Sh:1118

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 147, Number 4, April 2019, Pages 1719–1732
https://doi.org/10.1090/proc/14305
Article electronically published on January 8, 2019

LOCAL CHARACTER OF KIM-INDEPENDENCE

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(Communicated by Heike Mildenberger)

ABSTRACT. We show that NSOP_1 theories are exactly the theories in which Kim-independence satisfies a form of local character. In particular, we show that if T is NSOP_1, $M \models T$, and p is a complete type over M, then the collection of elementary substructures of size $|T|$ over which p does not Kim-fork is a club of $[M]^{|T|}$ and that this characterizes NSOP_1.

We also present a new phenomenon we call dual local-character for Kim-independence in NSOP_1 theories.

1. Introduction

A well-known theorem of Kim and Pillay characterizes the simple theories as those theories with an independence relation satisfying certain properties and shows that, moreover, any such independence relation must coincide with non-forking independence. As the theory of simple theories was being developed, work of Chatzidakis on ω-free PAC fields and Granger on vector spaces with bilinear forms furnished examples of non-simple theories for which there are nonetheless independence relations satisfying many of the fundamental properties of non-forking independence in simple theories. These properties include extension, symmetry, and the independence theorem. Chernikov and the second-named author proved an analogue of one direction of the Kim-Pillay theorem for NSOP_1 theories, showing essentially that the existence of an independence relation with these properties implies that a theory is NSOP_1 [CR16]. To establish the other direction, the first- and second-named authors introduced *Kim-independence* and showed that it is well-behaved in any NSOP_1 theory. The theory of Kim-independence provides an explanation for the simplicity-like phenomena observed in certain non-simple examples and a central issue of research concerning NSOP_1 theories is to determine the extent to which properties of non-forking independence in simple theories carry over to Kim-independence in NSOP_1 theories. This paper addresses the specific issue of local character for Kim-independence.

Simple theories are defined to be the theories in which forking satisfies local character. Local character of non-forking asserts that there is some cardinal $\kappa(T)$ so that, for any complete type p in finitely many variables over A, there is a set $B \subseteq A$ with $|B| < \kappa(T)$ over which p does not fork. An analogue of local character

Received by the editors July 14, 2017, and, in revised form, February 12, 2018, and June 19, 2018.

2010 *Mathematics Subject Classification.* Primary 03C45, 03C55, 03C80.

The first author would like to thank the Israel Science Foundation for partial support of this research (Grant no. 1533/14).

The third author was partially supported by European Research Council grant 338821, number 1118.

for Kim-independence in NSOP_1 theories was proved by the first- and second-named authors in [KR17, Theorem 4.5]. It was shown there that if T is NSOP_1 and $M \models T$, then for any $p \in S(M)$, there is $N \prec M$ with $|N| < \kappa = \left(2^{|T|}\right)^+$ such that p does not Kim-fork over N.

However, this result was an unsatisfactory generalization of local character in simple theories for three reasons. First, with respect to non-forking, it follows almost immediately that if $\kappa(T)$ exists at all, it can be taken to be $|T|^+$: given a type $p \in S(A)$ with no $B \subseteq A$ of size $< |T|^+$ over which p does not fork, one can find a chain of forking types of length $|T|^+$ and then by the pigeonhole principle, some formula must fork infinitely often with respect to the same disjunction of dividing formulas. This equivalence is no longer immediate when considering Kim-independence, because of the added constraint that the formulas must divide with respect to Morley sequences and it was asked [KR17, Question 4.7] if $\left(2^{|T|}\right)^+$ can be replaced by $|T|^+$ in an arbitrary NSOP_1 theory. Secondly, non-forking independence satisfies *base monotonicity*, which means that if $p \in S(B)$ does not fork over A, then p does not fork over B' whenever $A \subseteq B' \subseteq B$. In other words, local character of forking implies that every type does not fork over an entire *cone* of small subsets of its domain. However, in an NSOP_1 theory T, Kim-independence satisfies base monotonicity if and only if T is simple. One would like an analogue of local character for NSOP_1 theories that shows that types over models do not Kim-divide over *many* small submodels. Finally, local character of non-forking independence *characterizes* simple theories. Many tameness properties of Kim-independence are known to characterize NSOP_1 theories, e.g., symmetry and the independence theorem, so it is natural to ask if local character does as well.

Our main theorem is the following.

Theorem 1.1. *Suppose T is a complete theory with monster model $\mathbb{M} \models T$. The following are equivalent:*

(1) *T is $NSOP_1$.*
(2) *There is no continuous (see below) increasing sequence of $|T|$-sized models $\left\langle M_i \,\middle|\, i < |T|^+ \right\rangle$ with union M and $p \in S(M)$ such that $p \restriction M_{i+1}$ Kim-forks over M_i for all $i < |T|^+$.*
(3) *For any $M \models T$, $p \in S(M)$, the set of elementary substructures of M of size $|T|$ over which p does not Kim-divide is a stationary subset of $[M]^{|T|}$.*
(4) *For any $M \models T$, $p \in S(M)$, the set of elementary substructures of M of size $|T|$ over which p does not Kim-divide contains a club subset of $[M]^{|T|}$.*
(5) *For any $M \models T$, $p \in S(M)$, the set of elementary substructures of M of size $|T|$ over which p does not Kim-divide is a club subset of $[M]^{|T|}$.*
(6) *Suppose that $N \models T$, $M \prec N$, and $p \in S(N)$ does not Kim-divide over M. Then the set of elementary substructures of M of size $|T|$ over which p does not Kim-divide is a club subset of $[M]^{|T|}$.*

Continuous here means that the models indexed by limit ordinals are the union of their predecessors in the chain. The equivalence of (1) and (2) was noted in [KR17, Corollary 4.6] with $|T|^+$ replaced by $\left(2^{|T|}\right)^+$, which is considerably weaker than the theorem proved here.

In particular, this theorem implies that if T is NSOP_1, $M \models T$, and $p \in S(M)$, then the set of $N \prec M$ with $|N| = |T|$ such that p does not Kim-fork over N

is *non-empty*, answering a question asked by the first- and second-named authors [KR17, Question 4.7]. However, by demanding a stronger form of local character, we obtain a new characterization of NSOP_1.

Remark 1.2. In the first draft of this paper, published online in July 2017, we did not yet have (5) or (6) above. Shortly after that draft was available, in a private correspondence Pierre Simon found an easier proof of (1) implies (4), and we thank him for allowing us to include his proof here. Later we found a proof of (6). These proofs use symmetry of Kim-independence, and so are not as transparently connected to the syntactic definition of NSOP_1 theories as in the proof in simple theories, and our original proof.

Our original proof assumes towards contradiction that local character fails and reaches a contradiction to NSOP_1 as is done in, e.g., simple theories. For this approach to work we used stationary logic. This logic expands first-order logic by introducing a quantifier aa interpreted so that $M \models (\mathsf{aa}S)\,\varphi(S)$ if and only if the set of countable subsets $X \subseteq M$ such that M, when expanded with the predicate S interpreted as X, satisfies $\varphi(S)$ contains a club of $[M]^{\omega}$. This logic was introduced by the third-named author in [She75] and later studied by Mekler and the third-named author [MS86] who showed that the satisfiability of a theory in $L(\mathsf{aa})$ implies the satisfiability of a theory in a related logic, where the second-order quantifiers range over *uncountable* sets of a certain size. This theorem, which may be regarded as a version of the upward Lowenheim-Skolem theorem, provides a tool for "stretching" a family of counterexamples to local character in such a way that preserves the cardinality and continuity constraints needed to produce SOP_1.

After further review, we noticed that our original proof gives rise to a new phenomenon, which we call dual local-character.

In light of all this, we decided to rearrange the paper in the following way. After a short preliminaries section, we prove the main theorem. In Section 4 we discuss stationary logic and describe our original proof without giving details. In Section 5 we discuss the dual local-character. The details of all this can be found in an online version of this paper [KRS17].

2. Preliminaries

2.1. **NSOP_1 theories, invariant types, and Morley sequences.**

Throughout the paper, we will assume T is a complete theory with infinite models and a monster model $\mathbb{M}$.

Definition 2.1 ([DS04, Definition 2.2])**.** A formula $\varphi(x;y)$ has the 1-*strong order property* (SOP_1) if there is a tree of tuples $\langle a_\eta \mid \eta \in 2^{<\omega}\rangle$ so that

- For all $\eta \in 2^{\omega}$, the partial type $\{\varphi(x; a_{\eta\restriction n}) \mid n < \omega\}$ is consistent.
- For all $\nu, \eta \in 2^{<\omega}$, if $\nu \frown \langle 0\rangle \trianglelefteq \eta$, then $\left\{\varphi(x;a_\eta), \varphi\left(x; a_{\nu\frown\langle 1\rangle}\right)\right\}$ is inconsistent.

A theory T is *$NSOP_1$* if no formula has SOP_1 modulo T.

Fact 2.2 ([KR17, Proposition 2.4])**.** *T has NSOP_1 if and only if there is no formula $\varphi(x;y)$, $k<\omega$, and a sequence $\langle \bar{c}_i \mid i \in I\rangle$ with $\bar{c}_i = (c_{i,0}, c_{i,1})$ satisfying*

(1) *For all $i \in I$, $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$.*
(2) *$\{\varphi(x;c_{i,0}) \mid i \in I\}$ is consistent.*
(3) *$\{\varphi(x;c_{i,1}) \mid i \in I\}$ is k-inconsistent.*

We also use the following notation. Write $a \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{u}_{M} B$ for $\mathrm{tp}(a/MB)$ is finitely satisfiable in M; in other words it is a *coheir* of its restriction to M. A type $p \in S(M)$ is an *heir* of its restriction to $N \prec M$ if for every formula $\varphi(x;y) \in L(N)$ and every $b \in M$, if $\varphi(x;b) \in p$, then $\varphi(x;b') \in p$ for some $b' \in N$. We denote this by $c \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{h}_{N} M$. This is equivalent to saying that $M \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{u}_{N} c$.

Definition 2.3. A global type $q \in S(\mathbb{M})$ is called *A-invariant* if $b \equiv_A b'$ implies $\varphi(x;b) \in q$ if and only if $\varphi(x;b') \in q$. A global type q is *invariant* if there is some small set A such that q is A-invariant. If $q(x)$ and $r(y)$ are A-invariant global types, then the type $(q \otimes r)(x,y)$ is defined to be $\mathrm{tp}(a,b/\mathbb{M})$ for any $b \models r$ and $a \models q|_{\mathbb{M}b}$. It is also A-invariant. We define $q^{\otimes n}(x_0,\ldots,x_{n-1})$ by induction: $q^{\otimes 1} = q$ and $q^{\otimes n+1} = q(x_n) \otimes q^{\otimes n}(x_0,\ldots,x_{n-1})$.

Fact 2.4 ([She90, Lemma 4.1])**.** If T is any complete theory, $M \models T$, and $p \in S(M)$, then there is a complete global type q extending p which is, moreover, finitely satisfiable in M. In particular, q is M-invariant.

Definition 2.5. Suppose q is an A-invariant global type and I is a linearly ordered set. By a *Morley sequence in q over A of order type I*, we mean a sequence $\langle b_\alpha \mid \alpha \in I \rangle$ such that for each $\alpha \in I$, $b_\alpha \models q|_{Ab_{<\alpha}}$ where $b_{<\alpha} = \langle b_\beta \mid \beta < \alpha \rangle$. Given a linear order I, we will write $q^{\otimes I}$ for the A-invariant type in variables $\langle x_\alpha \mid \alpha < I \rangle$ so that for any $B \supseteq A$, if $\bar{b} \models q^{\otimes I}|_B$, then $b_\alpha \models q|_{Bb_{<\alpha}}$ for all $\alpha \in I$. If q is, moreover, finitely satisfiable in A, then we refer to a Morley sequence in q over A as a *coheir sequence* over A.

The above definition of $q^{\otimes I}$ generalizes the finite tensor product $q^{\otimes n}$ – given any global A-invariant type q and linearly ordered set I, one may easily show that $q^{\otimes I}$ exists and is A-invariant by compactness.

Definition 2.6. Suppose M is a model.

1. Given a formula $\varphi(x;b)$ and a global M-invariant type $q \supseteq \mathrm{tp}(b/M)$, we say that $\varphi(x;b)$ *k-Kim-divides over M via q* if, whenever $\langle b_i \mid i < \omega \rangle$ is a Morley sequence over M in q, then $\{\varphi(x;b_i) \mid i < \omega\}$ is k-inconsistent.
2. If q is a global M-invariant type with $q \supseteq \mathrm{tp}(b/M)$, we say $\varphi(x;b)$ *Kim-divides over M via q* if $\varphi(x;b)$ k-Kim-divides over M via q for some $k < \omega$.
3. We say $\varphi(x;b)$ Kim-divides over M if $\varphi(x;b)$ Kim-divides over M via q for some global M-invariant $q \supseteq \mathrm{tp}(b/M)$.
4. We say that $\varphi(x;b)$ Kim-forks over M if it implies a finite disjunction of formulas, each Kim-dividing over M.
5. We write $a \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{K}_{M} B$ if $\mathrm{tp}(a/MB)$ does not Kim-fork (or Kim independent) over M.

Note that if $a \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{u}_{M} B$, then $a \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{f}_{M} B$ (i.e., $\mathrm{tp}(a/BM)$ does not fork over M) which implies $a \mathop{\smile\!\!\!\!\!\!\!\;|}\,^{K}_{M} B$.

Fact 2.7 ([KR17, Theorem 3.15])**.** The following are equivalent for the complete theory T:

1. T is NSOP_1.
2. (Kim's lemma for Kim-dividing) Given any model $M \models T$ and formula $\varphi(x;b)$, $\varphi(x;b)$ Kim-divides via q for *some* global M-invariant $q \supseteq \mathrm{tp}(b/M)$ if and only if $\varphi(x;b)$ Kim-divides via q for *all* global M-invariant $q \supseteq \mathrm{tp}(b/M)$.

From this it easily follows that Kim-forking is equal to Kim-dividing [KR17, Proposition 3.19]. The notion of Kim independence, denoted by $\mathop{\smile\!\!\!\!\!\!\mid}^K$, satisfies many nice properties which turn out to be equivalent to NSOP$_1$.

Fact 2.8 ([KR17, Theorem 8.1])**.** The following are equivalent for the complete theory T:

(1) T is NSOP$_1$.
(2) Symmetry of Kim independence over models: $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$ iff $b \mathop{\smile\!\!\!\!\!\!\mid}^K_M a$ for any $M \models T$.
(3) Independence theorem over models: if $A \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$, $c \mathop{\smile\!\!\!\!\!\!\mid}^K_M A$, $c' \mathop{\smile\!\!\!\!\!\!\mid}^K_M B$, and $c \equiv_M c'$, then there is some $c'' \mathop{\smile\!\!\!\!\!\!\mid}^K_M AB$ such that $c'' \equiv_{MA} c$ and $c'' \equiv_{MB} c'$.

Fact 2.9 ([KR17, Lemma 7.6])**.** Suppose that T is NSOP$_1$ and that $\langle a_i \,|\, i < \omega \rangle$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley sequence over M in the sense that $a_i \mathop{\smile\!\!\!\!\!\!\mid}^K_M a_{<i}$ and the sequence is indiscernible. Then if $\varphi(x, a_0)$ does not Kim-divide over M, then $\{\varphi(x, a_i) \,|\, i < \omega\}$ does not Kim-divide over M, and in particular it is consistent.

2.2. **The generalized club filter.**

Definition 2.10. Let κ be a cardinal and let X be a set with $|X| \geq \kappa$. We write $[X]^\kappa$ to denote $\{Y \subseteq X \,|\, |Y| = \kappa\}$.

(1) A set $C \subseteq [X]^\kappa$ is *unbounded* if for every $Y \in [X]^\kappa$, there is some $Z \in C$ with $Y \subseteq Z$.
(2) A set $C \subseteq [X]^\kappa$ is *closed* if, whenever $\langle Y_i \,|\, i < \alpha \leq \kappa \rangle$ is a chain in C, i.e., each $Y_i \in C$ and $i < j < \alpha$ implies $Y_i \subseteq Y_j$, then $\bigcup_{i<\alpha} Y_i \in C$.
(3) A set $C \subseteq [X]^\kappa$ is *club* if it is closed and unbounded.
(4) A set $S \subseteq [X]^\kappa$ is *stationary* if $S \cap C \neq \emptyset$ for every club $C \subseteq [X]^\kappa$.

The intersection of two clubs is club. The *club filter* on $[X]^\kappa$ is the filter generated by the clubs. If $|X| = \kappa$, then the club filter on $[X]^\kappa$ is the principal ultrafilter consisting of subsets of $[X]^\kappa$ containing X.

Example 2.11. If M is an L-structure of size $\geq \kappa \geq |L|$, then the collection of elementary substructures of M of size κ is a club in $[M]^\kappa$.

Remark 2.12. In the literature, e.g., [Jec13, Definition 8.21], the above definitions are given instead for subsets of $\mathcal{P}_{\kappa^+}(X) = \{Y \subseteq X \,|\, |Y| < \kappa^+\}$ but note that $[X]^\kappa$ is a club subset of $\mathcal{P}_{\kappa^+}(X)$, hence all definitions relativize to this set in the natural way.

Fact 2.13. Let κ be a cardinal and let X be a set with $|X| \geq \kappa^+$.

(1) The club filter on $[X]^\kappa$ is κ^+-complete [Jec13, Theorem 8.22].
(2) For every club $C \subseteq [X]^\kappa$, there is a collection of finitary functions $\overline{f} = \langle f_i \,|\, i < \kappa \rangle$ with $f_i : X^{n_i} \to X$ such that
$$C_{\overline{f}} := \{Y \in [X]^\kappa \,|\, f_i(Y^{n_i}) \subseteq Y \text{ for all } i < \kappa\}$$
is contained in C. Equivalently, there is a function $F : X^{<\omega} \to [X]^\kappa$ such that $C_F \subseteq C$ [Jec13, Lemma 8.26].
(3) Conversely, given a collection of finitary functions $\overline{f} = \langle f_i \,|\, i < \kappa \rangle$ with $f_i : X^{n_i} \to X$, the set $C_{\overline{f}}$ is club in $[X]^\kappa$.
(4) When $\kappa = \omega$, for any club $C \subseteq [X]^\kappa$, there is a function $F : X^{<\omega} \to X$ such that $C_F \subseteq C$ [Jec13, Theorem 8.28].

We leave the proof of the next lemma to the reader.

Lemma 2.14. *Suppose λ is a cardinal, X is a set with $|X| = \lambda^+$, and $\langle Y_\alpha \,|\, \alpha < \lambda^+ \rangle$ is an increasing continuous sequence of sets of cardinality λ with union X. Then $\{Y_\alpha \,|\, \alpha < \lambda^+\}$ is a club of $[X]^\lambda$. In particular, if $X = \lambda^+$ and $C \subseteq \lambda^+ \setminus \lambda$ is a club of λ^+, then, recalling each ordinal in C is the set of its predecessors, we have C is a club of $[X]^\lambda$.*

3. Proof of Theorem 1.1

3.1. **A short proof of (1) implies (4) in Theorem 1.1 using heirs.** Here we give a short proof of (1) implies (4) in Theorem 1.1, due to Pierre Simon. We thank him for allowing us to include this proof.

Lemma 3.1. *Suppose $p(x) \in S(M)$, $M \models T$. Then the set of $N \prec M$ such that $|N| = |T|$ and p is an heir of $p|_N$ is a club subset of $[M]^{|T|}$.*

Proof. It is easy to verify that this set is closed under increasing unions, so it is closed. Therefore, to show it is unbounded, and hence club, it is enough to show that it contains a club.

Consider the L_p-structure M_p expanding M by forcing p to be definable — i.e., for every L-formula $\varphi(x;y)$ add a relation $R_\varphi(y)$ interpreted as $\{b \in M^{|y|} \,|\, \varphi(x,b) \in p\}$. Note that $|L_p| = |L|$. Then if $N' \prec M_p$, then its L-reduct N is such that p is an heir of $p|_N$. Thus we are done by Example 2.11. □

Theorem 3.2. *Suppose T is NSOP$_1$. If $M \models T$ and $p \in S(M)$, then the set of elementary substructures $N \prec M$ with $|N| = |T|$ such that p does not Kim-divide over N contains a club.*

Proof. By Lemma 3.1, it suffices to show that if p is an heir of $p|_N$, then p does not Kim-divide over N. But if p is an heir of $p|_N$, then, given $c \models p$, $M \mathop{\smile\!\!\!\!\!\!\;|}\,^u_N c$, hence $M \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N c$ by symmetry of Kim-independence (in fact one needs only a weak version of symmetry; see [KR17, Proposition 3.22]) which implies $c \mathop{\smile\!\!\!\!\!\!\;|}\,^K_N M$. This shows that p does not Kim-divide over N. □

3.2. **A proof of (1) implies (6) in Theorem 1.1.**

Lemma 3.3. *Suppose T is an arbitrary theory and $M \models T$ with $|M| \geq |T| = \kappa$. Given any global M-finitely satisfiable type q, let C_q denote the set of $N \prec M$ with $|N| = \kappa$ such that $q^{\otimes\omega}|_N = r^{\otimes\omega}|_N$ for some global N-finitely satisfiable type r. Then:*

1. *C_q is a club of $[M]^\kappa$.*
2. *Given any set $A \subseteq M$, there is some $N \prec M$ of size $\leq |T| + |A|$ such that $A \subseteq N$ and $q^{\otimes\omega}|_N$ is a type of a Morley sequence generated by some global type r finitely satisfiable in N and if $\varphi(x,c)$ Kim-divides over M via q, then $\varphi(x,c)$ Kim-divides over N via r.*

Proof. One proof of (1) essentially follows from the proof of [KR17, Lemma 4.4], so we also give an alternative one. Let $\bar{a} = \langle a_i \,|\, i < \omega \rangle$ be a coheir sequence generated by q over M. Then, $N \in C_q$ iff $N \prec M$ and $\bar{a}$ is a coheir sequence over N in the sense that $\operatorname{tp}(a_i/a_{<i}N)$ if finitely satisfiable in N. Thus it is easy to see that C_q is closed under unions.

Note that if $N \prec M$ is such that $\mathrm{tp}(\bar{a}/M)$ is an heir extension of its restriction to N, then $N \in C_q$: if $\varphi(a_i, a_{<i})$ holds when $\varphi(x, y)$ is some formula over N, then for some $c \in M$, $\varphi(c, a_{<i})$ holds, and by choice of N, we may assume that $c \in N$. Now Lemma 3.1 finishes the proof.

(2) is immediate from (1), applied to the theory $T(A)$ obtained from T by adding constants for the elements of A. $\square$

Theorem 3.4. *Suppose T is NSOP$_1$ with $|T| = \kappa$ and $M \models T$. Then for a finite tuple b and any set A, the following are equivalent:*

1. $A \mathop{\smile\!\!\!\!\!\mid}^K_M b$.
2. *There is a club $C \subseteq [M]^\kappa$ of elementary substructures of M such that $A \mathop{\smile\!\!\!\!\!\mid}^K_N b$ for all $N \in C$.*
3. *There is a stationary set $S \subseteq [M]^\kappa$ of elementary substructures of M such that $A \mathop{\smile\!\!\!\!\!\mid}^K_N b$ for all $N \in S$.*

Proof. (1) $\Longrightarrow$ (2) Suppose that $A \mathop{\smile\!\!\!\!\!\mid}^K_M b$. Let $q \supseteq \mathrm{tp}(b/M)$ be a global M-finitely satisfiable type and choose $\langle b_i \mid i < \omega \rangle \models q^{\otimes\omega}|_M$ with $b_0 = b$. By Lemma 3.3, there is a club C_q of elementary substructures $N \prec M$ with $|N| = |T|$ so that $q^{\otimes\omega}|_N = r^{\otimes\omega}|_N$ for some global N-finitely satisfiable type r. Fix $N \in C$, a a finite tuple from A and $\varphi(x; b, n) \in \mathrm{tp}(a/Nb)$. As $a \mathop{\smile\!\!\!\!\!\mid}^K_M b$, we know $\{\varphi(x; b_i, n) \mid i < \omega\}$ is consistent. As $\langle b_i \mid i < \omega \rangle$ is also a Morley sequence over N in a global N-finitely satisfiable type, it follows from Kim's lemma for Kim-dividing (Fact 2.7) that $\varphi(x; b, n)$ does not Kim-divide over N. As $\varphi(x; b, n)$ was arbitrary, we conclude $a \mathop{\smile\!\!\!\!\!\mid}^K_N b$. Since this was true for any a, we have that $A \mathop{\smile\!\!\!\!\!\mid}^K_N b$.

(2) $\Longrightarrow$ (3) is immediate.

(3) $\Longrightarrow$ (1) Suppose $a \not\mathop{\smile\!\!\!\!\!\mid}^K_M b$ for some finite tuple a from A. Let $\varphi(x; b, m) \in \mathrm{tp}(a/Mb)$ be a formula witnessing this. Fix $q \supseteq \mathrm{tp}(b/M)$ a global M-finitely satisfiable type and $\langle b_i \mid i < \omega \rangle \models q^{\otimes\omega}|_M$. Let $C' = \{N \prec M \mid |N| = |T| \text{ and } m \in N\}$. The set C' is clearly club so the intersection $C'' = C_q \cap C'$ is also a club of $[M]^\kappa$. If $N \in C''$ and $q^{\otimes\omega}|_N = r^{\otimes\omega}|_N$ for some global type r finitely satisfiable in N, then $\varphi(x; b, m) \in \mathrm{tp}(a/Nb)$ and $\langle b_i \mid i < \omega \rangle$ realizes $r^{\otimes\omega}|_N$. As $\{\varphi(x; b_i, m) \mid i < \omega\}$ is inconsistent, we have $a \not\mathop{\smile\!\!\!\!\!\mid}^K_N b$. As S is stationary, it must intersect C'', so we get a contradiction. $\square$

Corollary 3.5. *Suppose T is NSOP$_1$ with $|T| = \kappa$ and $M \models T$. Then for a finite tuple a and any set B, the following are equivalent:*

1. $a \mathop{\smile\!\!\!\!\!\mid}^K_M B$.
2. *There is a club $C \subseteq [M]^\kappa$ of elementary substructures of M such that $a \mathop{\smile\!\!\!\!\!\mid}^K_N B$ for all $N \in C$.*
3. *There is a stationary set $S \subseteq [M]^\kappa$ of elementary substructures of M such that $a \mathop{\smile\!\!\!\!\!\mid}^K_N B$ for all $N \in S$.*

Proof. Follows immediately from symmetry of Kim-independence and Theorem 3.4. $\square$

Lemma 3.6. *Suppose T is NSOP$_1$. Assume $M \prec N$. Suppose that $a \mathop{\smile\!\!\!\!\!\mid}^K_M N$ and $\varphi(x, a)$ Kim-divides over N for $\varphi(x, y) \in L(M)$. Then $\varphi(x, a)$ Kim-divides over M.*

Proof. Let $\langle a_i \mid i<\omega\rangle$ be an indiscernible sequence over N starting with $a_0 = a$ such that $a_i \mathop{\smile\!\!\!\!\!\!\mid}^h_N a_{<i}$ and $\{\varphi(x,a_i) \mid i<\omega\}$ is inconsistent (to construct it, let $\langle b_i \mid i\in\mathbb{Z}\rangle$ be a coheir sequence in the type of $\mathrm{tp}(a/N)$, so in particular $b_i \mathop{\smile\!\!\!\!\!\!\mid}^u_N b_{<i}$ for $i<0$, hence $b_{>i} \mathop{\smile\!\!\!\!\!\!\mid}^u_N b_i$ by transitivity of $\mathop{\smile\!\!\!\!\!\!\mid}^u$, and let $a_i = b_{-i}$ for $i<\omega$).

Then $\langle a_i \mid i<\omega\rangle$ is an $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley sequence over M in the sense that $a_i \mathop{\smile\!\!\!\!\!\!\mid}^K_M a_{<i}$. To see this, suppose not, i.e., by symmetry suppose that $a_{<i} \not\mathop{\smile\!\!\!\!\!\!\mid}^K_M a_i$. Then for some formula $\psi(z,x)$ over M, $\psi(a_{<i},a_i)$ holds and $\psi(z,a_i)$ Kim-divides over M. Since $a_{<i} \mathop{\smile\!\!\!\!\!\!\mid}^u_N a_i$, for some $n\in N$, $\psi(n,a_i)$ holds. However, since $a_i \equiv_N a$, by symmetry $N \mathop{\smile\!\!\!\!\!\!\mid}^K_M a_i$ — contradiction.

Suppose that $\varphi(x,a)$ does not Kim-divide over M. Then by Fact 2.9,

$$\{\varphi(x,a_i) \mid i<\omega\}$$

is consistent — contradiction. □

Lemma 3.7. *Suppose T is NSOP$_1$. Suppose that $\langle M_i \mid i\le\alpha\rangle$ is an increasing sequence of elementary substructures of a model N, that $M_\alpha = \bigcup\{M_i \mid i<\alpha\}$ and that $p\in S(N)$. Assume that p does not Kim-fork over M_i for all $i<\alpha$. Then p does not Kim-fork over M_α.*

Proof. Let $a\models p$. We want to show that $a \mathop{\smile\!\!\!\!\!\!\mid}^K_{M_\alpha} N$, so by symmetry it is enough to show that $N \mathop{\smile\!\!\!\!\!\!\mid}^K_{M_\alpha} a$. Suppose not. Then there is some formula $\varphi(x,y)$ in $L(M_\alpha)$ and some $b\in N$ such that $\varphi(b,a)$ holds and $\varphi(x,a)$ Kim-divides over M_α. Let $i<\alpha$ be such that $\varphi(x,y)\in L(M_i)$. Since $M_\alpha\subseteq N$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_{M_i} N$ by assumption, $a \mathop{\smile\!\!\!\!\!\!\mid}^K_{M_i} M_\alpha$. Hence by Lemma 3.6, $\varphi(x,a)$ Kim-divides over M_i. Hence $b \not\mathop{\smile\!\!\!\!\!\!\mid}^K_{M_i} a$. But this is a contradiction since $a \mathop{\smile\!\!\!\!\!\!\mid}^K_{M_i} N$ so by symmetry $b \mathop{\smile\!\!\!\!\!\!\mid}^K_{M_i} a$. □

We can now prove (1) $\implies$ (6) from Theorem 1.1.

Theorem 3.8. *Suppose that T is NSOP$_1$. Suppose that a is a finite tuple, $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M N$, and $M\prec N$. Then the set E of $M'\in[M]^{|T|}$ such that $M'\prec M$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_{M'} N$ is a club.*

Proof. The family E is closed under unions by Lemma 3.7. Hence to finish we only need to show that E contains a club, and this follows from Corollary 3.5 (1) $\implies$ (2). □

3.3. The equivalence (1)–(6).

We finish the proof of Theorem 1.1 with the following.

Theorem 3.9. *Suppose T is a complete theory. The following are equivalent:*

1. *T is NSOP$_1$.*
2. *There is no continuous increasing sequence of $|T|$-sized models $\langle M_i \mid i<|T|^+\rangle$ with union M and $p\in S(M)$ such that $p\restriction M_{i+1}$ Kim-forks over M_i for all $i<|T|^+$.*
3. *For any $M\models T$, $p\in S(M)$, the set of elementary substructures of M of size $|T|$ over which p does not Kim-divide is a stationary subset of $[M]^{|T|}$.*
4. *For any $M\models T$, $p\in S(M)$, the set of elementary substructures of M of size $|T|$ over which p does not Kim-divide contains a club subset of $[M]^{|T|}$.*

(5) *For any* $M \models T$, $p \in S(M)$, *the set of elementary substructures of* M *of size* $|T|$ *over which* p *does not Kim-divide is a club subset of* $[M]^{|T|}$.
(6) *Suppose that* $N \models T$, $M \prec N$, *and* $p \in S(N)$ *does not Kim-divide over* M. *Then the set of elementary substructures of* M *of size* $|T|$ *over which* p *does not Kim-divide is a club subset of* $[M]^{|T|}$.

Proof. (1) $\Longrightarrow$ (6) is Theorem 3.8.

(6) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) is trivial (for (6) implies (5), note that for $p \in S(M)$, p does not Kim-divide over M trivially).

(3) $\Longrightarrow$ (2) Assume (3) holds but (2) fails. By Lemma 2.14, $C = \{M_i \mid i < |T|^+\}$ is a club of $[M]^{|T|}$. As T is NSOP$_1$, by (3), there is a stationary set $S \subseteq [M]^{|T|}$ such that $N \in S$ implies p does not Kim-fork over N. Choose any $M_i \in C \cap S$ to obtain a contradiction.

(2) $\Longrightarrow$ (1). Suppose T has SOP$_1$ as witnessed by some formula $\varphi(x, y)$. Let T^{sk} be a Skolemized expansion of T. Then T^{sk} also has SOP$_1$ as witnessed by $\varphi(x, y)$. Thus by Fact 2.2, we can find a formula $\varphi(x, y)$ and an array $\langle c_{i,j} \mid i < \omega, j < 2\rangle$ such that $c_{i,0} \equiv_{\bar{c}_{<i}} c_{i,1}$ for all $i<\omega$, $\{\varphi(x, c_{i,0}) \mid i<\omega\}$ is consistent and $\{\varphi(x; c_{i,1}) \mid i<\omega\}$ is 2-inconsistent (all in $\mathbb{M}^{sk}$). By Ramsey and compactness we may assume that $\langle \bar{c}_i \mid i < \omega\rangle$ is indiscernible (with respect to $\mathbb{M}^{sk}$) and extend this sequence to length $|T|^+$.

For $i \leq |T|^+$, let $N_i = \mathrm{dcl}(\bar{c}_{<i})$ (in $\mathbb{M}^{sk}$). Then for every limit ordinal $\delta < |T|^+$, $\varphi(x, c_{\delta,1})$ Kim-divides over N_δ as the sequence $\langle c_{j,1} \mid \delta \leq j < |T|^+\rangle$ is indiscernible and for all $\delta \leq j$, $\bar{c}_j \mathop{\smile\!\!\!\!\!\!\,|}^{u}_{N_\delta} \bar{c}_{>j}$. As $c_{\delta,1} \equiv_{\bar{c}_{<\delta}} c_{\delta,0}$, it follows that $c_{\delta,1} \equiv_{N_\delta} c_{\delta,0}$, and hence $\varphi(x, c_{\delta,0})$ also Kim-divides. Let $p \in S(N_{|T|^+})$ be any complete type containing $\{\varphi(x, c_{\delta,0}) \mid \delta < \kappa\}$, which is possible as this partial type is consistent. The sequence $\langle N_\delta \mid \delta \in \lim(|T|^+)\rangle$ is an increasing and continuous sequence of elementary substructures of $N_{|T|^+}$ of size $|T|$ with union $N_{|T|^+}$ witnessing that (2) fails. $\square$

Remark 3.10. The proof of (1) implies (6) in Theorem 1.1 relies heavily on symmetry of Kim-independence, whose proof assumes that the whole theory is NSOP$_1$. However, a closer look at the proof of (1) implies (4) given in Section 3, or observing the proof using stationary logic sketched below, we see that for (1) implies (4), we only need that a particular formula $\varphi(x, y)$ does not have an SOP$_1$ array as in Fact 2.2.

Corollary 3.11. *Suppose* T *is NSOP*$_1$, $M \models T$, $M \prec N$, *and* $p \in S(N)$. *Then* p *does not Kim-fork over* M *iff for every* κ *with* $|T| \leq \kappa \leq |M|$, *the set of elementary substructures of* M *of size* κ *over which* p *does not Kim-divide is a club subset of* $[M]^\kappa$.

Proof. Suppose that p does not Kim-fork over M. Let $A \subseteq M$ be any subset of M of size κ and apply Theorem 1.1 to the theory $T(A)$ obtained from T by adding new constant symbols for the elements of A.

For the other direction, apply the left hand side with $\kappa = |T|$ and use Corollary 3.5. $\square$

Corollary 3.12. *Suppose* T *is NSOP*$_1$ *and* $M \models T$. *Then given any set* A, *there is a club* $E \subseteq [M]^{|T|+|A|}$ *such that* $N \in E$ *iff* $A \mathop{\smile\!\!\!\!\!\!\,|}^{K}_{N} M$.

Proof. Let $\kappa = |A| + |T|$. By Corollary 3.11, we know for each finite tuple a from A, there is a club $E_a \subseteq [M]^\kappa$ so that $N \in E_a$ iff $a \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N M$. Let $E = \bigcap_{a \in A} E_a$. As $|A| \leq \kappa$ and the club filter on $[M]^\kappa$ is κ^+-complete (Fact 2.13(1)), E is a club of $[M]^\kappa$. By the strong finite character of Kim-independence, we have $A \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N M$ iff $N \in E$. □

3.4. A sample application.

Proposition 3.13. *Suppose T is NSOP$_1$ and $A \models T$. Given any set C, there is some $C' \supseteq C$ with $|C'| = |C| + |T|$ such that $C' \cap A$ is a model and $C' \mathop{\smile\!\!\!\!\!\!\!\;|}^K_{A \cap C'} A$.*

Proof. Let $\kappa = |C| + |T|$. Let $C_0 = C$ and, by Corollary 3.12, we may let $E_0 \subseteq [A]^\kappa$ be a club of elementary substructures of A such that $N \in E_0$ implies $C_0 \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N A$. By induction, we will choose sets C_i, clubs $E_i \subseteq [A]^\kappa$, and models $X_i \prec A$ such that

(1) $X_i \in \bigcap_{j \leq i} E_i$ and $C_i \cap A \subseteq X_i$.
(2) $C_{i+1} = C_i \cup X_i$.
(3) For all $N \in E_i$, we have $C_i \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N A$.

Given $\langle C_i, X_i, E_i \,|\, i \leq n \rangle$, let $C_{n+1} = C_n \cup X_n$. By Corollary 3.12, we may let $E_{n+1} \subseteq [A]^\kappa$ be a club such that $N \in E_{n+1}$ implies $C_{n+1} \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N A$. As

$$\{X \in [A]^\kappa \,|\, C_{n+1} \cap A \subseteq X\}$$

is a club of $[A]^\kappa$, we may choose $X_{n+1} \in \bigcap_{i \leq n+1} E_i$ containing $C_{n+1} \cap A$. This completes the induction.

Let $C_\omega = \bigcup_{i<\omega} C_i$. By construction, $C_\omega \cap A = \bigcup_{i<\omega} X_i$. As $i < j$ implies $X_i \subseteq X_j$, and $i \geq n$ implies $X_i \in E_n$, it follows that

$$C_\omega \cap A = \bigcup_{i \geq n} X_i \in E_n$$

for all n, as E_n is club. Also as each X_i is a model, this additionally shows that $C_\omega \cap A$ is a model. Moreover, if $c \in C_\omega$ is a finite tuple, there is some n so that $c \in C_n$, hence $c \mathop{\smile\!\!\!\!\!\!\!\;|}^K_{C_\omega \cap A} A$, by the choice of E_n. Setting $C' = C_\omega$, we finish. □

3.5. Open questions.

Question 3.14. Is the dual of Lemma 3.6 also true? Namely, suppose that $a \mathop{\smile\!\!\!\!\!\!\!\;|}^K_M N$ and $\varphi(x, a)$ Kim-divides over M for $\varphi(x, y) \in L(M)$. Then is it true that $\varphi(x, a)$ Kim-divides over N?

If the answer to Question 3.14 is "yes", then we have the following weak form of transitivity (note that a full version of transitivity does not hold; see [KR17, Section 9.2]).

Claim 3.15 (Weak form of transitivity). Suppose the answer is "yes". Let $M \prec N$. Suppose that $a \mathop{\smile\!\!\!\!\!\!\!\;|}^K_M N$ and $a \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N B$. Then $a \mathop{\smile\!\!\!\!\!\!\!\;|}^K_M B$.

Proof. Suppose not. Then by symmetry there is a formula $\varphi(x, y)$ over M such that $\varphi(b, a)$ holds for some $b \in B$ and $\varphi(x, a)$ Kim-divides over M. However, since $b \mathop{\smile\!\!\!\!\!\!\!\;|}^K_N a$, $\varphi(x, a)$ does not Kim-divide over N. By assumption we arrive at a contradiction. □

Question 3.16. Does the weak form of transitivity hold in NSOP$_1$ theories?

Question 3.17. Is there a local counterpart to Lemma 3.7? Namely, under NSOP_1, assume that $\varphi(x,a)$ does not Kim-fork over M_i for $i<\alpha$ an increasing union. Is it true that $\varphi(x,a)$ does not Kim-fork over $\bigcup_{i<\alpha} M_i$?

Question 3.18. Is it true that T is NSOP_1 if and only if for every $M \models T$ and complete type $p \in S(M)$, there is some $N \prec M$ of cardinality $\leq |T|$ such that p does not Kim-fork over N?

4. A proof of (1) implies (4) in Theorem 1.1 using stationary logic

4.1. **More on clubs.** The club filter on $[X]^{\omega}$ was characterized by Kueker in terms of games of length ω [Kue72]. The natural analogue for games of length λ determines a filter on $\mathcal{P}_{\lambda^+}(X)$, which, in general, differs from the club filter. In generalizing stationary logic to quantification over sets of some uncountable size λ, it turns out that this filter provides a more useful analogue to the club filter on $[X]^{\omega}$ than the club filter on $[X]^{\lambda}$.

Definition 4.1. Suppose X is a set and λ is a regular cardinal. Given a subset $F \subseteq \mathcal{P}_{\lambda^+}(X)$, we define the game $G(F)$, to be the game of length λ where Players I and II alternate playing an increasing λ sequence of elements of $\mathcal{P}_{\lambda^+}(X)$. In this game, Player II wins if and only if the union of the sets played is in F. The filter $D_\lambda(X)$ is defined to be the filter generated by the sets $F \subseteq \mathcal{P}_{\lambda^+}(X)$ in which Player II has a winning strategy in $G(F)$. We say $Y \subseteq \mathcal{P}_{\lambda^+}(X)$ is $D_\lambda(X)$-*stationary* if Y intersects every set in $D_\lambda(X)$.

It is easy to check that every club $C \subseteq [X]^{\lambda}$ is an element of $D_\lambda(X)$ and, therefore, that every $S \subseteq [X]^{\lambda}$ that is $D_\lambda(X)$-stationary is also stationary with respect to the usual club filter on $[X]^{\lambda}$. It was remarked in [MS86] that if $\lambda = \lambda^{<\lambda}$, then $D_\lambda(\lambda^+)$ is just the filter generated by the clubs of λ^+ intersected with the set of ordinals of cofinality λ (considered as initial segments of λ^+). More precisely, we have the following fact. (We omit its proof since it is not necessary for the rest.)

Fact 4.2. Suppose λ is an infinite regular cardinal and write $S^{\lambda^+}_{\lambda}$ for the stationary set $\{\alpha < \lambda^+ \mid \mathrm{cf}(\alpha) = \lambda\}$.

1. If $C \subseteq \lambda^+$ is a club, then $C \cap S^{\lambda^+}_{\lambda} \in D_\lambda(\lambda^+)$.
2. Suppose $\lambda = \lambda^{<\lambda}$. Then $D_\lambda(\lambda^+)$ is generated by sets of the form $C \cap S^{\lambda^+}_{\lambda}$, where $C \subseteq \lambda^+$ is a club.

4.2. **Stationary logic.** The stationary logic $L(\mathsf{aa})$ was introduced in [She75] (where it was called $L(Q^{ss}_{\aleph_1})$). The logic is defined as follows: given a first-order language L, expand the language with countably many new unary predicates $\{S_i \mid i<\omega\}$ and a new quantifier aa. The formulas of L in $L(\mathsf{aa})$ are the smallest class containing the first-order formulas of L, closed under the usual first-order formation rules together with the rule that if φ is a formula, then $(\mathsf{aa}S_i)\,\varphi$ is also a formula, for any new unary predicate S_i. Satisfaction is defined as usual, together with the rule that $M \models (\mathsf{aa}S)\,\varphi(S)$ if and only if $M \models \varphi(S)$ when $S^M = X$ for "almost all" $X \in [M]^{\omega}$—that is, $\left\{X \in [M]^{\omega} \mid \text{ if } S^M = X \text{ then } M \models \varphi(S)\right\}$ contains a club of $[M]^{\omega}$. We define the quantifier stat dually: $M \models (\mathsf{stat}S)\,\varphi(S)$ if and only if $M \models \neg(\mathsf{aa}S)\neg\varphi(S)$. Note that $M \models (\mathsf{stat}S)\,\varphi(S)$ if and only if $\left\{X \in [M]^{\omega} \mid \text{ if } S^M = X \text{ then } M \models \varphi(S)\right\}$ is stationary. Given an L-structure M, we write $\mathrm{Th}_{\mathsf{aa}}(M)$ for the set of $L(\mathrm{aa})$-sentences satisfied by M. We refer the reader to [BKM78, Section 1] for a detailed treatment of stationary logic.

Later work by Mekler and the third-named author extended stationary logic, which quantifies over *countable sets*, to a logic that permits quantification over sets of higher cardinality [MS86]. For λ a regular cardinal, the logic $L(\mathsf{aa}^\lambda)$ is defined analogously to $L(\mathsf{aa})$, with semantics defined so that $M \models (\mathsf{aa}^\lambda S)\,\varphi(S)$ if and only if $\{X \in [M]^\lambda \mid \text{ if } S^M = X \text{ then } M \models \varphi(S)\} \in D_\lambda(M)$. The quantifier stat^λ is also understood dually: $M \models (\mathsf{stat}^\lambda S)\,\varphi(S)$ if and only if $M \models \neg(\mathsf{aa}^\lambda S)\,\neg\varphi(S)$. If T is an $L(\mathsf{aa})$-theory, one obtains an $L(\mathsf{aa}^\lambda)$-theory by replacing the quantifier aa with aa^λ. We call this theory the *λ-interpretation of T*. By working with $D_\lambda(M)$ instead of the full club filter on $[M]^\lambda$, one is able to relate satisfiability of an $L(\mathsf{aa})$-theory to the satisfiability of its λ-interpretation. Below, the "moreover" clause about λ-saturation is not stated in [MS86], but is immediate from the proof.

Fact 4.3 ([MS86, Theorem 1.3])**.** Suppose $\lambda = \lambda^{<\lambda}$ and T is a consistent $L(\mathsf{aa})$-theory of size at most λ. Then the λ-interpretation of T has a model of size at most λ^+. In fact, there is such a model which is, moreover, λ-saturated.

4.3. **Sketch of the proof of (1) implies (4).** The idea of the proof of (1) $\Longrightarrow$ (4) in Theorem 1.1 using stationary logic is as follows:

(1) Reduce the theorem to the case where the language is countable.
(2) Assume towards contradiction that $M \models T$, and there is $p \in S(M)$ so that the set
$$S_0 = \{N \prec M \mid |N| = \aleph_0 \text{ and } p \text{ Kim-divides over } N\}$$
is stationary. Using Fact 4.3 we can show that given any regular uncountable cardinal $\lambda = \lambda^{<\lambda}$, there is a model $M' \models T$, $|M'| = \lambda^+$, a formula $\varphi(x;y)$, and a type p_* over M' so that
$$S_0' = \{N' \prec M' \mid |N'| = \lambda, \text{ there is } \varphi(x;a_N') \in p_* \text{ that Kim-divides over } N'\}$$
is $D_\lambda(M')$-stationary.
(3) Now using Fact 4.2 and forcing (in order to find such a cardinal λ), we can get an increasing continuous chain of models and formulas forming a Kim-dividing chain, indexed by a stationary set.
(4) Apply the proof of [KR17, Theorem 4.5], to get some k such that for every $n < \omega$ there is a sequence $\langle e_i f_i \mid i < n\rangle$ such that $e_i \equiv_{e_{<i} f_{<i}} f_i$ for all $i < n$, $\{\varphi(x, e_i) \mid i < n\}$ is consistent while $\{\varphi(x, f_i) \mid i < n\}$ is k-inconsistent. as in Fact 2.2

For the full details, see [KRS17].

5. Dual local character

Definition 5.1 (T any theory)**.** Say that a formula $\varphi(x, a)$ *strongly Kim-divides* over a model M if for every global M-invariant type $q \supseteq \mathrm{tp}(a/M)$, $\varphi(x, a)$ Kim-divides *over M via q*.

Remark 5.2. By Fact 2.7, strong Kim-dividing = Kim-dividing iff T is NSOP_1.

Definition 5.3. A *dual type* (over A) in x is a set F of (A-)definable sets in x such that for some $k < \omega$, it is k-inconsistent. Say that F *dually divides* over a model N, if every $X \in F$ which is not definable over N divides over N. Similarly define when F *dually Kim-divides* over N and when F *strongly dually Kim-divides* over N.

Theorem 5.4. *The following are equivalent for a complete theory T:*

(1) *T is NSOP$_1$.*
(2) *There is no continuous increasing sequence of $|T|$-sized models $\langle M_i \mid i < |T|^+ \rangle$ with union M and a dual type F over M such that $F \restriction M_{i+1}$ does not strongly dually Kim-divide over M_i for all $i < |T|^+$.*
(3) *Assume that $M \models T$ and F is a dual type over M. Then there is a stationary subset S of $[M]^{|T|}$ such that if $N \in S$ then $N \prec M$ and F strongly dually Kim-divides over N.*
(4) *(Dual local character) Same as* (3) *but S is a club.*

Proof. The proof is essentially dualizing or inverting the proof (using stationary logic) of Theorem 1.1 (1) $\Longrightarrow$ (4), but we go into some details.

(1) $\Longrightarrow$ (4). We follow the proof of "(1) implies (4)" of Theorem 1.1 as described in Section 4. Namely, assume that (2) fails. This means that there is a stationary subset S of $[M]^{|T|}$ such that if $N \in S$, then $N \prec M$ and there is some $X \in F$ which is not definable over N but still does not Kim-divide over N. Using the same proof as in the first step in Section 4.3, we may assume that the language L is countable and that there is a single formula $\varphi(x, y)$ with $|x| = n$ such that if $N \in S$, then for some $b \in M \backslash N$, $\varphi(x, b)$ does not Kim-divide over N (and $\varphi(x, b)$ is not N-definable). Now we repeat the same procedure as in the proof described in Section 4.3 (2). Thus, for a regular uncountable cardinal $\lambda = \lambda^{<\lambda}$, we get a model $M' \models T$, $|M'| = \lambda^+$, a formula $\varphi(x, y)$, and a k-inconsistent family F_* of definable subsets over M' so that the set S'_0 of all $N' \prec M'$ of size λ such that for some $\varphi(x; a'_N) \in F_*$ which is not N'-definable and does not Kim-divide over N' is $D_\lambda(M')$-stationary. Now we repeat the proof in Section 4. The contradiction we will find at the end will be the same contradiction, but the roles of the two sequences e_i and f_j are reversed. Now $\{\varphi(x, e_i) \mid i < \omega\}$ is k-inconsistent (note that the formulas $\varphi(x, e_i)$ must define distinct definable sets from F_*) and $\langle \varphi(x, f_j) \mid j < n \rangle$ is consistent.

(4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) is exactly as in the proof of Theorem 3.9. The proof of (2) $\Longrightarrow$ (1) is just dualizing the proof of "(2) implies (1)" in Theorem 3.9 in the sense that the sequences $\langle c_{i,0} \mid i < \omega \rangle$ and $\langle c_{i,1} \mid i < \omega \rangle$ exchange places. □

Question 5.5. Is there a proof of the dual local-character which does not use stationary logic? Such a proof may reveal some new properties of Kim-dividing.

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