A characterization of $\mathrm{Ext}(G, \mathbb{Z})$ assuming $(V = L)$

by

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Abstract. We complete the characterization of $\mathrm{Ext}(G, \mathbb{Z})$ for any torsion-free abelian group G assuming Gödel's axiom of constructibility plus there is no weakly compact cardinal. In particular, we prove in $(V = L)$ that, for a singular cardinal ν of uncountable cofinality which is less than the first weakly compact cardinal and for every sequence $(\nu_p : p \in \Pi)$ of cardinals satisfying $\nu_p \leq 2^\nu$ (where Π is the set of all primes), there is a torsion-free abelian group G of size ν such that ν_p equals the p-rank of $\mathrm{Ext}(G, \mathbb{Z})$ for every prime p and 2^ν is the torsion-free rank of $\mathrm{Ext}(G, \mathbb{Z})$.

1. Introduction. Since the first author solved the well known Whitehead problem in 1977 (see [Sh1], [Sh2]) the structure of $\mathrm{Ext}(G, \mathbb{Z})$ for torsion-free abelian groups G has received much attention. Easy arguments show that $\mathrm{Ext}(G, \mathbb{Z})$ is always a divisible group for every torsion-free group G. Hence it is of the form

$$\mathrm{Ext}(G, \mathbb{Z}) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 ($p \in \Pi$) which are uniquely determined (and Π is the set of all prime numbers). Thus, the obvious question that arises is which sequences $(\nu_0, \nu_p : p \in \Pi)$ can appear as the cardinal invariants of $\mathrm{Ext}(G, \mathbb{Z})$ for some (which) torsion-free abelian group? On the one hand, there are a few results about possible sequences $(\nu_0, \nu_p : p \in \Pi)$ provable in ZFC. For instance, the trivial sequence consisting of zero entries only can be realized by any free abelian group. On the other hand, assuming Gödel's constructible universe $(V = L)$ plus there is no weakly compact car-

dinal it has been shown that almost all sequences (with natural restrictions) can be the cardinal invariants of $\mathrm{Ext}(G, \mathbb{Z})$ for some torsion-free abelian group G whenever the size of the group G is not a singular cardinal of uncountable cofinality (see Section 2 for details and [EkHu], [EkSh], [GrSh1], [GrSh2], [HiHuSh], [MeRoSh], [SaSh1], and [SaSh2] for references). However, the question of which sequences $(\nu_0, \nu_p : p \in \Pi)$ can occur is independent of ZFC. It is the purpose of this paper to deal with the remaining case, namely torsion-free abelian groups of cardinality ν where ν is singular of cofinality $\mathrm{cf}(\nu) > \aleph_0$. The idea is to use the construction principle from [MeRoSh] which holds under $(V = L)$ and to apply the main theorem from [MeRoSh] in our construction.

Our notation is standard and we write maps from the left. If H is a pure subgroup of the abelian group G, then we shall write $H \subseteq_* G$. For further details on abelian groups we refer to [Fu] and for set-theoretic methods to [EkMe], [Je] or [Ku].

2. The structure of $\mathrm{Ext}(G, \mathbb{Z})$.

In this section we recall the basic results on the structure of $\mathrm{Ext}(G, \mathbb{Z})$ for torsion-free G. It is easy to see that $\mathrm{Ext}(G, \mathbb{Z})$ is divisible for torsion-free G, hence it is of the form

$$\mathrm{Ext}(G, \mathbb{Z}) = \bigoplus_{p \in \Pi} \mathbb{Z}(p^\infty)^{(\nu_p)} \oplus \mathbb{Q}^{(\nu_0)}$$

for some cardinals ν_p, ν_0 $(p \in \Pi)$. Since the cardinals ν_p $(p \in \Pi)$ and ν_0 completely determine the structure of $\mathrm{Ext}(G, \mathbb{Z})$ we introduce the following terminology. We denote by $r_0^e(G)$ the *torsion-free rank* ν_0 of $\mathrm{Ext}(G, \mathbb{Z})$, which is the dimension of $\mathbb{Q} \otimes \mathrm{Ext}(G, \mathbb{Z})$, and by $r_p^e(G)$ the *p-rank* ν_p of $\mathrm{Ext}(G, \mathbb{Z})$, which is the dimension of $\mathrm{Ext}(G, \mathbb{Z})[p]$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$ for any prime number $p \in \Pi$. There are only a few results provable in ZFC when G is uncountable, but assuming Gödel's universe an almost complete characterization is known (if there is no weakly compact cardinal). The aim of this paper is to fill the remaining gap.

We first justify our restriction to torsion-free G. Let A be any abelian group and $t(A)$ its torsion subgroup. Then $\mathrm{Hom}(t(A), \mathbb{Z}) = 0$ and hence we obtain the short exact sequence

$$0 \to \mathrm{Ext}(A/t(A), \mathbb{Z}) \to \mathrm{Ext}(A, \mathbb{Z}) \to \mathrm{Ext}(t(A), \mathbb{Z}) \to 0$$

which must split since $\mathrm{Ext}(A/t(A), \mathbb{Z})$ is divisible. Thus

$$\mathrm{Ext}(A, \mathbb{Z}) \cong \mathrm{Ext}(A/t(A), \mathbb{Z}) \oplus \mathrm{Ext}(t(A), \mathbb{Z}).$$

Since the structure of $\mathrm{Ext}(t(A), \mathbb{Z}) \cong \prod_{p \in \Pi} \mathrm{Hom}(A, \mathbb{Z}(p^\infty))$ is well known in ZFC it is reasonable to assume that A is torsion-free and, of course, non-free. Using Pontryagin's theorem one proves

LEMMA 2.1. *Suppose G is a countable torsion-free group which is not free. Then $r_0^e(G) = 2^{\aleph_0}$.*

Proof. See [EkMe, Theorem XII 4.1]. ∎

Similarly, for the p-ranks of G we have the following result due to C. U. Jensen.

LEMMA 2.2. *If G is a countable torsion-free group, then for any prime p, either $r_p^e(G)$ is finite or $2^{\aleph_0}$.*

Proof. See [EkMe, Theorem XII 4.7]. ∎

This clarifies the structure of Ext$(G, \mathbb{Z})$ for countable torsion-free groups G since the existence of groups as in Lemmas 2.1 and 2.2 follows from Lemma 2.8 below. We now turn our attention to uncountable groups and assume Gödel's axiom of constructibility. The following is due to Hiller, Huber and Shelah.

LEMMA 2.3. $(V = L)$ *Suppose G is a torsion-free non-free group and let B be a subgroup of A of minimum cardinality ν such that A/B is free. Then $r_0^e(G) = 2^\nu$. In particular, $r_0^e(G)$ is uncountable and $r_0^e(G) = 2^{|G|}$ if $G^* = \mathrm{Hom}(G, \mathbb{Z}) = 0$.*

Proof. See [EkMe, Theorem XII 4.4, Corollary XII 4.5]. ∎

Note that the above lemma is not true in ZFC since for any countable divisible group D it is consistent that there exists an uncountable torsion-free group G with Ext$(G, \mathbb{Z}) \cong D$, hence $r_0^e(G) = 1$ is possible by taking $D = \mathbb{Q}$ (see Shelah [Sh3]).

Again we turn to p-ranks. There is a useful characterization of $r_p^e(G)$ using the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0.$$

The induced sequence

$$\mathrm{Hom}(G, \mathbb{Z}) \xrightarrow{\varphi^p} \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \to \mathrm{Ext}(G, \mathbb{Z}) \xrightarrow{p_*} \mathrm{Ext}(G, \mathbb{Z})$$

shows that the dimension of

$$\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})/\mathrm{Hom}(G, \mathbb{Z})\varphi^p$$

as a vector space over $\mathbb{Z}/p\mathbb{Z}$ is exactly $r_p^e(G)$.

The following result due to Mekler, Rosłanowski and Shelah shows that under the assumption of $(V = L)$ almost all possibilities for $r_p^e(G)$ can appear if the group is of regular cardinality.

LEMMA 2.4. $(V = L)$ *Let ν be an uncountable regular cardinal less than the first weakly compact cardinal. Suppose that $(\nu_p : p \in \Pi)$ is a sequence of cardinals such that for each p, $0 \le \nu_p \le 2^\nu$. Then there is an almost-free group G of cardinality ν such that $r_0^e(G) = 2^\nu$ and for all p, $r_p^e(G) = \nu_p$.*

Proof. See [MeRoSh, Main Theorem 3.9]. ∎

On the other hand, if the cardinality of G is singular, then the following holds, which was proved by Grossberg and Shelah.

LEMMA 2.5. *If ν is a singular strong limit cardinal of cofinality ω, then there is no torsion-free group G of cardinality ν such that $r_p^e(G) = \nu$ for any prime p.*

Proof. See [GrSh1, Theorem 1.0]. ∎

Note that Lemma 2.4 shows that the restriction in Lemma 2.5 is the only restriction for singular strong limit cardinals ν of cofinality ω. Namely, if $\varrho < \nu$ choose a regular cardinal $\varrho \leq \varrho' < \nu$ and apply Lemma 2.4 to obtain a torsion-free group G' with $r_p^e(G') = \varrho$ and $|G'| = \varrho'$. Since $\text{Ext}(-, \mathbb{Z})$ is a multiplicative functor we can now easily get a torsion-free group G from G' with $|G| = \nu$ and $r_p^e(G) = \varrho$.

Also the case of weakly compact cardinality was dealt with in [SaSh1] by Sageev and Shelah.

LEMMA 2.6. *If G is a torsion-free group of weakly compact cardinality ν and $r_p^e(G) \geq \nu$ for some prime p, then $r_p^e(G) = 2^\nu$.*

Proof. See [SaSh1, Main Theorem]. ∎

The above results show that under the assumption of $(V = L)$ the structure of $\text{Ext}(G, \mathbb{Z})$ for torsion-free groups G of cardinality ν is clarified for all cardinals ν except when ν is singular but not of cofinality ω. This will be the subject of the next section.

However, in a particular case, namely when $G^* = \text{Hom}(G, \mathbb{Z}) = 0$, a complete characterization of $\text{Ext}(G, \mathbb{Z})$ is known in Gödel's universe if there is no weakly compact cardinal. The following is due to Hiller–Huber–Shelah.

LEMMA 2.7. *If G is torsion-free such that $\text{Hom}(G, \mathbb{Z}) = 0$, then for all primes p, $r_p^e(G)$ is finite or of the form 2^{μ_p} for some infinite cardinal $\mu_p \leq |G|$.*

Proof. See [EkMe, Lemma XII 5.2]. ∎

Together with Lemma 2.3 and the next result due to Hiller, Huber and Shelah the characterization is complete if $\text{Hom}(G, \mathbb{Z}) = 0$.

LEMMA 2.8. *For any cardinal ν_0 of the form $\nu_0 = 2^{\mu_0}$ for some infinite μ_0 and any sequence $(\nu_p : p \in \Pi)$ of cardinals less than or equal to ν_0 such that each ν_p is either finite or of the form 2^{μ_p} for some infinite μ_p there is a torsion-free group G such that $\text{Hom}(G, \mathbb{Z}) = 0$ and $r_0^e(G) = \nu_0$, $r_p^e(G) = \nu_p$ for all primes $p \in \Pi$.*

Proof. See [HiHuSh, Theorem 3(b)]. ∎

3. The singular case. In this section we prove our main theorem which completes the characterization of Ext$(G, \mathbb{Z})$ for torsion-free groups G under the assumption of Gödel's axiom of constructibility plus there is no weakly compact cardinal. The idea of the proof is as follows: For a singular cardinal ν of uncountable cofinality we shall construct a torsion-free abelian group G, of size ν, as the union of pure subgroups G_α such that G has prescribed values for $r_0^e(G)$ and $r_p^e(G)$ ($p \in \Pi$). Together with the G_α's we also build homomorphisms f_γ^p for $\gamma < r_p^e(G)$ such that no non-trivial combination $\sum_{l < n} a_l f_{\gamma_l}^p \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ can be factored by the canonical homomorphism $\varphi^p : \mathrm{Hom}(G, \mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ to a homomorphism in $\mathrm{Hom}(G, \mathbb{Z})$. This is a typical application of the diamond principle which holds under $(V = L)$ for every regular uncountable cardinal. On the other hand, we also need that for every $f \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ there are $f_1 = \sum_{l < n} a_l f_{\gamma_l}^p \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ and $f_2 \in \mathrm{Hom}(G, \mathbb{Z})$ such that $f - f_1 = \varphi^p(f_2)$. The two demands seem to be hard to go together but the principle from [MeRoSh] allows us to carry out the construction.

THEOREM 3.1. $(V = L)$ *Let* ν *be an uncountable singular cardinal of cofinality* $\mathrm{cf}(\nu) > \aleph_0$ *which is less than the first weakly compact cardinal. If* $(\nu_p : p \in \Pi)$ *is a sequence of cardinals less than or equal to* 2^ν, *then there exists a torsion-free group* G *such that*

(i) $|G| = \nu$;

(ii) $r_0^e(G) = 2^\nu$;

(iii) $r_p^e(G) = \nu_p$ *for all* $p \in \Pi$.

Proof. First we note that by Lemma 2.8 for every prime $p \in \Pi$ there is a torsion-free group G_p such that $\mathrm{Hom}(G_p, \mathbb{Z}) = 0$, $|G_p| = \nu$, $r_0^e(G_p) = 2^\nu$ and $r_p^e(G_p) = 2^\nu = \nu^+$ but $r_q^e(G_p) = 0$ for all $q \neq p$. Since $\mathrm{Ext}(-, \mathbb{Z})$ commutes with direct sums it therefore suffices to assume $\nu_p \leq \nu$ for all $p \in \Pi$. Let $\kappa = \mathrm{cf}(\nu)$. Choose a continuous increasing sequence $\langle \mu_\alpha : \alpha < \kappa \rangle$ such that

- $\lim_{\alpha < \kappa} \mu_\alpha = \nu$;
- if α is a successor ordinal, then $\mu_\alpha = \lambda_\alpha^+$ for some strong limit singular cardinal $\lambda_\alpha > \kappa$ such that $\mathrm{cf}(\lambda_\alpha) = \aleph_0$ and $\lambda_{\alpha+1} > \mu_\alpha$.

Now, let $S \subseteq \{\alpha < \kappa : \mathrm{cf}(\alpha) = \aleph_0\}$ be stationary. Inductively we shall construct a torsion-free group $G = \bigcup_{\alpha < \kappa} G_\alpha$ such that the following conditions are satisfied:

(i) $|G_\alpha| = \mu_\alpha$ and G_α is μ_0-free; moreover, there is no $G' \subseteq G_\alpha$ of cardinality less than μ_α such that G_α/G' is free.

(ii) $G_\beta \subseteq_* G_\alpha$ if $\beta < \alpha$.

(iii) $\mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z}) = L_p^\alpha \oplus K_p^\alpha$ such that $f \restriction_{G_\alpha} \in K_p^\alpha$ for all $f \in K_p^\beta$, $\alpha < \beta$.

(iv) There are bases B_p^α of K_p^α as vector spaces over $\mathbb{Z}/p\mathbb{Z}$ and functions $T_\alpha : B_p^\alpha \to \mathrm{Hom}(G_\alpha, \mathbb{Z})$ such that

  (a) $f{\upharpoonright}_{G_\alpha} \in B_p^\alpha$ for all $f \in B_p^\beta$, $\alpha < \beta$;
  (b) $\varphi^p T_\alpha = \mathrm{id}_{B_p^\alpha}$, where φ^p is the canonical map $\varphi^p : \mathrm{Hom}(G_\alpha, \mathbb{Z}) \to \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$;
  (c) if $\alpha < \beta$ and $f \in B_p^\beta$, then $T_\alpha(f{\upharpoonright}_{G_\alpha}) = T_\beta(f){\upharpoonright}_{G_\alpha}$.

(v) There are bases $\langle f_\gamma^{\alpha,p} : \gamma < \nu_p \cap \mu_\alpha \rangle$ of L_p^α and $M_\alpha \subseteq \mathrm{Hom}(G_\alpha, \mathbb{Z})$ (for $\alpha \neq 0$) such that

  (a) $f_\gamma^{\alpha,p} \subseteq f_\gamma^{\beta,p}$ for all $\gamma < \nu_p \cap \mu_\alpha$, $\alpha < \beta$;
  (b) $M_\alpha = \bigoplus_{\gamma < \nu_p \cap \mu_\alpha, \delta \in [\alpha,\kappa)} \mathbb{Z} h_{\gamma,\delta}^{\alpha,p}$ and $\overline{p} h_{\gamma,\delta}^{\alpha,p} = f_\gamma^{\alpha,p}$ for all $\delta \in [\alpha,\kappa)$, where $\overline{p}$ is the canonical map $\overline{p} : \mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$;
  (c) for $\alpha < \beta \leq \delta < \kappa$ and $\gamma < \nu_p \cap \mu_\alpha$ we have $h_{\gamma,\delta}^{\alpha,p} \subseteq h_{\gamma,\delta}^{\beta,p}$;
  (d) if $\alpha \in S$ and $g \in \mathrm{Hom}(G_{\alpha+1}, \mathbb{Z})$ and $\overline{p}g \in L_p^{\alpha+1}\backslash\{0\}$, then $g \in M_{\alpha+1}$;
  (e) if $\nu_p \cap \mu_\alpha \leq \gamma < \nu_p \cap \mu_{\alpha+1}$, then $f_\gamma^{\alpha+1,p}{\upharpoonright}_{G_\alpha} = 0$ and $h_{\gamma,\varrho}^{\alpha+1,p}{\upharpoonright}_{G_\alpha} = 0$ for all $\varrho \in [\alpha+1, \kappa)$.

We first show that it is sufficient to carry out the inductive construction of G_α $(\alpha < \kappa)$. Assume that the torsion-free groups G_α $(\alpha < \kappa)$ are constructed satisfying conditions (i) to (v). Put $G = \bigcup_{\alpha<\kappa} G_\alpha$. Then G is a torsion-free group of cardinality ν. Moreover, $r_0^e(G) = 2^\nu$ follows from Lemma 2.3 and property (i). It remains to prove that $r_p^e(G) = \nu_p$ for all $p \in \Pi$. Let $p \in \Pi$ and $\alpha < \nu_p$. We define

$$f_\alpha^{\kappa,p} = \bigcup \{ f_\alpha^{\gamma,p} : \gamma \in [\delta_\alpha, \kappa) \}$$

where $\delta_\alpha = \min\{\delta < \kappa : \mu_\delta \geq \alpha\}$. By condition (v)(a) the function $f_\alpha^{\kappa,p}$ is a well defined homomorphism $f_\alpha^{\kappa,p} \in \mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ for every $\alpha < \nu_p$. We shall show that

- $\{f_\alpha^{\kappa,p} : \alpha < \nu_p\}$ are linearly independent as elements of $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$;
- $\{f_\alpha^{\kappa,p} : \alpha < \nu_p\}$ are linearly independent in $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$ modulo $\mathrm{Hom}(G, \mathbb{Z})\varphi^p$, i.e. no linear combination of them can be factored by $\overline{p}$ to a homomorphism from G to $\mathbb{Z}$;
- $\{f_\alpha^{\kappa,p} : \alpha < \nu_p\}$ together with $\mathrm{Hom}(G, \mathbb{Z})\varphi^p$ generate $\mathrm{Hom}(G, \mathbb{Z}/p\mathbb{Z})$.

Assume first that

$$\sum_{\alpha \in E} z_\alpha f_\alpha^{\kappa,p} = 0$$

for some finite subset $E \subseteq \nu_p$ and elements $z_\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then there exists β such that $\alpha < \nu_p \cap \mu_\beta$ and $f_\alpha^{\beta,p} = f_\alpha^{\kappa,p}\restriction_{G_\beta} \neq 0$ for all $\alpha \in E$. Hence

$$\sum_{\alpha \in E} z_\alpha f_\alpha^{\beta,p} = \Big(\sum_{\alpha \in E} z_\alpha f_\alpha^{\kappa,p} \Big) \restriction_{G_\beta} = 0.$$

But $\langle f_\alpha^{\beta,p} : \alpha < \nu_p \cap \mu_\beta \rangle$ is a basis of L_p^β and thus $z_\alpha = 0$ for all $\alpha \in E$. Therefore, the $f_\alpha^{\kappa,p}$'s ($\alpha < \nu_p$) are linearly independent.

Now, assume that there exists a finite linear combination $0 \neq \sum_{\alpha \in E} z_\alpha f_\alpha^{\kappa,p}$ which can be factored by $\overline{p}$ ($0 \neq z_\alpha \in \mathbb{Z}/p\mathbb{Z}$ for all $\alpha \in E$). Hence there is $0 \neq g \in \mathrm{Hom}(G, \mathbb{Z})$ such that

$$\sum_{\alpha \in E} z_\alpha f_\alpha^{\kappa,p} = \overline{p}g.$$

Since E is finite, there exists $\beta < \kappa$ such that $\alpha < \nu_p \cap \mu_\beta$ for all $\alpha \in E$. Therefore,

$$\overline{p}g\restriction_{G_\gamma} = \sum_{\alpha \in E} z_\alpha f_\alpha^{\kappa,p}\restriction_{G_\gamma} = \sum_{\alpha \in E} z_\alpha f_\alpha^{\gamma,p}$$

for every $\gamma \in [\beta, \kappa)$. By the linear independence of the $f_\alpha^{\kappa,p}$'s we may assume without loss of generality that $\overline{p}g\restriction_{G_\gamma} \neq 0$ for all $\gamma \in [\beta, \kappa)$ since otherwise we sufficiently enlarge β so that $f_\alpha^{\kappa,p}\restriction_{G_\beta} \neq 0$ for all $\alpha \in E$. We conclude that $\overline{p}g\restriction_{G_{\gamma+1}} \in L_p^{\gamma+1}\setminus\{0\}$ for all $\gamma \geq \beta$ and condition (v)(d) implies that $g\restriction_{G_{\gamma+1}} \in M_{\gamma+1}$ for all $\gamma \in S$, $\gamma \geq \beta$. Let $\delta = \gamma + 1$ for some $\gamma \in S$, $\gamma \geq \beta$. Then

$$g\restriction_{G_\delta} = \sum_{k < k_\delta} b_k^\delta h_{\alpha_k^\delta, j_k^\delta}^{\delta,p}$$

with $b_k^\delta \in \mathbb{Z}\setminus\{0\}$ and $\alpha_k^\delta < \nu_p \cap \mu_\delta$, $j_k^\delta \in [\delta, \kappa)$. Since the $h_{\beta,r}^{\delta,p}$ form a basis of M_δ, this representation is unique.

By a pigeon hole argument we may assume that $k_\delta = k_0$ and $b_k^\delta = b_k$ for arbitrarily large $\delta = \gamma + 1$, $\beta \leq \gamma \in S$. Let $\gamma_0 \in S$ be sufficiently large such that $k_0 = k_{\delta_0}$ with $\delta_0 = \gamma_0 + 1$. Thus

$$g\restriction_{G_{\delta_0}} = (g\restriction_{G_\delta})\restriction_{G_{\delta_0}} = \sum_{k < k_0} b_k h_{\alpha_k^{\delta_0}, j_k^{\delta_0}}^{\delta_0,p} = \sum_{k < k_0} b_k (h_{\alpha_k^\delta, j_k^\delta}^{\delta,p}\restriction_{G_{\delta_0}})$$

for all $\delta = \gamma + 1$, $\gamma_0 \leq \gamma \in S$. By condition (v)(e) and the two compatibility conditions (v)(c) and (v)(d) it easily follows that the following holds for all $\epsilon < \kappa$:

(v)(e$'$) if $\nu_p \cap \mu_\alpha \leq \gamma < \nu_p \cap \mu_{\epsilon+1}$, then $f_\gamma^{\epsilon+1,p}\restriction_{G_\alpha} = 0$ and $h_{\gamma,\varrho}^{\epsilon+1,p}\restriction_{G_\alpha} = 0$ for all $\varrho \in [\epsilon + 1, \kappa)$.

Thus, if $\alpha_k^\delta \geq \nu_p \cap \mu_{\delta_0}$ (and also $\alpha_k^\delta < \nu_p \cap \mu_\delta$), then (v)(e$'$) implies that $h_{\alpha_k^\delta, j_k^\delta}^{\delta,p}\restriction_{G_{\delta_0}} = 0$. Note that $\delta = \gamma + 1$ for some $\gamma_0 \leq \gamma \in S$. Hence $\alpha_k^\delta < \nu_p \cap \mu_{\delta_0}$

for all $k < k_0$ and therefore

$$h^{\delta,p}_{\alpha^\delta_k,j^\delta_k}\restriction_{G_{\delta_0}} = h^{\delta_0,p}_{\alpha^\delta_k,j^\delta_k}$$

for all j^δ_k. However, by uniqueness it follows that

$$\{\alpha^{\delta_0}_k : k < k_0\} = \{\alpha^\delta_k : k < k_0\}$$

and also

$$\{j^{\delta_0}_k : k < k_0\} = \{j^\delta_k : k < k_0\}$$

for all δ large enough. This contradicts the fact that $j^\delta_k \in [\delta,\kappa)$ for all $k < k_0$ and δ. Therefore we obtain $\nu_p \le r^e_p(G)$ for all $p \in \Pi$.

It remains to prove that $r^e_p(G) = \nu_p$ for all $p \in \Pi$. For this it suffices to show that $\langle f^{\kappa,p}_\alpha : \alpha < \nu_p \rangle$ generate $\mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$ modulo $\mathrm{Hom}(G,\mathbb{Z})\varphi^p$. Hence, let $0 \ne g \in \mathrm{Hom}(G,\mathbb{Z}/p\mathbb{Z})$. We have to prove that there is a finite linear combination $\sum_{\alpha \in E} z_\alpha f^{\kappa,p}_\alpha$ with $E \subseteq \nu_p$ and $0 \ne z_\alpha \in \mathbb{Z}/p\mathbb{Z}$ such that

$$g - \sum_{\alpha \in E} z_\alpha f^{\kappa,p}_\alpha = \overline{p}h$$

for some $h \in \mathrm{Hom}(G,\mathbb{Z})$. Let $g_\alpha = g\restriction_{G_\alpha}$ for all $\alpha < \kappa$. Hence, by (iii), there exist $k_\alpha \in K^p_\alpha$ and $l_\alpha \in L^p_\alpha$ such that $g_\alpha = k_\alpha + l_\alpha$ for every $\alpha < \kappa$ since $g_\alpha \in \mathrm{Hom}(G_\alpha,\mathbb{Z}/p\mathbb{Z})$. Thus

$$l_\alpha = \sum_{\beta \in E_\alpha} z^\alpha_\beta f^{\alpha,p}_\beta$$

for some finite subset $E_\alpha \subseteq \nu_p \cap \mu_\alpha$ and $0 \ne z^\alpha_\beta \in \mathbb{Z}/p\mathbb{Z}$. Since this representation is unique we may assume by a pigeon hole argument that $z^\alpha_\beta = z_\beta$ and $E_\alpha = E$ are independent of $\alpha < \kappa$. Note that $f^{\alpha,p}_\beta \restriction_{G_{\alpha'}} = f^{\alpha',p}_\beta$ if $\alpha' < \alpha$ and $\beta < \nu_p \cap \mu_{\alpha'}$ and 0 otherwise by (v)(e'). We conclude that

$$\overline{h} = g - \sum_{\beta \in E} z_\beta f^{\kappa,p}_\beta$$

satisfies $\overline{h}\restriction_{G_\alpha} = k_\alpha$ for all $\alpha < \kappa$. Since B^α_p forms a basis of K^α_p for all $\alpha < \kappa$ there is a finite subset $F_\alpha \subseteq B^\alpha_p$ and $0 \ne w^\alpha_b \in \mathbb{Z}/p\mathbb{Z}$ for $b \in F_\alpha$ such that

$$l_\alpha = \sum_{b \in F_\alpha} w^\alpha_b b$$

for all $\alpha < \kappa$. Again, a pigeon hole argument allows us to assume that $w_b = w^\alpha_b$ and $F = F_\alpha$ are independent of α by uniqueness. Note that $b\restriction_{G_\beta} \in B^\beta_p$ if $b \in B^\alpha_p$ and $\alpha > \beta$. Putting

$$h = \sum_{b \in F} w_b \bigcup_{\alpha < \kappa} T_\alpha(b)$$

it follows that $h \in \mathrm{Hom}(G, \mathbb{Z})$ is well defined by (iv) and hence $\overline{p}h = \overline{h}$. Therefore, $g - \sum_{\alpha \in E} z_\alpha f_\alpha^{\kappa, p} = \overline{h}$ has a lifting to $h \in \mathrm{Hom}(G, \mathbb{Z})$. This finishes the proof and it remains to show that we can carry on the induction as claimed, i.e. we have to construct groups G_α ($\alpha < \kappa$) such that (i)–(v) are satisfied. We shall distinguish four cases.

CASE A: $\alpha = 0$. Let $G_0 = \bigoplus_{\mu_0} \mathbb{Z}$. Moreover, put $L_p^0 = \{0\}$ and $K_p^0 = \mathrm{Hom}(G_0, \mathbb{Z}/p\mathbb{Z})$. Since G_0 is free, the existence of T_0 is obvious and if we choose $M_0 = \{0\}$, all conditions (i)–(v) are satisfied for G_0.

CASE B: α is a limit ordinal. Then we let $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$. As before we define $\delta_\gamma = \min\{\delta < \alpha : \mu_\delta \geq \gamma\}$ for $\gamma < \nu_p \cap \mu_\alpha$ and let

$$f_\gamma^{\alpha, p} = \bigcup\{f_\gamma^{\beta, p} : \delta_\gamma \leq \beta < \alpha\}$$

and similarly for $\delta \in [\alpha, \kappa)$ and $\gamma < \nu_p \cap \mu_\alpha$ we let

$$h_{\gamma, \delta}^{\alpha, p} = \bigcup\{h_{\gamma, \delta}^{\beta, p} : \delta_\alpha \leq \beta < \alpha\}.$$

By the continuity conditions (v)(a) and (v)(c) this is well defined. Hence, also $M_\alpha \subseteq \mathrm{Hom}(G_\alpha, \mathbb{Z})$ and $L_p^\alpha \subseteq \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z})$ are defined canonically. Finally, (iii) and the definition of L_p^α induce K_p^α as

$$K_p^\alpha = \{f \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z}) : f{\restriction}_{G_\beta} \in K_p^\beta \text{ for all } \beta < \alpha\}.$$

The corresponding set $B_p^\alpha = \{f \in \mathrm{Hom}(G_\alpha, \mathbb{Z}/p\mathbb{Z}) : f{\restriction}_{G_\beta} \in B_p^\beta \text{ for all } \beta < \alpha\}$ is a basis for K_p^α and the continuity condition (iv)(c) allows us to define $T_\alpha = \bigcup_{\beta < \alpha} T_\beta$. It is easy to check that (i)–(v) are now satisfied.

CASE C: $\alpha = \beta + 1$ and $\beta \notin S$. Then we let $G_\alpha = G_\beta \oplus \bigoplus_{\mu_\alpha} \mathbb{Z}$. In the obvious way we define L_p^α, K_p^α, B_p^α, T_α, $f_\gamma^{\alpha, p}$ for $\gamma < \nu_p \cap \mu_\alpha$, M_α, and $h_{\gamma, \delta}^{\alpha, p}$ for $\gamma < \nu_p \cap \mu_\alpha$ and $\delta \in [\alpha, \kappa)$.

CASE D: $\alpha = \beta + 1$ and $\beta \in S$. Here we imitate the proof of Main Theorem 3.9 of [MeRoSh]. We would like to avoid repeating the technical and lengthy construction from [MeRoSh] but instead point out the main changes for the convenience of the reader. It is then straightforward to modify the proof of Main Theorem 3.9 and its main ingredient Theorem 3.4 from [MeRoSh] and to adapt both to our setting.

We are in the following situation: λ_α is a strong limit singular cardinal strictly greater than $\kappa = \mathrm{cf}(\nu)$. Moreover, $\mu_\alpha = \lambda_\alpha^+ = 2^{\lambda_\alpha}$ is a regular cardinal, $\mathrm{cf}(\lambda_\alpha) = \aleph_0$ and $\lambda_\alpha = \lambda_{\beta+1} > \mu_\beta$ (μ_α plays the role of λ in [MeRoSh], so it is the successor of a strong limit singular cardinal). Since we are assuming $(V = L)$ the prediction principle from [MeRoSh] holds. In Main Theorem 3.9 from [MeRoSh] it is proved that we can find a torsion-free group $\widetilde{G}$ (denoted by G there) which has prescribed values $\widetilde{\nu}_p$ for $r_p^{\mathrm{e}}(\widetilde{G})$. The construction is very similar to ours, i.e. homomorphisms $\widetilde{f}_\gamma^p \in \mathrm{Hom}(\widetilde{G}, \mathbb{Z}/p\mathbb{Z})$ ($\gamma < \widetilde{\nu}_p$) are

constructed (denoted by $f_\lambda^{p,\xi}$ there) which cannot be factorized by $\bar{p}$ (see the proof of Main Theorem 3.9 in [MeRoSh, p. 346]). The main tool is Theorem 3.4 from [MeRoSh] which can be seen as a Step Lemma since it deals with the "killing" of only one undesired homomorphism. However, we are at stage α of our construction, hence we do not want that our homomorphisms $f_\gamma^{\beta,p}$ ($\gamma < \nu_p \cap \mu_\beta$) which we have dealt with so far have no extension to G_α, i.e. cannot be factorized by $\bar{p}$, but we just require that there are only some extensions, namely a set of extensions of $f_\gamma^{\alpha,p}$ which is assigned to each $f_\gamma^{\alpha,p}$. The role of this set is played by $\{h_{\gamma,\delta}^{\alpha,p} : \delta \in [\alpha,\kappa)\}$ (see (v)(b)). Thus the proof of Theorem 3.4 from [MeRoSh] carries over to our situation (for the case of successor cardinal of a strong limit singular cardinal of cofinality $\aleph_0$). Since the role of λ in [MeRoSh, Theorem 3.4] is played by μ_α in our setting only cases B and C in the proof of Theorem 3.4 remain. As usual one guesses the undesirable factorizations and kills them without affecting the work towards lifting that has already been done. The only difference is that we do not require that there is no lifting but we allow only the assigned ones. Now the adjusted version of Theorem 3.4 from [MeRoSh] is used in Main Theorem 3.9 from [MeRoSh] in the case of λ being a successor of a strong limit cardinal. The resulting group serves as our G_α. ∎

REMARK 3.2. We would like to remark that the only reason for the choice of μ_α as successor of a strong limit singular cardinal of cofinality $\aleph_0$ (if α is a successor ordinal) is that this is the easiest situation in the proof of [MeRoSh, Main Theorem 3.9]. However, the strategy described in Case D of the above proof of Theorem 3.1 (i.e. not killing all extensions of a homomorphism but allowing some of them to survive) works for every regular uncountable cardinal which is not weakly compact, e.g. $\aleph_1$. For instance it follows easily for $\aleph_1$ from [EkMe, Theorem XII 4.10] using [EkMe, Lemmas XII 4.8 and XII 4.9].

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