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ω-ELONGATIONS AND CRAWLEY'S PROBLEM

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ω-elongations of $Z(p)$ by separable p-primary groups are studied. Assuming $(V = L)$, direct sums of cyclic groups are characterized using ω-elongations. Also assuming $(V = L)$ much information is obtained about ω-elongations of $Z(p)$ by groups which are not direct sums of cyclic groups. Finally it is shown that it is consistent that there is an uncountable group B with a countable basic subgroup such that there is a unique ω-elongation of $Z(p)$ of B.

In this paper all groups are p-primary Abelian groups. Suppose B is a separable group; i.e. $p^\omega B = 0$. Here $p^n B = \{p^n x\colon x \in B\}$ and $p^\omega B = \bigcap p^n B$. A group H is said to be an *ω-elongation* of A by B if $A \simeq p^\omega H$ and $B \simeq H/p^\omega H$. In this paper we will study whether or not the Σ-cyclic groups (i.e. direct sum of cyclic groups) can be characterized by their elongations of $Z(p)$. Our main theorem is:

THEOREM 2.2. *Assume $(V = L)$. A group G is Σ-cyclic iff for every ω-elongation H of $Z(p)$ by G there is a homomorphism f from H to P such that $f(p^\omega H) \neq 0$. Here P is the Prüfer group generated by $\{x_n\colon n < \omega\}$ subject to the relations $px_0 = 0$ and $p^{n+1}x_{n+1} = x_0$. If we assume* MA + $\neg$CH *this criterion fails to characterize the Σ-cyclic groups.*

Following Megibben [**M**] we call a group B a *Crawley group*, if all elongations of $Z(p)$ by B are isomorphic (as groups). Crawley asked if all Crawley groups were Σ-cyclic. Megibben [**M**] showed, assuming MA + $\neg$Ch, there is a Crawley group which is not Σ-cyclic. Further he showed, assuming $(V = L)$, any Crawley group of cardinality $\aleph_1$ must be Σ-cyclic. Megibben's proof is somewhat indirect. In particular, the proof using $(V = L)$ involves valuated vector spaces. On aesthetic grounds it seems worthwhile to give a strictly group theoretic proof. In fact we get additional information on the structure of ω-elongations of $Z(p)$ by groups of cardinality $\aleph_1$ in L. The following result speaks of "rigid" systems of ω-elongations.

THEOREM 1.3. *Assume* $(V = L)$. *Suppose* B *is a separable group of cardinality* $\aleph_1$ *and* B *is not* Σ*-cyclic. There are* $2^{\aleph_1}$ ω*-elongations* $\{H_\alpha: \alpha < 2^{\aleph_1}\}$ *of* $Z(p)$ *by* B *such that: for* $\alpha \neq \beta$ *if* $f\colon H_\alpha \to H_\beta$ *and* $t \in p^\omega H_\alpha$ *then* $f(t) = 0$.

Also in §1 we give a proof of Megibben's result that if MA + $\neg$CH holds then any ω_1-separable group of cardinality $\aleph_1$ is a Crawley group. This proof is a run up for the more difficult result in §3:

THEOREM 3.2. *It is consistent that there is a Crawley group of cardinality* $\aleph_1$ *which is not* $\aleph_1$*-separable.*

Finally in §2, we use our methods to obtain additional information on Crawley groups in L. Perhaps the most interesting result is Corollary 2.3 which asserts if $V = L$ then any strongly Crawley group is Σ-cyclic. Here G is *strongly* Crawley if for all Σ-cyclic groups H, $G \oplus H$ is a Crawley group.

Eklof and Huber [**EH**] have studied ω-filtered vector spaces of cardinality $\aleph_1$. Eklof has found ω-filtered vector space versions of many of our theorems. We wish to thank him for his helpful comments on the first draft of this paper.

1. Crawley groups. We begin with a group theoretic lemma.

1.1. LEMMA. *Suppose* $Z(p) \rightarrowtail H \overset{\pi}{\twoheadrightarrow} B$ *is an* ω*-elongation. Further assume* $B \subseteq B'$, B' *is separable,* $|B'/B| = \aleph_0$ *and* B'/B *is not* Σ*-cyclic. Let* t *generate* $p^\omega H$. *There are* ω*-elongations* $Z(p) \rightarrowtail H_1 \overset{\pi_1}{\twoheadrightarrow} B'$ *and* $Z(p) \rightarrowtail H_2 \overset{\pi_2}{\twoheadrightarrow} B'$ *so that:* $\pi \subseteq \pi_1, \pi_2$, *and for all* $f\colon H \to G$, *if* $f(t) \neq 0$ *and* $p^\omega G \cong Z(p)$ *then* f *extends to at most one of* H_1 *and* H_2.

Proof. Let $Z(p) \rightarrowtail H_1 \overset{\pi_1}{\twoheadrightarrow} B'$ be any ω-elongation such that $\pi \subseteq \pi_1$. Since $|B'/B| = \aleph_0$, B'/B is not separable. Hence there are $\{x_n: n < \omega\} \subseteq B' \setminus B$ and $\{b_n: n < \omega\} \subseteq B$ so that $px_0 = b_0$ and $p^{n+1}x_{n+1} = x_0 + b_{n+1}$. Choose preimages x_n^1, b_n^1 $(n < \omega)$ of the x_n's and b_n's so that $px_0^1 = b_0^1$ and $p^{n+1}x_{n+1}^1 = x_0^1 + b_{n+1}^1$. There is another ω-elongation $Z(p) \rightarrowtail H_2 \overset{\pi_2}{\to} B'$ such that: $\pi \subseteq \pi_2$; and the x_n's have pre-images x_n^2 where $px_0^2 = b_0^1 + t$ and $p^{n+1}x_{n+1}^2 = x_0^2 + b_{n+1}^1$. (The existence of this second elongation can be established by direct computation. An elegant proof of a more general result can be found in [**EHM**].)

Suppose $f\colon H \to G$ and G as above and f extends to both H_1 and H_2. Then in G there are $\{y_n\colon n < \omega\}$ and $\{y'_n\colon n < \omega\}$ such that: $p^{n+1}y_{n+1} = y_0 + f(b^1_{n+1})$; $p^{n+1}y'_{n+1} = y'_0 + f(b^1_{n+1})$; $py_0 = f(b^1_0)$; and $py_0^1 = f(b'_0) + f(t)$. Subtracting we have $p(y'_0 - y_0) = f(t)$ and $p^{n+1}(y'_{n+1} - y_{n+1}) = (y'_0 - y_0)$. So $(y'_0 - y_0) \in p^\omega G$, but $p(y_0 - y_0) \neq 0$. This is a contradiction.

1.2. THEOREM. ([M]). *Assume* $(V = L)$. *Every Crawley group of cardinality* $\aleph_1$ *is* Σ*-cyclic.*

Proof. Assume B is a non-Σ-cyclic Crawley group and $|B| = \omega_1$. Let $Z(p) \rightarrowtail H \twoheadrightarrow B$ be any ω-elongation of $Z(p)$ by B. Let $\{B_\alpha\colon \alpha < \omega_1\}$ be an ω_1-filtration of B; i.e. for all α, $|B_\alpha| = \omega$; if λ is a limit ordinal, $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$; $B = \bigcup B_\alpha$; and for all α, $B_{\alpha+1}/B_\alpha$ is Σ-cyclic iff for all $\tau > \alpha$ B_τ/B_α is Σ-cyclic. We can assume B_0 has infinite final rank (i.e. for all n, $p^nB_0 \neq 0$). Since B is not Σ-cyclic, $W = \{\alpha\colon B_{\alpha+1}/B_\alpha$ is not Σ-cyclic$\}$ is stationary. Applying $\diamondsuit(W)$ choose $\{f_\alpha\colon \alpha < \omega_1\}$ such that: $f_\alpha\colon Z(p) \times B_\alpha \to H$; and for all $f\colon Z(p) \times B \to H$ there is $\alpha \in W$ so that $f \restriction Z(p) \times B_\alpha = f_\alpha$. (All the set-theoretic notation and results used are standard. See [E] for example.)

We construct an increasing sequence of ω-elongations $Z(p) \rightarrowtail G_\alpha \overset{\pi_\alpha}{\twoheadrightarrow} B_\alpha$ by induction on α. G_α will always be a group structure on $Z(p) \times B_\alpha$ and π_α is projection on the second coordinate. Let $Z(p) \rightarrowtail G_0 \overset{\pi_0}{\twoheadrightarrow} B_0$ be any ω-elongation. Let t be a generator of $p^\omega G_0$. The key case occurs when $\alpha \in W$, $f_\alpha\colon G_\alpha \to H$ is a homomorphism, and $f_\alpha(t) \neq 0$. Then by lemma 1.1 we can choose $G_{\alpha+1}$ so that f_α does not extend to a homomorphism from $G_{\alpha+1}$ to H. At other successor steps $G_{\alpha+1}$ can be chosen arbitrarily. At limit ordinals, we take unions. Let $G = \bigcup_{\alpha<\omega_1} G_\alpha$.

Suppose $f\colon G \to H$ is a homomorphism so that $f(t) \neq 0$. (An isomorphism would be such an f.) Then there is some $\alpha \in W$ so that $f \restriction G_\alpha = f_\alpha$. But then $f \restriction G_{\alpha+1}$ is an extension of f_α contradicting the choice of $G_{\alpha+1}$.

REMARKS. Lemma 1.1 is just what is needed to apply the "weak diamond" principle. So the hypothesis "$(V = L)$" can be weakened to "weak diamond holds for every stationary subset of ω_1". Further the proof can be used to produce $\aleph_2$ pairwise non-isomorphic ω-elongations of $Z(p)$ by B. All the above is known (cf. [EH]). In fact Eklof and Huber ([EH]) show CH and weak diamond for all stationary sets suffices to produce $2^{\aleph_1}$ pairwise non-isomorphic ω-elongations of $Z(p)$ by B.

The following result on "rigid" systems of elongations is new.

1.3. THEOREM. *Assume* ($V = L$). *Suppose* B *is a separable group of cardinality* $\aleph_1$ *and* B *is not* Σ*-cyclic. There are* $2^{\aleph_1}$ ω*-elongations* $\{H_\alpha\colon \alpha < 2^{\aleph_1}\}$ *of* $Z(p)$ *by* B *such that: for* $\alpha \neq \beta$ *if* $f\colon H_\alpha \to H_\beta$ *and* $t \in p^\omega H_\alpha$ *then* $f(t) = 0$.

Proof. If we wanted $\aleph_1$ ω-elongations the argument would be somewhat simpler. Let $\{B_\alpha\colon \alpha < \omega_1\}$ be an ω_1-filtration of B as is Theorem 1.2. We will define inductively a strictly monotone continuous function $g\colon \omega_1 \to \omega_1$ and for each $\eta \in {}^\alpha 2$ a group structure H_η on $Z(p) \times B_{g(\alpha)}$ so that $Z(p) \rightarrowtail H_\eta \overset{\pi_{g(\alpha)}}{\twoheadrightarrow} B_{g(\alpha)}$ is an ω-elongation. Here ${}^\alpha 2$ denotes the set of functions from α to 2 and $\pi_{g(\alpha)}$ is projection on the second coordinate. The construction will be done so that if for $\eta \in {}^{\omega_1}2$ we let $H_\eta = \bigcup H_{\eta \restriction \alpha}$ then $\{H_\eta\colon \eta \in {}^{\omega_1}2\}$ is the desired set of ω-elongations.

Let $W = \{\alpha\colon B_{\alpha+1}/B_\alpha$ is not Σ-cyclic$\}$. Apply $\diamondsuit(W)$ to get $\{(\eta_\alpha, \tau_\alpha, f_\alpha)\colon \alpha \in W\}$ so that: for all α, η_α, $\tau_\alpha \in {}^\alpha 2$; for all α, $f_\alpha\colon Z(p) \times B_\alpha \to Z(p) \times B_\alpha$; and for all $\eta, \tau \in {}^{\omega_1}2$, $f\colon Z(p) \times B \to Z(p) \times B$, $\{\alpha\colon \eta \restriction \alpha = \eta_\alpha,\ \tau \restriction \alpha = \tau_\alpha$ and $f \restriction Z(p) \times B_\alpha = f_\alpha\}$ is stationary. Let $g(0) = 0$ and $H_{\langle\,\rangle}$ be any group structure on $Z(p) \times B_0$ so that $Z(p) \rightarrowtail H_{\langle\,\rangle} \overset{B_0}{\twoheadrightarrow} B_0$ is an ω-elongation. The key case occurs when $g(\alpha) = \alpha$, $f_\alpha\colon H_{\eta_\alpha} \to H_{\tau_\alpha}$ is a homomorphism, and for some $t \in p^\omega H_{\eta_\alpha}$, $f_\alpha(t) \neq 0$. Let H^1, H^2 be ω-elongations of $Z(p)$ by $B_{\alpha+1}$ extending H_{η_α} as in Lemma 1.1. There are two subcases.

Subcase (i). Assume for all $\beta > \alpha$ and H an ω-elongation of $Z(p)$ by B_β extending H_{τ_α} there is no extension of f_α to a homomorphism from H^1 to H. In this case let $g(\alpha + 1) = \alpha + 1$. Let $H_{\eta_\alpha \frown \langle 0 \rangle} = H_{\eta_\alpha \frown \langle 1 \rangle} = H^1$. For all $\tau \neq \eta_\alpha$ and $i = 0, 1$, $H_{\tau \frown \langle i \rangle}$ can be chosen arbitrarily.

Subcase (ii). For some $\beta > \alpha$, there is H an ω-elongation of $Z(p)$ by B_β extending H_{τ_α} and an extension of f_α to a homomorphism from H^1 to H. In this case let $g(\alpha + 1) = \beta$ and $H_{\tau_\alpha \frown \langle 0 \rangle} = H_{\tau_\alpha \frown \langle 1 \rangle} = H$. Let $H_{\eta \frown \langle 0 \rangle} = H_{\eta \frown \langle 1 \rangle}$ be any ω-elongation of $Z(p)$ by B_β extending H^2. For all other τ and $i = 0, 1$, $H_{\tau \frown \langle i \rangle}$ can be chosen arbitrarily.

The point here is that if H and G are ω-elongations of $Z(p)$ by B extending $H_{\eta \frown \langle i \rangle}$ and $H_{\tau \frown \langle j \rangle}$ respectively ($i, j < 2$), then f_α does not extend to a homomorphism from H to G.

In the other successor case let $g(\alpha + 1) = g(\alpha) + 1$ and $H_{\tau\frown\langle i\rangle}$ be any allowable ω-elongation ($\tau \in {}^{\alpha}2$ and $i = 0, 1$). If λ is a limit ordinal we take unions; i.e. $g(\lambda) = \sup_1 \alpha < \lambda\, g(\alpha)$ and for $\eta \in {}^{\lambda}2 H_\eta = \bigcup_{\alpha<\lambda} H_{\eta\restriction\alpha}$. The verification that $\{H_\eta \colon \eta \in {}^{\omega_1}2\}$ is the desired set is routine.

In the proof of Theorem 1.3, the hypothesis can be weakened from "$(V = L)$" to "$\diamondsuit(W)$ holds for all stationary subsets W of ω_1". The latter statement is consistent with $2^{\aleph_1} > \aleph_2$. Eklof has shown the hypothesis can be weakened further to "CH and weak diamond holds for every stationary subset of ω_1". (His proof follows the proofs of Theorems 7.2 and 7.9 of **[EH]**.)

We now turn to the proof that it is consistent that there are non-Σ-cyclic Crawley groups. Recall a group is *ω_1-separable* if it is separable and every countable subset is contained in a countable direct summand. There are ω_1-separable groups of cardinality ω_1 which are not Σ-cyclic (**[F]** Thm. 75). The basic result we'll use is:

1.4. THEOREM. (**[M]**) *Assume* MA + $\neg$CH. *If B is an ω_1-separable group of cardinality $\aleph_1$ and A is a countable group, then* $P\mathrm{ext}(B, A) = 0$. *I.e. any pure exact sequence* $0 \to A \to C \to B \to 0$ *splits.*

Suppose B is an ω_1-separable group. A countable subgroup A is a *maximal* direct summand of B, if A is a direct summand of B, any Ulm invariant of B/A is either 0 or $\geq \omega$ and whenever an Ulm invariant of B is infinite so is the same invariant of a. Since we can calculate the Ulm invariants of A, A is determined up to isomorphism.

1.5. LEMMA. *Suppose B is an ω_1-separable group and A_1, A_2 are maximal countable direct summands. Then $B/A_1 \simeq B/A_2$.*

Proof. Choose K a countable direct summand of B such that $A_1, A_2 \subseteq K$ and $K/A_1 \simeq K/A_2$. (Such a K exists by the maximality of A_1 and A_2.) There is an automorphism of K which induces an isomorphism from A_1 to A_2.

1.6. THEOREM. (**[M]**) *Assume* MA + $\neg$CH. *There is a Crawley group which is not Σ-cyclic.*

Proof. Let B be an ω_1-separable group of cardinality $\aleph_1$ which is not Σ-cyclic. Suppose $Z(p) \rightarrowtail H \overset{\pi}{\twoheadrightarrow} B$ is an ω-elongation. Choose A a maximal countable direct summand of B such that $Z(p) \rightarrowtail H' \twoheadrightarrow A$ is an

ω-elongation. Here $H' = \pi^{-1}(A)$. So $H' \rightarrowtail H \twoheadrightarrow B/A$ is a pure exact sequence. Since B/A is ω_1-separable, $H \simeq H' \oplus B/A$. As both H' and B/A are determined up to isomorphism, so is H.

2. Characterizing Σ-cyclic groups in L. Let P denote the Prüfer group; i.e. the group generated by $\{x_n: n < \omega\}$ subject to the relations $px_0 = 0$ and $p^{n+1}x_{n+1} = x_0$.

2.1. LEMMA. *Assume $(V = L)$. Suppose $B \subseteq B'$, B' is separable and B'/B is not Σ-cyclic. Further suppose $Z(p) \rightarrowtail H \overset{\pi}{\twoheadrightarrow} B$ is an ω-elongation, f: $H \to P$ is a homomorphism and $f(p^\omega H) \neq 0$. Then there is an ω-elongation $Z(p) \rightarrowtail H' \twoheadrightarrow B'$ extending $Z(p) \rightarrowtail H \twoheadrightarrow B$ so that f does not extend to H'.*

Proof. Suppose $\kappa = |B'/B|$ is the minimum cardinality of a counterexample. By Lemma 1.1 $\kappa > \aleph_0$. Since the existence of an extension of f is an hereditary property, if $C \subseteq B'$ and $|C| < \kappa$ then $C + B/B$ is Σ-cyclic. By **[S1]**, κ must be regular. Choose an increasing continuous chain $\{C_\alpha: \alpha < \kappa\}$ of subgroups of B' so that: for all α, $|C_\alpha| < \kappa$; and $\{B + C_\alpha/B: \alpha < \kappa\}$ is a κ-filtration of B'/B. We can assume $C_0 = 0$. Since B'/B is not Σ-cyclic, $W = \{\alpha: C_{\alpha+1} + B/C_\alpha + B$ is not Σ-cyclic$\}$ is stationary in κ. Applying $\diamondsuit(W)$ we obtain $\{g_\alpha: \alpha \in W\}$ so that: g_α: $C_\alpha \to P$ and for all g: $C \to P$ there is $\alpha \in W$ so that $g \upharpoonright C_\alpha = g_\alpha$.

We can assume H is a group structure on $Z(p) \times B$ and π is projection on the second coordinate. We now define H_α by induction on α to be a group structure on $Z(p) \times (B + C_\alpha)$ extending H so that $Z(p) \rightarrowtail H_\alpha \overset{\pi_\alpha}{\twoheadrightarrow} B + C_\alpha$ is an ω-elongation and π_α is projection on the second coordinate. We identify C_α as a set with $\{0\} \times C_\alpha$. The key case occurs when $Z(p) \rightarrowtail H_\alpha \overset{\pi_\alpha}{\twoheadrightarrow} B + C_\alpha$ has been defined, $\alpha \in W$ and $f \cup g_\alpha$ induces a homomorphism from H_α to P.

Then use the inductive hypothesis to choose $Z(p) \rightarrowtail H_{\alpha+1} \overset{\pi_{\alpha+1}}{\twoheadrightarrow} B + C_{\alpha+1}$ so that $f \cup g_\alpha$ does not extend to a homomorphism from $H_{\alpha+1}$ to P. The proof can be finished as usual.

A, by now standard, argument using the $\diamondsuit$ principle yields the following theorem.

2.2. THEOREM. *Assume $(V = L)$. A group G is Σ-cyclic iff for every ω-elongation H of $Z(p)$ by G there is a homomorphism f from H to P such that $f(p^\omega H) \neq 0$.*

REMARK. The proof of Theorem 1.6 shows this characterization fails if we assume MA + $\neg$CH. The next result can be thought of as characterizing the *strongly Crawley* groups: i.e. those groups which remain Crawley when a Σ-cyclic group is added to them.

In Megibben's proof Crawley groups are attacked via Richman's criterion which links elongations of $Z(p)$ by G with codimension 1 dense subsocles of $G[p]$.

2.3. COROLLARY. *Assume* $(V = L)$. *Suppose* B *is a group of unbounded exponent.* B *is* Σ*-cyclic iff for all* Σ*-cyclic groups* $A, B \oplus A$ *is a Crawley group.*

Proof. Let A be $\oplus_{n<\omega} Z(p^n)$. Then there is an ω-elongation of $Z(p)$ by $B \oplus A$ isomorphic to $P \oplus B$. If $B \oplus A$ is Crawley, then any ω-elongation H of $Z(p)$ by $B \oplus A$ has a homomorphism f to P so that $f(p^\omega H) \neq 0$.

In Lemma 2.1, P can be replaced by any group, G, of cardinality $\leq \aleph_1$ such that $p^\omega G \simeq Z(p)$. This remark allows us to obtain additional information on Crawley groups in L. (The proofs are similar to the foregoing ones.)

2.4. THEOREM. *Assume* $(V = L)$. *Suppose* B *has a direct summand* C *of unbounded exponent and* $|C| \leq \aleph_1$. *Then* B *is a Crawley group iff* B *is* Σ*-cyclic.*

A group, A, is said to be *weakly* κ*-separable* if every subset of A of cardinality $< \kappa$ is contained in a pure subgroup B such that: B is Σ-cyclic; and for all $C \supseteq B$ if $|C/B| < \kappa$, then C/B is Σ-cyclic.

2.5. THEOREM. *Assume* $(V = L)$. *If* B *is weakly* κ*-separable of cardinality* κ, *then* B *is a Crawley group iff* B *is* Σ*-cyclic.*

Proof. By **[S1]** we can assume κ is regular and by **[M]** (or Theorem 1.2) that $\kappa > \aleph_1$. Let $\{B_\alpha\colon \alpha < \kappa\}$ be a κ-filtration of B by pure subgroups so that B_0 is countable, has unbounded exponent and B_α/B_0 is Σ-cyclic for all $\alpha < \kappa$. We can further assume $B_{\alpha+1}$ is a direct summand of B_ν for all $\nu < \alpha$.

Let $Z(p) \rightarrowtail H \twoheadrightarrow B$ be an ω-elongation extending an ω-elongation $Z \rightarrowtail H_0 \twoheadrightarrow B_0$. Let H_α be the preimage of B_α.

We now define a group structure H' on $Z(p) \times B$ so that $Z(p) \rightarrowtail H' \overset{\pi}{\twoheadrightarrow} B$ is an ω-elongation where π is projection on the second coordinate. Let $W = \{\alpha\colon B_{\alpha+1}/B_\alpha$ is not Σ-cyclic$\}$. Choose $\{g_\alpha\colon \alpha \in W\}$ so that: for all α, $g_\alpha\colon Z(p) \times B_\alpha \to H_\alpha$ and for all $g\colon Z(P) \times B \to H$ there is $\alpha \in W$ with $g \restriction Z(p) \times B = g_\alpha$. Choose H'_0 so that $Z(p) \rightarrowtail H'_0 \twoheadrightarrow B_0$ is an ω-elongation. The key case occurs when H'_α has been defined, $\alpha \in W$ and $g_\alpha\colon H'_\alpha \to H_\alpha$ is an isomorphism. Since $H_{\alpha+1}/H_0 \cong B_{\alpha+1}/B_0$, we can choose ρ a projection of $H_{\alpha+1}$ onto H_0. By Lemma 2.1, we can choose $H'_{\alpha+1}$ so that $\rho \circ g_\alpha$ does not extend to a homomorphism from $H'_{\alpha+1}$ to H_0. Let $H' = \bigcup H'_\alpha$.

Suppose $g\colon H' \to H$ is an isomorphism. Choose α so that $g \restriction H'_\alpha = g_\alpha$, an isomorphism. Choose $\nu > \alpha$ so that $g(H'_{\alpha+1}) \subseteq H_\nu$. Since $B_{\alpha+1}$ is a direct summand of B_ν, $H_\nu \cong H_{\alpha+1} \oplus B_\nu/B_\alpha$. So we can take $\rho'\colon H_\nu \to H_0$ extending ρ. Then $\rho' \circ g$ contradicts the choice of $H'_{\alpha+1}$.

3. Must Crawley groups be ω_1-separable? So far the examples given of Crawley groups are all ω_1-separable. In this section we will show that it is consistent that there is a Crawley group of cardinality ω_1 which is not ω_1-separable. It would be more interesting to produce, in some model of set theory, a Crawley group with no unbounded Σ-cyclic direct summand. This problem seems quite hard. The proof of our result is a mixture of the techniques in [S2] for producing a Whitehead group of class I with the proof of Theorem 1.6.

To start we will introduce the set theoretic machinery. A *tree* of height $\le \omega$ is a partially ordered set such that any element has finitely many predecessors and these predecessors are totally ordered. Suppose T is a tree of height ω (i.e. there are subchains of T of every finite length). A *branch* of T is a maximal chain of cardinality $\aleph_0$. Suppose $d_0\colon \omega \to \omega$, $d_1\colon \omega \to \omega + 1$ and $A \subseteq \omega$. A function $f\colon A \to \omega$ is said to be (d_0, d_1)-*admissible* if for all n $f(n) < d_1(n)$ and there is some k so that for all $n \ge k$, $f(n) < d_0(n)$.

Suppose T is a tree of height ω where underlying set is ω and that $\mathcal{S}$ is a set of branches. Let d_0 and d_1 be as above. $(T, \mathcal{S})$ satisfies (d_0, d_1)-*uniformization* if whenever $\{F_A\colon A \in \mathcal{S}\}$ is such that for all A, F_A is a countable collection of (d_0, d_1)-admissible functions from A to ω, and for all finite $X \subseteq A$, $f\colon X \to \omega$ if $f(x) < d_1(x)$ then f has an extension in F_A; then there is $f\colon \omega \to \omega$ so that for all A, $f \restriction A \in F_A$. Note if $(T, \mathcal{S})$ satisfies (d_0, d_1)-uniformization and for all n $d'_l(n) < d_l(n)$ $(l = 0, 1)$, then $(T, \mathcal{S})$ satisfies (d'_0, d'_1)-uniformization. A modification of the proof of Theorem 4.2 of [**S2**] yields:

3.1. THEOREM. *Fix* d_0; $\omega \to \omega$ *and* d_1: $\alpha \to \omega + 1$. *It is consistent that there is a tree* T *of height* ω *whose underlying set is* ω *and a collection* $\mathcal{S} = \{A_i : i \in \omega_1\}$ *of pairwise distinct branches such that* $(T, \mathcal{S})$ *satisfies* (d_0, d_1)*-uniformization.*

REMARK. Our proof of Theorem 3.2 could actually be done using the more usual uniformization property in Theorem 4.2 of [**S2**]. We state this stronger notion of uniformization in the hope it may prove useful.

Let d_0 and d_1 be defined by $d_0(n) = d_1(n) = p$. Assume T and $\{A_i : i < \omega_1\}$ satisfy (d_0, d_1)-uniformization where T is a tree of height ω whose underlying set is ω and $\{A_i : i < \omega_1\}$ is a collection of pairwise distinct branches. For each i, let $\langle a^i_n : n < \omega \rangle$ enumerate A in increasing order (in the sense of T). Note: if $a^i_m = a^j_n$ then $m = n$. Let $B = \{a^i_n : i < \omega_1,\ n < \omega\}$. Of course $B \subseteq \omega$. For clarity of notation we will distinguish between elements of ω, denoted by a^i_n, and integers.

Let G be the group freely generated by B and $\{x^i_n : i < \omega_1,\ n < \omega\}$ subject to the relations: $p^{n+1}a^i_n = 0$, for $i < \omega_1$ and $n < \omega$; $px^i_0 = 0$, for $i < \omega_1$; and for $i < \omega_1$ and $n < \omega$, $px^i_{n+1} = x^i_n + a^i_n$. Of course x^i_n can be thought of as the formal sum $\sum_{m \geq n} -p^{m-n}a^i_m$. G is contained in the torsion completion of a countable Σ-cyclic group. So G is separable but not ω_1-separable.

3.2. THEOREM. *It is consistent that there is a Crawley group* G *of cardinality* $\aleph_1$ *which is not* ω_1*-separable. In fact* G *has a countable basic subgroup.*

Proof. Let G be as above. Suppose $Z(p) \rightarrowtail H_0 \overset{\pi_0}{\twoheadrightarrow} G$ and $Z(p) \rightarrowtail H_1 \overset{\pi_1}{\twoheadrightarrow} G$ are two ω-elongations. Our first goal is to find subgroups G_1 and G_2, so that: $G = G_1 \oplus G_2$; G_1 is countable; and if we let $H'_0 = \pi_0^{-1}(G_1)$ and $H'_1 = \pi_1^{-1}(G_1)$ then $Z(p) \rightarrowtail H'_0 \overset{\pi_0}{\twoheadrightarrow} G_1$ and $Z(p) \rightarrowtail H'_1 \overset{\pi_1}{\twoheadrightarrow} G_1$ are ω-elongations. We will then show the pure exact sequences $H'_0 \rightarrowtail H_0 \twoheadrightarrow G_2$ and $H'_1 \rightarrowtail H_1 \twoheadrightarrow G_2$ split. Clearly this suffices.

To begin we wish to choose C such that: for some $i, j < \omega_1$ $C \subseteq B \cup \{x^i_n, x^j_n : n < \omega\}$; for all $n > 0$ and $l = 0, 1$ there is $n \leq m$ $c \in C$ and $c^* \in H_l$ so that $\pi_l(c^*) = c$, $p^m c^* \in p^\omega H_l \setminus \{0\}$; and for all but finitely many i, $A_i \cap C$ is finite. Note: if $\pi_l(c^*) = c$ and $p^m c^* \in p^\omega H_l$, then $p^m c = 0$. Also for all c^{**} if $\pi_l(c^{**}) = c$ then $p^m c^{**} = p^m c^*$.

Fix $l = 0, 1$. There are two cases. Suppose for all $n > 0$ there is $n \leq m$, $b \in B$, $b^* \in H_l$ so that $\pi_l(b^*) = b$ and $p^m b^* \in p^\omega H_l \setminus \{0\}$. In this case, let $D_l = \{b \in B$: for some m and $b^* \in H_l \pi_l(b^*) = b$ and $p^m b^* \in p^\omega H_l \setminus \{0\}\}$. If for some $i < \omega_1$, $D_l \cap A_i$ is infinite, let $C_l = A_i$. Otherwise let $C_l = D_l$.

In the second case pick n so that for all $b \in B$, $m \geq n$ and $b^* \in H_l$ if $\pi_l(b^*) = b$ and $p^m b = 0$ then $p^m b^* = 0$. Choose i and x_n^{i*} so that $\pi_l(x_n^{i*}) = x_n^i$ and $p^{n+1} x_n^{i*} \neq 0$. Such an i exists since $p^n G[p]$ is generated by $\{p^m b$: $b \in B$, $n \leq m$ and $p^{m+1} b = 0\} \cup \{p^n x_n^i$: $i < \omega_1\}$. Let a_k^{i*} be π_l-preimages of $a_k^i (n \leq k < m)$. Then

$$p^{m+1} x_m^{i*} = p^{n+1}\left(x_n^{i*} + \sum p^{k-n} a_k^{i*}\right) = p^{n+1} x_n^{i*}.$$

(The last equality follows from the choice of n.) In this case let $C_l = \{x_n^i$: $n < \omega\}$.

If we let $C = C_0 \cup c_1$, then C is as desired. We can modify C so that, if $x_n^i \in C$ then $a_n^i \in C$.

For convenience we can assume if $a_n^i \in C$ then $a_m^i \in C$ for all $m \leq n$. Let G_1 be the subgroup of G generated by C together with $\{x_n^i$: $n < \omega$ and $C \cap A_i$ is infinite$\}$. Let C_2 be the group generated by $B \setminus C$ together with $\{x_n^i$: $a_n^i \notin C$, $i < \omega_1\}$. It is clear that $G = G_1 \oplus G_2$, G_1 is countable and $Z(p) \rightarrowtail H_l' \twoheadrightarrow G_1$ is an ω-elongation.

We now wish to show for $l = 0, 1$, $H_l' \rightarrowtail H_l \twoheadrightarrow G_2$ splits. Since the proof is the same for each l we will drop the subscript l. Reindexing we can assume $\langle A_i$: $i < \omega_1\rangle$ enumerates the branches such that $A_i \cap C$ is finite. (The exceptional branches have been dropped.) For each i, let $n(i)$ denote the least n such that $a_n^i \notin C$. To find a splitting we must find preimages $\underline{x}_m^i$, $\underline{a}_m^i$ for x_m^i and a_m^i ($i < \omega_1$ and $n(i) \leq m < \omega$) such that: $p^{m+1} \underline{a}_m^i = p^{m+1} \underline{x}_m^i = 0$; and for all $m \geq n(i)$, $p \underline{x}_{m+1}^i = \underline{x}_m^i + \underline{a}_m^i$. (Of course, $a_m^i = a_m^j$, then $\underline{a}_m^i = \underline{a}_m^j$.) First for $i < \omega_1$ and $n(i) \leq m$ choose $\mathbf{x}_m^i$, $\mathbf{a}_m^i$ so that $\pi(\mathbf{x}_m^i) = x_m^i$ and $\pi(\mathbf{a}_m^i) = a_m^i$. Let t be a fixed element of $p^\omega H' \setminus \{0\}$. Since for all $i, n(i) \leq mp\mathbf{x}_{m+1}^i = \mathbf{x}_m^i + \mathbf{a}_m^i + kt$ and $pt = 0$, by modifying our choice of the $\mathbf{x}_m^i$, we assume $p\mathbf{x}_{m+1}^i = \mathbf{x}_m^i + \mathbf{a}_m^i$. For $m < \omega$ choose $b_m \in H'$ so that $p^{m+1} b_m = t$.

Claim. Fix $i < \omega_1$. For $m \geq n(i)$ choose $0 \leq k_m^i < p$ so that $p^{m+1}\mathbf{x}_m^i + p^{m+1} k_m^i b_m = 0$.

(1) Then $p^{m+1}(\mathbf{a}_m^i - k_m^i b_m - p k_{m+1}^i b_{m+1}) = 0$.

(2) Suppose $\underline{a}_m^i$ ($m \geq n(i)$), preimages of the a_m^i, are given such that for all m, $p^{m+1} \underline{a}_m^i = 0$; and for some n if $m \geq n$ then $\underline{a}_m^i = \mathbf{a}_m^i - k_m^i b_m + p k_{m+1}^i b_{m+1}$. Then there exists $\underline{x}_m^i$ ($m \geq n(i)$) preimages of the x_m^i so that: for all m: (i) $p^{m+1} \underline{x}_m^i = 0$ and (ii) $p \underline{x}_{m+1}^i = \underline{x}_m^i + \underline{a}_m^i$.

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Proof (*of claim*). (1) Since $\mathbf{a}^i_m = p\mathbf{x}^i_{m+1} - \mathbf{x}^i_m$,

$$p^{m+1}\left(\mathbf{a}^i_m - k^i_m b_m + pk^i_{m+1} b_{m+1}\right) = p^{m+2}\left(\mathbf{x}^i_{m+1} + k^i_{m+1} b_{m+1}\right) - p^{m+1}\left(\mathbf{x}^i_m + k^i_m b_m\right) = 0.$$

(2) Let $\underset{\sim}{x}^i_m = \mathbf{x}^i_m + k^i k_m b_m$, for $m \geq n$. For $n(i) \leq m < n$ there is only one choice of $\underset{\sim}{x}^i_m$ so that equation (ii) will be satisfied for all m. Just as in (1) we can verify equation (i) holds.

We can now apply our choice of T and $\{A_i\colon i < \omega_1\}$. For $i < \omega_1$, let $f \in F_i$ if (i) for $m \geq n(i)$, $f(a^i_m) = (r, s)$ where $0 \leq r$, $s < p$ and $p^{m+1}(\mathbf{a}^i_m - rb_m + psb_{m+1}) = 0$ and (ii) for all but finitely many m, $r = k^i_m$ and $s = k^i_{m+1}$. By the choice of t and $\{A_i\colon i < \omega_1\}$ there is $f\colon \{a^i_m\colon i < \omega_1,\ m \geq n(i)\} \to \omega \times \omega$ such that for all i $f \restriction A_i \in F_i$. For $i < \omega_1$ and $m \geq n(i)$, let $\underset{\sim}{a}^i_m = \mathbf{a}^i_m - rb_m + psb_{m+1}$ where $f(a^i_m) = (r, s)$. By claim (2) we can choose $\underset{\sim}{x}^i_m$ as desired.

REMARK. Warfield **[W]** has shown: if CH holds then any Crawley group with a countable basic subgroup is Σ-cyclic. This result can also be proved using weak diamond. See the remarks after Theorem 1.2 or Corollary 7.8 of **[EH]**.

If the following question has a positive answer, then Theorem 2.2 shows: if $(V = L)$ then every Crawley group in Σ-cyclic. A positive answer would also show our construction in §3 is in some sense best possible.

Question. Suppose G is a Crawley group and H is the ω-elongation of $Z(p)$ by G. Do there exist H_1 and H_2 so that: $H = H_1 \oplus H_2$; H_1 is countable and H_2 is separable?

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