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FORCING CLOSED UNBOUNDED SETS

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Abstract. We discuss the problem of finding forcing posets which introduce closed unbounded subsets to a given stationary set.

Introduction. A very interesting phenomenon, described by Baumgartner, Harrington and Kleinberg [B, H, K], shows that the notion of *stationary set* is not absolute: a stationary $S \subseteq \aleph_1$ can become nonstationary in a generic extension which preserves $\aleph_1$. More precisely, given any stationary $T \subseteq \aleph_1$, there is a poset P such that forcing with P does not add new countable sets to the ground model, but produces a closed unbounded subset of T. Our aim is to generalize this result and to present new problems. The paper is divided into three sections, each presenting a different approach for a generalization of [B, H, K].

In §1, $\aleph_1$ is changed to an arbitrary regular uncountable cardinal κ, S is a stationary subset of κ, and we want to find a generic extension which adds a closed unbounded subset to S, without adding new sets of size $< \kappa$. As it turns out, S has to be *fat* (this will be defined in 1.1) if such a generic extension can be found. In this part, we do not care about cardinals above κ—they might collapse. The definition of fat-stationarity (1.1), Lemma 1.2, and Theorem 1 (which deal with the case $\kappa = \mu^+$, $\mu^\mu = \mu$, or κ is strongly inaccessible) are due to J. Stavi. (See [N, S] where this material is applied to get results about the nontransitivity of the notion of *potential isomorphism* applied to models of $L_{\infty\lambda}$.) Independently, several other mathematicians were aware of some form of Theorem 1: Baumgartner, Fleissner and Kunen, Gregory and Harrington. In fact, the terminology *fat set* is adopted from [F, K] (p. 238, where κ-Baire spaces are discussed). Theorem 2, which deals with the case $\kappa = \mu^+$, μ singular, is due to Shelah. The argument used in the proof is further investigated in [S1, Chapter XIII].

In the second section, we concentrate on the requirement that no cardinals are collapsed, even those above κ. On the other hand, we allow new bounded subsets of κ. The posets described in §1 and the one in [B, H, K] work well if GCH is assumed. But if $2^{\aleph_0} > \aleph_1$, for example, then the forcing poset of [B, H, K] does collapse $\aleph_2$, so we need something else. Theorem 3 shows how to force a closed unbounded subset to a stationary $S \subseteq \omega_1$ without collapsing any cardinal. Baumgartner found how to force a closed unbounded subset of ω_1 with finite conditions; Shelah used this poset (restricted to a stationary set) to prove Theorem 3; the conditions used in the proof of Theorem 3 are a simplified version, due to

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Abraham, of Baumgartner's original ones. Theorems 4 and 5 are due to Abraham.

In §3 we try to replace $\aleph_1$ by $P_{\aleph_1}(\aleph_2)$—the collection of all countable subsets of $\aleph_2$. From our point of view (the absoluteness of the notion of "stationary set"), little is known about the club filter over $P_{\aleph_1}(\aleph_2)$ (see below). It is not clear even what should be the right generalization of [B, H, K] in this context. Theorems 7, 8 are due to Abraham, Theorems 6, 9 to Shelah.

Notation. For cardinals $\lambda < \kappa$, λ regular, $S^\kappa_\lambda = \{\alpha \in \kappa \mid \mathrm{cf}(\alpha) = \lambda\}$.

Closed unbounded is shortened to *club*.

${}^{\mu}\mu = \{f \mid \text{for some } \alpha < \mu, f\colon \alpha \to \mu\}$. $H(\lambda)$ is the collection of all sets hereditarily of cardinality $< \lambda$. If we say that we work in some universe W, then $H(\lambda)$, as any other concept, is to be interpreted in W. Jech [J] and Kueker [K] introduced the notions of club set and stationary set in $P_{\aleph_1}(\aleph_2)$. Kueker's theorem will be used frequently in §3: If $C \subseteq P_{\aleph_1}(\aleph_2)$ is club then there is $f\colon [\aleph_2]^{<\omega} \to P_{\aleph_1}(\aleph_2)$, a function taking finite subsets of $\aleph_2$ as arguments and countable subsets as values, such that if $X \in P_{\aleph_1}(\aleph_2)$ is closed under f then $X \in C$. (X closed under f means that $f(a) \subseteq X$ whenever $a \subseteq X$.) It is not difficult to ask for $f(a)$ to be a singleton.

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§1 Fat sets.

1.1 Definition. Let κ be a regular cardinal. A set $S \subseteq \kappa$ is called fat iff for every club $C \subseteq \kappa$, $S \cap C$ contains closed sets of ordinals of arbitrarily large order-types below κ.

1.2 Lemma. *Assume $\mu < \kappa$, κ regular, and $S \subseteq \kappa$ has the property that for every club $C \subseteq \kappa$, $S \cap C$ contains a closed set of ordinals of order-type $\mu + 1$. Then for any $\tau < \mu^+$ and every club $C \subseteq \kappa$, $S \cap C$ contains a closed set of order-type $\tau + 1$.*

Proof. The case $\kappa = \aleph_1$, due to H. Friedman [F], says that every stationary subset of $\aleph_1$ is fat.

Given a club $C \subseteq \kappa$ the proof of the lemma is by induction on τ. S being stationary, the case τ is successor is obvious. So assume τ is a limit ordinal and the lemma is true for ordinals below τ. We can easily find a club $D \subseteq \kappa$ such that for any $\alpha \in D$, $\beta < \alpha$, and $\zeta < \tau$, there is a closed subset of $S \cap C$ of order-type $\zeta + 1$, contained in the open interval (β, α). Put $\tau = \sum_{i<\mu'} \zeta_i$, where $\mu' \leq \mu$ and $\zeta_i < \tau$. Then find a closed subset E of $S \cap D \cap C$ of order-type $\mu' + 1$. In the i interval of E pick a closed subset of $S \cap C$ of order-type $\zeta_i + 1$. Putting everything together (including E) we get the desired closed subset of $S \cap C$ of order-type $\tau + 1$. □

A club subset of κ is surely fat. A fat $S \subseteq \kappa$ is said to be *nontrivial* iff $\kappa - S$ is stationary. In many cases nontrivial fat sets are easily obtained. For example, if $\kappa = \lambda^+$ and λ is regular, or if κ is Mahlo. For $\kappa = \lambda^+$ we use the theorem that any stationary set can be decomposed into two disjoint stationary sets. (See [J1].)

Clearly, if $S \subseteq \kappa$ contains a club set in some extension of the universe which does not add new bounded subsets of κ then S is fat in the ground universe. So fat sets are the only possible candidates for acquiring a club subset, if we make the requirement that no new bounded subsets of κ are added in the extension. The following theorem shows that in many cases fatness is all that is needed to obtain a club subset.

THEOREM 1. *Let κ be either a strongly inaccessible cardinal or the successor of a (regular) cardinal μ such that $\mu = \mu^{\mu}$. Let $S \subseteq \kappa$ be fat. Then there exists a poset P such that the following hold.*

(i) *Forcing with P adds a club $C \subseteq S$.*

(ii) *Forcing with P does not add new sets of size $< \kappa$ (hence cardinals and cofinalities $\leq \kappa$ remain unchanged in an extension by P).*

(iii) *Cardinality of P is 2^{κ}, so if $2^{\kappa} = \kappa$ cardinals above κ are not collapsed.*

PROOF. Given a fat $S \subseteq \kappa$, define the following poset P. $p \in P$ iff $p \subseteq S$ is a bounded and closed set of ordinals. P is partially ordered by end-extensions: $p \leq p'$ iff $p = p' \cap (\sup(p) + 1)$. (Note that if $p \in P$ then $\sup(p) = \bigcup p \in p$ as p is closed.) It is clear that 2^{κ} is the cardinality of P and that if $\dot{P}$ is a V-generic filter over P then $C = \bigcup\{p \mid p \in \dot{P}\}$ is a club subset of S. We have only to prove that no new sets of cardinality $<\kappa$ appear in $V[\dot{P}]$. In other words, given a regular cardinal $\tau < k$ and a sequence $\bar{D} = \langle D_i \mid i \in \tau\rangle$ of dense open subsets of P, it has to be shown that $\bigcap_{i<\tau} D_i$ is dense in P. Well, let $p \in P$ be given; we wish to find an extension of p in this intersection.

Let λ be big enough so that $H(\lambda)$ (the collection of all sets of cardinality hereditarily less than λ) contains P. Let $M = \langle H(\lambda), \in\rangle$. Define a sequence $\langle M_{\alpha} \mid \alpha < \kappa\rangle$ of elementary substructures of M such that:

(i) $P, p, \bar{D} \in M_0$, and some fixed well-order of $|P|$—the universe of P—is in M_0. Also $\tau + 1 \subseteq M_0$.

(ii) M_{α} is of cardinality $<\kappa$. If $\alpha < \beta$ then $M_{\alpha} \subset M_{\beta}$ and for limit δ, $M_{\delta} = \bigcup_{\eta<\delta} M_{\eta}$.

(iii) $c_{\alpha} = M_{\alpha} \cap \kappa$ (the intersection of the universe of M_{α} with κ) is an ordinal and $\langle c_{\alpha} \mid \alpha < \kappa\rangle$ is a continuous and increasing sequence cofinal in κ.

The M_{α} are easily defined. As $\beta^{\beta} < \kappa$ was assumed for all $\beta < \kappa$, we get that for $\beta < \alpha < \kappa$, M_{α} contains each subset of β of cardinality $< |\beta|$.

Now, $E = \{\alpha \mid \alpha = c_{\alpha}\}$ is a club subset of κ. S is fat; hence $S \cap E$ contains a closed subset of order-type $\tau + 1$, which we call A. Let $\alpha = \sup(A)$; then, even if $A \notin M_{\alpha}$, $A \cap \xi \in M_{\alpha}$ for each $\xi < \alpha$. Now we construct in M_{α} an increasing sequence in P of length τ, $\langle p_i \mid i < \tau\rangle$, such that $p_{i+1} \in D_i \cap M_{\alpha}$. Begin with $p_0 = p$. If $p_i \in P \cap M_{\alpha}$ is defined then p_{i+1} is the first member of D_i (in the fixed well-order of $|P|$) extending p_i, such that the ordinal interval $(\sup(p_i), \sup(p_{i+1}))$ has a nonempty intersection with A. For limit $\delta < \tau$, put simply

$$p_{\delta} = \bigcup_{i<\delta} p_i \cup \left\{\sup\left(\bigcup_{i<\delta} p_i\right)\right\}.$$

As only a proper initial segment of A is used in the definition of p_{δ}, one can conclude that $p_{\delta} \in M_{\alpha}$, and $p_{\delta} \subseteq S$ follows from the fact that $A \subseteq S$ is closed. Finally, $p_{\tau} = \bigcup_{i<\tau} p_i \cup \{\alpha\}$ is in $\bigcap_{i<\tau} D_i$ as required. □

1.3. When one tries to apply the proof of Theorem 1 to the case $\kappa = \mu^{+}$ and μ is a *singular* cardinal, difficulties arise even if GCH is assumed. For example, in the case $S \subseteq \aleph_\omega^+$ is a fat set: if we take structures M_α (as in the proof of Theorem 1) of cardinality $\aleph_\omega$ then we cannot demand that the M_α's should be closed under unions of countably many members of P (in case $\mathrm{cf}(\alpha) > \aleph_1$), because $\aleph_\omega^{\aleph_0} > \aleph_\omega$. The same problem occurs when $\tau^\xi \geq \kappa$ for some $\tau < \kappa$, even if κ is a successor of a regular cardinal or is a regular limit cardinal.

For the case $\kappa = \mu^+$, μ singular strong limit, there is a satisfactory answer that we exemplify with $\kappa = \aleph_\omega^+$.

THEOREM 2. *Assume $\aleph_\omega$ is a strong limit and $S \subseteq \aleph_\omega^+$ is fat. There is a forcing poset P which adds a club subset to S without adding new subsets of size $\leq \aleph_\omega$. The cardinality of P is $2^{\aleph_\omega}$.*

PROOF. Let S be a given fat subset of $\aleph_\omega^+$. Assume, w.l.o.g., that $S \cap \aleph_\omega = \varnothing$. P is defined, just like in the proof of Theorem 1, as the set of all bounded closed subsets of S. The question is why no new sets of size $\leq \aleph_\omega$ are added by forcing with P. It is enough to show that for every $n < \omega$ the intersection of $\aleph_n$ many dense open subsets of P is dense. So let dense open sets $\langle D_i \mid i < \aleph_n \rangle$ and $p \in P$ be given, we will find an extension of p in $\bigcap_{i<\aleph_n} D_i$. Fix a function $F(x, y)$ such that for $\aleph_\omega \leq \alpha < \aleph_\omega^+$, $\beta \mapsto F(\alpha, \beta)$, $\beta < \alpha$, is a one-to-one function of α onto $\aleph_\omega$. Now pick a sequence $\langle M_\alpha | \alpha < \aleph_\omega^+ \rangle$ of structures of cardinality $\aleph_\omega$, like in Theorem 1, requiring also that $F \in M_0$ and $\aleph_\omega \subset M_0$. Let $c_\alpha = M_\alpha \cap \aleph_\omega^+$ and $C = \{\alpha | \alpha = c_\alpha\}$ is a club set. Say $2^{\aleph_{n-1}} = \lambda < \aleph_\omega$. Use the fact that S is fat and obtain a closed $B \subset S \cap C$ of order-type λ^+ (B is closed in $\sup(B)$ but, somewhat inconsistently, we don't need $\sup(B) \in B$). Define a function $h: [B]^2 \to \omega$ as follows: for $a, b \in B$, $a < b$, let $h(a, b) = k$ iff k is the least integer such that $F(b, a) \in \aleph_k$. Using the partition relation $(2^{\aleph_{n-1}})^+ \to (\aleph_n)^2_{\aleph_{n-1}}$ (see [W]), find $A \subseteq B$ of order type $\aleph_n$ which is homogeneous, say for the color k. Put $\alpha = \sup(A)$. We construct now an increasing sequence $\langle p_i | i < \aleph_n \rangle$, just like in Theorem 1. As every ordinal which is limit of ordinals in A is in S, $p_\delta \in P$ for limit $\delta < \aleph_n$. Why $p_\delta \in M_\alpha$? Because every bounded subset X of A is in M_α, as we show now. Pick $b \in A$ bigger than all members of X; then $F(b, x) < \aleph_k$ for $x \in X$. But the set $\{F(b, x) | x \in X, x < b\}$, as any other subset of $\aleph_k$, is a member of M_α. Hence $X \in M_\alpha$. The proof ends just like that of Theorem 1. □

1.4. Now, in case GCH is not assumed, very little is known about forcing notions which introduce a club subset to fat $S \subseteq \kappa$ without adding new sets of cardinality $< \kappa$. In case $\kappa = \aleph_1$, [B, H, K] gives a positive answer to that question; but for $\kappa = \aleph_2$ even a simpler question is unanswered.

PROBLEM 1. Let $S \subseteq S^{\aleph_2}_{\aleph_1}$ be stationary. Is there a forcing notion which adds no new sets of cardinality $\aleph_1$ and adds a club $C \subseteq S \cup S^{\aleph_2}_{\aleph_0}$?

A positive answer to this problem follows from the existence of $\square_{\omega_1}$. In fact, Jensen's weak square sequence $\square^*_\mu$ (see 5.1 in [Jen]) is sufficient assumption to get the conclusion of Theorem 1 for fat $S \subseteq \mu^+$, even in case μ is singular.

§2. In the previous section we did not care about cardinals above κ; if the GCH is assumed then the posets described in §1 do not collapse cardinals. But if $2^{\aleph_0} > \aleph_1$ then the poset of all bounded closed subsets of a stationary $S \subseteq \aleph_1$ does collapse $\aleph_2$. In an earlier version of this paper we asked the following question: Let $S \subseteq \aleph_1$ be a stationary set. Is there a forcing notion that adds a club subset to

S, does not add reals and does not collapse any cardinals? A negative answer was provided by Todorčević [T]: assuming the consistency of ZFC + there is an inaccessible cardinal, he provides a model in which any poset which adds a new subset of ω_1 collapses $\aleph_1$ or $\aleph_2$.

PROBLEM 2. To what extent is the inaccessible necessary in the above? (compare [D, K].)

Concerning this problem, Abraham has shown that an inaccessible is not needed to get the consistency of: Every poset of cardinality $\aleph_1$ which adds a new subset to ω_1 collapses $\aleph_1$. (A subset of ω_1 is *new* if its intersection with every $\alpha < \omega_1$ is in the ground model.)

If we drop the requirement that no new countable sets are added, then, even if CH does not hold, it is possible to introduce a club subset to a stationary without collapsing cardinals.

THEOREM 3. *Let $S \subseteq \aleph_1$ be stationary. There is a poset P such that forcing with P adds a club subset to S, does not collapse $\aleph_1$, and P is of cardinality $\aleph_1$ (hence no cardinals are collapsed).*

PROOF. Define $p \in P$ iff p is a finite collection of closed intervals in $\aleph_1$ such that: (1) $[\alpha, \beta] \in p \Rightarrow \alpha \in S$, (2) if $[\alpha, \beta], [\alpha', \beta'] \in P$ then either $\alpha = \alpha'$ or $[\alpha, \beta] \cap [\alpha', \beta'] = \emptyset$. The intuitive meaning of $[\alpha, \beta] \in p$ is that α is a member of the generic club subset of S and that the closed interval $[\alpha, \beta]$ contains no other members of that club. So the partial order on P is simply inclusion: p' extends p iff $p \subseteq p'$.

If $\dot{P}$ is a V-generic filter over P, define $C = \{\alpha \mid \text{for some } \beta \text{ and } p \in \dot{P}, [\alpha, \beta] \in p\}$.

2.1 LEMMA. *$C \subseteq S$ is a club set.*

PROOF. We check only that C is closed. Suppose $p \Vdash$ "ζ is a limit point of C". If $[\zeta, \beta] \in p$ for some β ("ζ appears in p"), then $p \Vdash$ "$\zeta \in C$". If ζ does not appear in p, let α be the maximal ordinal appearing in P below ζ (or any ordinal in S below ζ if no ordinal below ζ appears in p). Let $p' = p \cup \{[\alpha, \zeta]\}$. Then $p' \in P$ extends p and $p' \Vdash$ "ζ is not a limit point of C, in fact no point of C is in the interval $(\alpha, \zeta]$". Contradiction. □

2.2 LEMMA. *Forcing with P does not collapse $\aleph_1$.*

PROOF. Assume $p \Vdash$ "$f: \aleph_0 \to \aleph_1$". We want an extension of p which forces f to be bounded. Let $N \prec H(\aleph_2)$ be a countable elementary substructure of the set of all sets of cardinality hereditarily $< \aleph_2$, such that $P, p, f \in N$, and such that $N \cap \omega_1 = \alpha \in S$. Let $p' = p \cup \{[\alpha, \alpha + 1]\}, p' \in P$.

CLAIM. $p' \Vdash$ "Range$(f) \subseteq \alpha$".

PROOF OF CLAIM. Suppose p'' extends p' and $p'' \Vdash$ "$f(n) = \eta$". If $[a, b] \in p''$ then either $b < \alpha$ or $a \geq \alpha$. Look at p^*, which consists of all pairs in p'' which are below α; then $p^* \in N$. Find $p^{**} \in N$ extending p^* such that $p^{**} \Vdash$ "$f(n) = \eta^*$" for some η^* (which is necessarily $< \alpha$). As all pairs of p^{**} are below α, $p'' \cup p^{**} \in P$. Hence $\eta = \eta^*$, so $p'' \Vdash$ "$f(n) < \alpha$". □

This theorem and proof can be easily generalized to the case $\kappa = \mu^+$, $\mu^\mu = \mu$ (but 2^μ is not restricted) and $S \subseteq S^\kappa_\mu$ and we wish to force a club subset to $S \cup S^\kappa_{<\mu}$ without adding sets of size $<\mu$ and without collapsing cardinals. But in the case $S \subseteq \kappa$ is an *arbitrary* fat set we need a different method. See [A1] for other applications of this method.

THEOREM 4. *Suppose $\kappa = \mu^+$, $\mu^\mu = \mu$, and $S \subseteq \kappa$ is fat. There is a poset such that* *forcing with it introduces a club subset of S, does not collapse any cardinals and does not add new sets of size* $<\mu$.

PROOF. *Again, the point is that* $2^\mu > \mu^+ = \kappa$ *and we do care about cardinals above* κ (*otherwise Theorem 1 can be applied*). The idea of the proof is to take only a limited amount of conditions so that the κ^+-antichain condition is satisfied; yet to make them rich enough so that no sets of size $<\mu$ are introduced. For this, we have to make a preparatory extension first, with the Cohen poset Q. Details follow.

Pick $A \subseteq \kappa$ such that in $L[A]$ κ is the successor of μ and any subset of κ of size $<\mu$ belongs to $L[A]$. The assumption that S is fat means that for every club $C \subseteq \kappa$ there is a closed $D \subseteq S \cap C$ of order-type $\mu + 1$. There might be 2^μ such sets D, but we make now a stronger assumption, (*), about S and A and later on show how to obtain this stronger assumption.

(*) For every club $C \subseteq \kappa$ there is a closed $D \subseteq S \cap C$ of order-type $\mu + 1$ such that $D \in L[A]$.

First we cultivate the ground and define a poset Q. $f \in Q$ iff f is a partial function on κ of cardinality $<\mu$ such that, for $\alpha \in \mathrm{Dom}(f)$, $f(\alpha) \in {}^{\underline{\mu}}\mu$. In other words, Q adds κ many functions from μ to μ with conditions of size $<\mu$. The ordering of Q is defined by: $f \leq f'$ iff $\alpha \in \mathrm{Dom}(f) \rightarrow \alpha \in \mathrm{Dom}(f')$ and $f(\alpha) \subseteq f'(\alpha)$. Q is μ closed, satisfies the κ-antichain condition (as $\mu^{\underline{\mu}} = \mu$), and $Q \in L[A]$.

Let $\dot{Q}$ be a V-generic filter over Q. $V[\dot{Q}]$ does not collapse cardinals or change cofinalities and does not add new sets of size $<\mu$. Let $W = L[A, S, \dot{Q}]$; observe that $\mu^{\underline{\mu}} = \mu$ and $2^\mu = \kappa$ hold in W.

Next, define P in W as the poset of all bounded closed subsets of S, partially ordered by end-extension. As $|P| = \kappa$, if we force over $V[\dot{Q}]$ with P, cardinals above κ are not collapsed. It is also clear that $Q*P$ (the iteration poset of Q followed by P) introduces a club subset to S. To show that no cardinals $\leq\kappa$ are collapsed and at the same time to prove that no new sets of size $<\mu$ are added by $Q*P$ it is enough to establish the following.

2.3. *If* $\dot{P}$ *is* $V[\dot{Q}]$*-generic over* P, *then any set of ordinals of size* μ *which belongs to* $V[\dot{Q}][\dot{P}]$ *is in fact in* $V[\dot{Q}]$.

PROOF OF 2.3. Let t be a name in $V[\dot{Q}]^P$ and $p \in P$ such that $p \Vdash^P$ "$t\colon \mu \rightarrow$ ordinals", in $V[\dot{Q}]$. We seek $p^* \in P$ extending p which decides all values of t. We work in $V[\dot{Q}]$. Pick a cardinal λ such that $\lambda > 2^\kappa$ and $t \in H(\lambda)$. Define an increasing and continuous sequence M_i, $i < \kappa$, of elementary substructures of $H(\lambda)$ of cardinality μ such that

1. $\mu + 1 \subseteq M_0$, and $P, p, t, \dot{Q} \in M_0$.
2. If we denote $\alpha_i = M_i \cap \kappa$ then $C = \{\alpha_i | i < \kappa\}$ is a club subset of κ.
3. M_{i+1} contains all subsets of M_i of cardinality $<\mu$.

Let $C' = \{\xi | \xi = \alpha_\xi\}$; then C' is a club set. Q satisfies the κ-a.c., so every club subset of κ in $V[\dot{Q}]$ contains a club set which is in V. So does C', and since (*) holds in V, there is $D \in L[A]$ such that $D \subseteq S \cap C'$ is closed and of order-type $\mu + 1$. Let β be the last member of D. Let $\pi\colon M_\beta \rightarrow M$ be the Mostowski collapse of M_β onto a transitive structure M. It is easy to check the following facts: $\pi(\kappa) = \beta$, $\pi(A) = A \cap \beta$, $\pi(S) = S \cap \beta$, $\pi(Q) = Q \restriction \beta = \{f \restriction \beta \,|\, f \in Q\}$, $\pi(\dot{Q}) = \dot{Q} \restriction \beta = \dot{Q} \cap (Q \restriction \beta)$, $\pi(P) = P \cap M_\beta = P'$, and $P' \in L[\pi(A), \pi(S), \pi(\dot{Q})]$ (because P' is defined in M as a member of $L_{M \cap \mathrm{Ord}}[\pi(A), \pi(S), \pi(\dot{Q})]$, just like P was defined in $L[A, S, \dot{Q}]$). Pick $\gamma \in \kappa$ such that $P' \in L[A, S, \dot{Q} \upharpoonright \gamma]$, and $M \in V[\dot{Q} \upharpoonright \gamma]$. P' is of cardinality μ in $L[A, S, \dot{Q} \upharpoonright \gamma]$ (as M is of such cardinality and $\kappa = \mu^+$ in $L[A]$ too). Let $h: \mu \to P'$ be a one-to-one correspondence there. Let $g: \mu \to \mu$ be the γ-generic function in Q, i.e., $g = \bigcup\{f(\gamma) | f \in \dot{Q}\}$. Then g is the $V[\dot{Q} \upharpoonright \gamma]$-generic function over the poset R of all functions from an ordinal $< \mu$ into μ.

Always in $V[\dot{Q}]$, define by induction on $\nu < \mu$ an increasing sequence $p_\nu \in P'$ such that the following hold.

1. $p_0 = p$ is the condition we want to extend.
2. If $\delta < \mu$ is limit then

$$p_\delta = \bigcup_{i<\delta} p_i \cup \left\{\sup\left(\bigcup_{i<\delta} p_i\right)\right\}.$$

3. Given $\nu < \mu$, if (i) $p_\nu \leq h(g(\nu)) = q$, and (ii) some member of D is in the interval $(\sup(p_\nu), \sup(q))$, then $p_{\nu+1} = q$. If those demands do not hold, then $p_{\nu+1} = p_\nu$.

Let us check that it is possible to construct such a sequence. By induction on $\nu < \mu$ we shall prove $p_\nu \in P'$. If ν is a limit ordinal then the sequence $\langle p_i | i < \nu\rangle$ is in $M \cap W$, since each $p_i \in M \cap W$, $i < \nu$. (Any subset of M of cardinality $< \mu$ is in M. Also W contains all bounded subsets of μ and P' has cardinality μ in W.) So $p_\nu \in M \cap W$, where

$$p_\nu = \bigcup_{i<\nu} p_i \cup \left\{\sup\left(\bigcup_{i<\nu} p_i\right)\right\}.$$

Moreover, p_ν is a closed subset of S (by (3) the interval $(\sup(p_\zeta), \sup(p_{\zeta+1}))$ contains a member of D, if nonempty). Hence $p_\nu \in P$ as $p_\nu \in W$. Even $p_\nu \in P'$ because $p_\nu \in M$. Now the case ν is successor is obvious.

Finally set $p^* = \bigcup_{\nu<\mu} p_\nu \cup \{\beta\}$. The sequence $\langle p_\nu | \nu \in \mu\rangle$ is definable in $L[A, S, \dot{Q}|\gamma + 1]$ using g, h, D, p as parameters. (We did not use t in the definition of the sequence!) Hence $p^* \in W$. Also p^* is a closed subset of S, so $p^* \in P$. Why does p^* decide all values of t? The answer is a density argument for R forcing in $V[\dot{Q}|\gamma]$ ($R = {}^{\mu}\mu$).

CLAIM. *For every $\alpha < \mu$ the following subset of R defined in $V[\dot{Q}|\gamma]$ is dense in R.*

$$\{f \in R | f \Vdash^R \text{“}p^* \textit{ extends some } p' \in P' \textit{ such that, in } M, p' \textit{ decides the value of } \pi(t)(\alpha)\text{”}\}.$$

PROOF OF THE CLAIM. Given $f \in R$ let $\mathrm{Dom}(f) = \nu < \mu$. There exists $p \in P'$ such that $f \Vdash$ “$p_\nu = p$”. (Because p_ν depends only on the first ν values of the generic function.) Find $p' \geq p$ in P' such that $D \cap (\sup(p), \sup(p')) \neq \emptyset$ and such that, in M, p' decides the value of $\pi(t)(\alpha)$ (this is a dense set in P'). $p' = h(\xi)$ for some $\xi < \mu$. Define f' extending f by setting $f'(\nu) = \xi$. Then $f' \Vdash^R$ “$h(g(\nu)) = p'$, and hence $p' = p_{\nu+1}$”.

Now that the claim is proved, observe that if in M p' decides the value of $\pi(f)(\alpha)$ then, by elementarity, $\pi^{-1}(p') = p'$ decides the value of f at $\pi^{-1}(\alpha) = \alpha$ in $V[\dot{Q}]$. Hence p^* decides $t(\alpha)$ for all $\alpha < \mu$. This proves 2.3—but not yet Theorem 4, because we have to show why the special assumption (∗) can be made.

The following poset was defined by Jensen and called “the club set forcing” in [D, J].

2.4 DEFINITION. $Z = \{\langle \nu, A\rangle \mid A \subseteq \mu$ is club and $\nu < \mu\}$, partially-ordered as follows: $\langle \nu, A\rangle \leq \langle \nu', A'\rangle$ iff $\nu \leq \nu'$ and $A' \subseteq A$ and $\nu \cap A = \nu \cap A'$.

As the intersection of $<\mu$ many club subsets of μ is club, Z is clearly μ closed. Also, because $\mu^\mu = \mu$, Z satisfies the κ-a.c. Let $\dot{Z}$ be a V-generic filter over Z and set $E = \bigcup\{\nu \cap A \mid \langle \nu, A\rangle \in \dot{Z}\}$. $E \subseteq \mu$ is club, and for every $C \in V$ club in μ there is $\beta < \mu$ such that $E - \beta \subseteq C$. Also $V[E] = V[\dot{Z}]$. See [D, J] for all of this.

2.5 LEMMA. *Let $S \in V$ be a fat subset of $\mu^+ = \kappa$; let $A \in V$, $A \subseteq \kappa$, be such that $(\mu^+)^{L[A]} = \kappa$; and let E be as above—a generic club set. Then for every club $C \subseteq \kappa$ in $V[E]$ there is $D \subseteq S \cap C$, closed of order-type $\mu + 1$, such that $D \in L[A, E]$.*

PROOF. First, it is clear why this lemma permits us to assume (*). Now, every club subset of κ in $V[E]$ contains a club set in V, because Z satisfies the κ-a.c. So it can be assumed that $C \in V$. As S is fat, there is $D' \subseteq S \cap C$ in V, closed and of order-type $\mu + 1$. Say $\alpha = \sup(D')$; then $\mathrm{cf}(\alpha) = \mu$ in $L[A]$. Let f: $\mu \to \alpha$ be an increasing continuous and cofinal function such that $f \in L[A]$. Put $B = \{\xi \in \mu \mid f(\xi) \in D'\}$. Then $B \subseteq \mu$ is club and $B \in V$. Hence for some $\beta < \mu$, $E - \beta \subseteq B$. So $\{f(\xi) \mid \xi \in E - \beta\} \cup \{\alpha\} = D \in L[A, E]$ is as required. □

2.6. Now that we have dropped the requirement that no new bounded subsets of κ appear in the extension, it is conceivable that a stationary subset of κ acquires a club subset even if it is not fat. We do not know of any characterization of those stationary sets which contain a club set in some extension. Let us only show that such a phenomenon is possible. We deal with the case $\kappa = \aleph_2$. First, we generalize the club set forcing 2.4.

2.7 LEMMA. *Assume $2^{\aleph_0} = \aleph_1$ and let D be a normal filter over ω_1. There exists a poset P, satisfying the $\aleph_2$-a.c., such that forcing with P adds no new countable sets and does introduce a club $C \subseteq \omega_1$ with the property that for any $E \in D$ there is $\gamma \in \omega_1$ such that $C - \gamma \subseteq E$.*

PROOF. Define $P = \{\langle a, E\rangle \mid E \in D, a \subseteq \omega_1$ is closed and countable$\}$. P is partially ordered by: $\langle a, E\rangle \leq \langle a', E'\rangle$ iff $E' \subseteq E$, $a = a' \cap (\sup(a) + 1)$ and $a' - (\sup(a) + 1) \subseteq E$. (Remark that $\sup(a) \in a$, as a is closed.) The meaning of a condition $\langle a, E\rangle$ is that a is an initial segment of the generic club set C and $C - (\sup(a) + 1) \subseteq E$.

It is obvious that $\langle a, E \cap E'\rangle$ lies above $\langle a, E\rangle$ and above $\langle a, E'\rangle$; hence P satisfies the $\aleph_2$-a.c. It is also clear that in a generic extension a club C as required is readily obtained. Why are no new countable sets added? In case D is the filter of club sets, P is countably closed; but in general we need a different argument.

Let $\bar{H} = \langle H_n \mid n < \omega\rangle$ be a sequence of dense open sets of P and $p \in P$. We shall find an extension of p which is in $\bigcap_{n<\omega} H_n$. Take λ so that $D, P, \bar{H} \in H(\lambda)$. Construct an increasing and continuous sequence of countable elementary substructures $M_\alpha \prec H(\lambda)$, $\alpha < \omega_1$, such that $D, P, \bar{H}, p \in M_0$. Of course $\{M_\alpha \cap \omega_1 \mid \alpha < \omega_1\}$ is a club subset of ω_1.

SUBLEMMA. *There exists $\alpha < \omega_1$ such that $M_\alpha \cap \omega_1 = \alpha$ and $\alpha \in \bigcap\{E \mid E \in M_\alpha \cap D\}$.*

The proof of the sublemma clearly follows from the normality of D and the continuity of the sequence of the M_α (which means that $M_\delta = \bigcup_{i<\delta} M_i$ for limit δ).

Now pick α as in the sublemma. Let $\langle \xi_n \mid n \in \omega\rangle$ be an increasing sequence cofinal in α. Define inductively $p_n \in P \cap M_\alpha$ such that (i)–(iii) hold:

(i) $p_0 = p$,
(ii) $p_n \in H_{n-1}$ for $n \geq 1$,
(iii) $p_n = \langle a_n, E_n \rangle$ satisfies $\sup(a_n) \geq \xi_n$.

Finally define $p^* = \langle a^*, E^* \rangle$ by $a^* = \bigcup_{n<\omega} a_n \cup \{\alpha\}$, $E^* = \bigcap_{n<\omega} E_n$. Then $p^* \in P$, $p^* \geq p_n$ for all n, and $p^* \in \bigcap_{n<\omega} H_n$. □

2.8 Definition. Given $S \subseteq \omega_2$, we say that *the initial segments of S form a normal filter over ω_1* iff $S^1 = S \cap S^{\aleph_2}_{\aleph_1}$ is stationary and for every $\alpha \in S^1$ there exists a club $C_\alpha \subseteq \alpha$ of order-type ω_1 such that if $f_\alpha\colon \omega_1 \to C_\alpha$ is order-preserving onto C_α, then the collection $\{f_\alpha^{-1}(C_\alpha \cap S) \mid \alpha \in S^1\}$ generates a normal filter over ω_1 (i.e. this collection is included in a nontrivial normal filter).

We shall see later on (2.9) that such an S need not be fat. But the converse is true, in the sense that if S is fat, then for some $S^1 \subseteq S$ the initial segments of S^1 form a normal filter over ω_1: the club sets filter.

Theorem 5. *Assume CH and let $S \subseteq \omega_2$ be such that the initial segments of S form a normal filter over ω_1. There is a cardinal preserving generic extension which adds no new countable sets in which S contains a club set.*

Proof. It is enough to find an extension which adds no new countable sets, collapses no cardinals, and in which S is fat; for then we can use Theorem 1. We let D be the normal filter over $\aleph_1$ generated by the initial segments of S. Introduce a generic club $C \subseteq \omega_1$ which is almost included in each set of D (Lemma 2.7). Observe that S^1, as well as any other stationary subset of ω_2 in the ground model, remains stationary in the extension $V[C]$: P of Lemma 2.7 satisfies the $\aleph_2$-a.c. It is not difficult to see that S is fat in $V[C]$.

2.9. We still have to show that the initial segments of a nonfat $S \subseteq \omega_2$ may form a normal filter over ω_1. Such an example is found in a generic extension of a universe that satisfies the GCH. For each $\delta \in S^{\aleph_2}_{\aleph_1}$ pick $C_\delta \subseteq \delta$, club of order-type ω_1, and $f_\delta\colon \omega_1 \to C_\delta$, order preserving onto C_δ. Let $Z \subseteq \omega_1$ be some stationary co-stationary subset of ω_1. Denote $Z_\delta = f_\delta[Z]$. Then Z_δ is a stationary co-stationary subset of δ. Our aim is to find $T \subseteq S^{\aleph_2}_{\aleph_0}$ with the following property.[1]

2.10. *For every $\delta \in S^{\aleph_2}_{\aleph_1}$ there is an $\alpha < \delta$ such that $Z_\delta - \alpha = T \cap c_\delta - \alpha$.*

For if we find such T, then $S = S^{\aleph_2}_{\aleph_1} \cup T$ is a stationary nonfat subset of $\aleph_2$ whose initial segments form a normal filter—the filter generated by Z. To obtain T, define a poset P by $p \in P$ iff $p\colon S^{\aleph_2}_{\aleph_1} \to \aleph_2$ is a countable partial pressing-down function such that for every $\delta, \delta' \in \mathrm{Dom}(p)$ and for every $\gamma \in (C_{\delta'} - p(\delta')) \cap (C_\delta - p(\delta))$ we have $\gamma \in Z_{\delta'}$ iff $\gamma \in Z_\delta$. P is partially ordered by inclusion. The meaning of $p(\delta) = \alpha$ is that the generic T will satisfy $Z_\delta - \alpha = T \cap C_\delta - \alpha$. It is clear that for any $p \in P$ and $\delta \in S^{\aleph_2}_{\aleph_1}$ there is $p' \in P$ such that $p \subseteq p'$ with $\delta \in \mathrm{Dom}(p')$ (find $\alpha < \delta$ with the property that $C_{\delta'} \cap (\alpha, \delta) = \varnothing$ for any $\delta' \in \mathrm{Dom}(p)$; then set $p'(\delta) = \alpha$). Hence there are $\aleph_2$ dense sets such that if $\dot{P}$ is a filter generic with respect to those dense sets, then $g = \bigcup \dot{P}$ is a function defined on $S^{\aleph_2}_{\aleph_1}$, and then $T = \bigcup \{Z_\delta - g(\delta) \mid \delta \in S^{\aleph_2}_{\aleph_1}\}$ satisfies 2.10. So it remains only to check that P is countably closed and satisfies the $\aleph_2$-a.c., in order to conclude that 2.10 holds in a

[1] S. Todorčević has pointed out that an argument of M. Magidor can show the existence of such T, assuming $\square_{\omega_1}$.

generic extension which adds no reals and collapses no cardinals. This is not difficult, and we leave it to the reader.

§3. Unlike the case in [B, H, K] where *every* stationary $S \subseteq \aleph_1$ can acquire a club subset in some generic extension, a stationary subset of $P_{\aleph_1}(\aleph_2)$ may have a much stronger absolute character, as Theorems 6 and 7 show.

The following theorem is much generalized in [S2].

THEOREM 6. *Let W be a transitive sub-universe of settheory such that $\aleph_2^W = \aleph_2$. Then $S = P_{\aleph_1}(\aleph_2)^W$ is a stationary subset of $P_{\aleph_1}(\aleph_2)$ (in the real world).*

PROOF. Observe first that $\aleph_1^W = \aleph_1$. Let $C \subseteq P_{\aleph_1}(\aleph_2)$ be a club set. By Kueker's theorem, there is a function f defined on $P_{\aleph_0}(\aleph_1)$ with values in $P_{\aleph_1}(\aleph_2)$ such that if $x \subseteq \aleph_2$ is countable and closed under f, then $x \in C$. One can easily find $\alpha < \aleph_2$ such that α (as a set) is closed under f. If α is countable, then as $\alpha \in W$ and $\aleph_1^W = \aleph_1$ we get $\alpha \in S \cap C$. If α is uncountable, then there is, in W, a bijective $h: \aleph_1 \to \alpha$. Since α is closed under f we can find a $\xi \in \aleph_1$ such that $h[\xi]$ is closed under f. Then $h[\xi] \in S \cap C$. □

In particular, if $\aleph_2^L = \aleph_2$ then $S = P_{\aleph_1}(\aleph_2)^L$ is a stationary subset of $P_{\aleph_1}(\aleph_2)$. Can we strengthen this and get that S contains a club set? No, as the following theorem says.

THEOREM 7. *Let $V[r]$ be a generic extension of V, $r \subseteq \omega$, $r \notin V$, obtained with a c.a.c. poset. Then, in $V[r]$, $P_{\aleph_1}(\aleph_2) \cap V$ and $P_{\aleph_1}(\aleph_2) - V$ are both stationary.*

PROOF. In view of Theorem 6 we have to show only that $P_{\aleph_1}(\aleph_2) - V$ is stationary. Let $C \subseteq P_{\aleph_1}(\aleph_2)$ be a club set. By Kueker's theorem, there is $f': P_{\aleph_0}(\aleph_2) \to P_{\aleph_1}(\aleph_2)$ such that $\{x \in P_{\aleph_0}(\aleph_2) | x \text{ is closed under } f'\} \subseteq C$. Since $V[r]$ is obtained via a countable-antichain-poset-extension, one can find $f \in V$, $f: P_{\aleph_0}(\aleph_2) \to P_{\aleph_1}(\aleph_2)$, such that $f'(a) \subseteq f(a)$ for all finite $a \subseteq \aleph_2$. If $x \in P_{\aleph_1}(\aleph_2)$ is closed under f then $x \in C$. We want to find x, closed under f, such that $x \notin V$. To this end, we follow word by word the proof of Theorem 3.2 of Baumgartner [B, T]. Work for a while in V. For $A \subseteq \omega_2$ let $\mathrm{cl}(A)$ denote the closure of A under f. Now $Z = \{\alpha \in \omega_2 | \alpha \text{ is closed under } f\}$ is club in ω_2. For $\alpha \in Z \cap S^{\aleph_2}_{\aleph_0}$ let $\langle \xi^\alpha_n | n \in \omega \rangle$ be an increasing sequence cofinal in α. Put $A_\alpha = \mathrm{cl}(\{\xi^\alpha_n | n \in \omega\})$. Next, for each $s \in {}^{\omega}2$, we will define an ordinal $\xi_s \in \omega_2$ and a stationary set $Z_s \subseteq Z \cap S^{\aleph_2}_{\aleph_0}$ such that (1)—(3) below hold.

(1) $\forall \alpha \in Z_s \exists n \in \omega (\xi_s = \xi^\alpha_n)$,

(2) $Z_{s\langle 0 \rangle} \cup Z_{s\langle 1 \rangle} \subseteq Z_s$,

(3) $\forall \alpha \in Z_{s\langle 0 \rangle} \forall \beta \in Z_{s\langle 1 \rangle} (\xi_{s\langle 0 \rangle} \notin A_\beta \text{ and } \xi_{s\langle 1 \rangle} \notin A_\alpha)$.

Let us see how this ends the proof. Working in $V[r]$, let $A = \mathrm{cl}(\{\xi_{r \restriction n} | n \in \omega\})$; then $A \in C$. We claim:

$$A \notin V \text{ and moreover: } r(n) = 0 \text{ iff } \xi_{(r \restriction n)\langle 0 \rangle} \in A.$$

The proof of this claim is easy, and we proceed to the inductive definition of the ξ_s and Z_s. To begin with, look at the pressing down function $\alpha \mapsto \xi^\alpha_0$ defined on $Z \cap S^{\aleph_2}_{\aleph_0}$; by Fodor's theorem there is a fixed $\xi_\varnothing$ and a stationary $Z_\varnothing \supseteq Z$ with $\xi_\varnothing = \xi^\alpha_0$ for all $\alpha \in Z_\varnothing$. Now suppose ξ_s and Z_s are defined; we claim that

$$K = \{\xi | \text{for some } n \ \{\alpha \in Z_s | \xi^\alpha_n = \xi\} \text{ is stationary}\}$$

is unbounded in ω_2. Indeed, if the set of these ξ's is *bounded* in ω_2 then a use of Fodor's theorem gives a quick contradiction. So $|K| = \aleph_2$; let K' consist of the first ω_1 elements of K. For every $\alpha \in Z_s$, because A_α is countable, there is some $\xi \in K'$ such that $\xi \notin A_\alpha$; hence we can pick $\xi_{s\langle 1\rangle} \in K'$ such that $\xi_{s\langle 1\rangle} \notin A_\alpha$ for stationary many $\alpha \in Z_s$, and we collect those α's to form $Z'_{s\langle 0\rangle}$. As $\xi_{s\langle 1\rangle} \in K'$ there exist n and a stationary set $Z'_{s\langle 1\rangle} \subseteq Z^{1}_{s_1}$ such that $\xi_{s\langle 1\rangle} = \xi^\beta_n$ for all $\beta \in Z'_{s\langle 1\rangle}$. Finally by a similar argument we can find a stationary $Z_{s\langle 1\rangle} \subseteq Z'_{s\langle 1\rangle}$ and $\xi_{s\langle 0\rangle}$ such that $\xi_{s\langle 0\rangle} \notin A_\beta$ for all $\beta \in Z_{s\langle 1\rangle}$, yet for some n

$$\{\alpha \in Z'_{s\langle 0\rangle} \mid \xi_{s\langle 0\rangle} = \xi^\alpha_n\} \overset{\text{Def}}{=} Z_{s\langle 0\rangle} \text{ is stationary. } \square$$

PROBLEM 3. Assume that there exists a nonconstructible real. Does it follow that $P_{\aleph_1}(\aleph_2) - L$ is stationary?[2]

3.1. *Forcing club subsets to* $P_{\aleph_1}(\aleph_2)$. By Theorem 6, if V' is a cardinal preserving extension of V, then $P_{\aleph_1}(\aleph_2)^{V'} \cap V$ is stationary in $P_{\aleph_1}(\aleph_2)^{V'}$; hence any club subset of $P_{\aleph_1}(\aleph_2)$ in V is stationary in V' (apply Kueker's theorem). Is it true that any *stationary* set in V remains stationary in V'? Well, if V' is obtained as a generic extension via a c.a.c. poset, then the answer is yes. This is because in a c.a.c. poset extension, for every club set $C \subseteq P_{\aleph_1}(\aleph_2)$ in V', there is a function $f \in V$ such that if $X \in P_{\aleph_1}(\aleph_2)$ is closed under f then $X \in C$. To get a negative answer we need a stationary co-stationary $S \subseteq P_{\aleph_1}(\aleph_2)$ in V, and a generic extension V' with a club $C \subseteq P_{\aleph_1}(\aleph_2)$ (without collapsing cardinals), such that $C \cap V \subseteq S$. This generic extension can be easily obtained as follows. Assume $V = L$, and let $T \subseteq \aleph_2$ be fat and such that $S^{\aleph_2}_{\aleph_0} - T$ is stationary. (In L there is a stationary $R \subseteq S^{\aleph_2}_{\aleph_0}$ such that for any $\alpha \in S^{\aleph_2}_{\aleph_1}$, $R \cap \alpha$ is nonstationary [Jen]; put $T = \aleph_2 - R$. On the other hand, in a model of Magidor [Ma, Theorem 1], no such T exists.) With Theorem 1, find a generic extension V' which contains a club $E \subseteq T$, does not collapse cardinals and does not add new countable sets. Now, in V, $S = \{X \in P_{\aleph_1}(\aleph_2) \mid \sup(X) \in T\}$ is a stationary co-stationary subset of $P_{\aleph_1}(\aleph_2)$ (use Kueker's theorem to check this). Yet in V', $C = \{X \in P_{\aleph_1}(\aleph_2) \mid \sup(X) \in E\}$ is a club set and $C \subseteq S$.

However, we feel uncomfortable with this easy solution: it solves the *new* problem of forcing a club subset to a stationary $S \subseteq P_{\aleph_1}(\aleph_2)$ by recourse to the established method of shooting a club set to a stationary subset of ω_2. The "real" problem seems to be to do it *without* adding new club subsets of $\aleph_2$. To be concrete, let us require that the poset giving the extension satisfies the $\aleph_2$-a.c. (and then any club subset of $\aleph_2$ in the extension contains an old club set). We give two examples where this can occur: Theorems 8 and 9.

3.2. Start with L. Let $P = \{p \mid p \text{ is a finite function from } \omega_1 \times \omega \text{ to } \omega\}$ be the poset for adding $\aleph_1$ many Cohen reals. Force with P and let r_i, $i \in \omega_1$, be the ith Cohen generic-real; put $V = L[(r_i \mid i \in \omega_1)]$. Let $T = P_{\aleph_1}(\aleph_2)^{L[r_0]} - L$, and $S = P_{\aleph_1}(\aleph_2)^V - T$. Now, T is stationary in $L[r_0]$, by Theorem 7; hence T is stationary in V (as V is obtained by a c.a.c. extension of $L[r_0]$). S is also stationary, since $S \supseteq P_{\aleph_1}(\aleph_2) \cap L$ and $P_{\aleph_1}(\aleph_2) \cap L$ is stationary by Theorem 6. It is obvious that if $X \in T$ and $Y \triangle X$ (the symmetric difference) is finite, then $Y \in T$ and X is infinite.

THEOREM 8. *In V there is an $\aleph_2$-a.c. poset Q which is (ω, ∞)-distributive (forcing*

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[2] Answered by M. Gitik—yes.

with Q adds no new countable sets) such that in any generic extension with Q there is a club subset of S.

PROOF.[3] Our aim is to force a function F: $[\aleph_2]^2 \to \aleph_2$ such that *any countable subset of $\aleph_2$, closed under F, is in S.*

3.3 DEFINITION. Define Q to be the set of all f such that:

(1) f is a countable two-place function, $\mathrm{Dom}(f) = D \subseteq \omega_2$ and f: $[D]^2 \to D$.

(2) For every $\alpha, \beta \in D$, $\alpha < \beta \Rightarrow \alpha \leq f(\alpha, \beta) \leq \beta$. (Call a function satisfying (1) and (2) a *middle function.*)

(3) $D \in L$ (but not necessarily $f \in L$).

(4) Whenever $E \subseteq D$ and $E \in T$, E is not closed under f.

Q is partially ordered by inclusion.

Remark that if $f \in Q$, $E \subseteq \aleph_2$ and $E \cap \mathrm{Dom}(f) \in T$, then E is not closed under f (because f: $[D]^2 \to D$, so E is not closed under f if $E \cap \mathrm{Dom}(f)$ is not closed under f). The *extension property*: if $f \in Q$ and $\alpha \in \omega_2$ then $\alpha \in \mathrm{Dom}(f')$ for some $f \subseteq f' \in Q$, is easily verified. Another easy property is the following: if f is a middle function, $D = \mathrm{Dom}(f)$, then, for every $\alpha \in \omega_2$: $f \in Q$ iff $f \restriction \alpha \in Q$ *and* $f \restriction (\omega_2 - \alpha) \in Q$. (Use the fact that if $E \in T$ then either $E \cap \alpha \in T$ or $E \cap (\omega_2 - \alpha) \in T$.) A standard application of the above is to use a $\triangle$-system argument and to prove that Q satisfies the $\aleph_2$-a.c. (even Q is $\aleph_2$-centered). The following is the principal lemma showing the (ω, ∞) distributivity.

3.4 LEMMA. *Let $\{D_n | n \in \omega\}$ be a collection of dense open subsets of Q. Then $\bigcap_{n \in \omega} D_n$ is dense open in Q.*

PROOF. Given $q_0 \in Q$ we have to find $q \geq q_0$, $q \in \bigcap_{n\in\omega} D_n$. $H(\aleph_3)$, the collection of all sets whose transitive closure is of cardinality $\leq \aleph_2$, is a model of ZF^-. Members of $H(\aleph_3)$ are T, $\langle r_i | i \in \omega_1\rangle$, Q, $\langle D_n | n \in \omega\rangle$ and q_0. Let $M \prec H(\aleph_3)$ be a countable elementary substructure of $H(\aleph_3)$ such that $\langle r_i | i \in \omega_1\rangle$, Q, $\langle D_n | n \in \omega\rangle$, $q_0 \in M$ and $M \cap \aleph_2 \in L$. (This is possible by Theorem 6; since there are countably many functions (Skolem functions) such that if $X \in P_{\aleph_1}(\aleph_2)$ is closed under these functions, then $X = M \cap \aleph_2$ for some substructure M as above.) Let π: $M \to \bar{M}$ be the Mostowski isomorphism, collapsing M onto a transitive $\bar{M}$. A well-known argument, which uses the transitivity of $\bar{M}$ and the c.a.c. of P, shows that there exists $\gamma < \omega_1$ $(\gamma > 1)$ with $\bar{M} \in L[\langle r_i | i \in \gamma\rangle] = W$. Of course, $\bar{M}$ is countable in W, so $\pi(Q) \in \bar{M}$ is countable too. Let j: $\aleph_0 \to \pi(Q)$, $j \in W$, be a bijection. r_γ, the γth Cohen real, is W-generic (function from ω to ω) over the Cohen poset of finite functions. Using r_γ and j, we will define inductively an increasing sequence of members of $\pi(Q)$, $\langle \bar{q}_i | i \in \omega\rangle$, as follows: $\bar{q}_0 = \pi(q_0)$ (q_0 is the given condition). Suppose $\bar{q}_k$ is defined: if $j(r_\gamma(k)) \supseteq \bar{q}_k$, set $\bar{q}_{k+1} = j(r_\gamma(k))$; otherwise, let $\bar{q}_{k+1} = \bar{q}_k$. Finally, set $q = \bigcup_{k\in\omega} \pi^{-1}(\bar{q}_k)$.

The following Claim ends the proof of Lemma 3.4, as $q_0 \subseteq q$.

3.5 CLAIM. $\mathrm{Dom}(q) = M \cap \aleph_2$, *$q \in Q$ and $q \in \bigcap_{n\in\omega} D_n$.*

PROOF. q is a countable function, defined on pairs, satisfying $\alpha \leq q(\alpha, \beta) \leq \beta$ for $\alpha < \beta$ in its domain. Clearly $q \in W[r_\gamma]$, so the following is about members of $W[r_\gamma]$:

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[3]S. Todorčević remarks that this is another example (see [S2, Chapter VII, §5]) of a nonproper poset which does not destroy stationary sets.

3.6. *The sequence* $\langle \bar{q}_k | k \in \omega \rangle$ *is W-generic over* $\pi(Q)$ *in the sense that if* $D \in W$ *is dense in* $\pi(Q)$, *then some* $\bar{q}_k$ *extends a member of* D.

The proof of 3.6 is by the following density argument for the Cohen poset $P_\gamma = {}^\omega\omega$. In W, given arbitrary $p \in P_\gamma$ and D dense in $\pi(Q)$, we will find $n \in \omega$ and an extension of p which forces $\bar{q}_n \in D$. Say $\text{Dom}(p) = k$; then for some $f \in \pi(Q)$, $p \Vdash^{P_\gamma}$ "$\bar{q}_k = f$". Pick $f' \in D$, $f \subseteq f'$; then $f' = j(l)$ for some $l \in \omega$. Define $p' = p \cup \{\langle k, l \rangle\}$, then $p' \Vdash^{P_\gamma}$ "$\bar{q}_{k+1} = f'$ and hence $\bar{q}_{k+1} \in D$"; proving 3.6.

Now because of the extension property (which holds in $\pi(Q)$ in $\bar{M}$), 3.6 shows that $\text{Dom}(\bigcup_{n \in \omega} \bar{q}_n) = \pi(\aleph_2)$ and hence $\text{Dom}(q) = M \cap \aleph_2$. So $\text{Dom}(q) \in L$. As $D_n \in M$, $\pi(D_n) \in \bar{M}$ is dense in $\pi(Q)$, so 3.6 implies that q extends some member of D_n for each $n \in \omega$. To prove $q \in Q$ we still have to check (4) in Definition 3.3; then $q \in \bigcap_{n \in \omega} D_n$ will follow.

Before doing that, let us prove that $\pi(Q) \subseteq Q$. Although $\pi \notin L$, from the fact that $M \cap \aleph_2 = D \in L$ it follows that $\pi \restriction \aleph_2 = \pi \restriction D \in L$. Hence, if $E \subseteq D$, $E \in T$ iff $\pi''E \in T$. Given any $\bar{f} \in \pi(Q)$, $\bar{f} = \pi(f)$ for some $f \in Q \cap M$ and $\text{Dom}(\bar{f}) = \pi(\text{Dom}(f))$, so $\text{Dom}(f) \in L$. Moreover, by what was said before, $\bar{f}$ satisfies all requirements (1)–(4) of 3.3, so $\bar{f} \in Q$.

Coming back to the proof that q satisfies (4), let $E \subseteq D = \text{Dom}(q)$, $E \in T$, be given; we will prove that E is not closed under q. Set $E' = \pi''E$, $E' \in T$. If we show that E' is not closed under $\bigcup_{k \in \omega} \bar{q}_k$, it will follow that E is not closed under q. In view of 3.6, it is enough to show that the set of $f \in \pi(Q)$ such that E' is not closed under f is dense in $\pi(Q)$ ($E' \in T \in L[r_0] \subseteq W$, so this set of f's is in W). So let $f \in \pi(Q)$ be given. In case $E' \cap \text{Dom}(f)$ is nonconstructible, $E' \cap \text{Dom}(f) \in T$; and so (as $f \in Q$) $E' \cap \text{Dom}(f)$ is not closed under f and hence E' is not closed under f. In the case where $E' \cap \text{Dom}(f)$ *is* constructible, it must be that $E' - \text{Dom}(f) \notin L$; and so $E' - \text{Dom}(f) \in T$ (because $\text{Dom}(f) \in L$). Let $a = \inf(E' - \text{Dom}(f))$ and $b = \text{Sup}(E' - \text{Dom}(f))$. So $a \in E' - \text{Dom}(f)$ and $b \notin E' - \text{Dom}(f)$. Let $[a, b)$ be the left-closed right-open ordinal-interval. $[a, b) \subseteq \bar{M}$ because $\bar{M}$ is transitive. We show that $[a, b) - E' \neq 0$. Well, if $[a, b) - E' = \emptyset$, then $E' - \text{Dom}(f) = [a, b) - \text{Dom}(f)$. Hence, as $E' = (E' \cap \text{Dom}(f)) \cup (E' - \text{Dom}(f))$, E' is constructible, contradicting $E' \in T$. Pick $\alpha \in [a, b) - E'$. As $\alpha \notin E'$ and $\alpha < b = \text{Sup}(E' - \text{Dom}(f))$, there is $\beta \in E' - \text{Dom}(f)$ with $\alpha < \beta$. Then we define $f' \in \pi(Q)$, $f \subseteq f'$, thus: set $\text{Dom}(f') = \text{Dom}(f) \cup \{a, \alpha, \beta\}$; as a, $\beta \in E' - \text{Dom}(f)$ we have freedom to define $f'(a, \beta) = \alpha$ and to define, for other arguments not in $\text{Dom}(f)$, $f'(x, y) = \max\{x, y\}$. It is easy to check $f' \in \pi(Q)$. E' is not closed under f' because $a, \beta \in E'$ but $\alpha \notin E'$.

This ends the proof of Lemma 3.4. To conclude the proof of Theorem 8, let $\dot{Q}$ be a V-generic filter over Q and put $F = \bigcup \dot{Q}$. Then $F: [\aleph_2]^2 \to \aleph_2$. Now if $E \in T$, then Lemma 3.4 and the extension property imply that $E \subseteq \text{Dom}(f)$ for some $f \in \dot{Q}$ (E is countable). Hence E is not closed under F. □

The technique of using generic reals as we did here is applied in [A1], [A2] and [A, S], §5.

3.7. We turn now to L and show there a stationary co-stationary set $S \subset P_{\aleph_1}(\aleph_2)$ and a poset P which satisfy the $\aleph_2$-a.c. such that forcing with P does not add new countable sets to L but introduces a club subset to S.

To begin with, let $T \subseteq S^{\aleph_2}_{\aleph_0}$ be a stationary set which does not reflect; i.e., $T \cap \alpha$ is nonstationary in α whenever $\mathrm{cf}(\alpha) = \aleph_1$.

$\diamondsuit_T$ holds in L, so an equivalent from gives us a sequence f_δ: $[\delta]^{<\omega} \to \delta$, $\delta \in T$, such that whenever f: $[\aleph_2]^{<\omega} \to \aleph_2$ is given, there is $\delta \in T$ with $f_\delta = f \restriction [\delta]^{<\omega}$.

For every $\delta \in T$ pick some countable $N_\delta \subseteq \delta$, closed under f_δ, with $\sup N_\delta = \delta$; moreover, we ask that if $\eta \in T \cap N_\delta$ then $N_\eta \subseteq N_\delta$, and if $\alpha \in T$, $\alpha < \delta$, is such that $\alpha = \sup(\alpha \cap N_\delta)$ then $\alpha \in N_\delta$.

In order to prove that such a countable N_δ exists we use the fact that any initial segment of T is nonstationary: so let C_η, $\eta \in S^{\aleph_2}_{\aleph_1}$, be closed unbounded, $C_\eta \subseteq \eta$, but $C_\eta \cap T = \varnothing$. Define a two-place function h on $\aleph_2$, such that if $\alpha < \eta$ and $\eta \in S^{\aleph_2}_{\aleph_1}$ then $h(\alpha, \eta)$ is the least member of C_η above α. Now let N_δ be cofinal in δ, closed under f_δ, closed under h, closed under $\alpha \mapsto N_\alpha$, and closed under the function that takes any successor ordinal to its predecessor and any countable-cofinality ordinal to a countable cofinal sequence. We have to check that if $\alpha \in T$, $\alpha = \sup(\alpha \cap N_\delta)$, $\alpha < \delta$, then $\alpha \in N_\delta$. Assume not, and let $\alpha < \beta < \delta$ be the first ordinal in N_δ above α. Necessarily, $\mathrm{cf}(\beta) = \aleph_1$ and $C_\beta \cap \alpha$ is unbounded in α. Hence $\alpha \in C_\beta$, so that $\alpha \notin T$.

It is not difficult to see that $T^* = \{N_\delta | \delta \in T\}$ is a stationary subset of $P_{\aleph_1}(\aleph_2)$. (Just use Kueker's theorem and the property of the diamond sequence to guess functions f: $[\aleph_2]^{<\omega} \to \aleph_2$ which corresponds to club sets.)

It is even easier to see that T^* is co-stationary. Put $S = P_{\aleph_1}(\aleph_2) - T^*$.

THEOREM 9. *There exists a poset P which satisfies the $\aleph_2$-a.c. such that forcing with P adds a club subset to S, but does not add new countable sets.*

PROOF. Members of P are all pairs (B, g) where g is a countable function, g: $B \to B$, and $B \subset \aleph_2$, satisfying the following:

(1) If $\delta \in T$ and $\delta = \sup(\delta \cap B)$ then $\delta \in B$.

(2) If $\delta \in B \cap T$ then $N_\delta \subseteq B$.

(3) For every $\delta \in B \cap T$ there is $a \in N_\delta$ with $g(a) \notin N_\delta$.

It is easy to check that for each $i < \aleph_2$ $\{(B, g) | i \in B\}$ is dense in P. It follows that a generic filter over P provides us with a function $\bar{g}$ on $\aleph_2$ such that no N_δ is closed under $\bar{g}$. If we show that P does not add new countable sets, then it follows that S acquires a club subset in any P-generic extension.

3.8 LEMMA. *Forcing with P does not add new countable sets.*

PROOF. Let τ be a name in L^P of a function from ω into the ordinals. Let $p_0 \in P$ be given; we want to find $p \geq p_0$ in P which decides all values $\tau(n)$, $n \in \omega$.

Let λ be a cardinal with P, $\tau \in H(\lambda)$. Pick a countable elementary submodel $M \prec H(\lambda)$, such that p_0, P, $\tau \in M$ and $\sup(M \cap \aleph_2) \notin T$. (Since T does not reflect, it is possible to find such an M: first find such a substructure of cardinality $\aleph_1$.)

CLAIM. *If $\alpha = \sup(M \cap \alpha)$ and $\alpha \in T$ then $\alpha \in M$.*

The proof, left to the reader, is like the one used to conclude that the N_δ exist.

Now we define an increasing sequence $p_n \in P \cap M$, $n \in \omega$, such that for every $D \in M$, dense and open in P, $p_n \in D$ for some n. Let $p = \bigcup p_n$ (i.e., if $p_n = (B_n, g_n)$ then $p = (B, g)$, where $B = \bigcup_{n<\omega} B_n$ and $g = \bigcup_{n<\omega} g_n$). Then $B = M \cap \aleph_2$ and $p \in P$ by the claim above, and p knows all the values of $\tau(n)$. □

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