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Advances in Cardinal Arithmetic

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Abstract

If $\mathrm{cf}\kappa = \kappa$, $\kappa^+ < \mathrm{cf}\lambda = \lambda$ then there is a stationary subset S of $\{\delta < \lambda : \mathrm{cf}(\delta) = \kappa\}$ in $I[\lambda]$. Moreover, we can find $\overline{C} = \langle C_\delta : \delta \in S\rangle$, C_δ a club of λ, $\mathrm{otp}(C_\delta) = \kappa$, guessing clubs and for each $\alpha < \lambda$ we have: $\{C_\delta \cap \alpha : \alpha \in \mathrm{nacc} C_\delta\}$ has cardinality $< \lambda$.

We prove that, for example, there is a stationary subset of $\mathcal{S}_{<\aleph_1}(\lambda)$ of cardinality $\mathrm{cf}(\mathcal{S}_{<\aleph_1}(\lambda), \subseteq)$.

We prove the existence of nice filters where instead of being normal filters on ω_1 they are normal filters with larger domains, which can increase during a play. They can help us transfer the situation on $\aleph_1$-complete filters to normal ones.

We consider ranks and niceness of normal filters, such that we can pass, say, from $\mathrm{pp}_{\Gamma(\aleph_1)}(\mu)$ (where $\mathrm{cf}\mu = \aleph_1$) to $\mathrm{pp}_{\mathrm{normal}}(\mu)$.

We consider some weakenings of G.C.H. and their consequences. Most have not been proved independent of ZFC.

1 $I[\lambda]$ is Quite Large and Guessing Clubs

On $I[\lambda]$ see [6], [5], [7, §4] (but this section is self-contained; see Definition 1.1 and Claim 1.2 below). We shall prove that for regular κ, λ, such that $\kappa^+ < \lambda$, there is a stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ in $I[\lambda]$. We then investigate "guessing clubs" in (ZFC).

Definition 1.1 For a regular uncountable cardinal λ, $I[\lambda]$ is the family of $A \subseteq \lambda$ such that $\{\delta \in A : \delta = \mathrm{cf}\delta\}$ is not stationary and for some $\langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$ we have:

(a) $\mathcal{P}_\alpha$ is a family of $< \lambda$ subsets of α;

(b) for every limit $\alpha \in A$ such that $\mathrm{cf}(\alpha) < \alpha$ there is $x \subseteq \alpha$, $\mathrm{otp}(x) < \alpha = \sup x$ such that
$$\bigwedge_{\beta<\alpha} x \cap \beta \in \bigcup_{\gamma<\alpha} \mathcal{P}_\gamma.$$

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We know (see [6], [5] or below)

Claim 1.2 *Let $\lambda > \aleph_0$ be regular.*

(1) *$A \in I[\lambda]$ iff (note: by (c) below the set of inaccessibles in A is not stationary and) there is $\langle C_\beta : \beta < \lambda \rangle$ such that:*

(a) *C_β is a closed subset of β;*

(b) *if $\alpha \in \mathrm{nacc} C_\beta$ then $C_\alpha = C_\beta \cap \beta$ (nacc stands for "non-accumulation");*

(c) *for some club E of λ, for every $\delta \in A \cap E$: $\mathrm{cf}\delta < \delta$ and $\delta = \sup C_\delta$, and $\mathrm{cf}(\delta) = \mathrm{otp}(C_\delta)$;*

(d) *$\mathrm{nacc}(C_\beta)$ is a set of successor ordinals.*

(2) *$I[\lambda]$ is a normal ideal.*

Proof.

1) <u>THE "IF" PART:</u>

Assume $\langle C_\beta : \beta < \lambda \rangle$ satisfy (a), (b), (c) with a club E for (c). For each limit $\alpha < \lambda$ choose a club e_α of order type $\mathrm{cf}(\alpha)$. We define, for $\alpha < \lambda$:

$$\mathcal{P}_\alpha =: \{C_\beta : \beta \le \alpha\} \cup \{e_\beta : \beta \le \alpha\} \cup \{e_\gamma \cap \alpha : \gamma \le \min(E \setminus (\alpha+1)\}.$$

It is easy to check that $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ exemplify "$A \in I[\lambda]$".

<u>THE "ONLY IF" PART:</u>

Let $\langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ exemplify "$A \in I[\lambda]$" (by Definition 1.1). Without loss of generality if $C \in \mathcal{P}_\alpha$, and $\zeta \in C$ then $C \backslash \zeta \in \mathcal{P}_\alpha$ and $C \cap \zeta \in \mathcal{P}_\alpha$.

For each limit $\beta < \lambda$ let e_β be a club of β, $\mathrm{otp}(e_\beta) = \mathrm{cf}(\beta)$ and $\mathrm{cf}\beta < \beta \Rightarrow \mathrm{cf}\beta < \min(e_\beta)$. Let $\langle \gamma_i : i < \lambda \rangle$ be strictly increasing continuous, each γ_i a non-successor ordinal $< \lambda$, $\gamma_0 = 0$, and $\gamma_{i+1} - \gamma_i \ge \aleph_0 + |\bigcup_{\alpha \le \gamma_i} \mathcal{P}_\alpha| + |\gamma_i|$ and $\gamma_i \in A \Rightarrow \mathrm{cf}(\gamma_i) < \gamma_i$.

Let F_i be a one to one function from $\left(\bigcup_{\alpha \le \gamma_i} \mathcal{P}_\alpha\right) \times \gamma_i$ into $\{\zeta + 1 : \gamma_i < \zeta + 1 < \gamma_{i+1}\}$. Now we define $C_\alpha \subseteq \alpha$ as follows.

Assume α is a successor ordinal, and let $i(\alpha)$ be such that $\gamma_{i(\alpha)} < \alpha < \gamma_{i(\alpha)+1}$. If $\alpha \notin \mathrm{Rang} F_{i(\alpha)}$, let $C_\alpha = \emptyset$. If $\alpha = F_{i(\alpha)}(x, \beta)$ (so $x \in \bigcup_{\varepsilon \le \gamma_{i(\alpha)}} \mathcal{P}_\varepsilon$, $\beta < \gamma_{i(\alpha)}$), let C_α be the closure (in the order topology on α) of:

$$F_j(x \cap \zeta, \beta) : \begin{cases} \text{(i)} & \zeta \in x, \\ \text{(ii)} & \mathrm{otp}(x \cap \zeta) \in e_\beta, \\ \text{(iii)} & j < i(\alpha) \text{ is minimal such that } x \cap \zeta \in \bigcup\limits_{\varepsilon \le \gamma_j} \mathcal{P}_\varepsilon, \\ \text{(iv)} & \text{if } \xi \in x \cap \zeta,\ \mathrm{otp}(x \cap \xi) \in e_\beta \text{ then} \\ & \qquad (\exists j(1) < j)[x \cap \xi \in \bigcup\limits_{\varepsilon \le \gamma_{j(1)}} \mathcal{P}_\varepsilon], \\ \text{(v)} & \beta < \min x. \end{cases}$$

Now for $\alpha < \lambda$ limit, choose C_α: if possible, $\mathrm{nacc} C_\alpha$ is a set of successor ordinals, C_α is a club of α, $[\beta \in \mathrm{nacc} C_\alpha \Rightarrow C_\beta = \beta \cap C_\alpha]$; if this is impossible, let $C_\delta = \emptyset$. Let $E =: \{\gamma_i : i \text{ limit} < \lambda\}$. Now we can check the condition in 1.2(1).

2) By Definition 1.1 $I[\lambda]$ is an ideal; by 1.2(1) $I[\lambda]$ includes the ideal of non-stationary subsets of λ. By the last phrase and Definition 1.1, clearly $I[\lambda]$ is normal. $\square_{1.2}$

Claim 1.3 *If κ, λ are regular, $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$, $S \in I[\lambda]$, S stationary, $\kappa^+ < \lambda$ then we can find $\overline{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ such that for $\delta(*) =: \kappa$ we have:*

$$\bigoplus_{\overline{\mathcal{P}},S}^{\lambda,\delta(*)} : \begin{cases} \text{(i)} & \mathcal{P}_\alpha \textit{ is a family of closed subsets of } \alpha, |\mathcal{P}_\alpha| < \lambda; \\ \text{(ii)} & \mathrm{otp}C \leq \delta(*) \textit{ for } C \in \cup_\alpha \mathcal{P}_\alpha; \\ \text{(iii)} & \textit{for some club } E \textit{ of } \lambda, \textit{ we have:} \\ & [\alpha \notin E \Rightarrow \mathcal{P}_\alpha = \emptyset] \textit{ and} \\ & [\alpha \in E \Rightarrow (\forall C \in \mathcal{P}_\alpha)(\mathrm{otp}C \leq \delta(*))]; \\ & [\alpha \in E \backslash (S \cap \mathrm{acc}E) \Rightarrow (\forall C \in \mathcal{P}_\alpha)(\mathrm{otp}C < \delta(*))]; \\ & [\alpha \in S \cap \mathrm{acc}E \Rightarrow (\exists ! C \in \mathcal{P}_\alpha)(\mathrm{otp}C = \delta(*))]; \\ & [\alpha \in S \cap \mathrm{acc}E \;\&\; C \in \mathcal{P}_\alpha \;\&\; \mathrm{otp}C = \delta(*) \Rightarrow \alpha = \sup C)]; \\ \text{(iv)} & C \in \mathcal{P}_\alpha \;\&\; \beta \in \mathrm{nacc}C \Rightarrow \beta \cap C \in \mathcal{P}_\beta; \\ \text{(v)} & \textit{for any club } E \textit{ of } \lambda, \textit{ for some } \delta \in S \cap E \textit{ and } C \in \mathcal{P}_\delta \textit{ we have} \\ & C \subseteq E \;\&\; \mathrm{otp}C = \delta(*). \end{cases}$$

Proof. Let $\langle C_\alpha : \alpha < \lambda \rangle$ witness "$S \in I[\lambda]$" as in 1.2(1); without loss of generality $\mathrm{otp}C_\alpha \leq \delta(*)$. For any club E let us define $\mathcal{P}^\alpha_E$ by induction on $\alpha < \lambda$:

$$\begin{aligned} \mathcal{P}^\alpha_E =: & \{\alpha \cap g\ell(C_\beta, E) : \alpha \in E \text{ and } \alpha \leq \beta < \min[E \setminus (\alpha+1)]\} \\ & \cup \{C \cup \{\beta\} : \text{ for some } \beta \in E, \beta < \alpha, C \in \mathcal{P}^\beta_E \text{ and } \mathrm{otp}(C) < \delta(*)\} \end{aligned}$$

where

$$g\ell(C_\beta, E) =: \{\sup(E \cap (\gamma+1)) : \gamma \in C_\beta \text{ and } \gamma > \min E\}.$$

Note that $|\mathcal{P}^\alpha_E| \leq |\min(E \setminus (\alpha+1))| < \lambda$. We can prove that for some club E of λ $\langle \mathcal{P}^\alpha_E : \alpha < \lambda \rangle$ is as required except (v) which can be corrected (just by trying successively κ^+ clubs $E_\zeta (\zeta < \kappa^+)$ decreasing with ζ, see [13]) and (iv) which is guaranteed by demanding E to consist of limit ordinals only and the second set in the union defining $\mathcal{P}^\alpha_E$. $\square_{1.3}$

The following lemma gives a sufficient condition for the existence of "quite large" stationary sets in $I[\lambda]$ of almost any fixed cofinality.

Lemma 1.4 *Suppose*

(i) $\lambda > \kappa > \aleph_0$, *$\lambda$ and κ are regular,*

(ii) $\overline{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$, *$\mathcal{P}_\alpha$ a family of $< \lambda$ closed subsets of α,*

(iii) $I_{\overline{\mathcal{P}}} =: \{S \subseteq \kappa$: *for some club E of κ, for no $\delta \in S \cap E$ is there a club C of δ, such that $C \subseteq E$ and $[\alpha \in \mathrm{nacc}C \Rightarrow C \cap \alpha \in \bigcup_{\beta<\alpha} \mathcal{P}_\beta]\}$ is a proper ideal on κ.*

Then there is $S^* \in I[\lambda]$ such that for stationarily many $\delta < \lambda$ of cofinality κ, $S^* \cap \delta$ is stationary in δ; moreover for some club E of δ of order type κ,

$$\{\mathrm{otp}(\alpha \cap E) : \alpha \in E \setminus S^*\} \in I_{\overline{\mathcal{P}}}.$$

Remark 1.4A: The "for stationarily many" in the conclusion can be strengthened to: a set whose complement is in the ideal defined in [13], §2.

Proof. Let χ be regular large enough, N^* be an elementary submodel of $(H(\chi), \in, <^*_\chi)$ of cardinality λ such that $(\lambda+1) \subseteq N^*$, $\overline{\mathcal{P}} \in N$. Let $\overline{C} = \langle C_i : i < \lambda \rangle$ list $N^* \cap \{A \subseteq \lambda : |A| < \kappa\}$ and let

$$\begin{aligned} S^* &= \{\delta < \lambda : \text{cf}(\delta) < \kappa \text{ and for some } A \subseteq \delta,\ \delta = \sup A, \\ &\quad \text{otp}A < \kappa \text{ and } (\forall \alpha < \delta)[A \cap \alpha \in \{C_i : i < \delta\}]\}. \end{aligned}$$

Clearly $S^* \in I[\lambda]$; so we should only find enough $\delta < \lambda$ of cofinality κ as required. So let E^a_0 be a club of λ. We can choose inductively $M_\zeta(\zeta \le \kappa)$ such that:

(a) $M_\zeta \prec (H(\chi), \in, <^*_\chi)$,

(b) $\|M_\zeta\| < \lambda$, $M_\zeta \cap \lambda$ an ordinal,

(c) M_ζ is increasing continuous,

(d) $N, \kappa, \overline{\mathcal{P}}, \overline{C}, E^a_0$ belongs to M_0,

(e) $\langle M_\varepsilon : \varepsilon \le \zeta \rangle \in M_{\zeta+1}$.

Let $\delta_\zeta = \sup(M_\zeta \cap \lambda)$, so $\langle \delta_\zeta : \zeta \le \kappa \rangle$ is strictly increasing continuous, so $\delta =: \delta_\kappa$ has cofinality κ. Hence there is a strictly increasing continuous sequence $\langle \alpha_\zeta : \zeta < \kappa \rangle \in N^*$ with limit δ, and clearly $E = \{\zeta < \kappa : \alpha_\zeta = \delta_\zeta\}$ is a club of κ. We know that

$$T =: \{\zeta < \kappa : \zeta \text{ limit and for some club } C \text{ of } \zeta,\ C \subseteq E \text{ and } \bigwedge_{\varepsilon<\zeta} [C \cap \varepsilon \in \bigcup_{\xi<\zeta} \mathcal{P}_\xi]\}$$

is stationary; moreover, $\kappa \backslash T \in I_{\overline{\mathcal{P}}}$ (see assumption (iii)) and clearly $T \subseteq E$. Clearly it suffices to show

(*) $\zeta \in T \Rightarrow \delta_\zeta \in S^*$.

Suppose $\zeta \in T$, so there is C, a club of ζ such that $C \subseteq E$ and $\bigwedge_{\varepsilon<\zeta}[C \cap \varepsilon \in \bigcup_{\xi<\zeta}\mathcal{P}_\xi]$. Let $C^* = \{\delta_\varepsilon : \varepsilon \in C\}$, so C^* is a club of δ_ζ of order type $\le \zeta < \kappa$ (which is $< \delta_0 \le \delta_\zeta$). It suffices to show for $\xi \in C$ that $\{\delta_\varepsilon : \varepsilon \in \xi \cap C\} \in \{C_i : i < \delta_\zeta\}$. For this end we shall show

(α) $\{\delta_\varepsilon : \varepsilon \in C \cap \xi\} \in \{C_i : i < \lambda\}$,

(β) $\{\delta_\varepsilon : \varepsilon \in C \cap \xi\} \in M_{\xi+1}$.

This suffices as $\langle C_i : i < \lambda \rangle \in M_0 \prec M_{\xi+1}$ and $M_{\xi+1} \cap \{C_i : i < \lambda\} = \{C_i : i \in \lambda \cap M_{\xi+1}\} = \{C_i : i < \delta_{\xi+1}\}$.

PROOF OF (α): Remember $\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle \in N^*$. Also $\langle \mathcal{P}_\varepsilon : \varepsilon < \kappa \rangle \in N^*$ hence $\bigcup_{\varepsilon<\kappa} \mathcal{P}_\varepsilon \subseteq N^*$ (as $\kappa < \lambda$, $|\mathcal{P}_\varepsilon| < \lambda, \text{cf}\lambda = \lambda$) and $C \cap \xi \in \bigcup_{\varepsilon<\kappa} \mathcal{P}_\varepsilon$; hence $C \cap \xi \in N^*$. Together $\{\alpha_\varepsilon : \varepsilon \in \xi \cap C\} \in N^*$; as $\varepsilon \in C \Rightarrow \varepsilon \in E \Rightarrow \alpha_\varepsilon = \delta_\varepsilon$ (from $C \subseteq E$ and the definition of E), and from the definition of $\langle C_i : i < \lambda \rangle$, we finish.

PROOF OF (β): We know $\overline{\mathcal{P}} \in M_0$; as $|\mathcal{P}_\varepsilon| < \lambda$, $\kappa < \lambda$ and $M_\varepsilon \cap \lambda$ is an ordinal, clearly $\bigcup_{\varepsilon<\kappa} \mathcal{P}_\varepsilon \subseteq M_0$ (remember $|\mathcal{P}_\varepsilon| < \lambda$, $\kappa < \lambda$). So for $\varepsilon < \zeta$, $C \cap \varepsilon \in \bigcup_{\gamma<\zeta} \mathcal{P}_\gamma \subseteq M_0 \subseteq M_{\xi+1}$.

As $\langle M_i : i \leq \xi \rangle \in M_{\xi+1}$ clearly $\langle \delta_i : i \leq \xi \rangle \in M_{\xi+1}$ hence by the previous sentence $\langle \delta_i : i \in C \cap \xi \rangle \in M_{\xi+1}$, as required. $\square_{1.4}$

Conclusion 1.5 *If κ, λ are regular, $\kappa^+ < \lambda$ then there is a stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ in $I[\lambda]$.*

Proof. If $\lambda = \kappa^{++}$ — use [9], 4.1. So assume $\lambda > \kappa^{++}$. By [9], 4.1 the pair (κ, κ^{++}) satisfies the assumption of 1.3 for $S = \{\delta < \kappa^{++} : \mathrm{cf}\delta = \kappa\}$; (i.e. κ, λ there stands for κ, κ^{++} here). Hence the conclusion of 1.3 holds for some $\overline{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \kappa^{++} \rangle$, $|\mathcal{P}_\alpha| < |\kappa^{++}|$. Now apply 1.4 with (κ^{++}, λ) here standing for (κ, λ) there (we have just proved $I_{\overline{\mathcal{P}}}$ is a proper ideal, so assumption (ii) holds). Note:

$(*)$ $\{\delta < \kappa^{++} : \mathrm{cf}\delta = \kappa\} \notin I_{\overline{\mathcal{P}}}$.

Now the conclusion of 1.4 (see the "moreover" and choice of $\overline{\mathcal{P}}$, i.e.$(*)$) gives the desired conclusion. $\square_{1.5}$

Conclusion 1.6 *If $\lambda > \kappa$ are uncountable regular, $\kappa^+ < \lambda$, then for some stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$ and some $\overline{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ we have: $\bigoplus^{\lambda,\kappa}_{\overline{\mathcal{P}},S}$ from the conclusion of 1.3 holds.*

Proof. As κ is regular apply 1.5 and then 1.3. $\square_{1.6}$

Now 1.6 was a statement I have long wanted to know, still sometimes we want to have "$C_\delta \subseteq E, \mathrm{otp}C = \delta(*)$", $\delta(*)$ not a regular cardinal. We shall deal with such problems.

Claim 1.7 *Suppose*

(i) *$\lambda > \kappa > \aleph_0$, λ and κ are regular cardinals,*

(ii) *$\overline{\mathcal{P}}_\ell = \langle \mathcal{P}_{\ell,\alpha} : \alpha < \kappa \rangle$ for $\ell = 1, 2$, where $\mathcal{P}_{1,\alpha}$ is a family of $< \lambda$ closed subsets of α, $\mathcal{P}_{2,\alpha}$ is a family of $\leq \lambda$ clubs of α and $[C \in \mathcal{P}_{2,\alpha}\ \&\ \beta \in C \Rightarrow C \cap \beta \in \bigcup_{\gamma<\alpha} \mathcal{P}_{1,\gamma}]$,*

(iii) *$I_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2} =: \{S \subseteq \kappa :$ for some club E of κ, for no $\delta \in S \cap E$ is there $C \in \mathcal{P}_{2,\alpha}$, $C \subseteq E\}$ is a proper ideal on κ.*

Then we can find $\overline{\mathcal{P}}^*_\ell = \langle \mathcal{P}^*_{\ell,\alpha} : \alpha < \lambda \rangle$ for $\ell = 1, 2$ such that:

(A) *$\mathcal{P}^*_{1,\alpha}$ is a family of $< \lambda$ closed subsets of α;*

(B) *$\beta \in \mathrm{nacc}C\ \&\ C \in \mathcal{P}^*_{1,\alpha} \Rightarrow C \cap \beta \in \mathcal{P}^*_{1,\beta}$;*

(C) *$\mathcal{P}^*_{2,\delta}$ is a family of $\leq \lambda$ clubs of δ (for δ limit $< \lambda$) $[\beta \in \mathrm{nacc}C\ \&\ C \in \mathcal{P}^*_{2,\delta} \Rightarrow C \cap \beta \in \mathcal{P}^*_{1,\beta}]$;*

(D) *for every club E of λ, for some strictly increasing continuous sequence $\langle \delta_\zeta : \zeta \leq \kappa \rangle$ of ordinals $< \lambda$ we have*

$\{\zeta < \kappa : \zeta$ limit, and for some $C \in \mathcal{P}_{2,\zeta}$ we have:

$$\{\delta_\varepsilon : \varepsilon \in C\} \in \mathcal{P}^*_{2,\delta_\zeta} \text{ (hence } [\xi \in \mathrm{nacc}C \Rightarrow \{\delta_\varepsilon : \varepsilon \in C \cap \xi\} \in \mathcal{P}^*_{1,\delta_\xi}])\} \equiv \kappa \bmod I_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2};$$

(E) *we have* e_δ *a club of* δ *of order type* $\mathrm{cf}(\delta)$ *for any limit* $\delta < \lambda$*; such that for any* $C \in \bigcup_{\alpha<\lambda} \mathcal{P}^*_{2,\alpha}$ *for some* $\delta < \lambda$, $\mathrm{cf}\delta = \kappa$ *and* $C' \in \bigcup_{\beta<\kappa} \mathcal{P}_{2,\beta}$ *we have* $C = \{\gamma \in e_\delta : \mathrm{otp}(e_\delta \cap \gamma) \in C'\}$.

Proof. Same proof as 1.4. (Note that without loss of generality $[C \in \mathcal{P}_{1,\alpha}\ \&\ \beta < \alpha < \kappa \Rightarrow C \cap \beta \in \mathcal{P}_{1,\beta}]$). $\square_{1.7}$

Conclusion 1.8: *If* $\delta(*)$ *is a limit ordinal and* $\lambda = \mathrm{cf}\lambda > |\delta(*)|^+$ *then we can find* $\overline{\mathcal{P}}^*_\ell = \langle \mathcal{P}^*_{\ell,\alpha} : \alpha < \lambda\rangle$ *for* $\ell = 1,2$ *and stationary* $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \mathrm{cf}\delta(*)\}$ *such that:*

$\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}^*_1,\overline{\mathcal{P}}^*_2,S}$:

- (A) $\mathcal{P}^*_{1,\alpha}$ *is a family of* $< \lambda$ *closed subsets of* α *each of order type* $< \delta(*)$;
- (B) $\beta \in \mathrm{nacc}C\ \&\ C \in \mathcal{P}^*_{1,\alpha} \Rightarrow C \cap \beta \in \mathcal{P}^*_{1,\beta}$;
- (C) $\mathcal{P}^*_{2,\delta}$ *is a family of* $\leq \lambda$ *clubs of* δ *(yes, maybe* $= \lambda$*) of order type* $\delta(*)$, *and* $[\beta \in \mathrm{nacc}C\ \&\ C \in \mathcal{P}^*_{2,\delta} \Rightarrow C \cap \beta \in \mathcal{P}^*_{1,\beta}]$;
- (D) *for every club* E *of* λ*, for some* $\delta \in E \cap S$, $\mathrm{cf}\delta = \mathrm{cf}(\delta(*))$ *and there is* $C \in \mathcal{P}^*_{2,\beta}$ *such that* $C \subseteq E$.

Proof. If $\lambda = |\delta(*)|^{++}$ (or any successor of regulars) use [3], III, 6.4(2)] or [13], 2.14(2)(c)&(d)). If $\lambda > |\delta(*)|^{++}$ let $\kappa = |\delta(*)|^{++}$ and let $S_1 = \{\delta < \kappa^{++} : \mathrm{cf}\delta = \mathrm{cf}\delta(*)\}$; applying the previous sentence we get $\overline{\mathcal{P}}^*_1, \overline{\mathcal{P}}^*_2$ satisfying $\bigoplus^{\kappa^{++},\delta(*)}_{\overline{\mathcal{P}}^*_1,\overline{\mathcal{P}}^*_2,S_1}$, hence satisfying the assumption of 1.7 so we can apply 1.7. $\square_{1.8}$

Definition 1.9 ${}^+\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2,S}$ is defined as in 1.8 except that we replace (C) by:

(C)$^+$ $\mathcal{P}^*_{2,\delta}$ is a family of $< \lambda$ clubs of δ of order type $\delta(*)$.

Remark 1.9A Note that if $\mathcal{P}_\alpha = \mathcal{P}_{1,\alpha} \cup \mathcal{P}_{2,\alpha}$, $|\mathcal{P}_{2,\alpha}| \leq 1$, $\mathcal{P}_{1,\alpha} = \{C \in \mathcal{P}_\alpha : \mathrm{otp}C < \delta(*)\}$, $\mathcal{P}_{2,\alpha} = \{C \in \mathcal{P}_\alpha : \mathrm{otp}C = \delta(*)\}$ then ${}^+\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2,S} \Leftrightarrow \bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}},S}$.

Claim 1.10 *Suppose* $\lambda = \mathrm{cf}\lambda > |\delta(*)|^+$, $\delta(*)$ *a limit ordinal, additively indecomposable (i.e.* $\alpha < \delta(*) \Rightarrow \alpha + \alpha < \delta(*)$*),* $\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2,S}$ *from 1.8 and*

$(*)$ $\alpha \in S \Rightarrow |\mathcal{P}_{2,\alpha}| \leq |\alpha|$.

(Note: a non-stationary subset of S *does not count; e.g. for* λ *successor cardinal the* α *with* $|\alpha|^+ < \lambda$*. Note:* ${}^+\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2,S}$ *holds by* $(*)$ *and if* λ *is successor then* ${}^+\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}_1,\overline{\mathcal{P}}_2,S}$ *suffices).*

Then for some stationary $S_1 \subseteq S$ *and* $\overline{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda\rangle$ *we have:* $\mathcal{P}_\alpha \subseteq \mathcal{P}_{1,\alpha} \cup \mathcal{P}_{2,\alpha}$ *and:*

${}^*\bigotimes^{\lambda,\delta(*)}_{\overline{\mathcal{P}},S_1}$:

- (i) $\mathcal{P}_\alpha$ *is a family of closed subsets of* α, $|\mathcal{P}_\alpha| < \lambda$;
- (ii) $\mathrm{otp}C < \delta(*)$ *if* $C \in \mathcal{P}_\alpha$, $\alpha \notin S_1$;
- (iii) *if* $\alpha \in S_1$ *then:* $\mathcal{P}_\alpha = \{C_\alpha\}$, $\mathrm{otp}C_\alpha = \delta(*)$, C_α *a club of* α *disjoint to* S_1;
- (iv) $C \in \mathcal{P}_\alpha\ \&\ \beta \in \mathrm{nacc}C \Rightarrow \beta \cap C \in \mathcal{P}_\beta$;
- (v) *for any club* E *of* λ *for some* $\delta \in S_1$ *we have* $C_\delta \subseteq E$.

Remark: Note there are two points we gain: for $\alpha \in S_1$, $\mathcal{P}_\alpha$ is a singleton (as in 1.3), and an ordinal α cannot have a double role — C_α a guess (i.e. $\alpha \in S_1$) and C_α is a proper initial segment of such C_δ . When $\delta(*)$ is a regular cardinal this is easier.

Proof. Let $\mathcal{P}_{2,\alpha} = \{C_{\alpha,i} : i < \alpha\}$ (such a list exists as we have assumed $|\mathcal{P}_{2,\alpha}| \le |\alpha|$, we ignore the case $\overline{\mathcal{P}}_{2,\alpha} = \emptyset$). Now

$(*)_0$ for some $i < \lambda$ for every club E of λ for some $\delta \in S \cap E$ we have $C_{\delta,i} \setminus E$ is bounded in α. [Why? If not, for every $i < \lambda$ there is a club E_i of λ such that for no $\delta \in S \cap E$ is $C_{\delta,i} \setminus E$ bounded in α . Let $E^* = \{j < \lambda : j \text{ a limit ordinal}, j \in \bigcap_{i<j} E_i\}$, it is a club of λ, hence for some $\delta \in S \cap E^*$ and $C \in \mathcal{P}_{2,\delta}$ we have $C \subseteq E^*$. So for some $i < \alpha$, $C = C_{\delta,i}$, so $C \subseteq E^* \subseteq E_i \cup i$ hence $C_{\delta,i} \setminus i \subseteq E_i$, contradicting the choice of E_i.]

$(*)_1$ for some $i < \lambda$ and $\gamma < \delta(*)$, letting $C_\delta =: C_{\delta,i} \setminus \{\zeta \in C_{\delta,i} : \mathrm{otp}(\zeta \cap C_{\delta,i}) < \gamma\}$ we have: for every club E of λ, for some $\delta \in S \cap E$ we have: $C_\delta \subseteq E$. [Why? Let $i(*)$ be as in $(*)_0$, and for each $\gamma < \delta(*)$ suppose E_γ exemplify the failure of $(*)_1$ for $i(*)$ and γ, now $\bigcap_{\gamma<\delta(*)} E_\gamma$ is a club of λ exemplifying the failure of $(*)_0$ for $i(*)$, contradiction. So for some $\gamma < \delta(*)$ we succeed.]

$(*)_2$ Without loss of generality $|\mathcal{P}_{2,\alpha}| \le 1$, so let $\mathcal{P}_{2,\alpha} = \{C_\alpha\}$. [Why? Let i, γ and C_δ (for $\delta \in S$) be as in $(*)_1$ and use $\mathcal{P}'_{1,\alpha} = \{C \setminus \{\zeta \in C : \mathrm{otp}(\zeta \cap C) < \gamma\} : C \in \mathcal{P}_{1,\alpha}\}$, $\mathcal{P}'_{2,i} = \{C_\delta\}$.]

$(*)_3$ for some $h : \lambda \to |\delta(*)|^+$, for every $\alpha \in S$ we have $h(\alpha) \notin \{h(\beta) : \beta \in C_\alpha\}$. [Why? Choose $h(\alpha)$ by induction on α.]

$(*)_4$ for some $\beta < |\delta(*)|^+$, for every club E of λ, for some $\delta \in S \cap h^{-1}(\{\beta\})$, $C_\delta \subseteq E$. [Why? If for each β there is a counterexample E_β then $\bigcap\{E_\beta : \beta < |\delta(*)|^+\}$ is a counterexample for $(*)_2$.]

Now we have gotten the desired conclusion. $\square_{1.10}$

Claim 1.11 *If $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$, $S \in I[\lambda]$, $\kappa^+ < \lambda = \mathrm{cf}\lambda$, then for some stationary $S_1 \subseteq S$ and $\overline{\mathcal{P}}_1$ we have ${}^*\bigoplus^{\lambda,\delta(*)}_{\overline{\mathcal{P}}_1,S_1}$.*

Proof. Same proof as 1.3 (plus $(*)_3$, $(*)_4$ in the proof of 1.8). $\square_{1.11}$

Claim 1.12 *Assume $\lambda = \mu^+$, $|\delta(*)| < \mu$, $\mathrm{cf}(\delta(*)) \neq \mathrm{cf}\mu$. Then we can find stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \mathrm{cf}\delta(*)\}$ and $\overline{\mathcal{P}}$ such that ${}^*\bigotimes^{\lambda,\delta(*)}_{\overline{\mathcal{P}},S}$.*

Remark: This strengthens 1.8.

Proof.

CASE (α): μ REGULAR.

By [3], III, 6.4(2)] or [13], 2.14(2) ((c)&(d)).

CASE β: μ SINGULAR.

Let $\theta =: \mathrm{cf}\mu$, $\sigma =: |\delta(*)|^+ + \theta^+$ and $\mu = \sum_{\zeta<\theta} \mu_\zeta$, $\langle \mu_\zeta : \zeta < \theta \rangle$ strictly increasing, $\mu_0 > \sigma$ and for each $\alpha < \lambda$ let $\alpha = \bigcup_{\zeta<\theta} A_{\alpha,\zeta}$, $\langle A_{\alpha,\zeta} : \zeta < \theta \rangle$ increasing, $|A_{\alpha,\zeta}| \le \mu_\zeta$.

By 1.6 there is a sequence $\overline{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha < \lambda \rangle$ and stationary $S_1 \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) = \sigma\}$ such that $\bigoplus^{\lambda,\sigma}_{\overline{\mathcal{P}},S_1}$ of 1.3 holds. Let $\bigcup\{\mathcal{P}_\alpha : \alpha < \lambda\} \cup \{\emptyset\}$ be $\{C_\alpha : \alpha < \lambda\}$ such that $C_\alpha \subseteq \alpha$,

$[\alpha \in S_1 \Rightarrow \alpha = \sup C_\alpha \ \&\ C_\alpha \in \mathcal{P}_\alpha \ \&\ \mathrm{otp} C_\alpha = \sigma]$ and $[\alpha \notin S_1 \Rightarrow \mathrm{otp} C_\alpha < \sigma]$. For some club E_1^* of λ, $[\alpha \in E_1^* \Rightarrow \bigcup_{\beta<\alpha} \mathcal{P}_\beta = \{C_\beta : \beta < \alpha\}]$.

Looking again at $\oplus^{\lambda,\sigma}_{\mathcal{P},S_1}$, we can assume $S_1 \subseteq E_1^*$ & $(\forall\delta)[\delta \in S_1 \Rightarrow C_\delta \subseteq E_1^*]\}$, hence because we can replace every C_α by $\{\beta \in C_\alpha : \mathrm{otp}(\beta \cap C_\alpha)$ is even $\}$, without loss of generality

$$(*)\ [\delta \in S_1 \ \&\ \alpha \in \mathrm{nacc} C_\delta \Rightarrow \alpha \cap C_\delta \in \{C_\beta : \beta < \mathrm{Min}(C_\delta \cap \alpha)\}].$$

Without loss of generality $[\beta \in A_{\alpha,\zeta} \Rightarrow C_\beta \subseteq A_{\alpha,\zeta}]$ (just note $|C_\beta| \leq \sigma < \mu_\zeta$) and $\alpha \in A_{\beta,\zeta} \Rightarrow A_{\alpha,\zeta} \subseteq A_{\beta,\zeta}$. For $\alpha \in S_1$ let $C_\alpha = \{\beta_{\alpha,\varepsilon} : \varepsilon < \sigma\}$ ($\beta_{\alpha,\varepsilon}$ increasing in ε) and let $\beta^*_{\alpha,\varepsilon} \in [\beta_{\alpha,\varepsilon}, \beta_{\alpha,\varepsilon+1})$ be minimal such that $C_\alpha \cap \beta_{\alpha,\varepsilon+1} = C_{\beta^*_{\alpha,\varepsilon}}$ (exists by (*) above). Without loss of generality every C_α is an initial segment of some C_β, $\beta \in S_1$ (if not, we redefine it as $\emptyset$).

$(*)_1$ there are $\gamma = \gamma(*) < \theta$ and stationary $S_2 \subseteq S_1$ such that for every club E of λ, for some $\delta \in S_2$ we have: $C_\delta \subseteq E$, and for arbitrarily large $\varepsilon < \sigma$, $\beta^*_{\delta,\varepsilon} \in A_{\beta_{\delta,\varepsilon+1},\gamma}$. [Why? If not, for every $\gamma < \theta$ (by trying $\gamma(*) = \gamma$) there is a club E_γ of λ exemplifying the failure of $(*)_1$ for γ . Let $E = \bigcap_{\gamma<\theta} E_\gamma \cap E_1^*$, so E is a club of λ, hence

$$S' =: \{\delta : \delta < \lambda, \delta \in S_1 (\text{so } \mathrm{cf}\delta = \sigma) \text{ and } C_\delta \subseteq E\}$$

is a stationary subset of λ. For each $\delta \in S'$ and $\varepsilon < \sigma$, for some $\gamma = \gamma(\delta,\varepsilon) < \theta$ we have $\beta^*_{\delta,\varepsilon} \in A_{\beta_{\delta,\varepsilon+1},\gamma}$, but as $\sigma = \mathrm{cf}\sigma \neq \mathrm{cf}\theta = \theta$ for some $\gamma(\delta)$, $\{\varepsilon < \sigma : \varepsilon\gamma(\delta,\varepsilon) = \gamma(\delta)\}$ is unbounded in σ. But $\delta \in E_{\gamma(\delta)}$, contradiction].

$(*)_2$ Without loss of generality: if $\beta \in \mathrm{nacc} C_\alpha$, $\alpha < \lambda$ then $(\exists \xi \in A_{\beta,\gamma(*)})[\beta > \xi > \sup(\beta \cap C_\alpha) \ \&\ \beta \cap C_\alpha = C_\xi]$. [Why? Define C'_α for $\alpha < \lambda$:

$$C^0_\alpha = \{\beta : \beta \in \mathrm{nacc} C_\alpha \text{ and } (\exists \xi \in A_{\beta,\gamma(*)})[\beta > \xi \geq \sup(\beta \cap C_\alpha) \ \&\ \beta \cap C_\alpha = C_\xi]\}.$$

$$C'_\alpha \text{ is: } \begin{cases} \emptyset & \text{if } \alpha \in S_2, \alpha > \sup C^0_\alpha, \\ \alpha \cap \text{closure of } C^0_\alpha & \text{otherwise.} \end{cases}$$

Now $\langle C_\alpha : \alpha < \lambda \rangle$ can be replaced by $\langle C'_\alpha : \alpha < \lambda \rangle$].

$(*)_3$ For some $\gamma_1 = \gamma_1(*) < \theta$, for every club E of λ, for some $\delta \in E : \mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))$, and there is a club e of δ satisfying: $e \subseteq E$, $\mathrm{otp}(e)$ is $\delta(*)$, and for arbitrarily large $\beta \in \mathrm{nacc}(e)$ we have $e \cap \beta \in \{C_\zeta : \zeta \in A_{\delta,\gamma_1}\}$. [Why? If not, for each $\gamma_1 < \theta$ there is a club E_{γ_1} of λ for which there is no δ as required. Let $E =: \bigcap_{\gamma_1<\theta} E_{\gamma_1}$, so E is a club of λ, hence for some $\alpha \in \mathrm{acc}(E) \cap S_2$, $C_\alpha \subseteq E$. Letting again $C_\alpha = \{\beta_{\alpha,\varepsilon} : \varepsilon < \sigma\}$ (increasing), $C_\alpha \cap \beta_{\alpha,\varepsilon} = C_{\delta,\beta^*_{\delta,\varepsilon}}$ where $\beta^*_{\delta,\varepsilon} \in A_{\beta_{\delta,\varepsilon+1},\gamma(*)}$ clearly $\delta =: \beta_{\alpha,\delta(*)}$, $e = \{\beta_{\delta,\varepsilon} : \varepsilon < \delta(*)\}$ satisfies the requirements except the last. As $\mathrm{cf}(\delta(*)) \neq \mathrm{cf}(\mu)$, for some $\gamma_1(*) < \theta, \gamma_1(*) \geq \gamma(*)$ and $\{\varepsilon < \delta(*) : \beta^*_{\delta,\varepsilon} \in A_{\beta_{\delta,\delta(*)},\gamma_1(*)}\}$ is unbounded in $\delta(*)$. Clearly $\delta =: \beta_{\alpha,\delta(*)}$, $e =: C_\alpha \cap \delta$ satisfies the requirement. Now this contradicts the choice of $E_{\gamma_1(*)}$].

$(*)_4$ For some club E^a of λ, for every club $E^b \subseteq E^a$ of λ, for some $\delta \in E^b$ we have:

(a) $\mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))$;

(b) for some club e of $\delta : e \subseteq E^b$, $\mathrm{otp}(e) = \delta(*)$, and for arbitrarily large $\beta \in \mathrm{nacc}(e)$ we have $e \cap \beta \in \{C_\xi : \varepsilon \in A_{\delta,\gamma_1(*)}\}$;

(c) for every $\beta \in A_{\delta,\gamma_1(*)}$ we have: $C_\beta \subseteq E^a \Rightarrow C_\beta \subseteq E^b$ (we could have demanded $C_\beta \cap E^a = C_\beta \cap E^b$). [Why? If not we choose E_i for $i < \mu^+_{\gamma_1(*)}$ by induction on i, $[j < i \Rightarrow E_i \subseteq E_j]$, E_i a club of λ, and E_{i+1} exemplify the failure of E_i as a candidate for E^a. So $\bigcap_i E_i$ is a club of λ hence by $(*)_3$ there are δ and e as there. Now $\langle\{\beta \in A_{\delta,\gamma_1(*)} : C_\beta \subseteq E_i\} : i < \mu^+_{\gamma_1(*)}\rangle$ is a decreasing sequence of subsets of $A_{\delta,\gamma_1(*)}$ of length $\mu^+_{\gamma_1(*)}$, and $|A_{\delta,\gamma_1(*)}| \leq \mu_{\gamma_1(*)}$, hence it is eventually constant. So for every i large enough, δ contradicts the choice of E_{i+1}].

Let $S = \{\delta < \lambda : \mathrm{cf}(\delta) = \mathrm{cf}(\delta(*))$, and there is a club $e = e_\delta$ of δ satisfying: $e \subseteq E^a$, $\mathrm{otp}(e) = \delta(*)$, $\alpha \in \mathrm{nacc}e \Rightarrow e \cap \alpha \in A_{\alpha,\gamma(*)}$ and for arbitrarily large $\beta \in \mathrm{nacc}(e)$ we have $e \cap \beta \in \{C_\xi : \xi \in A_{\delta,\gamma(*)}\}\}$. So S is stationary, let for $\delta \in S$, C^*_δ be an e as above. For $\alpha < \lambda$ let $\mathcal{P}_{1,\alpha} = \{C_\beta : \beta \leq \alpha, \beta \in A_{\alpha,\gamma_2(*)}\}$.

$(*)_5$

(a) for every club E of λ, for some $\delta \in S$, $C^*_\delta \subseteq E$;

(b) C^*_δ is a club of δ, $\mathrm{otp}(C^*_\delta) = \delta(*)$;

(c) if $\beta \in \mathrm{nacc}C^*_\delta(\delta \in S)$ then $C^*_\delta \cap \beta \in \mathcal{P}_{1,\beta}$;

(d) $|\mathcal{P}_{1,\beta}| \leq \mu_{\gamma(*)}$, $\mathcal{P}_{1,\beta}$ is a family of closed subsets of β of order type $< \delta(*)$.

[Why? This is what we have proved.]

Now repeating $(*)_3$, $(*)_4$ of the proof of 1.10, and we finish. $\square_{1.12}$

Claim 1.13

(1) *Assume $\lambda = \mu^+$, $|\delta(*)| < \mu$, $\aleph_0 < \mathrm{cf}(\delta(*)) = \mathrm{cf}(\mu)(< \mu)$; then we can find stationary $S \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \mathrm{cf}(\delta(*))\}$ and $\overline{\mathcal{P}}$ such that ${}^*\otimes^{\lambda,\delta(*)}_{\overline{\mathcal{P}},S}$, except when:*

$\oplus$ *for every regular $\sigma < \mu$, we can find $h : \sigma \to \mathrm{cf}(\mu)$ such that for no δ, ϵ do we have: if $\delta < \sigma$, $\mathrm{cf}(\delta) = \mathrm{cf}(\mu)$, $\epsilon < \mathrm{cf}\mu$ then $\{\alpha < \delta : h(\alpha) < \epsilon\}$ is not a stationary subset of δ.*

(2) *In* 1.12 *and* 1.13(1) *we can have $\mu > \sup_{\alpha<\lambda} |\mathcal{P}_\alpha|$.*

(3) *If* 1.13(2), *if μ is strong limit we can have $|\mathcal{P}_\alpha| \leq 1$ for each α.*

Remark Compare with [7], §3.

Proof. Left to the reader (reread the proof of 1.12 and [7], §3).

Claim 1.14 *Let κ be regular uncountable. We can choose for each regular λ, $\overline{\mathcal{P}}^\lambda = \langle \mathcal{P}^\lambda_\alpha : \alpha < \lambda\rangle$ (assuming global choice) such that:*

(a) *for each λ, $\mathcal{P}^\lambda_\alpha$ is a family of $\leq \lambda$ of closed subsets of α of order type $< \kappa$.*

(b) *if χ is regular, F is the function $\lambda \mapsto \overline{\mathcal{P}}^\lambda$ (for λ regular $< \chi$), $\aleph_0 < \kappa = \mathrm{cf}\kappa$, $\kappa^{++} < \chi$, $x \in H(\chi)$ then we can find $\overline{N} = \langle N_i : i \leq \kappa \rangle$, an increasing continuous chain of elementary submodels of $(H(\chi), \in, <^*_\chi, F)$, $\langle N_j : j \leq i \rangle \in N_{i+1}$, $\|N_i\| = \aleph_0 + |i|$, $x \in N_0$ such that:*

(*) *if $\kappa^+ < \theta = \mathrm{cf}\theta \in N_i$, then for some club C of $\sup(N_\kappa \cap \theta)$ of order type κ, for any $j_1^i < j < \kappa$ we have:*

$$C \cap \sup(N_j \cap \theta) \in N_{j+1} \text{ and } \mathrm{otp}(C \cap \sup(N_j \cap \theta)) = j.$$

Proof. Let $\langle C_\alpha : \alpha \in S \rangle$ be such that $S \subseteq \{\alpha \leq \kappa^{++} : \mathrm{cf}\alpha \leq \kappa\}$ is stationary, $\mathrm{otp} C_\alpha \leq \kappa$, $[\beta \in C_\alpha \Rightarrow C_\beta = \beta \cap C_\alpha]$, C_α a closed subset of α, $[\alpha \text{ limit} \Rightarrow \alpha = \sup C_\alpha]$, $\{\alpha \in S : \mathrm{cf}\alpha = \kappa\}$ stationary, and for every club E of κ^{++} there is $\delta \in S$, $\mathrm{cf}(\delta) = \kappa$, $C_\delta \subseteq E$. For $i \in \kappa^{++} \setminus S$ let $C_i = \emptyset$. Now for every regular $\lambda > \kappa^+$ and $\alpha \leq \lambda$, let $e^\lambda_\alpha \subseteq \alpha$ be a club of α for $\alpha \leq \lambda$ limit and let

$$\overline{\mathcal{P}}^\lambda_\alpha = \{\{i \in e_\delta : i < \alpha, \mathrm{otp}(e_\delta \cap i) \in C_\beta\} : \delta < \lambda \text{ has cofinality } \kappa^{++}, \text{ and } \beta \in S\}.$$

Given $x \in H(\chi)$, we choose by induction on $i < \kappa^{++}$, M_i, N_i such that:

$N_i \prec M_i \prec (H(\chi), \in, <^*_\chi, F)$,

$\|M_i\| = |i| + \aleph_0$,

$\|N_i\| = |C_i| + \aleph_0$,

$M_i (i < \kappa^{++})$ is increasing continuous,

$x \in M_0$,

$\langle M_j : j \leq i \rangle \in M_{i+1}$,

N_i is the Skolem Hull of $\{\langle N_j : j \in C_\zeta \rangle : \zeta \in C_i\}$.

We leave the checking to the reader. $\square_{1.14}$

2 Measuring $S_{<\kappa}(\lambda)$

We prove that two natural ways to measure $S_{<\kappa}(\lambda)$ (κ regular uncountable) give the same cardinal: the minimal cardinality of a cofinal subset; i.e. its cofinality (i.e. $\mathrm{cov}(\lambda, \kappa, \kappa, 2)$) and the minimal cardinality of a stationary subset. The theorem is really somewhat stronger: for appropriate normal ideal on $S_{<\kappa}(\lambda)$, some member of the dual filter has the right cardinality.

The problem is natural and I did not trace its origin, but until recent years it seems (at least to me) it surely is independent, and I find it gratifying we get a clean answer. I thank P. Matet and M. Gitik for reminding me of the problem.

We then find applications to Δ-systems and largeness of $I[\lambda]$.

Definition 2.1

(1) $(\overline{C}, \overline{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ if

(i) $\aleph_0 < \kappa = \mathrm{cf}\kappa < \theta = \mathrm{cf}\theta$,

(ii) $\overline{C} = \langle C_\delta : \delta \in S\rangle$, $\overline{\mathcal{P}} = \langle \mathcal{P}_\delta : \delta \in S\rangle$,

(iii) $S \subseteq \theta$, S is stationary (we shall write $S = S(\overline{C})$),

(iv) C_δ is an unbounded subset of δ (not necessarily closed),

(v) $id^a(\overline{C})$ is a proper ideal (i.e. for every club E of θ for some $\delta \in S$, $C_\delta \subseteq E$),

(vi) $\bigwedge_{\delta\in S} \operatorname{otp} C_\delta < \kappa$ (hence $[\delta \in S \Rightarrow \operatorname{cf}(\delta) < \kappa]$),

(vii) $\mathcal{P}_\delta$ is a directed family of bounded subsets of C_δ, $\bigcup_{x\in\mathcal{P}_\delta} x = C_\delta$, and $|\mathcal{P}_\delta| < \kappa$,

(viii) for every $\alpha < \theta$ the set

$$\mathcal{P}^*_\alpha =: \{a \cap \alpha : \text{for some } \delta \in S \text{ we have } \alpha < \delta \in S, a \in \mathcal{P}_\delta \text{ and } \alpha \in C_\delta\}$$

has cardinality $< \theta$ or at least

(viii)$^-$ for some list $\langle a_i : i < \theta\rangle$ of $\bigcup_\alpha \mathcal{P}_\alpha$ we have: $\mathcal{P}_\alpha \subseteq \{a_j : j < \alpha\}$,

(ix) for $x \in \bigcup_{\delta\in S} \mathcal{P}_\delta$, $|\{y \in \bigcup_{\delta\in S} \mathcal{P}_\delta : y \subseteq x\}| < \kappa$.

(2) $\overline{C} \in \mathcal{T}^0[\theta,\kappa]$ if $(\overline{C},\overline{\mathcal{P}}) \in \mathcal{T}^*[\theta,\kappa]$ with $\mathcal{P}_\delta = \{C_\delta \cap \alpha : \alpha \in C_\delta\}$ or at least $\mathcal{P}_\delta = \{C_\delta \cap \alpha : C_\delta \cap \alpha \text{ has a least element}\}$.

(3) $\overline{C} \in \mathcal{T}^1[\theta,\kappa]$ if $(\overline{C},\overline{\mathcal{P}}) \in \mathcal{T}^*[\theta,\kappa]$ with $\mathcal{P}_\delta = S_{<\aleph_0}(C_\delta)$.

Note that:

Claim 2.2

(1) *If $\theta = \operatorname{cf}\theta > \kappa = \operatorname{cf}\kappa > \sigma = \operatorname{cf}\sigma$, then there is $\overline{C} \in \mathcal{T}^1[\theta,\kappa]$ such that:*

$$\{\delta \in S(\overline{C}) : \operatorname{cf}\delta = \sigma\} \neq \emptyset \bmod \operatorname{id}^a(\overline{C}).$$

(2) *If $S \subseteq \{\delta < \theta : \operatorname{cf}\delta < \kappa\}$ is stationary, $\overline{C}$ an S-club system, $|C_\delta| < \kappa$, and $\operatorname{id}^a(\overline{C})$ a proper ideal, then $\overline{C} \in \mathcal{T}^1[\theta,\kappa]$.*

(3) *In (2) if in addition $|\{C_\delta \cap \alpha : \alpha \in C_\delta, \delta \in S\}| < \theta$ then $\overline{C} \in \mathcal{T}^0[\theta,\kappa]$.*

(4) *In part (1) if θ is a successor of regular then we can demand $\overline{C} \in \mathcal{T}^0[\theta,\kappa]$ each C_δ closed.*

(5) *In part (1) if $\theta = \operatorname{cf}\theta > \kappa = \operatorname{cf}\kappa > \sigma = \operatorname{cf}\sigma$ then there is $\overline{C} \in \mathcal{T}^0[\theta,\kappa]$ such that: $\{\delta \in S(\overline{C}) : \operatorname{cf}\delta = \sigma\} \neq \emptyset \bmod id^a(\overline{C})$.*

Proof.

(1) By [13], §2 and then part (2).

(2) Check.

(3) Check.

(4) By [3], III, 6.4(2) (or [13], 2.14(2) ((c)&(d)).

(5) By 1.5 and 1.11 (so we use the non-accumulation points).

Remember (see [Sh52], §3)

Definition 2.3 (1) $\mathcal{D}^{\kappa}_{<\kappa}(\lambda)$ is the filter on $\mathcal{S}_{<\kappa}(\lambda)$ defined by:
for $X \subseteq \mathcal{S}_{<\kappa}(\lambda)$:
$X \in \mathcal{D}^{\kappa}_{<\kappa}(\lambda)$ iff there is a function F from $\bigcup_{\zeta<\kappa} {}^{\zeta}[\mathcal{S}_{<\kappa}(\lambda)]$ to $\mathcal{S}_{<\kappa}(\lambda)$ such that: if $a_\zeta \in \mathcal{S}_{<\kappa}(\lambda)$ for $\zeta < \kappa$, is increasing continuous and for each $\zeta < \kappa$ we have $F(\langle \ldots, a_\xi, \ldots \rangle)_{\xi \le \zeta} \subseteq a_{\zeta+1}$ then $\{\zeta < \kappa : a_\zeta \in X\} \in \mathcal{D}_\kappa$ ($\mathcal{D}_\kappa$ the filter generated by the family of clubs of κ).

Similarly

Definition 2.4 For $\lambda \ge \theta = \mathrm{cf}\theta > \kappa = \mathrm{cf}\kappa > \aleph_0$, $(\overline{C}, \overline{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$ we define a filter $\mathcal{D}_{(\overline{C},\overline{\mathcal{P}})}(\lambda)$ on $\mathcal{S}_{<\kappa}(\lambda)$; (let $\chi = \beth_{\omega+1}(\lambda)$):
$Y \in \mathcal{D}_{(\overline{C},\overline{\mathcal{P}})}(\lambda)$ iff $Y \subseteq \mathcal{S}_{<\kappa}(\lambda)$ and for some $x \in H(\chi)$, for every $\langle N_\alpha, N^*_a : \alpha < \theta, a \in \bigcup_{\delta\in S} \mathcal{P}_\delta \rangle$ satisfying $\otimes$ below, and also $[a \in \mathcal{P}_\delta \;\&\; \delta \in S \;\&\; \alpha < \kappa \Rightarrow x \in N^*_a \;\&\; x \in N_\alpha]$, there is $A \in \mathrm{id}^a(\overline{C})$ such that: $\delta \in S(\overline{C}) \setminus A \Rightarrow \bigcup_{a\in\mathcal{P}_\delta} N^*_a \cap \lambda \in Y$, where

$$\otimes : \begin{cases} \text{(i)} & N_\alpha \prec (H(\chi), \in, <^*_\chi); \\ \text{(ii)} & \|N_\alpha\| < \theta,\ N_\alpha \cap \theta \text{ an initial segment;} \\ \text{(iii)} & \langle N_\beta : \beta \le \alpha \rangle \in N_{\alpha+1}; \\ \text{(iv)} & N_\alpha \text{ increasing continuous;} \\ \text{(v)} & N^*_a \prec (H(\chi), \in, <^*_\chi) \text{ for } a \in \bigcup_{\delta\in S} \mathcal{P}_\delta; \\ \text{(vi)} & \|N^*_a\| < \kappa,\ N^*_a \cap \kappa \text{ an initial segment;} \\ \text{(vii)} & b \subseteq a \text{ (both in } \bigcup_{\delta\in S} \mathcal{P}_\delta) \text{ implies } N^*_b \prec N^*_a; \\ \text{(viii)} & \text{if } \alpha \in a \in \bigcup_{\delta\in S} \mathcal{P}_\delta \text{ then } \langle N_\beta, N^*_b : \beta \le \alpha, b \subseteq \alpha, b \in \bigcup_{\delta\in S} \mathcal{P}_\delta \rangle \\ & \quad \text{belongs to } N^*_a; \\ \text{(ix)} & \langle N_\beta, N^*_b : \beta \le \alpha, b \subseteq \alpha+1, b \in \bigcup_{\delta\in S} \mathcal{P}_\delta \rangle \text{ belongs to } N_{\alpha+1}; \\ \text{(x)} & a \subseteq N^*_a \text{ and } \alpha \in a \Rightarrow \alpha \cap a \in N^*_a; \\ \text{(xi)} & a \subseteq \alpha, a \in \bigcup_{\delta\in S} \mathcal{P}_\delta \text{ implies } N^*_a \in N_{\alpha+1} \text{ (remember (viii) of 2.1).} \end{cases}$$

Clearly

Claim 2.5

(1) *Any $\chi \ge 2^\lambda$ can serve, and $x = (Y, \lambda, \overline{C}, \overline{\mathcal{P}})$ is enough.*

(2) *$\mathcal{D}_{(\overline{C},\overline{\mathcal{P}})}(\lambda)$ is a fine normal filter on $\mathcal{S}_{<\kappa}(\lambda)$ when $(\overline{C}, \overline{\mathcal{P}}) \in \mathcal{T}^*[\theta, \kappa]$, $\lambda \ge \theta$, hence it extends $\mathcal{D}_{<\kappa}(\lambda)$. (Remember $\mathrm{id}^a(\overline{C})$ is a proper ideal.)*

Theorem 2.6 *Suppose $\lambda > \kappa = \mathrm{cf}\kappa > \aleph_0$. Then the following three cardinals are equal for $(\overline{C}, \overline{\mathcal{P}}) \in \mathcal{T}^*[\kappa^+, \kappa]$:*

$\mu(0) = \mathrm{cf}(\mathcal{S}_{<\kappa}(\lambda), \subseteq)$,

$\mu(1) = \mathrm{cov}(\lambda, \kappa, \kappa, 2) = \min\{|\mathcal{P}| : \mathcal{P} \subseteq \mathcal{S}_{<\kappa}(\lambda)$, *and for every $a \subseteq \lambda$, $|a| < \kappa$, there is $b \in \mathcal{P}$, $a \subseteq b\}$,*

$\mu(2) = \min\{|S| : S \subseteq \mathcal{S}_{<\kappa}(\lambda)$ *is stationary*$\}$,
$\mu(3) = \mu_{(\overline{C},\overline{\mathcal{P}})} = \min\{|Y| : Y \in \mathcal{D}_{(\overline{C},\overline{\mathcal{P}})}(\lambda)\}$.

Remark 2.6A

(1) It is well known that if $\lambda > 2^{<\kappa}$ then the equality holds.

(2) This is close to "strong covering".

(3) In the proof we may replace "$\theta = \kappa^{+}$" by "$\lambda > \theta = \mathrm{cf}\theta > \kappa$" if $\alpha < \theta \Rightarrow \mathrm{cov}(\alpha, \kappa, \kappa, 2) < \theta$.

(4) Note if $\lambda = \kappa$, then $\mu(1) = \mu(2)$ trivially.

(5) Note that only $\mu(3)$ has $(\overline{C}, \overline{\mathcal{P}})$ in its definition, so actually $\mu(3)$ does not depend on $(\overline{C}, \overline{\mathcal{P}})$.

Remark 2.6B

(1) We can weaken in Definition 2.1(1) demand (ix) as follows:

(ix) there is a sequence $\langle a_i, \mathcal{P}^*_i : i < \lambda\rangle$ such that

(a) $|a_i| < \kappa$, $\mathcal{P}^*_i$ is a family of $< \kappa$ subsets of a_i;

(b) for every δ and $x \in \mathcal{P}_\delta$, for some $i < \delta$, $a_i = x$ and

$$(\forall b)[b \in \mathcal{P}_\delta \ \& \ b \subseteq a \Rightarrow b \in \mathcal{P}^*_i].$$

In this case 2.6, 2.6A(3) (and 2.5) remain true and we can strengthen 2.2.

(2) We can even use another order on $\mathcal{P}_\delta$ (not $\subseteq$).

Proof. Clearly $\lambda \le \mu(0) = \mu(1) \le \mu(2) \le \mu(3)$ (the last by 2.5(2)). So we shall prove $\mu(3) \le \mu(1)$, (suffices by 2.2(1)) and let $\mathcal{P}$ exemplify $\mu(1) = \mathrm{cov}(\lambda, \kappa, \kappa, 2)$.

Let χ be e.g. $\beth_3(\lambda)^+$, M^*_λ be the model with universe $\lambda + 1$ and all functions definable in $(H(\chi), \in, <^*_\chi, \lambda, \kappa, \mu(1))$. Let M^* be an elementary submodel of $(H(\chi), \in, <^*_\chi)$ of cardinality $\mu(1)$, $\mathcal{P} \in M^*$, $M^*_\lambda \in M^*$, $(\overline{C}, \overline{\mathcal{P}}) \in M^*$ and $\mu(1) + 1 \subseteq M^*$ hence $\mathcal{P} \subseteq M^*$. It is enough to prove that $M^* \cap \mathcal{S}_{<\kappa}(\lambda)$ belongs to $\mathcal{D}_{(\overline{C},\overline{\mathcal{P}})}(\lambda)$.

So let N_i (for $i < \kappa^+$), N^*_a (for $a \in \bigcup_{\delta \in S} \mathcal{P}_\delta$) be such that: they satisfy $\otimes$ of 2.4 and M^*_λ, M^*, $\mathcal{P}$, λ , κ , $\overline{C}$, $\overline{\mathcal{P}}$ belong to every N_α, N^*_a. It is enough to prove that $\{\delta < \kappa^+ : \lambda \cap \bigcup_{a \in \mathcal{P}_\delta} N^*_x \in M^*\} = \kappa^+ \bmod \mathrm{id}^a(\overline{C})$.

For each $i \in S$ there is a set a_i such that $(\bigcup_{y \in \mathcal{P}_i} N^*_y) \cap \lambda \subseteq a_i \in \mathcal{P}$; so without loss of generality $a_i \in N_{i+1}$. Let $\mathfrak{a}_i =: \mathrm{Reg} \cap a_i \cap \lambda^+ \backslash \kappa^{++}$, so $\mathfrak{a}_i$ is a set of $< \kappa$ regular cardinals $> \kappa^+$ and $\mathfrak{a}_i \in N_{i+1}$ too, so there is $\langle \mathfrak{b}_\lambda[\mathfrak{a}_i] : \lambda \in \mathrm{pcf}\mathfrak{a}_i\rangle$ as in [13], 2.6, without loss of generality it is definable from $\mathfrak{a}_i$ (in $(H(\chi), \in, <^*_\chi)$). Also $a \in \mathcal{P} \subseteq M^*$ so $a \in M^*$, so $\mathfrak{a} \in M^*$. Hence $\langle \mathfrak{b}_\lambda[\mathfrak{a}_i] : \lambda \in \mathrm{pcf}\mathfrak{a}_i\rangle \in N_{i+1} \cap M^*$, and also there is $\langle f^{\mathfrak{a}_i}_{\theta,\alpha} : \alpha < \theta, \theta \in \mathrm{pcf}\mathfrak{a}_i\rangle$ as in [13], 1.2, and again without loss of generality it belongs to $N_{i+1} \cap M^*$. As $\max \mathrm{pcf}\mathfrak{a}_i \le \mathrm{cov}(\lambda, \kappa, \kappa, 2) \le \mu(1)$ (first inequality by [10], 5.4), clearly each $f^{\mathfrak{a}_i}_{\theta,\alpha} \in M^*$. Let h be the function with domain $\bigcup_{i \in S} \mathfrak{a}_i$, $h(\theta) = \sup(\theta \cap \bigcup_{i<\kappa^+} N_i)$. So by [13], 2.3(1) each $h{\restriction}\mathfrak{a}_i$ has the form $\max\{f^{\mathfrak{a}_i}_{\theta_\ell,\alpha_\ell} : \ell < n\}$ hence belongs to M^*. Let e be a definable function in $(H(\chi), \in, <^*_\chi, \lambda, \kappa)$, Dom $e = \lambda + 1$, e_α is a club of α of order type cfα, enumerated as $\langle e_\alpha(\zeta) : \zeta < \text{cf}\alpha\rangle$. Now for each $\theta \in \bigcup_{i<\kappa^+} \mathfrak{a}_i$,

$$E_\theta =: \{i < \kappa^+ : (\forall \zeta < \kappa^+)[[e_{h(\theta)}(\zeta) \in N_i \Leftrightarrow \zeta < i],\ i \text{ is limit},$$
$$\theta \in \bigcup_{j<i} \mathfrak{a}_j \text{ and } \sup(N_i \cap \theta) = \sup\{e_{h(\theta)}(\zeta) : \zeta < i\}]\}$$

is a club of κ^+, hence

$$E = \{\delta < \kappa^+ : \delta \text{ limit and } [\theta \in (\bigcup\{N_y : \text{ for some } \alpha \in S \text{ and } \zeta \in C_\alpha \cap \delta$$
$$\text{we have } \sup y < \zeta,\ y \in \mathcal{P}_\alpha\})\ \&\ \theta \in \text{Reg} \cap \lambda^+ \backslash \kappa^{++} \Rightarrow \delta \in E_\theta] \text{ and } N_\delta \cap \kappa^+ = \delta\}$$

is a club of κ^+ (note: we use (viii) of Definition 2.1(1)). For each $\delta \in E \cap S$ such that $C_\delta \subseteq E$, let $\delta^* = \sup(\kappa \cap \bigcup_{y \in \mathcal{P}_\delta} N^*_y)$ so $\delta^* < \kappa$, and we define by induction on n models $M_{y,\delta,n}$ for every $y \in \mathcal{P}_\delta$ (really, they do not depend on δ). Now $M_{y,\delta,0}$ is the Skolem Hull in M^*_λ of $\{i : i \in y\} \cup \{j : j < \delta^*\}$. $M_{y,\delta,n+1}$ is the Skolem Hull in M^*_λ of

$$M_{y,\delta,n} \cup \{e_\theta(\zeta) : \theta \in (\text{Reg} \cap \lambda^+ \backslash \kappa^{++}) \cap M_{y,\delta,n} \text{ and } \zeta \in y\}.$$

Now (A), (B), (C), (D), (E) below suffice to finish.

(A) We can easily prove by induction on n that:

  (a) for $y \in \mathcal{P}_\delta$ we have $M_{y,\delta,n} \subseteq \bigcup_{y \in \mathcal{P}_\delta} N^*_y$;
  (b) for $z \subseteq y$ in $\mathcal{P}_\delta$ we have $M_{z,\delta,n} \subseteq M_{y,\delta,n}$;
  (c) for $y \in \mathcal{P}_\delta$ and $m < n$ we have $M_{y,\delta,m} \subseteq M_{y,\delta,n}$;
  (d) assume $i \in y$ (hence $i \in E$), $\{y, z\} \subseteq \mathcal{P}_\delta$, $\sup z < i$, $z \subseteq y$ and $\theta \in \bigcup_{i<\kappa^+} N_i \cap \text{Reg} \cap \lambda^+ \backslash \kappa^{++}$; then:
  $\theta \in N^*_z \cap N_i \Rightarrow e_{h(\theta)}(i) = \sup(\theta \cap N_i) \in N^*_y$ and
  $\theta \in M_{z,\delta,n} \cap N_i \Rightarrow \sup(M_{z,\delta,n} \cap \theta) \le e_{h(\theta)}(i) \le \sup(M_{y,\delta,n} \cap \theta)$.

(B) We can also prove that $\langle M_{y,\delta,n} : n < \omega, y \in \mathcal{P}_\delta\rangle$ is definable in $(H(\chi), \in, <^*_\chi)$ from the parameters δ, M^*_λ, $(\overline{C}, \overline{\mathcal{P}})$ and $h \restriction \mathfrak{a}_i$, all of them belong to M^*, hence the sequence, and $\bigcup_{n<\omega, y \in \mathcal{P}_\delta} M_{y,\delta,n}$ belongs to M^*.

(C) $(\bigcup_{n<\omega, y\in\mathcal{P}_\delta} M_{y,\delta,n}) \cap \text{Reg} \cap (\kappa^+, \lambda^+)$ is a subset of $\mathfrak{a}_i$ (use (A)(a) and definition of $a_i, \mathfrak{a}_i$).

(D) if $\sigma \in \bigcup_{n<\omega, y\in\mathcal{P}_\delta} M_{y,\delta,n}$, $\sigma \in \text{Reg} \cap \lambda^+ \backslash \kappa$ then $\sigma \cap \bigcup_{n<\omega} M_{y,\delta,n}$ is unbounded in $\sigma \cap \bigcup_{y\in\mathcal{P}_\delta} N^*_\delta$ [when $\sigma > \kappa^+$ use $(*)$, for $\sigma = \kappa^+$ as C_δ is equal to $\bigcup_{y\in\mathcal{P}_\delta} y$ and $\delta = \sup C_\delta$, for $\sigma = \kappa$ see (d), choice of $M_{y,\delta,0}$].

(E) $\bigcup_{n<\omega, y\in\mathcal{P}_\delta} M_{y,\delta,n} \cap \lambda = \bigcup_{y\in\mathcal{P}_\delta} N^*_y \cap \lambda$. (See [14], 3.3A, 5.1A). $\square_{2.6}$

Conclusion 2.7 *Suppose $\lambda > \kappa > \aleph_0$ are regular cardinals and $(\forall \mu < \lambda)[\text{cov}(\mu, \kappa, \kappa, 2) < \lambda]$. If for $\alpha < \lambda$, a_α is a subset of λ of cardinality $< \kappa$ and $S \in \mathcal{D}_{<\kappa}(\lambda)$ (or just $S \neq \emptyset$*

mod $\mathcal{D}^{\kappa}_{<\kappa}(\lambda)$) *then we can find a stationary* $T \subseteq \{\delta < \lambda : \mathrm{cf}\delta = \kappa\}$, $c \subseteq \lambda$ *and* $\langle b_\delta : \delta \in T\rangle$ *such that:*

$$a_\delta \subseteq b_\delta \in S \text{ for } \delta \in T$$

and

$$b_\delta \cap \delta = c \text{ for } \delta \in T.$$

Remark: See on this and on 2.9 Rubin Shelah [2] and [12], §6.

Conclusion 2.8 *If* $\lambda > \kappa > \aleph_0$, λ *and* κ *are regular cardinals and* $[\kappa < \mu < \lambda \Rightarrow \mathrm{cov}(\mu, \kappa, \kappa, 2) < \lambda]$ *then* $\{\delta < \lambda : \mathrm{cf}(\delta) < \kappa\} \in I[\lambda]$.

Proof. Use $\mu(3)$ of 2.6.

Claim 2.9 *Let* $(*)_{\mu,\lambda,\kappa}$ *mean: if* $a_i \in \mathcal{S}_{<\kappa}(\lambda)$ *for* $i \in S$, $S \subseteq \{\delta < \mu : \mathrm{cf}\delta = \kappa\}$ *is stationary, then for some* $b \in \mathcal{S}_{<\kappa}(\lambda)$, $\{i \in S : a_i \cap i \subseteq b\}$ *is stationary. Let* $(*)^-_{\mu,\lambda,\kappa}$ *be defined similarly but* $\{i \in S : a_i \subseteq b\}$ *only unbounded. Then for* $\aleph_0 < \kappa < \lambda < \mu$ *regular we have:*

$$\mathrm{cov}(\lambda, \kappa, \kappa, 2) < \mu \Rightarrow (*)_{\mu,\lambda,\kappa} \Rightarrow (*)^-_{\mu,\lambda,\kappa}$$
$$\Rightarrow (\forall \lambda' \leq \lambda)[\kappa < \lambda' \leq \lambda \;\&\; \mathrm{cf}\lambda' < \kappa \Rightarrow \mathrm{pp}_{<\kappa}\lambda' < \mu].$$

Remark So it is conceivable that the $\Rightarrow$ are $\Leftrightarrow$. See [12], §3.

Proof. Straightforward.

3 Nice Filters Revisited

This generalizes [11] (and see there).

See [15], §5 on this generalization of normal filters.

Conventions 3.1

(1) We use $\aleph_1$ rather than an uncountable regular κ for simplicity.

(2) Let μ^* be $> \aleph_1$ and $\mathcal{Y}_i = \{i\} \times (\bigcup_{\mu<\mu^*} \mu)$, $\mathcal{Y} = \bigcup_{i<\omega_1} \mathcal{Y}_i$, $\iota(y) = i$ when $y \in \mathcal{Y}_i$.

(3) Let Eq denote a set of equivalence relations e on $\mathcal{Y}$ refining $\bigcup_{i<\omega_1} \mathcal{Y}_i \times \mathcal{Y}_i$ with $< \mu^*$ equivalence classes, each class of cardinality $|\mathcal{Y}|$. We say $e_1 \leq e_2$ if e_2 refines e_1. If not said otherwise, every e is in Eq. Let Eq_μ be the set of all such equivalence relations with $< \mu$ equivalence classes. Let $\iota(x/e) = \iota(x)$.

Definition 3.2

(1) Let FIL(e) = FIL$(e, \mathcal{Y})$ denote the set of D such that:

  (a) D is a filter on $\mathcal{Y}/e$,

  (b) for any club C of ω_1, $\bigcup_{i\in C} \mathcal{Y}_i/e \in D$,

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(c) (*normality*) if $X_i \in D$ for $i < \omega_1$ then $\{(\delta,j)/e : (\delta,j) \in \mathcal{Y}$, δ limit and $i < \delta \Rightarrow (\delta,j) \in X_i\}$ belongs to D.

(2) $\mathrm{FIL}(\mathcal{Y}) = \mathrm{FIL}(Eq,\mathcal{Y})$ is $\bigcup_{e\in Eq} \mathrm{FIL}(e,\mathcal{Y})$. For $D \in \mathrm{FIL}(\mathcal{Y})$, let $e = e[D]$ be such that $D \in \mathrm{FIL}(e,\mathcal{Y})$.

(3) For $D \in \mathrm{FIL}(e)$ let $D^{[*]} = \{X \subseteq \mathcal{Y} : \{y/e : y/e \subseteq X\} \in D\}$.

(4) For $D \in \mathrm{FIL}(\mathcal{Y})$ and $e(1) \geq e(D)$, let $D^{[e(1)]} = \{X \subseteq \mathcal{Y}/e(1) : X^{[*]} \in D^{[*]}\}$.

(5) For $A \subseteq \mathcal{Y}/e$, $A^{[*]} = \bigcup_{(x/e)\in A} x/e$, and for $e(1) \geq e$ let

$$A^{[e(1)]} = \{y/e(1) : y/e \in A\}.$$

Definition 3.2A For $D \in \mathrm{FIL}(e,\mathcal{Y})$, let D^+ be $\{Y \subseteq \mathcal{Y}/e : Y \neq \emptyset \bmod D\}$.

Definition 3.3

(0) For $f : \mathcal{Y}/e \to X$ let $f^{[*]} : \mathcal{Y} \to X$ be $f^{[*]}(x) = f(x/e)$. We say $f : \mathcal{Y} \to X$ is supported by e if it has the form $g^{[*]}$ for some $g : \mathcal{Y}/e \to X$. Let $e_1, e_2 \in Eq$, $f_\ell : \mathcal{Y}/e_\ell \to X$; we say $f_1 = f_2^{[e_1]}$ if $f_1^{[*]} = f_2^{[*]}$.

(1) Let $F_c({}^\omega\omega, e) = F_c({}^\omega\omega, e, \mathcal{Y})$ be the family of $\bar{g}$, a sequence of the form $\langle g_\eta : \eta \in u\rangle$, $u \in f_c({}^\omega\omega)$ = the family of non-empty finite subsets of ${}^{\omega>}\omega$ closed under initial segment, and for each $\eta \in u$ we have $g_\eta \in {}^{\mathcal{Y}}\mathrm{Ord}$ is supported by e. Let $\mathrm{Dom}\,\bar{g} = u$, $\mathrm{Range}\,\bar{g} = \{g_\eta : \eta \in u\}$. We let $e = e(\bar{g})$, an abuse of notation.

(2) We say $\bar{g}$ is decreasing for D or D-decreasing (for $D \in \mathrm{FIL}(e,I)$) if $\eta \triangleleft \nu \Rightarrow g_\nu <_D g_\eta$.

(3) If $u = \{<>\}$, $g = g_{<>}$ we write g instead of $\langle g_\eta : \eta \in u\rangle$.

Definition 3.4

(1) For $e \in Eq$, $D \in \mathrm{FIL}(e,\mathcal{Y})$ and D-decreasing $\bar{g} \in F_c({}^\omega\omega, e)$ we define a game $G^*(D,\bar{g},e,\mathcal{Y})$ (we may omit $\mathcal{Y}$). In the nth move (stipulating $e_{-1} = e$, $D_{-1} = D$, $\bar{g}_{-1} = \bar{g}$):

> player I chooses $e_n \geq e_{n-1}$ and $A_n \subseteq \mathcal{Y}/e_n$, $A_n \neq \emptyset \bmod D_{n-1}^{[e_n]}$ and he chooses $\bar{g}^n \in F_c({}^\omega\omega, e_n)$ extending $\bar{g}_{n-1}$ (i.e. $\bar{g}^{n-1} = \bar{g}^n \restriction \mathrm{Dom}\,\bar{g}_{n-1}$), $\bar{g}^n$ supported by e_n and $\bar{g}^n$ is $(D_n^{[e_n]} + A_n)$-decreasing, player II chooses $D_n \in \mathrm{FIL}(e_n,\mathcal{Y})$ extending $D_{n-1}^{[e_n]} + A_n$.

In the end, the second player wins if $\bigcup_{n<\omega} \mathrm{Dom}\,\bar{g}^n$ has no infinite branch.

(2) $G^{\bar{\gamma}}(D,\bar{g},e,\mathcal{Y})$ is defined similarly to $G^*(D,\bar{g},e,\mathcal{Y})$ ($\mathrm{Dom}\,\bar{\gamma} = \mathrm{Dom}\,\bar{g}$) but the second player has, in addition, to choose an ordinal α_η for $\eta \in \mathrm{Dom}\,\bar{g}^n \setminus \bigcup_{\ell<n} \mathrm{Dom}\,\bar{g}^\ell$ such that $[\eta \triangleleft \nu \;\&\; \nu \in \mathrm{Dom}\,\bar{g}^{n-1} \Rightarrow \alpha_\nu < \alpha_\eta]$ and $\alpha_\eta = \gamma_\eta$ for $\eta \in \mathrm{Dom}\,\bar{g}$.

(3) $wG^*(D,\bar{g},e,\mathcal{Y})$ and $wG^{\bar{\gamma}}(D,\bar{g},e,\mathcal{Y})$ are defined similarly but e is not changed during a play.

(4) If $\bar{\gamma} = \langle\gamma_{<>}\rangle$, $\bar{g} = \langle g_{<>}\rangle$ we write $\gamma_{<>}$ instead of $\bar{\gamma}$, $g_{<>}$ instead of $\bar{g}$.

(5) If $E \subseteq \mathrm{FIL}(\mathcal{Y})$ the games G^*_E, $G^{\bar{\gamma}}_E$ are defined similarly, but player II can choose filters only from E (so we like to have $A \in D^+$, $D \in E \Rightarrow D + A \in E$).

Remark 3.4A Denote the above games $G^*_0, G^{\bar{\gamma}}_0$. Another variant is

(3) For $e \in Eq$, $D \in \mathrm{FIL}(e,\mathcal{Y})$ and D-decreasing $\bar{g} \in F_c({}^\omega\omega)$ we define a game $G^*_1(D,\bar{g},e,\mathcal{Y})$. We stipulate $e_{-1} = e$, $D_{-1} = D$.

In the nth move first player chooses $e_n, e_{n-1} \leq e_n \in Eq$ and $D'_n \in \mathrm{FIL}(e_n,\mathcal{Y})$ such that:

($*$) for some $A_n \subseteq \mathcal{Y}/e_{n-1}$, $A_n \neq \emptyset \bmod D_{n-1}$ we have:

(i) $(D_{n-1} + A_n)^{[e_n]} \subseteq D_n$;

(ii) D'_n is the normal filter on $\mathcal{Y}/e_n$ generated by $(D_{n-1} + A_n)^{[e_n]} \cup \{A^n_\zeta : \zeta < \zeta^*_n\}$ where for some $\langle C_\zeta : \zeta < \zeta_n\rangle$ we have:

(a) each C_ζ is a club of ω_1,

(b) if $\zeta_\ell < \zeta^*_n$ for $\ell < \omega$, $i \in \bigcap_{\ell<\omega} C_{\zeta_\ell}$, $x \in \mathcal{Y}/e_{n-1}$, and $\iota(x) = i$, then for some $x' \in \mathcal{Y}/e_n$, we have $x' \subseteq x$, $x' \in \bigcap_{\ell<\omega} A^n_{\zeta_\ell}$.

First player also chooses $\bar{g}^n$ extending $\bar{g}^{n-1}$ D'_n-decreasing and the second player chooses D_n, $D'_n \subseteq D_n \in \mathrm{FIL}(e_n,\mathcal{Y}_n)$.

(4) We define $G^{\bar{\gamma}}_1(D,\bar{g},e,\mathcal{Y})$ as in (2) using G^*_1 instead of G^*_0.

(5) If player II wins, e.g. $G^{\bar{\gamma}}_E(D,\bar{f},e,\mathcal{Y})$ this is true for $E' =: \{D' \in G :$ player II wins $G^{\bar{\gamma}}_{E^*}(D',\bar{f},e,Y)\}$.

Definition 3.5

(1) We say $D \in \mathrm{FIL}(\mathcal{Y})$ is nice to $\bar{g} \in F_c({}^\omega\omega,e,\mathcal{Y})$, $e = e(D)$, if player II wins the game $G^*(D,\bar{g},e)$ (so in particular $\bar{g}$ is D-decreasing, $\bar{g}$ supported by e).

(2) We say $D \in \mathrm{FIL}(\mathcal{Y})$ is nice if it is nice to $\bar{g}$ for every $\bar{g} \in F_c({}^\omega\omega,e,\mathcal{Y})$.

(3) We say D is nice to α if it is nice to the constant function α. We say D is nice to $g \in {}^{\aleph_1}\mathrm{Ord}$ if it is nice to $g^{[e(D)]}$.

(4) "Weakly nice" is defined similarly but e is not changed.

Remark "Nice" in [11] is the weakly nice here, but formally they act on different objects; but if $x\ e\ y \Leftrightarrow \iota(x) = \iota(y)$ we get a situation isomorphic to the old one.

Claim 3.6 *Let $D \in \mathrm{FIL}(\mathcal{Y})$ and $e = e(D)$.*

(1) *If D is nice to f, $f \in F_c({}^\omega\omega,e,\mathcal{Y})$, $g \in F_c({}^\omega\omega,e,\mathcal{Y})$ and $g \leq f$ then D is nice to f.*

(2) *If D is nice to f, $e = e(D) \le e(1) \in Eq$ then $D^{[e(1)]}$ is nice to $f^{[e(1)]}$.*

(3) *The games from 3.4(2) are determined and winning strategies do not need memory.*

(4) *D is nice to $\bar{g}$ iff D is nice to $g_{<>}$ (when $\bar{g} \in F_c({}^{\omega}\omega, e, \mathcal{Y})$ is D-decreasing).*

(5) *If $Eq' \subseteq Eq$ and for simplicity $\bigcup_{i<\omega_1}\{i\} \times \mathcal{Y}_i \in Eq'$ and for every $e \in Eq'$, $e \le e(1) \in Eq$ for some permutation π of $\mathcal{Y}$, $\pi(e) = e$, $\pi(e(1)) \le e(2) \in Eq'$ then we can replace Eq by Eq'.*

(6) *For $Eq = Eq_\mu$ (where $\mu \le \mu^*$) there is Eq' as above with: $|Eq'|$ countable if μ is a successor cardinal ($> \aleph_1$), $|Eq'| = \mathrm{cf}\mu$ if μ is a limit cardinal.*

Proof. Left to the reader. (For part (4) use 3.7(2) below.) $\square_{3.6}$

Claim 3.7

(1) *Second player wins $G^*(D, \bar{g}, e)$ iff for some $\bar{\gamma}$ second player wins $G^{\bar{\gamma}}(D, \bar{g}, e)$.*

(2) *If second player wins $G^{\gamma}(D, f, e)$ then for any D-decreasing $\bar{g} \in F_c({}^{\omega}\omega, e, \mathcal{Y})$, $\bar{g}$ supported by e and $\bigwedge_{\eta,x} g_\eta(x) \le f(x)$, the second player wins in $G^{\bar{\gamma}}(D, \bar{g}, e)$, when we let*

$$\gamma_\eta = \gamma \times [\max_{\eta \trianglelefteq \nu \in \mathrm{Dom}\,\bar{g}} (\ell g(\nu) - \ell g(\nu) + 1)].$$

(3) *If $u_1, u_2 \in f_c({}^{\omega >}\omega)$, $h : u_1 \to u_2$ satisfies $[\eta \triangleleft \nu \Leftrightarrow h(\eta) \triangleleft h(\nu)]$ and for $\ell = 1, 2$ we have $\bar{g}^\ell \in F_2({}^{\omega >}\omega, e_2, \mathcal{Y})$, $g^1_\eta = g^2_{h(\eta)}$ (for $\eta \in u_1$), $\bar{\gamma}^\ell = \langle \gamma^\ell_\eta : \eta \in u_2 \rangle$ is $\triangleleft$-decreasing sequence of ordinals, $\gamma^1_\eta \ge \gamma^2_{h(\eta)}$ and the second player wins in $G^{\bar{\gamma}^2}(D, \bar{g}^2, e, \mathcal{Y})$ then the second player wins in $G^{\bar{\gamma}^1}(D, \bar{g}^1, e, \mathcal{Y})$.*

Proof.

(1) The "if" part is trivial, the "only if" as in [11].

The following is a consequence of a theorem of Dodd and Jensen [DoJ]:

Theorem 3.8 *If λ is a cardinal, $S \subseteq \lambda$ then:*

(1) *$K[S]$, the core model, is a model of ZFC $+ (\forall \mu \ge \lambda) 2^\mu = \mu^+$.*

(2) *If in $K[S]$ there is no Ramsey cardinal $\mu > \lambda$ (or much weaker condition holds) then $(K[S], V)$ satisfies the μ-covering lemma for $\mu \ge \lambda + \aleph_1$; i.e. if $B \in V$ is a set of ordinals of power $\le \mu$ then there is $B' \in K[S]$, $B \subseteq B'$, $V \models |B'| \le \mu$.*

(3) *If $V \models (\exists \mu \ge \lambda)(\exists \kappa)[\mu^\kappa > \mu^+ > 2^\kappa]$ then in $K[S]$ there is a Ramsey cardinal $\mu > \lambda$.*

Lemma 3.9 *Suppose $f \in {}^{\aleph_1}\mathrm{Ord}$, $\lambda > \lambda_0 =: \sum_{\alpha<\mu^*} 2^{|\alpha|^{\aleph_0}} + \prod_{i<\omega_1} |f(i) + 1| + |Eq|$, and for every $A \subseteq \lambda_0$, in $K[A]$ there is a Ramsey cardinal $> \lambda_0$, then for every normal filter $D \in \mathrm{FIL}(e, \mathcal{Y})$, D is nice to f.*

Remark: The point in the proof is that via forcing we translate the filters from FIL$(e, \mathcal{Y})$ to normal filters on ω_1 [for higher κ's cardinal restrictions are better].

Proof. Without loss of generality $(\forall i) f(i) \geq 2$.

Let $S \subseteq \lambda_0$ be such that $[\alpha < \mu^*$ & $A \subseteq 2^{|\alpha|^{\aleph_0}} \Rightarrow A \in L[S]]$, $Eq \in L[S]$ and: if $g \in {}^{\aleph_1}\mathrm{Ord}$, $(\forall i < \omega_1) g(i) \leq f(i)$ then $g \in L[S]$ (possible as $\prod_{i<\omega_1} |f(i)+1| \leq \lambda_0$). We work for awhile in $K[S]$. In $K[S]$ there is a Ramsey cardinal $\mu > \lambda_0$ (see 3.8(3)). Let, in $K[S]$,

$$I = \{X : X \subseteq \mu, X \cap \omega_1 \text{ a countable ordinal } > 0, \{\omega_1, \mu\} \subseteq X, \text{ moreover } X \cap \lambda_0 \text{ is countable}\}.$$

Let

$$J = \{X \in I : X \text{ has order type } \geq f(X \cap \omega_1)\}.$$

Now for $g \in {}^{\aleph_1}\mathrm{Ord}$ such that $\bigwedge_{i<\omega_1} g(i) < f(i)$ let $\hat{g}$ be the function with domain J, $\hat{g}(X) =$ the $g(X \cap \omega_1)$-th member of X.

Let $D = \{A_i : \omega_1 \leq i \leq 2^{|\mathcal{Y}/e|}\}$ and we arrange $\langle A_i : \omega_1 \leq i < 2^{|\mathcal{Y}/e|}\rangle \in L[S]$, (as $\mathcal{Y}/e$ has cardinality $< \mu^*$, so $2^{|\mathcal{Y}/e|} \leq \lambda_0$).

Let F be the minimal fine normal filter on I (in $K[S]$) to which J_D belongs where

$$J_D = \{X : X \in J \text{ and } i \in (\omega_1, 2^{|\mathcal{Y}/e|}) \cap X \Rightarrow X \cap \omega_1 \in A_i\}.$$

Clearly it is a proper filter as $K[S] \models$ "μ is a Ramsey cardinal".

Observation 3.9A [in $K[S]$]. Assume P is a proper forcing notion of cardinality $\leq |\alpha|^{\aleph_0}$ for some $\alpha < \mu^*$ (or just P, MAC$(P) \in K[S]$ and $\{X \in I : X \cap |\mathrm{MAC}(P)|$ is countable$\} \in F$ where MAC(P) is the set of maximal antichains of P) and let F^P be the normal fine filter which F generates in V^P. Then

(1) F-positiveness is preserved; i.e. if $X \in V$, $X \subseteq I$, $F \in$ FIL$(\mathcal{Y})$ and $V \models$ "$X \neq \emptyset \bmod F$" then $\Vdash_P$"$X \neq \emptyset \bmod F^P$".

(2) Moreover, if $Q < P$, (Q proper and) P/Q is proper then forcing with P/Q preserve F^Q-positiveness.

Let $\mathcal{P}(\mathcal{Y}/e) = \{A^e_\zeta : \zeta < 2^{|\mathcal{Y}/e|}\}$.

Now we describe a winning strategy for the second player. In the side we choose also $(p_n, \Gamma_n, \underset{\sim}{f}_n)$, $\overline{\gamma}^n$, $\underset{\sim}{W}_n$ such that[1] (where e_n, A_n are chosen by the second player):

(A) (i) $P_n = \prod_{\ell \leq n} Q_\ell$, Q_ℓ is Levy$(\aleph_1, \mathcal{Y}/e_n)$ (we could use iterations, too, here it does not matter);

(ii) $p_n \in P_n$;

(iii) p_n increasing in n;

(iv) $\underset{\sim}{f}_n$ is a P_n-name of a function from ω_1 to $\mathcal{Y}/e_n$;

(v) $p_n \Vdash_{P_n}$ "$\underset{\sim}{f}_n(i) \in \mathcal{Y}_i/e_n$";

[1]By the homogeneity of the forcing notion the value of p_n is immaterial.

(vi) $p_{n+1} \Vdash$ "$\underset{\sim}{f}_{n+1}(i) \subseteq \underset{\sim}{f}_n(i)$ for every $i < \omega_1$";

(vii) $\underset{\sim}{f}_n$ is given naturally — it can be interpreted as the generic object of Q_n except trivialities.

(B) (i) $\bar{\gamma}^n, \bar{g}^n$ has the same domain, $\gamma^n_\eta < \mu$;

(ii) $p_n \Vdash_{P_n}$ "$\underset{\sim}{W}_n \subseteq J_D$, $\underset{\sim}{W}_{n+1} \subseteq \underset{\sim}{W}_n$";

(iii) $\bar{\gamma}^n = \gamma^{n+1} \restriction \mathrm{Dom}\, \bar{\gamma}^n$, $\mathrm{Dom}\, \bar{\gamma}^n = \mathrm{Dom}\, \bar{g}^n$;

(iv) $p_n \Vdash_{P_n}$ "$\{X \in J_D :$ for $\ell \in \{0, ..., n\}, \underset{\sim}{f}_\ell(X \cap \omega_1) \in A_\ell$ and $\bigwedge_{\eta \in \mathrm{Dom}\, \bar{g}^n} \hat{g}_\eta(X) = \gamma_\eta$ and for $\ell \in \{-1, 0, ..., n-1\}, \zeta \in X \cap 2^{|\mathcal{Y}/e_\ell|}$ we have: $A^{e_\ell}_\zeta \in D_\ell \Rightarrow \underset{\sim}{f}_\ell(X \cap \omega_1) \in A^{e_\ell}_\zeta\} \supseteq \underset{\sim}{W}_n \neq \emptyset \mod F^{P_n}$"

(C) $D_n = \{Z \subseteq \mathcal{Y}/e_n : p_n \Vdash_{P_n}$ "$\{X \in J_D : \underset{\sim}{f}_n(X \cap \omega_1) \notin Z\} = \emptyset \mod D^{P_n}_n + \underset{\sim}{W}_n$"$\}$.

Note that $D_n \in K[S]$, so every initial segment of the play (in which the second player uses this strategy) belongs to $K[S]$. $\square_{3.9}$

Remark 3.9B

(1) From the proof, instead $K[S] \models$"λ is Ramsey", $K[S] \models$"$\mu \rightarrow (\alpha)^{<\omega}_2$ for $\alpha < \lambda_0$" is enough for showing 3.9.

(2) Also if $\prod(|f(i)| + 1) < \mu_0$, $[\alpha < \mu_0 \Rightarrow |\alpha|^{\aleph_0} < \mu_0]$, it is enough: $S \subseteq \alpha < \mu_0 \Rightarrow$ in $K[S]$ there is $\mu \rightarrow (\alpha)^{<\omega}_2$.

Theorem 3.10 *Let $D^* \in \mathrm{FIL}(e, \mathcal{Y})$ be a normal ideal on $\aleph_1$. If for every $f : \aleph_1 \rightarrow (\sum_{\chi<\mu^*} \chi^{\aleph_1})^+$, D^* is nice to f, then for every $f \in {}^{\aleph_1}\mathrm{Ord}$, D is nice to f.*

Proof. As in [11], 1.7.

Remark 3.10A So, the existence of μ, $\mu \rightarrow (\alpha)^{<\omega}_{\aleph_0}$ for every $\alpha < (\sum_{\chi<\mu^*} \chi^{\aleph_1})^+$, is enough for "$D^*$ is nice".

Conclusion 3.11 *Let $\lambda_0 = \sum_{\chi<\mu^*} 2^{\chi^{\aleph_0}} + |Eq|$, $\mu^* \geq \aleph_2$; if for every $S \subseteq \lambda_0$ there is a Ramsey cardinal in $K[S]$ above λ_0 then every $D \in \mathrm{FIL}(\mathcal{Y})$ is nice.*

Proof. By 3.9, 3.10.

Concluding Remark 3.12

(1) We could have used other forcing notions, not Levy$(\aleph_1, \mathcal{Y}/e_n)$. E.g. if $\mu = \aleph_2$ we could use finite iterations of the forcing of Baumgartner to add a club of ω_1, by finite conditions. (So this forcing notion has cardinality $\aleph_1$.) Then in 3.9 we can weaken the demands on $\lambda_0 : \lambda_0 = \sum_{\chi<\mu_0} 2^\chi + \prod_{i<\omega_1} |1 + f(i)| + |Eq|$, hence also in 3.11, $\lambda_0 = \sum_{\chi<\mu^*} 2^\chi$ is O.K.

(2) Concerning $|Eq|$ remember 3.6(5), (6).

(3) Similarly to (1). If $\bigwedge_{\theta<\mu} \mathrm{cov}(\theta, \aleph_1, \aleph_1, 2) < \mu$ then by 2.6 we can use forcing notions of Todorčevič for collapsing $\theta < \mu$ which has cardinality $< \mu$.

(4) If we want to have $\lambda_0 =: \prod_{i<\omega_1} |f(i)+2|$ (or even $T_D(f+2)$), we can get this by weakening further the first player letting him choose only A_n which are easily definable from the $\bar{g}^{n-1}$, we shall return to it in a subsequent paper.

4 Ranks

Convention 4.1 Like 3.1 and: $\bar{g} \in F_c({}^{\omega}\omega, e^*, \mathcal{Y})$, $\eta^* \in \mathrm{Dom}\ \bar{g}^*$, ν^* an immediate successor of η^* not in Dom g^*, $D^* \in \mathrm{FIL}(e^*, \mathcal{Y})$ is such that in $G^{\bar{\gamma}^*}(D^*, \bar{g}^*, e^*)$ second player wins (all constant). $\mathrm{FIL}^*(e, \mathcal{Y})$ will be the set of $D \in \mathrm{FIL}(e, \mathcal{Y})$ such that $e \geq e^*$, $(D^*)^{[e]} \subseteq D$ and in $G^{\bar{\gamma}^*}(D^*, \bar{g}^*, e^*)$ second player wins. (So actually $\mathrm{FIL}(e^*, \mathcal{Y})$ depends on $D^*, \bar{g}^*, e^*$, too.)

Definition 4.2

(1) $rk^5_D(f)$ for $D \in \mathrm{FIL}^*(e, \mathcal{Y})$, $f \in {}^{\mathcal{Y}/e}\mathrm{Ord}$, $f <_D \bar{g}^*_{\eta^*}$ will be: the minimal ordinal α such that for some $D_1, e_1, \bar{\gamma}^1$ we have $D^{[e_1]} \subseteq D_1 \in \mathrm{FIL}(e_1, \mathcal{Y})$, $\bar{\gamma}^1 = \bar{\gamma}^{*\wedge}\langle \nu^*, \alpha\rangle$ (i.e. Dom $\bar{\gamma}^1 = (\mathrm{Dom}\ \bar{\gamma}^*) \cup \{\nu^*\}$, $\bar{\gamma}^1 \restriction \mathrm{Dom}\ \bar{\gamma}^* = \bar{\gamma}^*$, $\gamma^1_{\nu^*} = \alpha$) and in $G^{\bar{\gamma}^1}(D, \bar{g}^{*\wedge}\langle \nu^*, f\rangle)$ second player wins and ∞ if there is no such α .

(2) $rk^4_D(f)$ is $\sup\{rk^5_{D+A}(f) : A \in D^+\}$.

Claim 4.3

(1) $rk^5_D(f)$ *is (under the circumstances of* 4.1, 4.2*) an ordinal* $< \gamma^*_{\eta^*}$.

(2) $rk^4_D(f)$ *is an ordinal* $\leq \gamma^*_{\eta^*}$.

Claim 4.4 *If* $D \in \mathrm{FIL}^*(e, \mathcal{Y})$, $h <_D f <_D g^*_{\eta^*}$ *then* $rk^5_D(h) < rk^5_D(f)$.

Proof. Let e_1, D_1 witness $rk^5_D(f) = \alpha$ so $e(D) \leq e_1$, $D \subseteq D_1 \in \mathrm{FIL}^*(e_1, \mathcal{Y})$ and in $G^{\bar{\gamma}^\wedge\langle\nu^*,\alpha\rangle}(D_1, \bar{g}^{*\wedge}\langle \nu^*, f\rangle, e)$ second player wins. We play for the first player: $e = e_1$, $A_0 = \mathcal{Y}/e_1$, $\bar{g}^0 = \bar{g}^{*\wedge}\langle \nu^*, f\rangle^\wedge\langle \nu^{*\wedge}\langle 0\rangle, g\rangle$, now the first player should be able to answer say e_2, D_2, $\bar{\gamma}^2$. So $\gamma^2_{\nu^{*\wedge}\langle 0\rangle} < \gamma^2_{\nu^*} = \alpha$, and by 3.7(3), we know that in $G^{\bar{\gamma}'}(D_2, \bar{g}^{*\wedge}\langle \nu^*, g\rangle, e_2)$ where $\bar{\gamma}' = \bar{\gamma}^\wedge\langle \nu^*, \gamma^2_{\nu^{*\wedge}\langle 0\rangle}\rangle$, second player wins. $\square_{4.4}$

Claim 4.5 *Let* $e \geq e^*$, $D \in \mathrm{FIL}^*(e, \mathcal{Y})$.

(1) *For* $e \geq e(D)$, $A \in (D^{[e]})^+$, $f \in {}^{\mathcal{Y}/e}\mathrm{Ord}$, $f <_D g^*_{\eta^*}$ *we have:*

$$rk^5_D(f) \leq rk^5_{D^{[e]}+A}(f) \leq rk^4_{D^{[e]}+A}(f) \leq rk^4_D(f).$$

(2) *If* $e_2 \geq e_1 \geq e(D)$, $f_\ell \in {}^{\mathcal{Y}}\mathrm{Ord}$ *is supported by* e_ℓ, $f_1 \leq_D f_2 <_D g^*_{\eta^*}$ *then* $rk^\ell_D(f_1) \leq rk^\ell_D(f_2)$ *for* $\ell = 4, 5$.

5 More on Ranks and Higher Objects

Convention 5.1

(a) μ^* is a cardinal $> \aleph_1$ (using $\aleph_1$ rather than an uncountable regular κ is to save parameters).

(b) $\mathcal{Y}$ is a set of cardinality $\sum_{\kappa<\mu^*} \kappa$.

(c) ι is a function from $\mathcal{Y}$ onto ω_1, $|\iota^{-1}(\{\alpha\})| = |\mathcal{Y}|$ for $\alpha < \omega$.

(d) Eq is the set of equivalence relations e on $\mathcal{Y}$ such that:

> (α) $y\ e\ z \Rightarrow \iota(y) = \iota(z)$,
> (β) each equivalence class has cardinality $|\mathcal{Y}|$,
> (γ) e has $< \mu^*$ equivalence classes.

(e) D denotes a normal filter on some $\mathcal{Y}/e$ ($e \in Eq$), we write $e = e(D)$. The set of such D's is FIL($\mathcal{Y}$).

(f) E denotes a set of D's as above, such that:

> (α) for some $D = \min E \in E$,
>
> $$(\forall D')[D' \in E \Rightarrow (e, D) \leq (e(D'), D')],$$
>
> (β) if $D \in E$, $A \subseteq \mathcal{Y}/e_1$, $e_1 \geq e(D)$, $A \neq \emptyset \bmod D$ then $D^{[e_1]}+A \in E$.

(g) $E^{[e]} =: \{D \in E : e(D) = e\}$.

(h) $\mathcal{E}$ denotes a set of E's as above, such that:

> (α) there is $E = \text{Min } \mathcal{E} \in \mathcal{E}$ satisfying
>
> $$(\forall E')(E' \in E \Rightarrow E' \subseteq E),$$
>
> (β) if $D \in E \in \mathcal{E}$ then
>
> $$E_{[D]} = \{D' : D' \in E \text{ and } (e(D), D) \leq (e(D'), D')\} \in \mathcal{E}.$$

Definition 5.2

(1) We say E is λ-divisible when: for every $D \in E$, and Z a set of cardinality $< \lambda$, there are D', j such that:

> (α) $D' \in E$;
> (β) $(e(D), D) \leq (e(D'), D')$;
> (γ) $j : \mathcal{Y}/e(D') \to Z$;
> (δ) for every function $h : \mathcal{Y}/e(D) \to Z$,
>
> $$\{y/e(D') : h(y/e(D)) = j\,(y/e(D'))\} \neq \emptyset \bmod D'.$$

(2) We say E has λ-sums when: for every $D \in E \in \mathcal{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda \rangle$ of subsets of $\mathcal{Y}/e(D)$ there is $Z^* \subseteq \mathcal{Y}/e(D)$ such that: $Z^* \cap Z_\zeta = \emptyset \mod D$ and: [if $(e(D), D) \le (e', D')$, $e' = e(D')$, $D' \in E_{[D]}$ and $\bigwedge_\zeta Z_\zeta^{[e']} = \emptyset \mod D'$, then $Z^* \in D'$].

(3) We say E has weak λ-sums if for every $D \in E \in \mathcal{E}$ and sequence $\langle Z_\zeta : \zeta < \zeta^* < \lambda \rangle$ of subsets of $\mathcal{Y}/e(D)$ there is D^*, $D^* \in E_{[D]}$ such that:

(α) if $(e(D), D) \le (e', D')$, $D' \in E_{[D]}$ and $Z_\zeta = \emptyset \mod D'$ for $\zeta < \zeta^*$, $e(D^*) \le e(D')$, then $D^* \subseteq D'$, and

(β) $Z_\zeta = \emptyset \mod D^*$ for $\zeta < \zeta^*$.

(4) If $\lambda = \mu^*$ we omit it. We say $\mathcal{E}$ is λ-divisible if every $E \in \mathcal{E}$ is. Similarly we define "$\mathcal{E}$ has [weak] λ-sums" by modifying clause [(3)] (2), replacing E by $\mathcal{E}$ and D by E.

We now define variants of the games from §3.

Definition 5.3 For a given $\mathcal{E}$, for every $E \in \mathcal{E}$:

(1) We define a game $G_2^*(E, \overline{g})$.

In the n-th move first player chooses $D_n \in E_{n-1}$ (stipulating $E_{-1} = E$) and choose $\overline{g}_n \in F_c({}^{\omega}\omega, e(D_n), \mathcal{Y})$ extending $\overline{g}_{n-1}$ (stipulating $\overline{g}_{-1} = \overline{g}$) such that $\overline{g}_n$ is D_n-decreasing. Then the second player chooses E_n, $(E_{n-1})_{[D_n]} \subseteq E_n \in \mathcal{E}$.

In the end the second player wins if $\bigcup_{n<\omega} \mathrm{Dom}\, \overline{g}_n$ has no infinite branch.

(2) We define a game $G_2^{\overline{\gamma}}(E, \overline{g})$ where $\mathrm{Dom}\, \overline{\gamma} = \mathrm{Dom}\, \overline{g}$, each γ_n an ordinal, $[\eta \lhd \nu \Rightarrow \gamma_\eta > \gamma_\nu]$ similarly to $G_2^*(D, \overline{g})$ but the second player in addition chooses an indexed set $\overline{\gamma}_n$ of ordinals, $\mathrm{Dom}\, \overline{\gamma}_n = \mathrm{Dom}\, \overline{g}_n$, $\overline{\gamma}_n \restriction \mathrm{Dom}\, \overline{\gamma}_{n-1} = \overline{\gamma}_{n-1}$ and $[\eta \lhd \nu \Rightarrow \gamma_{n,\eta} > \gamma_{n,\nu}]$.

Definition 5.4

(1) We say $\mathcal{E}$ is nice to $\overline{g} \in F_c({}^{\omega}\omega, e, \mathcal{Y})$ if for every $E \in \mathcal{E}$ with $e \le e(E)$ the second player wins the game $G_2^*(E, \overline{g})$.

(2) We say $\mathcal{E}$ is nice if it is nice to $\overline{g}$ whenever $E \in \mathcal{E}$, $e \le e(E)$, $\overline{g} \in F_c({}^{\omega}\omega, e, \mathcal{Y})$, $\overline{g}$ is $(\min E)$-decreasing, we have: $\mathcal{E}_{[E]}$ is nice to $\overline{g}$.

(3) If $\mathrm{Dom}\, \overline{g} = \{<>\}$ we write $g_{<>}$ instead of $\overline{g}$.

(4) We say $\mathcal{E}$ is nice to α if it is nice to the constant function α.

Claim 5.5

(1) *If $\mathcal{E}$ is nice to f, $f \in F_c({}^{\omega}\omega, e, \mathcal{Y})$, $g \in F_c({}^{\omega}\omega, e, \mathcal{Y})$, $g \le f$ then $\mathcal{E}$ is nice to f.*

(2) *The games from* 5.4 *are determined, and the winning side has winning strategy which does not need memory.*

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(3) *The second player wins* $G_2^*(E,\overline{g})$ *iff for some* $\overline{\gamma}$ *second player wins* $G_2^{\overline{\gamma}}(E,g)$.

(4) *If the second player wins* $G_2^{\gamma}(E,f)$, $\overline{g} \in F_c({}^{\omega}\omega, e(E))$, $g_\eta \leq f$ *for* $\eta \in \mathrm{Dom}\,(\overline{g})$ *then the second player wins in* $G_2^{\overline{\gamma}}(E,\overline{g})$ *when we let*

$$\gamma_\eta = \gamma + [\max_{\eta \trianglelefteq \nu \in \mathrm{Dom}\,\overline{g}} (\ell g \nu - \ell g \eta + 1)].$$

Lemma 5.6 *Suppose* $f_0 \in {}^{(\mathcal{Y}/e)}\mathrm{Ord}$, $e \in Eq$, $\lambda_0 =: \sup_{e_0 \leq e \in Eq} \prod_{x \in \mathcal{Y}/e}(f_0^{[e]}(x)+1)$.

(1) *If there is a Ramsey cardinal* $\geq \bigcup\{f(x)+1 : x \in \mathrm{Dom}\, f_0\}$ *then there is a* μ^**-divisible* $\mathcal{E}$ *nice to* f_0 *having weak* μ^**-sums.*

(2) *If for every* $A \subseteq \lambda_0$ *there is in* $K[A_0]$ *a Ramsey cardinal* $> \lambda_0$, *then there is a* μ^**-divisible* $\mathcal{E}$ *which has weak* μ^**-sums and is nice to* f.

(3) *In part 2 if* $\lambda_0 = 2^{<\mu_0}$ *then there is a* μ^**-divisible nice* $\mathcal{E}$ *which has weak* μ^**-sums.*

Remark: This enables us to pass from "$\mathrm{pp}_{\Gamma(\theta,\aleph_1)}$ large" to "$\mathrm{pp}_{\mathrm{normal}}$ is large".

Proof. (1) Define $f_1 \in {}^{(\aleph_1)}\mathrm{Ord}$, $f_1(i) = \sup\{f_0(y/e) : \iota(y) = i\}$, let λ be such that: $\lambda \to (\sup_{i<\aleph_1} f_1(i))_2^{<\omega}$ (or just $\emptyset \notin D_n^*$ — see below), let $\lambda_n = (\lambda^{\mu^*})^{+n}$,

$$I_n = \{s : s \subseteq \lambda_n, s \cap \omega_1 \text{ a countable ordinal}\},$$
$$J_n = \{s \in I_n : s \cap \lambda \text{ has order type} \geq f_0(s \cap \omega_1)\}.$$

Let D_n^* be the minimal fine normal filter on J_n.

Let for $n < \omega$ and $e \in Eq$, $H_{n,e} = \{h : h$ a function from J_n into $\mathcal{Y}/e$ such that $\iota(h(s)) = s \cap \omega_1\}$.

Let $P_n = \{p : p \subseteq J_n, p \neq \emptyset \bmod D_n^*\}$, $P = \bigcup_{n<\omega} P_n$ and for $p \in P$ let $n(p)$ be the unique n such that $p \in P_n$.

Let $p \leq q$ (in P) if $n(p) \leq n(q)$ and $\{s \cap \lambda_{n(p)} : s \in q\} \subseteq p$. Now for every $e \in Eq$, $n < \omega$, $p \in P_n$, $h \in H_{n,e}$ we let:

$$D_p^{n,e,h} = \{A \subseteq \mathcal{Y}/e : h^{-1}(A) \supseteq p \bmod D_{n(p)}^*\},$$

$$E_p^{n,e,h} = \{D_q^{n^1,e^1,h^1} : p \leq q \in P, n^1 = n(q), \text{ and } (n^1,e^1,h^1) \geq (n,e,h)\},$$

where $(n^1,e^1,h^1) \geq (n,e,h)$ means: $n \leq n^1 < \omega$, $e \leq e^1 \in Eq$, $h^1 \in H_{n^1,e^1}$ and for $s \in J_{(n^1)}$, $h^1(s)^{[e]} = h(s \cap \lambda_n)$. We define $(p^1,n^1,e^1,h^1) \geq (p,n,e,h)$ similarly and let

$$\mathcal{E}_p^{n,e,h} = \{E_q^{n^1,e^1,h^1} : p \leq q \in P, n^1 = n(q), (n^1,e^1,h^1) \geq (n,e,h)\}.$$

[Note: $(p^1,n^1,e^1,h^1) \geq (p,n,e,h)$ implies $D_{p^1}^{n^1,e^1,h^1} \supseteq D_p^{n,e,h}$, $E_{p^1}^{n^1,e^1,h^1} \subseteq E_p^{n,e,h}$ and $\mathcal{E}_{p^1}^{n^1,e^1,h^1} \subseteq \mathcal{E}_p^{n,e,h}$.] Now any $\mathcal{E} = \mathcal{E}_p^{n,e,h}$ $(p \in P)$ is as required.

A new point is "$\mathcal{E}$ is μ-divisible". So suppose $E \in \mathcal{E} = \mathcal{E}_p^{n,e,h}$ so $E = E_q^{n^1,e^1,h^1}$ for some $(q,n^1,e^1,h^1) \geq (p,n,e,h)$. Let Z be a set of cardinality $< \mu^*$, so $(\lambda_{n^1})^{|Z|} = \lambda_{n_1}$; let $\{h_\zeta : \zeta < \zeta^* = |\mathcal{Y}/e_1|^{|Z|} \leq 2^\mu \leq \lambda_{n^1}\}$ list all functions h from $\mathcal{Y}/e_1$ to Z. Let $\langle S_\zeta : \zeta <$

$|\mathcal{Y}/e_1|^{|Z|}\rangle$ list a sequence of pairwise disjoint stationary subsets of $\{\delta < \lambda_{n^1+1} : \mathrm{cf}\delta = \aleph_0\}$. Let $e_2 \in Eq$ be such that $e_1 \le e_2$ and for every $y \in \mathcal{Y}$, $\{z/e_2 : z\, e_1\, y\} = \{x(y/e,t) : t \in Z\}$; we let q_2, $q \le q_2 \in P$ be: $q_2 = \{s \in J_{n^1+1} : s \cap \lambda_{n^1} \in q$ and $\sup s \in \bigcup_\zeta S_\zeta\}$; lastly we define $h^2 : J_{n^1+1} \to \mathcal{Y}/e_1$ by: $h^2(s) = x(h^1(s \cap \lambda_{n^1}), h_\zeta(s \cap \lambda_{n^1}))$ if $s \in q_2$, $\sup s \in S_\zeta$ (for $s \in J_{n^1+1} \setminus q_2$ it does not matter).

The proof that q_2, e_2, h^2 are as required is as in [2] and more specifically [8].

As for proving "$\mathcal{E}^{n,e,h}_p$ has weak μ^*-sums" the point is that the family of fine normal filters on J_n has μ^*-sum.

(2) Similar to 3.9 (and 3.6(5),(6)).

(3) Similar to [11], 1.7. $\square_{5.6}$

6 Hypotheses: Weakening of GCH

We define some hypotheses; except for the first we do not know now whether their negations are consistent with ZFC.

Hypothesis 6.1

(A) $pp(\lambda) = \lambda^+$ for every singular λ .

(B) If $\mathfrak{a}$ is a set of regular cardinals, $|\mathfrak{a}| < \mathrm{Min}\,\mathfrak{a}$ then $|\mathrm{pcf}\mathfrak{a}| \le |\mathfrak{a}|$.

(C) If $\mathfrak{a}$ is a set of regular cardinals, $|\mathfrak{a}| < \mathrm{Min}\,\mathfrak{a}$ then $\mathrm{pcf}\mathfrak{a}$ has no accumulation point which is inaccessible (i.e.: λ inaccessible $\Rightarrow \sup(\lambda \cap \mathrm{pcf}\mathfrak{a}) < \lambda$).

(D) For every λ, $\{\mu < \lambda : \mu$ singular and $\mathrm{pp}\mu \ge \lambda\}$ is countable.

(E) For every λ, $\{\mu < \lambda : \mu$ singular and $\mathrm{cf}\mu = \aleph_0$ and $\mathrm{pp}\mu \ge \lambda\}$ is countable.

(F) For every λ, $\{\mu < \lambda : \mu$ singular of uncountable cofinality, $\mathrm{pp}_{\Gamma(\mathrm{cf}\mu)}(\mu) \ge \lambda\}$ is finite.

$(D)_{\theta,\sigma,\kappa}$ For every λ, $\{\mu < \lambda : \mu > \mathrm{cf}\mu \in [\sigma,\theta)$ and $\mathrm{pp}_{\Gamma(\theta,\sigma)}(\mu) \ge \lambda\}$ has cardinality $< \kappa$.

$(A)_\Gamma$ If $\mu > \mathrm{cf}\mu$ then $\mathrm{pp}_\Gamma(\mu) = \mu^+$ (or in the definition of $\mathrm{pp}_\Gamma(\mu)$ the supremum is on the empty set).

$(B)_\Gamma$, $(C)_\Gamma$ Similar versions (i.e. use pcf_Γ).

We concentrate on the parameter free case.

Claim 6.2 *In* 6.1, *we have:*

(1) (A) $\Rightarrow$ (B) $\Rightarrow$ (C)*;*

(2) (A) $\Rightarrow$ (D) $\Rightarrow$ (E), (A) $\Rightarrow$ (F);

(3) (E) + (F) $\Rightarrow$ (D) $\Rightarrow$ (B). *[Last implication — by the localization theorem* [13], §2.*]*

Theorem 6.3 *Assume Hypothesis* 6.1A.

(1) *For every* $\lambda > \kappa$, *cov*$(\lambda, \kappa^+, \kappa^+, 2) = \begin{cases} \lambda^+ & \text{if } \mathrm{cf}(\lambda) \le \kappa, \\ \lambda & \text{if } \mathrm{cf}(\lambda) > \kappa. \end{cases}$

(2) *For every* $\lambda > \kappa = \mathrm{cf}\kappa > \aleph_0$, *there is a stationary* $S \subseteq \mathcal{S}_{\le\kappa}(\lambda)$, $|S| = \lambda^+$ *if* $\mathrm{cf}(\lambda) \le \kappa$ *and* $|S| = \lambda$ *if* $\mathrm{cf}(\lambda) > \kappa$.

(3) *For* μ *singular, there is a tree with* $\mathrm{cf}\mu$ *levels, each level of cardinality* $< \mu$, *and with* $\ge \mu^+$ $(\mathrm{cf}(\mu))$*-branches.*

(4) *If* $\kappa \le \mathrm{cf}\mu < \mu \le 2^\kappa$ *then there is an entangled linear order* $\mathcal{T}$ *of cardinality* μ^+.

Proof.

(1) By [14], §1.

(2) By part (1) and 2.6.

(3), (4) By [10], §4.

Theorem 6.4 [Hypothesis 6.1(D)]. *If* $\lambda > 2^{\aleph_0}$, *and* $\lambda > \theta \ge \mathrm{cf}\lambda + 2^{\aleph_0}$ *then* $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) =^+ \mathrm{pp}_\theta(\lambda)$.

Remark See [14], §3, §5 on earlier results; [16] for later results.

Proof. We prove by induction on $\mathrm{pp}_\theta(\lambda)$ (not on λ!) for fixed θ. For a given λ, let

$$\Theta_1 =: \{\mu : \lambda \le \mu < \mathrm{pp}_\theta^+(\lambda), \mathrm{cf}\mu \le \theta, \mathrm{pp}_\theta^+(\mu) = \mathrm{pp}^+(\lambda)\},$$

$$\Theta_2 =: \{\mu : \lambda \le \mu < pp_\theta^+(\lambda), \mathrm{cf}\mu \le \theta \text{ and } \mathrm{cov}(\mu, \mu, \theta^+, 2) \ge \mathrm{pp}_\theta^+(\mu)\}.$$

As we know that $[\lambda \le \mu < pp_\theta^+(\lambda)\ \&\ \mathrm{cf}\mu \le \theta \Rightarrow pp_\theta^+(\mu) \le pp_\theta^+(\lambda)]$ (by [10], 2.3) and by the induction hypothesis clearly $\Theta_2 \subseteq \Theta_1$. But by Hypothesis 6.1(D) we have Θ_1 countable hence Θ_2 is countable (really $|\Theta_1| \le \theta$ suffices). By [10], 5.3(10) Θ_2 is closed hence it has a last element σ. By [10], proof of 5.4(1)—first part $\mathrm{cov}(\alpha, \sigma^+, \theta^+, 2) < \mathrm{pp}^+(\lambda)$ for $\alpha < \mathrm{pp}^+(\lambda)$ (and as said above $\sigma \in \Theta_1$). Now apply 6.5 below (we have Hypothesis 6.1(C) by 6.2(3) + 6.2(1) with $\lambda, \chi, \theta, \kappa$ there standing for $\sigma, \mathrm{pp}^+(\lambda), \theta, \mathrm{cf}\lambda$ here). $\square_{6.4}$

Claim 6.5 *Suppose*

(a) $\lambda > \mathrm{cf}\lambda = \kappa$, $\lambda > \theta \ge \kappa$,

(b) $\chi = \mathrm{cf}\chi > \lambda$ *and* $\mathrm{cov}(\alpha, \lambda^+, \theta^+, 2) < \chi$ *for* $\alpha < \chi$,

(c) $\mathrm{pp}_\theta^+(\lambda) \le \chi$,

(d) $\lambda > 2^{\aleph_0}$ *if* $\kappa = \aleph_0$,

(e) *if* χ *is inaccessible then Hypothesis* 6.1(C).

Then

(α) $\mathrm{cov}(\lambda, \lambda, \theta^+, 2) < \chi$;

(β) *moreover, for some* $\lambda_0 < \lambda$, $\mathrm{cov}(\chi, \lambda_0^+, \theta^+, 2) = \chi$.

Proof. We concentrate on the case $\mathrm{cf}\lambda = \aleph_0$, which is harder [if $\mathrm{cf}\lambda > \aleph_0$ it suffices to choose f_ξ for $\xi < \omega$]. Note that in the conclusion, (β) follows from (α) (by [10], 5.3 (10). Let $\chi^* = \beth_3(\lambda)^+$, and choose by induction on $\zeta \leq (2^{\aleph_0})^+$ a model $M_\zeta^* \prec (H(\chi)^*, \in, <^*_{\chi^*})$, $\|M_\zeta^*\| < \chi$, $M_\zeta^* \cap \chi$ an ordinal, M_ζ^* increasing continuous in ζ, $\{\kappa, \chi, \lambda, \theta\} \in M_0^*$ and $\langle M_\xi^* : \xi \leq \zeta \rangle \in M_{\zeta+1}^*$. Let $M^* = M^*_{(2^{\aleph_0})^+}$. Let $\mathcal{P}_\zeta =: \mathcal{S}_{<\lambda}(\lambda) \cap M_\zeta^*$, and $\mathcal{P} = \mathcal{P}_{(2^{\aleph_0})^+}$. Clearly $\mathcal{P}$ is a family of $< \chi$ subsets of λ each of cardinality $< \lambda$, so it suffices to prove:

($*$) if $a \subseteq \lambda$, $|a| \leq \theta$ then for some $A \in \mathcal{P}$, $a \subseteq A$.

Given $a \subseteq \lambda$, $|a| \leq \theta$, we define by induction on $\zeta < \omega_1$, f_ζ such that:

(a) $f_\zeta \in M^*$, f_ζ belongs to $\prod(\lambda \cap \mathrm{Reg})$.

(b) For $w \subseteq \zeta$ satisfying $(\exists A \in M^*)[\{f_\xi : \xi \in w\} \subseteq A \ \&\ |A| < \lambda]$, let A_w be the $<^*_{\chi^*}$-first such A of minimal cardinality and we let N_w^a be the Skolem Hull of $\{f_\xi : \xi \in w\}$ in $(H(\chi^*), \in, <^*_{\chi^*})$ and N_w^b be the Skolem Hull of $A_w = A_w \cup \{f_\xi : \xi \in w\}$ in $(H(\chi^*), \in, <^*_{\chi^*})$. We demand for every such w that: for every large enough $\sigma \in \lambda \cap \mathrm{Reg} \cap N_w^a$ we have $\sup(\sigma \cap N_w^b) < f_\zeta(\sigma)$.

For defining f_ζ, let $W_\zeta = \{w \subseteq \zeta : A_w \text{ well defined}\}$ so $W_\zeta \subseteq M^*$, $|W_\zeta| \leq 2^{\aleph_0}$ hence for some $\xi(\zeta) < (2^{\aleph_0})^+$, $W_\zeta \subseteq M^*_{\xi(\zeta)}$. For $w \in W_\zeta$, let N_w^+ be the Skolem Hull of A_w in $(H(\chi^*), \in, <^*_{\chi^*})$, so $N_w^+ \in M^*_{\xi(\zeta)+1}$ (see its definition) and $\|N_w^+\| = |A_w|$ hence

$$\mathfrak{a}_w = \{\sigma : \sigma \in N_w^+ \cap \lambda \cap \mathrm{Reg} \cap N_w^+ \backslash |A_w|^+\}$$

belongs to $M^*_{\xi(\zeta)+1}$, and it includes an end segment of $\lambda \cap \mathrm{Reg} \cap N_w^+$. Now by [14], 3.2, $\mathrm{cf}_{\leq\theta}(\prod \mathfrak{a}_w / J_\lambda^{bd}) < \chi$ (we use Hypothesis 6.1(C) if χ is inaccessible).

As $\mathfrak{a}_w \in M_{\xi(\zeta)+1}$ there is $f_w^\zeta \in (\prod \mathfrak{a}_w) \cap M^*_{\xi(\zeta)+1}$ such that:

($*$) for every large enough $\sigma \in N_w^+ \cap \mathrm{Reg} \cap \lambda$ we have $\sup(\sigma \cap N_w^+) < f_w^\zeta(\sigma)$,

but $N_w^a \subseteq N_w^+$ hence

($*$)′ for every large enough $\sigma \in N_w^a \cap \mathrm{Reg} \cap \lambda$ we have $\sup(\sigma \cap N_w^+) < f_w^\zeta(\sigma)$.

Now $M_{\xi(\zeta)+1} \in M_{\xi(\zeta)+2}$, $\|M_{\xi(\zeta)+1}\| < \chi$ hence there is a cofinal $\mathcal{P}' \subseteq \mathcal{S}_{\leq\lambda}(|M_{\xi(\zeta)+1}|)$ of cardinality $< \chi$ in $M_{\xi(\zeta)+2}$; as $M_{\xi(\zeta)+2} \cap \chi$ is an ordinal, necessarily $\mathcal{P}' \subseteq M_{\xi(\zeta)+2}$ hence there is $A^\zeta \in M_{\xi(\zeta)+2}$ such that $\bigwedge_{w \in W_\zeta} f_w^\zeta \in A^\zeta$ and $|A^\zeta| \leq \lambda$. So there is $f_\zeta \in \prod(\mathrm{Reg} \cap \lambda)$ in $M_{\xi(\zeta)+2}$ satisfying $(\forall f)[f \in A^\zeta \ \&\ (\exists \theta)[\theta < \lambda \ \&\ f \restriction (\mathrm{Reg} \cap \lambda \backslash \theta) < f_\zeta]]$.

Now there is $A \in M^*$, $|A| \leq \lambda$, $\{f_\xi : \xi < \omega_1\} \subseteq A$ (by assumption (b) of the claim), hence for some $A \in M^*$, $|A| < \lambda$ and $w^* = \{\xi < \omega_1 : f_\xi \in A\}$ is uncountable. For each $\xi \in w$, for some $\lambda_\xi < \lambda$,

$$\lambda_\xi < \sigma \in \lambda \cap \mathrm{Reg} \cap N^a_{w^* \cap \xi} \Rightarrow \sup(N^b_{w^* \cap \xi} \cap \sigma) < f_\xi(\sigma).$$

As we assume $\mathrm{cf}\lambda = \aleph_0$, for some $\lambda(*) < \lambda$, there are $\xi_0 < \xi_1 < \ldots < \xi_n < \ldots$ in w^* such that $\lambda_{\xi_n} \leq \lambda(*)$.

Let N^*= Skolem Hull of $A \cup (\lambda(*)+1)$ in $(H(\chi^*), \in, <^*_{\chi^*})$; it belongs to M^*, hence $N^* \cap \lambda \in \mathcal{P}$. So it suffices to show that $N^b_{\{\xi_n : n<\omega\}}$ is a subset of N^*, which is done as in [14], 3.3A, 5.1A. $\square_{6.5}$

Remark 6.5A

(1) We may want to omit the "$\lambda > 2^{\aleph_0}$ and $\theta \geq \mathrm{cf}\lambda + 2^{\aleph_0}$" in 6.4, 6.5. Of course, this is used only in 6.5, and we may replace it by: for some $\lambda_0 < \lambda$

$(*)_{\lambda_0}$ if c is a two place function from λ_0 to κ such that $[\alpha < \beta < \gamma \Rightarrow c(\alpha,\gamma) \leq \max\{c(\alpha,\beta), c(\beta,\gamma)\}$, then for some $n_0 < \omega$ and infinite $w \subseteq \lambda_0$ we have $\alpha \in w \ \&\ \beta \in w \ \&\ \alpha < \beta \Rightarrow c(\alpha,\beta) \leq n_0$.

Unfortunately, this is equivalent to

$(*)'_{\lambda_0}$ there are functions $f_\alpha \in {}^\omega\mathrm{Ord}$ for $\alpha < \lambda_0$ such that: $\alpha < \beta \Rightarrow f_\beta <_{J^{bd}_\omega} f_\alpha$

[why? $(*)'_{\lambda_0} \Rightarrow (*)_{\lambda_0}$ using $c(\alpha,\beta) = \min\{n : (\forall m)[m \geq n \Rightarrow f_\alpha(m) > f_\beta(m)]\}$.
$(*)_{\lambda_0} \Rightarrow (*)'_{\lambda_0}$ as for each $\alpha < \lambda_0$ and n we define when $f_\alpha(n) \geq \zeta$:

$$f_\alpha(n) \geq \zeta \Leftrightarrow \bigwedge_{\xi<\zeta} (\exists\beta)[\alpha < \beta \ \&\ c(\alpha,\beta) \leq n \ \&\ f_\beta(n) \geq \xi].$$

Now $f_\alpha(n)$ is the minimal value; if it is ∞ we get contradiction to the choice of c, and $[\alpha < \beta \ \&\ c(\alpha,\beta) = n \leq m \Rightarrow f_\alpha(m) > f_\beta(m)]$ is as required.

Claim 6.6 *Assume* (E) *(or just* $(\mathrm{D})_{\theta,\aleph_0,\theta}$*).*
If $\kappa \leq \theta = \mathrm{cf}\mu < \mu < 2^\mu$ *then there is an entangled linear order of cardinality* μ^+.

Proof. By [10], 2.1 for some strictly increasing continuous $\langle \mu_i : i < \theta \rangle$, $\mu = \bigcup_{i<\kappa} \mu_i$ and $\mu^+ = \mathrm{tcf} \prod \mu_i^+ / J^{bd}_\theta$. Now note

$(*)$ for some $i < \kappa$, for every $j \in (i, \kappa)$, $\mu^+ \notin \mathrm{pcf}\{\mu^+_\alpha : i < \alpha < j\}$.

Now we can choose by induction on $\zeta < \theta$, $i(\zeta) < \theta$ such that $i(\zeta)$ strictly increasing and $\mu_{i(\zeta)} > \max\mathrm{pcf}\{\lambda_j : i < j < \bigcup_{\xi<\zeta} i(\xi)\}$. Now to $\langle \mu^+_{i(\zeta)} : \zeta < \theta \rangle$ apply [10], 4.12.

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