| = \lambda$, there exist $p_i \in S^1(|M|) (i < \lambda)$ such that for each $p \in S^1(|M|)$, if $|p| < \kappa$, then p is included in some p_i .

We shall prove more:

- $*_3$ If $|M| \leq 2^\mu$, there are $p_i \in S^1(|M|) (i < \mu)$ such that if $A \subseteq |M|$, $|A| < \kappa$, and $p \in S^1(A)$ is not algebraic, then $p \subseteq p_i$ for some i .

Now we translate $*_3$ to a set-theoretic problem. Let $p \in S^1(A)$, $A \subseteq |M|$. Since T_{ind}^* has elimination of quantifiers, it suffices to know just the atomic formulas, and there are two alternatives, $R(a, x)$ or $\sim R(a, x)$. Define $f_p: A \rightarrow \{0, 1\}$ by $f_p(a) = 0$ if $R(a, x) \in p$, $f_p(a) = 1$ if $\sim R(a, x) \in p$. For all $p, q \in S^1(A)$, $f_p = f_q$ implies that $p = q$. By elimination of quantifiers, for each function f from A to $\{0, 1\}$, there is some p such that $f_p = f$.

Without loss of generality, we can assume that in $*_3$, the universe of M is 2^μ , and $*_3$ can be translated to the following statement:

- $*_4$ There are functions $f_i: 2^\mu \rightarrow \{0, 1\} (i < \mu)$ such that for all $A \subseteq 2^\mu$ with $|A| < \kappa$, for each $f: A \rightarrow \{0, 1\}$, there exists $i < \mu$ such that $f \subseteq f_i$.

We prove $*_4$ by changing $\{0, 1\}$ to μ . More precisely, the functions f_i in $*_4$ will be given by $f_i(\alpha) = g_\alpha(i)$, where the functions g_α satisfy the following:

- $*_5$ There are $g_\alpha: \mu \rightarrow \mu (\alpha < 2^\mu)$ such that for all $\zeta < \kappa$, $\alpha_0, \dots, \alpha_\zeta < 2^\mu$, distinct, and $j_0, \dots, j_\zeta < \mu$, there is some $i < \mu$ such that $g_{\alpha_0}(i) = j_0, \dots, g_{\alpha_\zeta}(i) = j_\zeta$.

This final statement, $*_5$, is proved in [1], in the appendix as Theorem 1.5(1), and is due to Engelkind and Karlowich.

Just as we defined P -stable, we can define P -simple as in [2], and we get similar results. Restricting ourselves to $\lambda \geq 2^{|T|}$, $\kappa > |T|$, we get at least the following possibilities:

- (A) $Q_T^P(\lambda, \kappa)$ iff either $\lambda = \lambda^{<\kappa}$ or else T is (λ, P) -stable
 (B) $Q_T^P(\lambda, \kappa)$ iff $\lambda = \lambda^{<\kappa}$
 (C) $Q_T^P(\lambda, \kappa)$ iff $Q_{T_{\text{ind}}^*}^P(\lambda, \kappa)$.

Now by results in this section, (A) occurs when T is P -stable, and by the remark above, (B) occurs when T is not P -simple; e.g., T is the theory of linear orderings T_{ord} . By the previous discussion, if we assume GCH , then (A) and (B) are the only possibilities. However, when there is $\mu^{<\kappa} = \mu < \lambda < 2^\mu$, then (C) is distinct from (A) and (B) (which are always distinct). By [2], it is consistent that there are $\mu^{<\kappa} = \mu < \lambda < 2^\mu$, however there are no more possibilities, at least for $\kappa > |T|$.

2 Answer to the problem We shall try to show that there are T, P such that $Q_T^P(\lambda, \kappa)$ does not behave in any of the previous ways (in some universe of set theory.)

Definition 2.1 Let T_3 be the following theory, in a language with the unary relation symbols P and Q , the binary relation symbol ϵ , and the ternary relation symbol R . The axioms say that Q and P are a partition of the universe,

$$\begin{aligned} x \in y &\rightarrow Q(x) \wedge P(y) \\ R(x, y, z) &\rightarrow \sim(\exists u)[x \in u \wedge y \in u \wedge z \in u] \\ R(x, y, z) &\rightarrow Q(x) \wedge x \neq y \\ R &\text{ is symmetric, so } R \text{ is a set of triangles from } Q \\ R(x, y, z_1) \wedge R(x, y, z_2) &\rightarrow z_1 = z_2, \text{ so no two triangles have an edge in} \\ &\text{common} \end{aligned}$$

Finally, except for the above conditions, everything else which is possible occurs; i.e., let T_3 be the model completion of the above. Then T_3 almost has elimination of quantifiers, i.e., every formula is equivalent to a formula of the form $\phi(y_1 \dots y_n) = (\exists y_{n+1} \dots y_k) \left(\left[\bigwedge_{\ell=n+1}^k \bigvee_{i,j < \ell} R(y_i, y_j, y_\ell) \right] \wedge \psi \right)$ where ψ is quantifier free. Also T_3 is P -unstable.

Theorem 2.2 If $M \models T_3$ and $\phi_i(x, \vec{a}_i) \wedge P(x) (i < \omega_1)$ are nonalgebraic, then they cannot be pairwise contradictory. (Hence, T_3 is P -simple.)

Remark The theory T_3 is between T_{ind}^* and T_{ord} in the sense of $Q_T^P(\lambda, \kappa)$.

Proof of Theorem 2.2: We can assume that $\phi_i = \phi$, so the length of \vec{a}_i is fixed. We can also assume that $\langle \vec{a}_i : i < \omega \rangle$ is 3-indiscernible (by Ramsey's Theorem). We can assume that $\vec{a}_i = \langle a_{\ell}^i : \ell < m \rangle$, and that any fixed components come first; i.e., $a_{\ell}^i = a_{\ell}$ for $\ell < n$, and $a_{\ell_1}^{i_1} = a_{\ell_2}^{i_2}$ if either $i_1 = i_2$ and $\ell_1 = \ell_2$ or else $\ell_1 = \ell_2 < n$. If ϕ_i is not algebraic, we can assume $\phi_i(x, \vec{a}_i) \vdash \bigwedge_{\ell < m} (x \neq a_{\ell}^i) \wedge P(x)$, and ϕ_i is a

conjunction of basic formulas. By the almost elimination of quantifiers we can assume ϕ_i is quantifier free.

What happens if $\phi_1 \wedge \phi_2$ is contradictory? (The only contradiction is between the $a_{\ell}^1 \in x$ and $a_{\ell}^2 \notin x$.) We do not have contradictions of the form $a_{\ell} \in x \wedge a_{\ell} \notin x$. So only R can cause problems. We can assume $\phi_1 \vdash a_k^1 \in x, a_{\ell}^1 \in x$, and $\phi_2 \vdash a_m^2 \in x$, where $M \models R(a_k^1, a_{\ell}^1, a_m^2)$. By 2-indiscernibility, $R(a_k^1, a_{\ell}^1, a_m^3)$. Then by the axioms, $a_m^2 = a_m^3$. However, the a_m^i are distinct, for $i < \omega_1$, so there can be no such ϕ_i . So no matter what size the model is, its Boolean algebra of definable infinite sets satisfies the \aleph_1 -chain condition.

Theorem 2.3 It is consistent with ZFC that $Q_{T_3}^P$ is strictly weaker than $Q_{T_{\text{ord}}}^P$ (both are statements about pairs of cardinals λ, κ), and it is also consistent with ZFC that $Q_{T_3}^P$ is strictly stronger than $Q_{T_{\text{ind}}^*}^P$.

Proof: The fact that $Q_{T_3}^P$ is weaker than $Q_{T_{\text{ord}}}^P$ and stronger than $Q_{T_{\text{ind}}^*}^P$ is clear. We shall show below that each "strictly" is consistent. In each case, the pair of cardinals to be considered is $\lambda = \aleph_{\omega}, \kappa = \aleph_1$. We can get $CH + 2^{\aleph_1} > \aleph_{\omega}$ and a generalization of MA , such that every BA of power less than 2^{\aleph_1} which satisfies the \aleph_1 -chain condition is the union of \aleph_1 ultrafilters with each set of \aleph_0 elements with the finite intersection property included in one (see [2], Theorem 3.10). Then $Q_{T_3}^P(\aleph_{\omega}, \aleph_1)$ holds, but $Q_{T_{\text{ord}}}^P(\aleph_{\omega}, \aleph_1)$ fails.

For the other case we begin with a model of $V = L$ and add \aleph_ω subsets of \aleph_1 by the usual \aleph_1 -complete conditions. Then, in the forcing extension we have $\aleph_1^{\aleph_0} = \aleph_1$, since the forcing is \aleph_1 complete, and by hypothesis, $\aleph_1 < \aleph_\omega \leq 2^{\aleph_1}$. Thus, by Theorem 1.18, $Q_{T_3}^P(\aleph_\omega, \aleph_1)$ holds in the extension.

In order to show that $Q_{T_3}^P(\aleph_\omega, \aleph_1)$ fails in the extension we will show that its negation is implied by the following property (*). We then prove that (*) holds in the extension.

(*) For each $\lambda \leq \aleph_\omega$, $\lambda > \aleph_0$ there is a set of triples R on λ^{++} , no two with a common edge, such that λ^{++} is not the union of λ sets with no triangles.

Theorem 2.4 (*) implies $\sim Q_{T_3}^P(\aleph_\omega, \aleph_1)$.

Proof: Let (A_n, R_n) exemplify (*) for $\lambda = \aleph_n$. Then $|A_n| = \aleph_{n+2}$, R_n is a set of triples from A_n , and without loss of generality, we can assume the A_n are pairwise disjoint. Let $(A, R) = \bigcup_n (A_n, R_n)$. We find M , of cardinality \aleph_ω , such that $M \models T_3$, $A \subseteq Q^M$, $R^M \upharpoonright A = R$. Suppose M is \aleph_1 -saturated. We will get a contradiction.

Let $M = \bigcup_n M_n$, where $|M_n| = |P_n| = \aleph_n$. For every $a \in P^M$, n , let $C(a, n) = \{b \in A_n : b \in a\}$. Then $C(a, n) \subseteq A_n$ is triangle-free, and there are \aleph_n such a 's. Hence, $A_n \neq \bigcup_{a \in P} (M_n \cap C(a, n))$ (from (*)). The fact that there exist $y_n \in A_n$ such that $y_n \notin \bigcup_{a \in P} C(a, n)$ means: $y_n \in A_n \subseteq Q^M$, and for all $a \in P^M$, n , $y_n \notin a$. Now $\{y_n : n < \omega\} \subseteq Q^M$, and this set is triangle-free. (The A_n are disjoint, R_n is a set from A_n , and $R = \bigcup_n R_n$.)

Then the set of formulas $\{y_n \in x : n < \omega\}$ is consistent. No element of $\bigcup_n M_n = M$ realizes it. Therefore, M is not \aleph_1 -saturated.

Proof of ():* Recall that we are forcing with a product X of copies X_α of the usual \aleph_1 -complete conditions for adding a subset of \aleph_1 , viz., mappings from a countable subset of ω_1 to 2. There is a copy for each $\alpha < \aleph_\omega$ and a condition in X may be nontrivial in countably many coordinates. Let G be X -generic. Then $V[G]$ contains no new reals and has the same cardinals as V since X is \aleph_1 -complete and satisfies the \aleph_2 -chain condition since $V \models CH$. Fix $\lambda < \aleph_\omega$ for the rest of the discussion. We will construct in V itself R as described in (*), except that R will be a set of triples on $\lambda^{++} \times \lambda$ instead of λ^{++} . Of course, R must work for unions of λ sets in $V[G]$, not only in V .

Suppose first that $R \in V$ is some possible candidate, but that R does not work in $V[G]$. Then $V[G] \models \exists P \text{ “} P: \lambda^{++} \times \lambda^+ \rightarrow \lambda \wedge \forall x, y, z \in \lambda^{++} \times \lambda^+ [P(x) = P(y) = P(z) \rightarrow \sim R(x, y, z)] \text{”}$. Now, for any $P: \lambda^{++} \times \lambda^+ \rightarrow \lambda$ in $V[G]$, there is a set $S \subseteq \aleph_\omega$, $\bar{S} = \lambda^{++}$ such that $P \in V[G_S]$ where G_S is generic over the product of the X_α 's from $\alpha \in S$. Now, since $R \in V$, using an automorphism argument, it is clear that we could find some P satisfying “ ” above in $V[G_{\lambda^{++}}]$ where $G_{\lambda^{++}}$ is generic over the product of the first λ^{++} coordinates, which we will call $Y_{\lambda^{++}}$. Thus, in constructing R we will only worry about those $P \in V[G_{\lambda^{++}}]$.

First, since the *GCH* holds in V we may enumerate all subsets of $\lambda^{++} \times \lambda^+$ of power λ^+ in a list $\{C_\gamma\}_{\gamma < \lambda^{++}}$. We will make use of the following known fact, a proof of which is included for the reader's convenience.

Lemma 2.5 *For each $\alpha < \lambda^{++}$ there is a function $f_\alpha: \lambda^+ \times \lambda^+ \rightarrow \alpha \times \lambda^+$ such that for $\gamma < \alpha$, if $C_\gamma \subseteq \alpha \times \lambda^+$, and A is a subset of λ^+ of power λ^+ , then there is a pair of ordinals $(i, j) \in A$ such that $f_\alpha(i, j) \in C_\gamma$.*

Proof: Let $\{B_\xi: \xi < \lambda^+\}$ be a list of all subsets of λ^+ of power λ . Let $\{C^i: i < \lambda^+\}$ be a list of all C_γ , $\gamma < \alpha$, $C_\gamma \subseteq \alpha \times \lambda^+$.

We will define f_α by induction. Assume that so far we have only defined $f_\alpha(i, j)$ for $i < j < \zeta$. Now we define $f(i, \zeta)$ for $i < \zeta$ so that if $i, \xi < \zeta$, $B_\xi \subseteq \zeta$, then for some $j \in B_\xi$, $f(j, \zeta) \in C^i$. This is easily shown.

Now, if $A \subseteq \lambda^+$ has power λ^+ , and $i < \lambda^+$, then for some ξ , $B_\xi \subseteq A$, and for some $\zeta \in A$, $i, \xi < \zeta$, $B_\xi \subseteq \zeta$. Hence, for some $j \in B_\xi \subseteq A$, $f(j, \zeta) \in C^i$.

We now return to the central argument. We define R to consist of all triangles of the form

$$((\alpha, i), (\alpha, j), f_\alpha(i, j))$$

for $\alpha < \lambda^{++}$, $i < j < \lambda^+$. It is immediate that R contains no triangles with an edge in common.

Now assume $q_0 \Vdash "h: \lambda^{++} \times \lambda^+ \rightarrow \lambda"$. For each $(\alpha, \beta) \in \lambda^{++} \times \lambda^+$ there is an $r_{\alpha\beta} \geq q_0$ such that $r_{\alpha\beta} \Vdash "h(\alpha, \beta) = \dot{\delta}(\delta, \beta)"$ for some $\delta(\alpha, \beta) < \lambda$.

For a fixed α there is a $D_\alpha \subseteq \lambda^+$, $\overline{D}_\alpha = \lambda^+$ and fixed $\delta_\alpha < \lambda$ such that

$$\{r_{\alpha\beta}: \beta \in D_\alpha\}$$

forms a Δ -system with "heart" r_α , and for each $\beta \in D_\alpha$, $r_{\alpha\beta} \Vdash h(\alpha, \beta) = \dot{\delta}_\alpha$.

Similarly there is an $E \subseteq \lambda^{++}$, $\overline{E} = \lambda^{++}$ and $\delta < \lambda$ such that $\{r_\alpha: \alpha \in E\}$ forms a Δ -system and $\forall \alpha \in E \forall \beta \in D_\alpha r_{\alpha\beta} \Vdash h(\alpha, \beta) = \dot{\delta}$. We need the following fact about our forcing conditions.

(**) *If we have a Δ -system of length λ^+ with a "heart", then any extension of the "heart" is compatible with all but at most \aleph_0 members of the system.*

Now, for each $\alpha \in E$ we partition D_α into λ^+ pairwise disjoint subsets, $\{Z_{\alpha, \xi}\}_{\xi < \lambda^+}$, each of power λ^+ . Each $Z_{\alpha, \xi}$ appears in our original list $\{C_\gamma\}_{\gamma < \lambda^{++}}$. List $E = \{\beta_i: i < \lambda^{++}\}$ in increasing order. Choose $\alpha^* \in E$ such that for each $i < \aleph_2$:

- (i) $\alpha^* > \beta_i$,
- (ii) $Z_{\beta_i, \xi}$ appears before the α^* th place in the list of C_γ 's.

For each $i < \aleph_2$, consider $\{j \in D_{\alpha^*}: r_{\alpha^*, j} \text{ and } r_{\beta_i} \text{ are compatible}\}$.

By (**), for some i this set has power λ^+ since for each j , all but $\leq \aleph_0$ i 's work. Let $\beta^* = \beta_i$ for such an i .

For each $\zeta < \lambda^+$ we want to know for how many j 's in D_{α^*} is $r_{\alpha^*, j}$ compatible with all $r_{\beta^*, k}$, for $k \in Z_{\beta^*, \zeta}$.

For a fixed j such that $r_{\alpha^*, j}$ is compatible with r_{β^*} , $r_{\alpha^*, j}$ is compatible with all but $\leq \mu$ $r_{\beta^*, k}$, $k \in D_{\beta^*}$, by (**). Hence, all but $\leq \aleph_0$ ζ 's work for it. So, for

some ζ , $W = \{j \in D_{\alpha^*} : r_{\alpha^*,j} \text{ is compatible with } r_{\beta^*,k} \text{ for every } k \in Z_{\beta^*,\zeta}\}$ has power λ^+ .

Now $\{\beta^*\} \times Z_{\beta^*,\zeta}$ appears in the list before the α^{th} entry, so by the definition of R , there are $j_1 \neq j_2$ in W and $k \in Z_{\beta^*,\zeta}$ such that $((\alpha^*, j_1), (\alpha^*, j_2), (\beta^*, k))$ is in R .

Now, $r_{\beta^*,k}$ and r_{α^*,j_1} are compatible since $j_1 \in W$. Similarly, $r_{\beta^*,k}$ and r_{α^*,j_2} are compatible since $j_2 \in W$. In addition, r_{α^*,j_1} and r_{α^*,j_2} are compatible since $j_1, j_2 \in D_{\alpha^*}$. Then, $r^* = r_{\alpha^*,j_1} \cup r_{\alpha^*,j_2} \cup r_{\beta^*,k}$ is a forcing condition and extends each of the three. Thus,

$$\begin{aligned} r^* &\Vdash h(\dot{\alpha}^*, \dot{j}_1) = \dot{\delta} \\ r^* &\Vdash h(\dot{\alpha}^*, \dot{j}_2) = \dot{\delta} \\ r^* &\Vdash h(\dot{\beta}^*, \dot{k}) = \dot{\delta} \end{aligned}$$

and we are done, since r^* , an extension of q_0 forces each vertex of a triangle in R to be in the same part of the partition with name h .

Concluding remarks

(1) There was nothing special, of course, about the cardinals we used. By doing the two forcing arguments for cardinals far apart, we can get a universe of set theory in which $Q_{T_3}^P$ is strictly weaker than $Q_{T_{\text{ord}}}$ and strictly stronger than $Q_{T_{\text{ind}}}^P$.

(2) We can define a theory $T_{n,\ell}$, $3 \leq n < \omega$, $2 \leq \ell < n$ in analogy with T_3 : R will be a symmetric n -place relation on Q , $P(x) \leftrightarrow \sim Q(x)$, $x \in y \rightarrow Q(x) \wedge P(y)$, $R(x_1, \dots, x_n) \rightarrow \sim(\exists u) \bigwedge_{i=1}^n x_i \in u$, $R(x_1, \dots, x_n) \rightarrow \bigwedge_{i < j \leq n} x_i \neq x_j$,

and $R(x_1, \dots, x_\ell, y_{\ell+1}, \dots, y_n) \wedge R(x_1, \dots, x_\ell, z_{\ell+1}, \dots, z_n) \rightarrow \bigwedge_{i=\ell+1}^n \bigvee_{j=\ell+1}^n y_i = z_j$,

i.e., any two n -gons intersect in $< \ell$ points. $T_{n,\ell}$ is the model completion of the theory just described.

We now repeat the proof of the first part of Theorem 2.3 without using the full power of the relevant generalization of MA (cf. [3]). We can start with $V = L$ and iterate just the forcing needed to make $Q_{T_{n,\ell}}^P(\aleph_\omega, \aleph_1)$ true.

If M is a model of $T_{n,\ell}$ and B_M the Boolean algebra of nonalgebraic formulas over M , then B_M satisfies:

(+) For any \aleph_1 nonzero elements of a_i of B , there are $[n/\ell]$ which have a common lower bound, and $a'_i \leq a_i$, $a'_i \neq 0$ such that any finite set of them has nonzero intersection iff any subset of at most n elements has a nonzero intersection.

We have to iterate forcing which makes such a Boolean algebra the union of few filters. The required set of forcing conditions will satisfy a corresponding strong chain condition. The hope is then to imitate the proof of the second part of Theorem 2.3 for showing that in this universe $Q_{T_{n_1,\ell_1}}^P$ fails for the corresponding pair of cardinals. In this way we can hope to prove that there

may be infinitely many distinct Q_T^P (T simple), and even ones incomparable under inclusion. However, this suggestion has not been carried out and checked.

(3) In section 1 we have restricted ourselves to $\kappa > |T|$, and here we use $\kappa = \aleph_0 = |T_3|$, but these restrictions are inessential to the overall argument.

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