



A strong failure of \aleph_0 -stability for atomic classes

Michael C. Laskowski¹ · Saharon Shelah^{2,3}

Received: 19 January 2017 / Accepted: 28 March 2018 / Published online: 25 April 2018
© Springer-Verlag GmbH Germany, part of Springer Nature 2018

Abstract We study classes of atomic models \mathbf{At}_T of a countable, complete first-order theory T . We prove that if \mathbf{At}_T is not pcl-small, i.e., there is an atomic model N that realizes uncountably many types over $\text{pcl}_N(\bar{a})$ for some finite \bar{a} from N , then there are 2^{\aleph_1} non-isomorphic atomic models of T , each of size \aleph_1 .

Keywords Atomic models · Pseudo-algebraic · Non-structure

Mathematics Subject Classification Primary 03C50; Secondary 03C35

1 Introduction

In a series of papers [3–5], Baldwin and the authors have begun to develop a model theory for complete sentences of $L_{\omega_1, \omega}$ that have fewer than 2^{\aleph_1} non-isomorphic models of size \aleph_1 . By well known reductions, see e.g., Sect. 6.1 of [1], one can replace the reference to infinitary sentences by restricting to the class of *atomic*¹ models of a

¹ A model M is *atomic* if, for every finite tuple \bar{a} from M , $\text{tp}_M(\bar{a})$ is *principal* i.e., is uniquely determined by a single formula $\varphi(\bar{x}) \in \text{tp}_M(\bar{a})$.

M. C. Laskowski: Partially supported by NSF Grant DMS-1308546.
S. Shelah: Partially supported by European Research Council Grant 338821 and NSF Grant DMS-1362974. Publication no. 1099.

✉ Michael C. Laskowski
mcl@math.umd.edu

¹ Department of Mathematics, University of Maryland, College Park, USA

² Hebrew University, Jerusalem, Israel

³ Rutgers University, New Brunswick, USA

countable, complete first-order theory. Specifically, for every complete sentence Φ of $L_{\omega_1, \omega}$, there is a complete first-order theory T in a countable vocabulary containing the vocabulary of Φ such that the models of Φ are precisely the reducts of the class \mathbf{At}_T of atomic models of T to the smaller vocabulary.

The whole of this paper concerns complete theories T in a countable language such that the class \mathbf{At}_T of atomic models of T has at least one uncountable element. By theorems of Vaught, these restrictions on T are well understood. A countable, complete T has an atomic model if and only if every consistent formula can be extended to a complete formula. Furthermore, any two countable, atomic models of T are isomorphic, and a model is prime if and only if it is countable and atomic. Using a well-known union of chains argument, T has an atomic model of size \aleph_1 if and only if the countable atomic model is not minimal, i.e., it has a proper elementary substructure.

To date, the analysis of uncountable atomic models in \mathbf{At}_T has followed the first-order setting. Recall that if T is a complete theory in a countable language that is not \aleph_0 -stable, then there are 2^{\aleph_1} non-isomorphic models of size \aleph_1 , see e.g., VIII Conclusion 1.7(2) of [8]. The proof of this splits into two cases. One first establishes the result for unsuperstable theories, and then invokes a separate argument for theories that are superstable, but not \aleph_0 -stable.

Superstability itself does not make a good dividing line for atomic models. This can be seen by considering a two-sorted structure $M = (U, V)$, where U denotes an infinite set with no structure, and V consists of a single copy of (\mathbb{Z}, \leq) . Even though $T = Th(M)$ is unstable, \mathbf{At}_T is κ -categorical for every infinite cardinal κ – the point being that the (\mathbb{Z}, \leq) sort cannot be increased in any atomic model.

To adjust for this, in [3], Baldwin and the authors defined the notion of *pseudo-algebraicity*, which was introduced in [3], that is the correct analog of algebraicity in the context of atomic models. Suppose M is an atomic model, and b, \bar{a} are from M . We say $b \in \text{pcl}_M(\bar{a})$ if $b \in N$ for every elementary submodel $N \preceq M$ that contains \bar{a} .

By analogy to weak minimality, call a formula $\varphi(x, \bar{a})$ *pseudo-minimal* if it is not pseudo-algebraic, yet pseudo-algebraic closure pcl_M satisfies the exchange axiom on the set of solutions $\varphi(M, \bar{a})$. [Weakly minimal formulas can be characterized as the non-algebraic formulas for which the relation of algebraic closure satisfies exchange.] In the first order context, if T is superstable, then every non-algebraic formula extends to a weakly minimal formula. By analogy, in [3] we prove that if \mathbf{At}_T is an atomic class and there is some non-pseudo-algebraic formula that cannot be extended to a pseudo-minimal formula, then there are 2^{\aleph_1} non-isomorphic atomic models of size \aleph_1 .

This paper seeks an atomic model analogue of the superstable, non- \aleph_0 -stable many-models result in first order. To begin, it is natural to restrict our attention to types that can be realized in an atomic model. Suppose M is atomic and $A \subseteq M$. We let $S_{at}(A)$ denote the set of complete types p over A for which Ab is an atomic set for some (equivalently, for every) realization b of p . It is easily checked that when A is countable, $S_{at}(A)$ is a G_δ subset of the Stone space $S(A)$, hence $S_{at}(A)$ is Polish with respect to the induced topology. By analogy with the first order case, we call an atomic class \mathbf{At}_T *\aleph_0 -stable* if $S_{at}(M)$ is countable, where M denotes the unique countable model in \mathbf{At}_T .

The grail, which remains open, would be to prove that non- \aleph_0 -stability of an atomic class \mathbf{At}_T implies many atomic models in \aleph_1 . Here, we content ourselves with some-

what less. We repeatedly use the fact that any countable, atomic set A is contained in a countable, atomic model M . However, unlike the first-order case, some types in $S_{at}(A)$ need not extend to types in $S_{at}(M)$. Indeed, there are examples where the space $S_{at}(A)$ is uncountable (hence contains a perfect set) while $S_{at}(M)$ is countable. Thus, for analyzing types over countable, atomic sets $A \subseteq M$, we are led to consider

$$S_{at}^+(A, M) := \{p \mid A : p \in S_{at}(M)\}.$$

Equivalently, $S_{at}^+(A, M)$ is the set of $q \in S_{at}(A)$ that can be extended to a type $q^* \in S_{at}(M)$.

We repeatedly use the following observations. Suppose $\bar{a} \subseteq M \leq M'$ and $f : M \rightarrow M'$ is an isomorphism fixing \bar{a} pointwise. Then $\text{pcl}_M(\bar{a}) = \text{pcl}_{M'}(\bar{a})$. Moreover, f induces an elementary permutation of $D = \text{pcl}_M(\bar{a})$, which in turn induces a bijection between the spaces of types $S_{at}^+(D, M)$ and $S_{at}(D, M')$.

We now give the major new definition of this paper.

Definition 1.1 An atomic class \mathbf{At}_T with an uncountable model is *pcl-small* if, for every atomic model N and for every finite \bar{a} from N , N realizes only countably many complete types over $\text{pcl}_N(\bar{a})$.

The name of this notion is by analogy with the first-order case – A complete, first-order theory T is small if and only if for every model N and every finite \bar{a} from N , N realizes only countably many complete types over \bar{a} . The following proposition relates pcl-smallness with the spaces of types $S_{at}^+(D, M)$.

Proposition 1.2 *The atomic class \mathbf{At}_T is pcl-small if and only if the space of types $S_{at}^+(\text{pcl}_M(\bar{a}), M)$ is countable for every countable, atomic model M and every finite \bar{a} from M .*

Proof First, assume that some atomic model N and finite sequence \bar{a} from N witness that \mathbf{At}_T is not pcl-small. Choose $\{c_i : i \in \omega_1\} \subseteq N$ realizing distinct complete types over $D = \text{pcl}_N(\bar{a})$. Also, choose a countable $M \leq N$ that contains \bar{a} , and hence D . Then $\{\text{tp}(c_i/D) : i \in \omega_1\}$ witness that $S_{at}^+(D, M)$ is uncountable.

For the converse, choose a countable, atomic model M and \bar{a} from M such that $S_{at}^+(D, M)$ is uncountable, where $D = \text{pcl}_M(\bar{a})$. We will inductively construct a continuous, increasing elementary chain $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable, atomic models with $M = M_0$ and, for each ordinal α , there is an element $c_\alpha \in M_{\alpha+1}$ such that $\text{tp}(c_\alpha/D)$ is not realized in M_α . Given such a sequence, it is evident that $N = \bigcup_{\alpha < \omega_1} M_\alpha$ and \bar{a} witness that \mathbf{At}_T is not pcl-small. To construct such a sequence, we have defined M_0 to be M and take unions at limit ordinals. For the successor step, assume M_α has been defined. As M and M_α are each countable atomic models that contain \bar{a} , choose an isomorphism $f : M \rightarrow M_\alpha$ fixing \bar{a} pointwise. As noted above, f fixes D setwise. As M_α is countable, so is the set $\{\text{tp}(c/D) : c \in M_\alpha\}$. As $S_{at}^+(D, M)$ is uncountable, choose an atomic type $p \in S_{at}(M)$, whose restriction to D is distinct from $\{f^{-1}(\text{tp}(c/D)) : c \in M_\alpha\}$. Now choose c_α to realize $f(p)$. Then, as $M_\alpha c_\alpha$ is a countable atomic set, choose a countable elementary extension $M_{\alpha+1} \geq M_\alpha$ containing c_α . \square

Recall that an atomic class \mathbf{At}_T is \aleph_0 -stable² if $S_{at}(M)$ is countable for all (equivalently, for some) countable atomic models M . As $S_{at}^+(A, M)$ is a set of projections of types in $S_{at}(M)$, it will be countable whenever $S_{at}(M)$ is. This observation makes the following corollary to Proposition 1.2 immediate:

Corollary 1.3 *If an atomic class \mathbf{At}_T is \aleph_0 -stable, then \mathbf{At}_T is pcl-small.*

The converse to Corollary 1.3 fails. For example, the theory $T = REF(bin)$ of countably many, binary splitting equivalence relations is not \aleph_0 -stable, yet $\text{pcl}_M(\bar{a}) = \bar{a}$ for every model M and \bar{a} from M . Thus, $S_{at}(\text{pcl}_M(\bar{a}))$ and hence $S_{at}^+(\text{pcl}(\bar{a}), M)$ is countable for every finite tuple \bar{a} inside any atomic model M . The main theorem of this paper is:

Theorem 1.4 *Let T be a countable, complete theory T with an uncountable atomic model. If the atomic class \mathbf{At}_T is not pcl-small, then there are 2^{\aleph_1} non-isomorphic models in \mathbf{At}_T , each of size \aleph_1 .*

Section 2 sets the stage for the proof. It describes the spaces of types $S_{at}^+(A, M)$, states a transfer theorem for sentences of $L_{\omega_1, \omega}(Q)$, and details a non-structural configuration arising from non-pcl-smallness. In Sect. 3, the non-structural configuration is exploited to give a family of 2^{\aleph_0} non-isomorphic structures (N, \bar{b}^*) , where each of the reducts N is in \mathbf{At}_T and has size \aleph_1 . Theorem 1.4 is finally proved in Sect. 4. It is remarkable that whereas it is a ZFC theorem, the proof is non-uniform depending on the relative sizes of the cardinals 2^{\aleph_0} and 2^{\aleph_1} .

2 Preliminaries

In this section, we develop some general tools that will be used in the proof of Theorem 1.4.

2.1 On $S_{at}^+(A, M)$

In this subsection we explore the space of types

$$S_{at}^+(A, M) = \{p|A : p \in S_{at}(M)\}$$

where A is a subset of a countable, atomic model M .

Fix a countable, atomic model M and an arbitrary subset $A \subseteq M$. Let \mathcal{P} denote the space of complete types in one free variable over finite subsets of M . As M is atomic, \mathcal{P} can be identified with the set of complete formulas $\varphi(x, m)$ over M . Implication gives a natural partial order on \mathcal{P} , namely $p \leq q$ if and only if $\text{dom}(p) \subseteq \text{dom}(q)$ and $q \vdash p$. One should think of elements of \mathcal{P} as ‘finite approximations’ of types in $S_{at}^+(A, M)$. We describe two conditions on $p \in \mathcal{P}$ that identify extreme behaviors in this regard.

² Sadly, this usage of ‘ \aleph_0 -stability’ is analogous, but distinct from, the familiar first-order notion.

Definition 2.1 We say a type $p^* \in S_{at}^+(A, M)$ lies above $p \in \mathcal{P}$ if there is some $\bar{p} \in S_{at}(M)$ extending $p \cup p^*$. As every $p \in \mathcal{P}$ extends to a type in $S_{at}(M)$, it follows that at least one $p^* \in S_{at}^+(A, M)$ lies above p .

- An element $p \in \mathcal{P}$ determines a type in $S_{at}^+(A, M)$ if exactly one $p^* \in S_{at}^+(A, M)$ lies above p .
- An element $p \in \mathcal{P}$ is *A-large* if $\{p^* \in S_{at}^+(A, M) : p^* \text{ lies above } p\}$ is uncountable.

To understand these extreme behaviors, we define a rank function $\text{rk}_A : \mathcal{P} \rightarrow (\omega_1 + 1)$ as follows:

- $\text{rk}_A(p) \geq 0$ for all $p \in \mathcal{P}$;
- For $\alpha \leq \omega_1$, $\text{rk}_A(p) \geq \alpha$ if and only if for every $\beta < \alpha$ and for all finite F , $\text{dom}(p) \subseteq F \subseteq M$, there is $q \in S_{at}(F)$ with $q \geq p$ that β -*A splits*, where:
 - A type $q \in S_{at}(F)$ *A-splits* if, for some $\varphi(x, \bar{a})$ with \bar{a} from A , there are $q_1, q_2 \geq q$ with $q \cup \varphi(x, \bar{a}) \subseteq q_1$ and $q \cup \neg\varphi(x, \bar{a}) \subseteq q_2$; and $q \in S_{at}(F)$ β -*A splits* if, in addition, $\text{rk}_A(q_1), \text{rk}_A(q_2) \geq \beta$.
- For $\alpha < \omega_1$, we say $\text{rk}_A(p) = \alpha$ if $\text{rk}_A(p) \geq \alpha$, but $\text{rk}_A(p) \not\geq \alpha + 1$.

Proposition 2.2 If $p \in \mathcal{P}$ and $\text{rk}_A(p) = \alpha < \omega_1$, then some $r \geq p$ determines a type in $S_{at}^+(A, M)$.

Proof We prove this by induction on α . We begin with $\alpha = 0$. Suppose $\text{rk}_A(p) = 0$. As $\text{rk}_A(p) \not\geq 1$, there is a finite F , $\text{dom}(p) \subseteq F \subseteq M$ for which there is no $q \in S_{at}(F)$ and $\varphi(x, \bar{a})$ with \bar{a} from A for which $q \geq p$ and both $q \cup \{\varphi(x, \bar{a})\}$ and $q \cup \{\neg\varphi(x, \bar{a})\}$ are consistent. So fix any $r \in S_{at}(F)$ with $r \geq p$. Any such r determines a type in $S_{at}^+(A, M)$.

Next, choose $0 < \alpha < \omega_1$ and assume the Proposition holds for all $\beta < \alpha$. Choose $p \in S_{at}(E)$ with $\text{rk}_A(p) = \alpha$. As $\text{rk}_A(p) \geq \alpha$, while $\text{rk}_A(p) \not\geq \alpha + 1$, there is a finite F , $E \subseteq F \subseteq M$ for which there is no $q \in S_{at}(F)$ that both extends p and α -*A splits*. So choose any $q \in S_{at}(F)$ with $q \geq p$. If q determines a type in $S_{at}^+(A, M)$, then we finish, so assume otherwise. Thus, there is some $\varphi(x, \bar{a})$ with \bar{a} from A such that both $q \cup \{\varphi(x, \bar{a})\}$ and $q \cup \{\neg\varphi(x, \bar{a})\}$ are consistent. Choose complete types $q_1, q_2 \in S_{at}(F\bar{a})$ extending these partial types. Clearly, both $q_1, q_2 \geq q$, but since q does not α -*A split*, at least one of them has $\text{rk}_A(q_\ell) < \alpha$. But then by our inductive hypothesis, there is $r \geq q_\ell$ that determines a type in $S_{at}^+(A, M)$ and we finish. \square

Next, we turn our attention to *A-large* types and types of rank at least ω_1 and see that these coincide. We begin with two lemmas, the first involving types of rank at least ω_1 and the second involving *A-large* types.

Lemma 2.3 Assume that $E \subseteq M$ is finite and $p \in S_{at}(E)$ has $\text{rk}_A(p) \geq \omega_1$. Then:

1. For every finite F , $E \subseteq F \subseteq M$, there is $q \in S_{at}(F)$, $q \geq p$, with $\text{rk}_A(q) \geq \omega_1$; and
2. There is some formula $\varphi(x, \bar{a})$ with \bar{a} from A and $q_1, q_2 \in \mathcal{P}$ with $p \cup \{\varphi(x, \bar{a})\} \subseteq q_1$, $p \cup \{\neg\varphi(x, \bar{a})\} \subseteq q_2$, and both $\text{rk}_A(q_1), \text{rk}_A(q_2) \geq \omega_1$.

Proof (1) Fix a finite F satisfying $E \subseteq F \subseteq M$. As $\text{rk}_A(p) \geq \omega_1$, for every $\beta < \omega_1$ there is some $q \geq p$ with $q \in S_{\text{at}}(F)$ for which certain extensions of q have rank at least β . It follows that $\text{rk}_A(q) \geq \beta$ for any such witness. However, as $S_{\text{at}}(F)$ is countable, there is some $q \in S_{\text{at}}(F)$ which serves as a witness for uncountably many β . Thus, $\text{rk}_A(q) \geq \omega_1$ for any such $q \geq p$.

(2) Assume that there were no such formula $\varphi(x, \bar{a})$. Then, for any formula $\varphi(x, \bar{a})$, since \mathcal{P} is countable, there would be an ordinal $\beta^* < \omega_1$ such that **either** every $q \in \mathcal{P}$ extending $p \cup \{\varphi(x, \bar{a})\}$, $\text{rk}_A(q) < \beta^*$ **or** every $q \in \mathcal{P}$ extending $p \cup \{\neg\varphi(x, \bar{a})\}$ has $\text{rk}_A(q) < \beta^*$. Continuing, as there are only countably many formulas $\varphi(x, \bar{a})$, there would be an ordinal $\beta^{**} < \omega_1$ that works for all formulas $\varphi(x, \bar{a})$. Restating this, p does not β^{**} - A split, so no extension of p could β^{**} - A split either. This contradicts $\text{rk}_A(p) \geq \beta^{**} + 1$. \square

Lemma 2.4 *Suppose $q \in S_{\text{at}}(F)$ is A -large. Then:*

1. *For every finite F' , $F \subseteq F' \subseteq M$, there is some A -large $r \in S_{\text{at}}(F')$ with $r \geq q$; and*
2. *For some $\varphi(x, \bar{a})$ with parameters from A , there are A -large extensions $r_1 \supseteq q \cup \{\varphi(x, \bar{a})\}$ and $r_2 \supseteq q \cup \{\neg\varphi(x, \bar{a})\}$.*

Proof Fix such a q and let $\mathcal{S} = \{p^* \in S_{\text{at}}^+(A, M) : p^* \text{ lies above } q\}$. (1) is immediate, since \mathcal{S} is uncountable, while $S_{\text{at}}(F')$ is countable.

For (2), first note that if there is no such $\varphi(x, \bar{a})$, then there is at most one $p^* \in \mathcal{S}$ with the property that:

For any formula $\varphi(x, \bar{a})$ with parameters from A , $\varphi(x, \bar{a}) \in p^*$ if and only if there is an A -large $r \in S_{\text{at}}(F\bar{a})$ extending $q \cup \{\varphi(x, \bar{a})\}$.

It follows that for any $q^* \in \mathcal{S} - \{p^*\}$, q^* lies over some $r \geq q$ that is not A -large. That is, using the fact that there are only countably many $r \geq q$, $\mathcal{S} - \{p^*\}$ is contained in the union of countably many countable sets. But this contradicts q being A -large. \square

Proposition 2.5 *For $p \in \mathcal{P}$, $\text{rk}_A(p) \geq \omega_1$ if and only if p is A -large.*

Proof First, assume that $\text{rk}_A(p) \geq \omega_1$. Fix an enumeration $\{c_n : n \in \omega\}$ of M . Using Clauses (1) and (2) of Lemma 2.3, we inductively construct a tree $\{p_\nu : \nu \in 2^{<\omega}\}$ of elements of \mathcal{P} satisfying:

1. $\text{rk}_A(p_\nu) \geq \omega_1$ for all $\nu \in 2^{<\omega}$;
2. If $\text{lg}(\nu) = n$, then $\{c_i : i < n\} \subseteq \text{dom}(p_\nu)$;
3. $p_\emptyset = p$;
4. For $\nu \sqsubseteq \mu$, $p_\nu \leq p_\mu$;
5. For each ν there is a formula $\varphi(x, \bar{a})$ with \bar{a} from A such that $\varphi(x, \bar{a}) \in p_{\nu 0}$ and $\neg\varphi(x, \bar{a}) \in p_{\nu 1}$.

Given such a tree, for each $\eta \in 2^\omega$, let $\bar{p}_\eta := \bigcup \{p_{\eta|n} : n \in \omega\}$ and let $p_\eta^* := \bar{p}_\eta|A$. By Clauses (2) and (4), each $\bar{p}_\eta \in S_{\text{at}}(M)$, so each $p_\eta^* \in S_{\text{at}}^+(A, M)$. By Clause (5), $p_\eta^* \neq p_{\eta'}^*$ for distinct $\eta, \eta' \in 2^\omega$. Finally, each of these types lies over p by Clause (3). Thus, p is A -large.

Conversely, we argue by induction on $\alpha < \omega_1$ that:

$(*)_\alpha$: If $p \in \mathcal{P}$ is A -large, then $\text{rk}_A(p) \geq \alpha$.

Establishing $(*)_0$ is trivial, and for limit $\alpha < \omega_1$, it is easy to establish $(*)_\alpha$ given that $(*)_\beta$ holds for all $\beta < \alpha$. So assume $(*)_\alpha$ holds and we will establish $(*)_{\alpha+1}$. Choose any A -large $p \in \mathcal{P}$. Towards showing $\text{rk}_A(p) \geq \alpha + 1$, choose any finite F , $\text{dom}(p) \subseteq F \subseteq M$. As $S_{\text{at}}(F)$ is countable and uncountably many types in $S_{\text{at}}^+(A, M)$ lie above p , there is some A -large $q \in S_{\text{at}}(F)$ with $q \geq p$.

Next, by Lemma 2.4 choose a formula $\varphi(x, \bar{a})$ with \bar{a} from A such that there are A -large extensions $r_1 \supseteq q \cup \{\varphi(x, \bar{a})\}$ and $r_2 \supseteq q \cup \{\neg\varphi(x, \bar{a})\}$. Applying $(*)_\alpha$ to both r_1, r_2 gives $\text{rk}_A(r_1), \text{rk}_A(r_2) \geq \alpha$. Thus, q α - A splits. Thus, by definition of the rank, $\text{rk}_A(p) \geq \alpha + 1$. \square

We obtain the following Corollary, which is analogous to the statement ‘If T is small, then the isolated types are dense’ from the first-order context.

Corollary 2.6 *If $S_{\text{at}}^+(A, M)$ is countable, then every $p \in \mathcal{P}$ has an extension $q \geq p$ that determines a type in $S_{\text{at}}^+(A, M)$.*

Proof If $S_{\text{at}}^+(A, M)$ is countable, then no $p \in \mathcal{P}$ is A -large. Thus, every $p \in \mathcal{P}$ has $\text{rk}_A(p) < \omega_1$ by Proposition 2.5, so has an extension determining a type in $S_{\text{at}}^+(A, M)$ by Proposition 2.2. \square

We close with a complementary result about extensions of A -large types.

Definition 2.7 *A type $r \in S_{\text{at}}(M)$ is A -perfect if $r \upharpoonright_A$ is omitted in M and for every finite \bar{m} from M , the restriction $r \upharpoonright_{\bar{m}}$ is A -large.*

The name *perfect* is chosen because, relative to the usual topology on $S_{\text{at}}(M)$, there are a perfect set of A -perfect types extending any A -large $p \in \mathcal{P}$. However, for what follows, all we need to establish is that there are uncountably many, which is notationally simpler to prove.

Proposition 2.8 *Suppose $p \in \mathcal{P}$ is A -large. Then there are uncountably many A -perfect $r \in S_{\text{at}}(M)$ extending p .*

Proof Fix an A -large $p \in \mathcal{P}$. Choose a set $R \subseteq S_{\text{at}}(M)$ of representatives for $\{p^* \in S_{\text{at}}^+(A, M) : p^* \text{ lies above } p\}$, i.e., for every such p^* , there is exactly one $\bar{p} \in R$ whose restriction $\bar{p} \upharpoonright_A = p^*$. As p is A -large, R is uncountable. Now, for each finite \bar{m} from M , there are only countably many complete $q \in S_{\text{at}}(\bar{m})$, and if some $q \in S_{\text{at}}(\bar{m})$ is A -small, then only countably many $\bar{p} \in R$ extend q . As M is countable, there are only countably many \bar{m} , hence all but countably many $\bar{p} \in R$ satisfy $\bar{p} \upharpoonright_{\bar{m}}$ A -large for every \bar{m} . Further, again since M is countable, at most countably many $\bar{p} \in R$ have restrictions to A that are realized in M . Thus, all but countably many $\bar{p} \in R$ are A -perfect. \square

2.2 A transfer result

In this brief subsection we state a transfer result that follows immediately by Keisler’s completeness theorem for the logic $L_{\omega_1, \omega}(Q)$, given in [7]. Recall that $L_{\omega_1, \omega}(Q)$ is

the logic obtained by taking the (usual) set of atomic L formulas and closing under boolean combinations, existential quantification, the ‘ Q -quantifier,’ i.e., if $\theta(y, \bar{x})$ is a formula, then so is $Qy\theta(y, \bar{x})$; and countable conjunctions of formulas involving a finite set of free variables, i.e., if $\{\psi_i(\bar{x}) : i \in \omega\}$ is a set of formulas, then so is $\bigwedge_{i \in \omega} \psi_i(\bar{x})$. We are only interested in *standard interpretations* of these formulas, i.e., $M \models \bigwedge_{i \in \omega} \psi_i(\bar{a})$ if and only if $M \models \psi_i(\bar{a})$ for every $i \in \omega$; and $M \models Qy\theta(y, \bar{a})$ if and only if the solution set $\theta(M, \bar{a})$ is uncountable.

Throughout the discussion let ZFC^* denote a sufficiently large, finite subset of the ZFC axioms. In the notation of [10], Proposition 2.9 states that sentences of $L_{\omega_1, \omega}(Q)$ are *grounded*.

Proposition 2.9 *There is a sufficiently large, finite subset ZFC^* of ZFC such that whenever a countable language L and a sentence $\Phi \in L_{\omega_1, \omega}(Q)$ are given, IF there is a countable, transitive model $(\mathcal{B}, \epsilon) \models ZFC^*$ with $L, \Phi \in \mathcal{B}$ and*

$$(\mathcal{B}, \epsilon) \models \text{‘There is } M \models \Phi \text{ and } |M| = \aleph_1 \text{’}$$

THEN (in $V!$) there is $N \models \Phi$ and $|N| = \aleph_1$.

Proof This follows immediately from Keiser’s completeness theorem for $L_{\omega_1, \omega}$, given that provability is absolute between transitive models of set theory. More modern, ‘constructive’ proofs can be found in [2] and [3]. These use the existence \mathcal{B} -normal ultrafilters. Given an arbitrary language $L^* \in \mathcal{B}$ and any countable L^* -structure (\mathcal{B}, E, \dots) where the reduct (\mathcal{B}, E) is an ω -model of ZFC^* , for any \mathcal{B} -normal ultrafilter \mathcal{U} , the ultrapower $Ult(\mathcal{B}, \mathcal{U})$ is a countable ω -model that is an L^* -elementary extension of (\mathcal{B}, E, \dots) . It has the additional property that for any L^* -definable subset D , $D^{Ult(\mathcal{B}, \mathcal{U})}$ properly extends $D^{\mathcal{B}}$ if and only if $(\mathcal{B}, E, \dots) \models \text{‘}D \text{ is uncountable’}$.

Using this, one constructs (in $V!$) a continuous, L^* -elementary ω_1 -sequence $(\mathcal{B}_\alpha : \alpha < \omega_1)$ of ω -models, where each $\mathcal{B}_{\alpha+1} = Ult(\mathcal{B}_\alpha, \mathcal{U}_\alpha)$. Then the interpretation $M^{\mathcal{C}}$ where $\mathcal{C} = \bigcup_{\alpha < \omega_1} \mathcal{B}_\alpha$ will be a suitable choice of N . More details of this construction are given in [2] or [3]. \square

2.3 A configuration arising from non-pcl-smallness

The goal of this subsection is to prove the following Proposition, the data from which will be used throughout Sect. 3.

Proposition 2.10 *Assume T is a countable, complete theory for which \mathbf{At}_T has an uncountable atomic model, but is not pcl-small. Then there are a countable, atomic $M^* \in \mathbf{At}_T$, finite sequences $\bar{a}^* \subseteq \bar{b}^* \subseteq M^*$, and complete 1-types $\{r_j(x, \bar{b}^*) : j \in \omega\}$ such that, letting $D^* = \text{pcl}_{M^*}(\bar{a}^*)$, $A_n = \bigcup \{r_j(M^*, \bar{b}^*) : j < n\}$ and $A^* = \bigcup \{A_n : n \in \omega\}$ we have:*

1. $A^* \subseteq D^*$;
2. $S_{at}^+(A_n, M^*)$ is countable for every $n \in \omega$; but
3. $S_{at}^+(A^*, M^*)$ is uncountable.

Proof Fix any countable, atomic $M^* \in \mathbf{At}_T$. Using Proposition 1.2 and the non-pcl-smallness of \mathbf{At}_T , choose a finite tuple $\bar{a}^* \subseteq M^*$ such that $S_{at}^+(D^*, M^*)$ is uncountable, where $D^* = \text{pcl}_{M^*}(\bar{a}^*) \subseteq M^*$.

Fix any finite tuple $\bar{b} \supseteq \bar{a}^*$ from M^* and look at the complete 1-types $\mathcal{Q}_{\bar{b}} := \{r \in S_{at}(\bar{b}) \text{ such that } r(M^*) \subseteq D^*\}$. These types visibly induce a partition of D^* , and it is easily seen that if $\bar{b}' \supseteq \bar{b}$, the partition induced by \bar{b}' refines the partition induced by \bar{b} . Let $\mathcal{Q} := \bigcup \{\mathcal{Q}_{\bar{b}} : \bar{a}^* \subseteq \bar{b} \subseteq M^*\}$.

Define a rank function $\text{rk} : \mathcal{Q} \rightarrow \mathcal{ON} \cup \{\infty\}$ as follows:

- $\text{rk}(c/\bar{b}) \geq 0$ if and only if $\text{tp}(c/\bar{b}) \in \mathcal{Q}$;
- $\text{rk}(c/\bar{b}) \geq 1$ if and only if $\text{tp}(c/\bar{b}) \in \mathcal{Q}$ and there are infinitely many $c' \in D^*$ realizing $\text{tp}(c/\bar{b})$; and
- for an ordinal $\alpha \geq 2$, $\text{rk}(c/\bar{b}) \geq \alpha$ if and only if for every $\beta < \alpha$ and every \bar{b}' from M^* , there is $c' \in D^*$ realizing $\text{tp}(c/\bar{b})$ such that $\text{rk}(c'/\bar{b}\bar{b}') \geq \beta$.
- $\text{rk}(c/\bar{b}) = \alpha$ if and only if $\text{rk}(c/\bar{b}) \geq \alpha$ but $\text{rk}(c/\bar{b}) \not\geq \alpha + 1$.

Claim 1. For every $r \in \mathcal{Q}$, $\text{rk}(r)$ is a countable ordinal.

Proof Assume by way of contradiction that $\text{rk}(c/\bar{b}) \geq \omega_1$ for some type c/\bar{b} . Then, for any \bar{b}' from M , as D^* is countable, there is an element $c' \in D^*$ such that $\text{rk}(c'/\bar{b}\bar{b}') \geq \beta$ for uncountably many β 's, hence $\text{rk}(c'/\bar{b}\bar{b}') \geq \omega_1$ as well. Using this idea, if we let $\langle \bar{b}_n : n \in \omega \rangle$ be an increasing sequence of finite sequences from M^* whose union is all of M^* , then we can find a sequence $\langle c_n : n \in \omega \rangle$ of elements from D^* such that, for each n , $\text{rk}(c_n/\bar{b}_n) \geq \omega_1$ and $\text{tp}(c_n/\bar{b}_n) \subseteq \text{tp}(c_{n+1}/\bar{b}_{n+1})$. The union of these 1-types yields a complete, atomic 1-type $q \in S_{at}(M^*)$ all of whose realizations are in $\text{pcl}_{M^*}(\bar{a})$. However, since the type asserting that ‘ $x = c$ ’ has rank 0 for each $c \in D^*$, q is omitted in M^* . To obtain a contradiction, choose a realization e of q and, as M^*e is a countable, atomic set, construct a countable, atomic elementary extension $M' \geq M^*$ with $e \in M'$. But now, q implies that $e \in \text{pcl}_{M'}(\bar{a})$, yet this is contradicted by the fact that M^* contains \bar{a} but not e . \square

As notation, for a subset $\mathcal{S} \subseteq \mathcal{Q}_{\bar{b}}$, let $A_{\mathcal{S}} = \bigcup \{r(M^*) : r \in \mathcal{S}\}$, which is always a subset of D^* . Define the set of ‘candidates’ as

$$\mathcal{C} = \{(\mathcal{S}, \bar{b}) : \bar{b} \supseteq \bar{a}^*, \mathcal{S} \subseteq \mathcal{Q}_{\bar{b}}, \text{ and } S_{at}^+(A_{\mathcal{S}}, M^*) \text{ uncountable}\}$$

Note that \mathcal{C} is non-empty as $(\mathcal{S}_0, \bar{a}^*) \in \mathcal{C}$, where \mathcal{S}_0 is an enumeration of all the complete, pseudo-algebraic types over \bar{a}^* . Among all candidates, choose $(\mathcal{S}^*, \bar{b}^*) \in \mathcal{C}$ such that

$$\alpha^* := \sup\{\text{rk}(r) + 1 : r \in \mathcal{S}^*\}$$

is as small as possible. Enumerate $\mathcal{S}^* = \{r_j : j \in \omega\}$ and put $A^* := A_{\mathcal{S}^*}$ and $A_n := \bigcup \{r_j(M^*, \bar{b}^*) : j < n\}$ for each $n \in \omega$. As Clauses (1) and (3) are immediate, it suffices to prove the following Claim:

Claim 2. For each $n \in \omega$, $S_{at}^+(A_n, M^*)$ is countable.

Proof Fix any $n \in \omega$. First, note that if $\text{rk}(r_j) = 0$ for every $j < n$, then A_n would be finite, which would imply $S_{at}(A_n)$ is countable. As $S_{at}(A_n)$ contains $S_{at}^+(A_n, M^*)$, the result follows.

Now assume $\text{rk}(r_j) > 0$ for at least one $j < n$. Let $\beta := \max\{\text{rk}(r_j) : j < n\}$ and let $F = \{j < n : \text{rk}(r_j) = \beta\}$. Clearly, $\beta < \alpha^*$. For each $j \in F$, as $\beta > 0$ but $\text{rk}(r_j) \not\geq \beta + 1$, there is a finite tuple \bar{b}_j such that $\text{rk}(c/\bar{b}^*\bar{b}_j) < \beta$ for all $c \in r_j(M^*)$.

Let \bar{b}' be the concatenation of \bar{b}^* with each \bar{b}_j for $j \in F$ and let

$$S' := \{r' \in Q_{\bar{b}'} : r' \text{ extends some } r_j \text{ with } j < n\}$$

Subclaim. $\text{rk}(r') < \beta$ for every $r' \in S'$.

Proof Fix $r' \in S'$ and choose $c \in r'(M^*, \bar{b}')$. There are two cases. On one hand, if r' extends some r_j with $j \in F$, then $\text{rk}(c/\bar{b}') \leq \text{rk}(c/\bar{b}^*\bar{b}_j) < \beta$. On the other hand, if r' extends some r_j with $r_j \notin F$, then as $\text{rk}(r_j) < \beta$, $\text{rk}(c/\bar{b}') \leq \text{rk}(c/\bar{b}^*) < \beta$. \square

Clearly $A_{S'} = A_n$, so $S_{at}^+(A_n, M^*) = S_{at}^+(A_{S'}, M^*)$. Thus, if $S_{at}^+(A_n, M^*)$ were uncountable, then (S', \bar{b}') would be a candidate, i.e., an element of \mathcal{C} . But, as $\beta < \alpha^*$, this is impossible by the Subclaim and the minimality of α^* . \square

3 A family of 2^{\aleph_0} atomic models of size \aleph_1

Throughout the whole of this section, we assume that T is a complete theory in a countable language for which \mathbf{At}_T has an uncountable atomic model, but is not pcl-small. Appealing to Proposition 2.10,

Fix, for the whole of this section, a countable atomic model M^* , tuples $\bar{a}^* \subseteq \bar{b}^* \subseteq M^*$ and sets A^* and A_n for each $n \in \omega$ as in Proposition 2.10.

We work with this fixed configuration for the whole of this section and, in Sect. 3.3 eventually prove:

Proposition 3.1 *There is a family $\{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of atomic models of T , each of size \aleph_1 , that are pairwise non-isomorphic over \bar{b}^* .*

3.1 Colorings of models realizing many types over A^*

Definition 3.2 Call a structure (N, \bar{b}^*) *rich* if $N \in \mathbf{At}_T$ has size \aleph_1 , $M^* \preceq N$, and N realizes uncountably many 1-types over A^* .

Lemma 3.3 *For each $n \in \omega$, a rich (N, \bar{b}^*) realizes only countably many distinct 1-types over A_n .*

Proof Fix any (N, \bar{b}^*) and $n < \omega$ as above. If $\{c_i : i \in \omega_1\}$ realize distinct types over A_n , then the types $\{\text{tp}_N(c_i/M^*) : i \in \omega_1\}$ would be distinct, contradicting $S_{at}^+(A_n, M^*)$ countable. \square

How can we tell whether rich structures are non-isomorphic? We introduce the notion of \mathcal{U} -colorings and Corollary 3.6 gives a sufficient condition.

Definition 3.4 Fix a subset $\mathcal{U} \subseteq \omega$ and a rich (N, \bar{b}^*) .

- For elements $d, d' \in N$, define the *splitting number* $\text{spl}(d, d') \in (\omega + 1)$ to be the least $k < \omega$ such that $\text{tp}(d/A_k) \neq \text{tp}(d'/A_k)$ if such exists; and $\text{spl}(d, d') = \omega$ if $\text{tp}(d/A^*) = \text{tp}(d'/A^*)$.
- A \mathcal{U} -coloring of a rich (N, \bar{b}^*) is a function

$$c : N \rightarrow \omega$$

such that for all pairs $d, d' \in N$, at least one of the following hold:

1. $\text{tp}(d/A^*) = \text{tp}(d'/A^*)$; or
 2. $c(d) \neq c(d')$; or
 3. $\text{spl}(d, d') \in \mathcal{U}$.
- The *color filter* $\mathcal{F}(N, \bar{b}^*) := \{\mathcal{U} \subseteq \omega : \text{a } \mathcal{U}\text{-coloring of } (N, \bar{b}^*) \text{ exists}\}$.

Lemma 3.5 Fix a rich (N, \bar{b}^*) . Then:

1. $\mathcal{F}(N, \bar{b}^*)$ is a filter;
2. $\mathcal{F}(N, \bar{b}^*)$ contains the cofinite subsets of ω ; but
3. No finite $\mathcal{U} \subseteq \omega$ is in $\mathcal{F}(N, \bar{b}^*)$.

Proof (1) First, note that if $\mathcal{U} \subseteq \mathcal{U}' \subseteq \omega$, then every \mathcal{U} -coloring c is also a \mathcal{U}' -coloring. Thus, $\mathcal{F}(N, \bar{b}^*)$ is upward closed. Next, suppose $\mathcal{U}_1 \in \mathcal{F}(N, \bar{b}^*)$ via the coloring $c_1 : N \rightarrow \omega$ and $\mathcal{U}_2 \in \mathcal{F}(N, \bar{b}^*)$ via the coloring $c_2 : N \rightarrow \omega$. Fix any bijection $t : \omega \times \omega \rightarrow \omega$. It is easily checked that $c^* : N \rightarrow \omega$ defined by $c^*(d) = t(c_1(d), c_2(d))$ is a $\mathcal{U}_1 \cap \mathcal{U}_2$ -coloring of (N, \bar{b}^*) . Thus, $\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{F}(N, \bar{b}^*)$. So $\mathcal{F}(N, \bar{b}^*)$ is a filter.

(2) As $\mathcal{F}(N, \bar{b}^*)$ is a filter, it suffices to show that for every $n \in \omega$, $B_n \in \mathcal{F}(N, \bar{b}^*)$, where $B_n = (\omega - \{0, \dots, n-1\})$. Fix such an n . By Lemma 3.3, N realizes at most countably many types over A_n . Thus, we can produce a map $c : N \rightarrow \omega$ such that $c(d) = c(d')$ if and only if $\text{tp}(d/A_n) = \text{tp}(d'/A_n)$. As any such c is a B_n -coloring, $B_n \in \mathcal{F}(N, \bar{b}^*)$.

(3) It suffices to show that no $n = \{0, \dots, n-1\}$ is in $\mathcal{F}(N, \bar{b}^*)$. To see this, let $c : N \rightarrow \omega$ be an arbitrary map. We will show that c is not an $\{0, \dots, n-1\}$ -coloring. As N realizes \aleph_1 distinct types over A^* , there is some $m^* \in \omega$ and an uncountable subset $\{d_\alpha : \alpha < \omega_1\} \subseteq N$ that realize distinct types over A^* , yet $c(d_\alpha) = m^*$ for each α . However, as N realizes only countably many types over A_n , there are $\alpha \neq \beta$ such that $n \leq \text{spl}(d_\alpha, d_\beta) < \omega$. Thus, c is not an $\{0, \dots, n-1\}$ -coloring. \square

We close with a sufficient condition for non-isomorphism of rich models.

Corollary 3.6 Suppose that for $\ell = 1, 2$, (N_ℓ, \bar{b}^*) is a \mathcal{U}_ℓ -colored rich model, and $\mathcal{U}_1 \cap \mathcal{U}_2$ is finite. Then there is no isomorphism $f : N_1 \rightarrow N_2$ fixing \bar{b}^* pointwise.

Proof If there were such an isomorphism, then (N_2, \bar{b}^*) would be both \mathcal{U}_1 -colored and \mathcal{U}_2 -colored. Thus, both $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{F}(N_2, \bar{b}^*)$, which contradicts Lemma 3.5. \square

3.2 Constructing a colored rich model via forcing

Arguing as in the proof of Proposition 1.2, from the data of Proposition 2.10 we can construct a rich (N, \bar{b}^*) as the union of a continuous, elementary chain $\langle M_\alpha : \alpha \in \omega_1 \rangle$ of countable, atomic models with $M_0 = M^*$ such that, for each $\alpha \in \omega_1$ there is a distinguished $b_\alpha \in M_{\alpha+1}$ such that $\text{tp}(b_\alpha/A^*)$ is omitted in M_α .

Our goal is to construct a sufficiently generic rich (N, \bar{b}^*) , along with a coloring $c : N \rightarrow (\omega + 1)$ via forcing. Our forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ encodes finite approximations of such an (N, \bar{b}^*) and c . A fundamental building block is the notion of a *striated type* over a finite subset \bar{a} satisfying $\bar{b}^* \subseteq \bar{a} \subseteq M^*$. As an atomic type over a finite subset is generated by a complete formula, we use the terms interchangeably.

Definition 3.7 Choose a finite tuple \bar{a} with $\bar{b}^* \subseteq \bar{a} \subseteq M^*$. A *striated type over \bar{a}* is a complete formula $\theta(\bar{x}) \in S_{\text{at}}(\bar{a})$ whose variables are partitioned as $\bar{x} = \langle \bar{x}_j : j < \ell \rangle$ where, for each j , $\bar{x}_j = \langle x_{j,n} : n < n(j) \rangle$ is an $n(j)$ -tuple of variable symbols such that $\theta(\bar{x})$ implies $\text{tp}(x_{j,0}/\bar{a} \cup \{\bar{x}_i : i < j\})$ is A^* -large. The integer ℓ is the *length* of the striated type.

A *simple realization* of a striated type $\theta(\bar{x})$ of length ℓ is a sequence $\bar{b} = \langle \bar{b}_j : j < \ell \rangle$ of tuples from M^* such that $M^* \models \theta(\bar{b})$. A *perfect chain realization* of $\theta(\bar{x})$ is a pair (\bar{M}, \bar{b}) , consisting of a chain $M_0 \leq M_1 \leq M_{\ell-1} \leq M^*$ of ℓ elementary submodels of M^* and a simple realization $\bar{b} = \langle \bar{b}_j : j < \ell \rangle$ from M^* that satisfy: For each $j < \ell$,

1. $\bar{a} \cup \{\bar{b}_i : i < j\} \subseteq M_j$; and
2. $\text{tp}(b_{j,0}/M_j)$ is A^* -perfect (see Definition 2.7).

Lemma 3.8 *Every striated type $\theta(\bar{x}) \in S_{\text{at}}(\bar{a})$ has a perfect chain realization.*

Proof We argue by induction on ℓ , the length of the striation. For striations of length zero there is nothing to prove, so assume the Lemma holds for striated types of length ℓ and choose an $(\ell + 1)$ -striation $\theta(\bar{x}) \in S_{\text{at}}(\bar{a})$. Let $\theta \upharpoonright_\ell$ be the truncation of θ to the variables $\bar{x} \upharpoonright_\ell = \langle \bar{x}_j : j < \ell \rangle$. As $\theta \upharpoonright_\ell$ is clearly an ℓ -striation, it has a perfect chain realization, i.e., a chain $M_0 \leq M_1 \leq M_{\ell-1} \leq M^*$ and a tuple $\bar{b} = \langle \bar{b}_j : j < \ell \rangle$ from M^* realizing $\theta \upharpoonright_\ell$ such that $\bar{a} \cup \{\bar{b}_i : i < j\} \subseteq M_j$ and $\text{tp}(b_{j,0}/M_j)$ is A^* -perfect for each $j < \ell$.

Now, since $\text{tp}(x_{\ell,0}/\bar{a}\bar{b})$ is A^* -large, by applying Proposition 2.8 there is an A^* -perfect type $\bar{p} \in S_{\text{at}}(M^*)$ (in a single variable $x_{\ell,0}$) extending $\text{tp}(x_{\ell,0}/\bar{a}\bar{b})$. Choose a countable, atomic $N \geq M^*$ and $e \in N$ realizing \bar{p} . As N and M^* are both countable and atomic, choose an isomorphism $f : N \rightarrow M^*$ that fixes $\bar{a}\bar{b}$ pointwise. Then $f(M_0) \leq f(M_1) \leq \dots \leq f(M_{\ell-1}) \leq f(M^*) \leq M^*$ is a chain. Let $b_{\ell,0} := f(e)$ and choose $\langle b_{\ell,1}, \dots, b_{\ell,n(\ell)-1} \rangle$ arbitrarily from M^* so that, letting $\bar{b}_\ell = \langle \bar{b}_{\ell,n} : n < n(\ell) \rangle$, $\bar{b} \frown \bar{b}_\ell$ realizes $\theta(\bar{x})$. This chain and this sequence form a perfect chain realization of θ . \square

The following Lemma is immediate, and indicates the advantage of working with A^* -perfect types.

Lemma 3.9 *Let (\bar{M}, \bar{b}) be any perfect chain realization of a striated type $\theta(\bar{x}) \in S_{\text{at}}(\bar{a})$. Then for every $\bar{c} \subseteq M_0$, $\text{tp}(\bar{b}/\bar{a}\bar{c}) \in S_{\text{at}}(\bar{a}\bar{c})$ is a striated type extending $\theta(\bar{x})$, and (\bar{M}, \bar{b}) is a perfect chain realization of it.*

The Lemma below, whose proof simply amounts to unpacking definitions, demonstrate that striated types are rather malleable.

Lemma 3.10 1. If $\text{tp}(\bar{c}/\bar{a})$ is a striated type of length k and $\text{tp}(\bar{d}/\bar{a}\bar{c})$ is a striated type of length ℓ , then $\text{tp}(\bar{c}\bar{d}/\bar{a})$ is a striated type of length $k + \ell$.
 2. Suppose $\text{tp}(\bar{b}/\bar{a})$ is a striated type of length ℓ and $k < \ell$. Let $\bar{b}_{<k}$ and $\bar{b}_{\geq k}$ be the induced partition of \bar{b} . Then $\text{tp}(\bar{b}_{<k}/\bar{a})$ is a striated type of length ℓ and $\text{tp}(\bar{b}_{\geq k}/\bar{a}\bar{b}_{<k})$ is a striated type of length $(\ell - k)$. Moreover, if (\bar{M}, \bar{b}) is a perfect chain realization of $\text{tp}(\bar{b}/\bar{a})$, then $(\bar{M}_{<k}, \bar{b}_{<k})$ is a perfect chain realization of $\text{tp}(\bar{b}_{<k}/\bar{a})$ and $(\bar{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{a}\bar{b}_{<k})$.

We begin by defining a partial order $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$ of ‘preconditions’. Then our forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ will be a dense suborder of these preconditions.

Definition 3.11 \mathbb{Q}_0 is the set of all $\mathbf{p} = (\bar{\mathbf{a}}_{\mathbf{p}}, u_{\mathbf{p}}, \bar{n}_{\mathbf{p}}, \theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}}), k_{\mathbf{p}}, \mathcal{U}_{\mathbf{p}}, c_{\mathbf{p}})$, where

1. $\bar{\mathbf{a}}_{\mathbf{p}}$ is a finite subset of M^* containing \bar{b}^* ;
2. $u_{\mathbf{p}}$ is a finite subset of ω_1 ;
3. $\bar{n}_{\mathbf{p}} = \langle n_t : t \in u_{\mathbf{p}} \rangle$ is a sequence of positive integers;
4. $\bar{x}_{\mathbf{p}} = \langle \bar{x}_{t,\mathbf{p}} : t \in u_{\mathbf{p}} \rangle$, where each $\bar{x}_{t,\mathbf{p}} = \langle x_{t,n} : n < \bar{n}_t \rangle$ is a finite sequence from the set $X = \{x_{t,n} : t \in \omega_1, n \in \omega\}$ of variable symbols;
5. $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}}) \in \text{Sat}(\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $|u_{\mathbf{p}}|$ (see Definition 3.7);
6. $k_{\mathbf{p}} \in \omega$;
7. $\mathcal{U}_{\mathbf{p}} \subseteq k_{\mathbf{p}} = \{0, \dots, k_{\mathbf{p}} - 1\}$;
8. $c_{\mathbf{p}} : \bar{x}_{\mathbf{p}} \rightarrow \omega$ is a function such that for all pairs $x_{t,n}, x_{s,m}$ from $\bar{x}_{\mathbf{p}}$ with $c_{\mathbf{p}}(x_{t,n}) = c_{\mathbf{p}}(x_{s,m})$
 - (a) either $\text{spl}(b_{t,n}, b_{s,m}) \geq k_{\mathbf{p}}$ for all perfect chain realizations (\bar{M}, \bar{b}) of $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$;
 - (b) or there is some $k \in \mathcal{U}_{\mathbf{p}}$ such that $\text{spl}(b_{t,n}, b_{s,m}) = k$ for all perfect chain realizations (\bar{M}, \bar{b}) of $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$.

We order elements of \mathbb{Q}_0 by: $\mathbf{p} \leq_{\mathbb{Q}_0} \mathbf{q}$ if and only if

- $\bar{\mathbf{a}}_{\mathbf{p}} \subseteq \bar{\mathbf{a}}_{\mathbf{q}}$;
- $u_{\mathbf{p}} \subseteq u_{\mathbf{q}}$ and $n_{t,\mathbf{p}} \leq n_{t,\mathbf{q}}$ for all $t \in u_{\mathbf{p}}$, hence $\bar{x}_{t,\mathbf{p}}$ is a subsequence of $\bar{x}_{t,\mathbf{q}}$;
- $\theta_{\mathbf{q}}(\bar{x}_{\mathbf{q}}) \upharpoonright \theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$;
- $k_{\mathbf{p}} \leq k_{\mathbf{q}}$;
- $\mathcal{U}_{\mathbf{p}} = \mathcal{U}_{\mathbf{q}} \cap k_{\mathbf{p}}$ (hence, for $j < k_{\mathbf{p}}$, $j \in \mathcal{U}_{\mathbf{p}}$ if and only if $j \in \mathcal{U}_{\mathbf{q}}$);
- $c_{\mathbf{p}} = c_{\mathbf{q}} \upharpoonright \bar{x}_{\mathbf{p}}$.

Visibly, $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$ is a partial order. As notation, for $\mathbf{p} \in \mathbb{Q}_0$ and $x_{t,n} \in \bar{x}_{\mathbf{p}}$, let $p(x_{t,n}) \in \text{Sat}(\bar{\mathbf{a}}_{\mathbf{p}})$ be $\text{tp}(e_{t,n}/\bar{\mathbf{a}}_{\mathbf{p}})$ for any realization $\bar{e}_{\mathbf{p}}$ in M^* of $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$. Call a precondition $\mathbf{p} \in \mathbb{Q}_0$ *unarily decided* if, for every $x_{t,n} \in \bar{x}_{\mathbf{p}}$, $p(x_{t,n})$ determines a type in $\text{Sat}^+(A_{k_{\mathbf{p}}}, M^*)$ (see Definition 2.1). That the unarily decided preconditions are dense follows easily from the fact that $\text{Sat}^+(A_{k_{\mathbf{p}}}, M^*)$ is countable.

Lemma 3.12 *The set $\{\mathbf{p} \in \mathbb{Q}_0 : \mathbf{p} \text{ is unarily decided}\}$ is dense in $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$. Moreover, given any $\mathbf{p} \in \mathbb{Q}_0$, there is a unarily decided $\mathbf{q} \geq_{\mathbb{Q}_0} \mathbf{p}$ with $\bar{x}_{\mathbf{q}} = \bar{x}_{\mathbf{p}}$ and $k_{\mathbf{q}} = k_{\mathbf{p}}$ (hence $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$).*

Proof Fix $\mathbf{p} \in \mathbb{Q}_0$ and let $k := k_{\mathbf{p}}$. Arguing by induction on the size of the finite set $\bar{x}_{\mathbf{p}}$, it is enough to strengthen $p(x_{t,n})$ individually for each $x_{t,n} \in \bar{x}_{\mathbf{p}}$. So fix $x_{t,n} \in \bar{x}_{\mathbf{p}}$. By Corollary 2.6 there is an $\bar{a}' \supseteq \bar{a}_{\mathbf{p}}$ and a 1-type $q_1(x_{t,n}) \in S_{at}(\bar{a}')$ extending $p(x_{t,n})$ that determines a type in $S_{at}^+(A_{k_{\mathbf{p}}}, M^*)$. Then, using Lemma 3.9 we can choose a striated type $\theta'(\bar{x}_{\mathbf{p}}) \in S_{at}(\bar{a}')$ extending $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}}) \cup q_1$.

We iterate the above procedure for each of the (finitely many) elements of $\bar{x}_{\mathbf{p}}$, thereby getting a unarily decided precondition $\mathbf{p}' \geq_{\mathbb{Q}_0} \mathbf{p}$ whose type $\theta_{\mathbf{p}'}(\bar{x}_{\mathbf{p}})$ still has the same free variables, and each of $k_{\mathbf{p}}, \mathcal{U}_{\mathbf{p}}, c_{\mathbf{p}}$ are unchanged. \square

Next, call a precondition $\mathbf{p} \in \mathbb{Q}_0$ *fully decided* if it is unarily decided and for each pair $x_{t,n}, x_{s,m}$ from $\bar{x}_{\mathbf{p}}$ with $c_{\mathbf{p}}(x_{t,n}) = c_{\mathbf{p}}(x_{s,m})$, if $\text{spl}(b_{t,n}, b_{s,m}) \geq k_{\mathbf{p}}$ for some perfect chain realization (\bar{M}, \bar{b}) , then $\text{tp}(b_{t,n}/A^*) = \text{tp}(b_{s,m}/A^*)$ for all perfect chain realizations (\bar{M}, \bar{b}) of $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$.

Lemma 3.13 *The set $\{\mathbf{p} \in \mathbb{Q}_0 : \mathbf{p} \text{ is fully decided}\}$ is dense in $(\mathbb{Q}_0, \leq_{\mathbb{Q}_0})$. Moreover, given any $\mathbf{p} \in \mathbb{Q}_0$, there is a fully decided $\mathbf{q} \geq_{\mathbb{Q}_0} \mathbf{p}$ with $\bar{x}_{\mathbf{q}} = \bar{x}_{\mathbf{p}}$.*

Proof It suffices to handle each pair $x_{t,n}, x_{s,m}$ from $\bar{x}_{\mathbf{p}}$ with $c(x_{t,n}) = c(x_{s,m})$ separately. Given such a pair, suppose there is some perfect chain realization (\bar{M}, \bar{b}) of $\theta(\bar{x}_{\mathbf{p}}) \in S_{at}(\bar{\mathbf{a}}_{\mathbf{p}})$ with $k_{\mathbf{p}} \leq \text{spl}(b_{t,n}, b_{s,m}) < \omega$. Among all such perfect chain realizations, choose one that minimizes $k^* = \text{spl}(b_{t,n}, b_{s,m})$. Choose a formula $\varphi(x, \bar{c})$ with \bar{c} from A_{k^*+1} witnessing that $\text{tp}(b_{t,n}/A_{k^*+1}) \neq \text{tp}(b_{s,m}/A_{k^*+1})$. As $A_{k^*+1} \subseteq M_0$, by applying Lemma 3.9, let $\theta^*(\bar{x}_{\mathbf{p}})$ be a complete formula over $\bar{\mathbf{a}}_{\mathbf{p}}\bar{c}$ isolating $\text{tp}(\bar{b}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{c})$. Form the precondition $\mathbf{p}' \in \mathbb{Q}_0$ by putting $\bar{\mathbf{a}}_{\mathbf{p}'} = \bar{\mathbf{a}}_{\mathbf{p}}\bar{c}$; $\theta_{\mathbf{p}'} = \theta^*$; $k_{\mathbf{p}'} = k^* + 1$; and $\mathcal{U}_{\mathbf{p}'} = \mathcal{U}_{\mathbf{p}} \cup \{k^*\}$; while leaving $\bar{x}_{\mathbf{p}}$ and $c_{\mathbf{p}}$ unchanged. It is evident that $\text{spl}(b'_{t,n}, b'_{s,m}) = k^* \in \mathcal{U}_{\mathbf{p}'}$ for all perfect chain realizations (\bar{M}, \bar{b}') of $\theta_{\mathbf{p}'}$. Continuing this process for each of the (finitely many) relevant pairs gives us a fully decided extension of \mathbf{p} . \square

Definition 3.14 The forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ is the set of fully decided $\mathbf{p} \in \mathbb{Q}_0$ with the inherited order.

Lemma 3.15 *The forcing $(\mathbb{Q}, \leq_{\mathbb{Q}})$ has the countable chain condition (c.c.c.).*

Proof Suppose $\{\mathbf{p}_i : i \in \omega_1\}$ is an uncountable subset of \mathbb{Q} . In light of Lemma 3.13, it suffices to find $i \neq j$ for which there is some precondition $\mathbf{q} \in \mathbb{Q}_0$ satisfying $\mathbf{p}_i \leq_{\mathbb{Q}_0} \mathbf{q}$ and $\mathbf{p}_j \leq_{\mathbb{Q}_0} \mathbf{q}$. First, by the Δ -system lemma applied to the finite sets $\{u_{\mathbf{p}_i}\}$, we may assume that $|u_{\mathbf{p}_i}|$ is constant and there is some fixed u^* that is an initial segment of each $u_{\mathbf{p}_i}$ and, moreover, whenever $i < j$, every element of $(u_{\mathbf{p}_i} \setminus u^*)$ is less than every element of $(u_{\mathbf{p}_j} \setminus u^*)$. By further trimming, but preserving uncountability, we may assume that the integer $k_{\mathbf{p}}$, the subset $\mathcal{U}_{\mathbf{p}} \subseteq k_{\mathbf{p}}$, and the parameter $\bar{\mathbf{a}}_{\mathbf{p}}$ remain constant. As notation, for $i < j$, let $f : u_{\mathbf{p}_i} \rightarrow u_{\mathbf{p}_j}$ be the unique order-preserving bijection. We may additionally assume that $n_{\mathbf{p}_i}(t) = n_{\mathbf{p}_j}(f(t))$, hence f has a natural extension (also called f): $\bar{x}_{\mathbf{p}_i} \rightarrow \bar{x}_{\mathbf{p}_j}$ given by $f(x_{t,n}) = x_{f(t),n}$. With this identification, we may assume $\theta_{\mathbf{p}_i}(\bar{x}_{\mathbf{p}_i}) = \theta_{\mathbf{p}_j}(f(\bar{x}_{\mathbf{p}_i}))$. As well, we may also assume $\text{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \text{tp}(x_{f(t),n}/A_{k_{\mathbf{p}}})$ for every $x_{t,n} \in \bar{x}_{\mathbf{p}_i}$. As well, the colorings match up as well, i.e., $c(x_{t,n}) = x_{f(t),n}$.

Now fix $i < j$. Define \mathbf{q} by $k_{\mathbf{q}} := k_{\mathbf{p}}; \mathcal{U}_{\mathbf{q}} := \mathcal{U}_{\mathbf{p}};$ and $\bar{\mathbf{a}}_{\mathbf{q}} := \bar{\mathbf{a}}_{\mathbf{p}}$ (the common values). Let $u_{\mathbf{q}} := u_{\mathbf{p}_i} \cup u_{\mathbf{p}_j}$, and, for $t \in u_{\mathbf{p}_i}$, $n_{t, \mathbf{q}} = n_{t, \mathbf{p}_i}$ while $n_{t, \mathbf{q}} = n_{t, \mathbf{p}_j}$ for $t \in u_{\mathbf{p}_j}$. To produce the striated type $\theta_{\mathbf{q}} \in S_{at}(\bar{\mathbf{a}}_{\mathbf{q}})$, first choose a perfect chain realization (\bar{M}, \bar{b}) of $\theta_{\mathbf{p}_i}(\bar{x}_{\mathbf{p}_i})$. Say $|u_{\mathbf{p}_i}| = \ell = |u_{\mathbf{p}_j}|$, while $|u^*| = k < \ell$. By Lemma 3.10(2), $\text{tp}(\bar{b}_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length k and $(\bar{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of the striated type $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ of length $(\ell - k)$. Choose \bar{d} from M_k such that $\text{tp}(\bar{d}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}) = \text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. Then by Lemma 3.9 (with M_k playing the role of M_0 there), $(\bar{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of the striated type $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}\bar{d})$. So, by Lemma 3.10(1), $\text{tp}(\bar{d}\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ is a striated type of length $2(\ell - k)$. Thus, a second application of Lemma 3.10(1) implies that $\text{tp}(\bar{b}_{<k}\bar{d}\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $2\ell - k$. Let $\theta_{\mathbf{q}}$ be a complete formula over $\bar{\mathbf{a}}_{\mathbf{p}}$ generating this type.

In order to show that \mathbf{q} is a precondition (i.e., an element of \mathbb{Q}_0) only Clause (8) requires an argument. Fix any $x_{t,n}, x_{s,m}$ in $\bar{x}_{\mathbf{q}}$ with $c_{\mathbf{q}}(x_{t,n}) = c_{\mathbf{q}}(x_{s,m})$. As both $\mathbf{p}_i, \mathbf{p}_j \in \mathbb{Q}_0$, the verification is immediate if $\{t, s\}$ is a subset of either $u_{\mathbf{p}_i}$ or $u_{\mathbf{p}_j}$, so assume otherwise. By symmetry, assume $t \in u_{\mathbf{p}_i} - u^*$ and $s \in u_{\mathbf{p}_j} - u^*$. The point is that by our trimming, $x_{f(t),n} \in \bar{x}_{\mathbf{p}_j}$, $c_{\mathbf{p}_j}(x_{f(t),n}) = c_{\mathbf{p}_i}(x_{t,n})$, and $\text{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \text{tp}(x_{f(t),n}/A_{k_{\mathbf{p}}})$. There are now two cases: First, if $\text{tp}(x_{f(t),n}/A^*) = \text{tp}(x_{s,m}/A^*)$, then it follows that $\text{tp}(x_{t,n}/A_{k_{\mathbf{p}}}) = \text{tp}(x_{s,m}/A_{k_{\mathbf{p}}})$, hence $\text{spl}(e_{t,n}, e_{s,m}) \geq k_{\mathbf{p}}$ for any perfect chain realization (\bar{N}, \bar{v}) of $\theta_{\mathbf{q}}$. On the other hand, if $\theta_{\mathbf{p}_j}$ ‘says’ $\text{spl}(x_{f(t),n}, x_{s,m}) = k \in \mathcal{U}_{\mathbf{p}_j}$, then $\theta_{\mathbf{q}}$ ‘says’ $\text{spl}(x_{t,n}, x_{s,m}) = k \in \mathcal{U}_{\mathbf{q}}$ as well. Thus, $\mathbf{q} \in \mathbb{Q}_0$, which suffices by Lemma 3.13. \square

Lemma 3.16 *Each of the following sets are dense and open in $(\mathbb{Q}, \leq_{\mathbb{Q}})$.*

1. For every $t \in \omega_1$, $D_t = \{\mathbf{p} \in \mathbb{Q} : t \in u_{\mathbf{p}}\}$;
2. For every $(t, n) \in \omega_1 \times \omega$, $D_{t,n} = \{\mathbf{p} \in \mathbb{Q} : x_{t,n} \in \bar{x}_{\mathbf{p}}\}$; and
3. **Henkin witnesses:** For all $t \in \omega_1$, all $\langle x_{s_i, n_i} : i < m \rangle$ with each $s_i \leq t$ and all $\varphi(y, v_i : i < m)$, the set $\{\mathbf{p} \in \mathbb{Q} : \text{either } \theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}}) \vdash \forall y \neg \varphi(y, x_{s_i, n_i} : i < m) \text{ or for some } n^*, \theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}}) \vdash \varphi(x_{t, n^*}, x_{s_i, n_i} : i < m)\}$.
4. For all $e \in M^*$, $D_e = \{\mathbf{p} \in \mathbb{Q} : e \in \bar{\mathbf{a}}_{\mathbf{p}} \text{ and } \theta(\bar{x}_{\mathbf{p}}) \vdash x_{0,n} = e \text{ for some } n \in \omega\}$.

Proof That each of these sets is open is immediate. As for density, in all four clauses we will show that given some $\mathbf{p} \in \mathbb{Q}$, we will find an extension $\mathbf{q} \geq_{\mathbb{Q}} \mathbf{p}$ with $\bar{x}_{\mathbf{q}}$ a one-point extension of $\bar{x}_{\mathbf{p}}$. In all cases, we will put $k_{\mathbf{q}} := k_{\mathbf{p}}, \mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$ and since $\bar{x}_{\mathbf{p}}$ is finite, we can choose the color $c_{\mathbf{q}}$ of the ‘new element’ to be distinct from the other colors. Because of that, Clause (8) for \mathbf{q} follows immediately from the fact $\mathbf{p} \in \mathbb{Q}$. Thus, for all four clauses, all of the work is in finding a striated type $\theta_{\mathbf{q}}$ extending $\theta_{\mathbf{p}}$.

(1) Fix $t \in \omega_1$ and choose an arbitrary $\mathbf{p} \in \mathbb{Q}$. If $t \in u_{\mathbf{p}}$ then there is nothing to prove, so assume otherwise. Let $\ell = |u_{\mathbf{p}}|$ and let $k = |\{s \in u_{\mathbf{p}} : s < t\}|$. Assume that $k < \ell$, as the case of $k = \ell$ is similar, but easier. Choose a perfect chain realization (\bar{M}, \bar{b}) of $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$. By Lemma 3.10(2), $\text{tp}(\bar{b}_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length k . By Lemma 2.4(1), choose an A^* -large type $r \in S_{at}(\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$ and choose a realization e of r in M_k . One checks immediately that $\text{tp}(\bar{b}_{<k}e/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $(k + 1)$. Now, also by Lemma 3.10(2), $(\bar{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. So, by Lemma 3.9, $(\bar{M}_{\geq k}, \bar{b}_{\geq k})$ is also a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}e)$. In particular, $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}e)$ is a striated type of length $(\ell - k)$. Thus,

by Lemma 3.10(1), $\text{tp}(\bar{b}_{<k}e\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length $(\ell + 1)$. Take $\bar{\mathbf{a}}_{\mathbf{q}} := \bar{\mathbf{a}}_{\mathbf{p}}$, $\bar{x}_{\mathbf{q}} := \bar{x}_{\mathbf{p}} \cup \{x_{t,0}\}$, and take $\theta_{\mathbf{q}}(\bar{x}_{\mathbf{q}})$ to be a complete formula in $\text{tp}(\bar{b}_{<k}e\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{q}})$.

The proofs of (2) and (3) are extremely similar. We prove (2) and indicate the adjustment necessary for (3). Fix $(t, n) \in \omega_1 \times \omega$. By (1) and an inductive argument, we may assume we are given $\mathbf{p} \in \mathbb{Q}$ with $t \in u_{\mathbf{p}}$ and $x_{t,n-1} \in \bar{x}_{\mathbf{p}}$. Say $|u_{\mathbf{p}}| = \ell$ and assume t is the $(k - 1)$ st element of $u_{\mathbf{p}}$ in ascending order. Choose a perfect chain realization (\bar{M}, \bar{b}) of $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$. By Lemma 3.10(2), $\text{tp}(\bar{b}_{<k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is striated of length k . Choose an arbitrary $e \in M_k^3$ and adjoin it to \bar{b}_{k-1} . More formally, let $\bar{b}_{<k}^* := \langle \bar{b}_j^* : j < k \rangle$, where $\bar{b}_j^* = \bar{b}_j$ for $j < k - 2$, while $\bar{b}_{k-1}^* := \bar{b}_{k-1}e$. Note that $\text{tp}(\bar{b}_{<k}^*/\bar{\mathbf{a}}_{\mathbf{p}})$ remains a striated type of length k . By Lemma 3.10(2), $(\bar{M}_{\geq k}, \bar{b}_{\geq k})$ is a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k})$. So, by Lemma 3.9 it is also a perfect chain realization of $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}^*)$. In particular, $\text{tp}(\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}}\bar{b}_{<k}^*)$ is a striated type of length $(\ell - k)$, so $\text{tp}(\bar{b}_{<k}^*\bar{b}_{\geq k}/\bar{\mathbf{a}}_{\mathbf{p}})$ is a striated type of length ℓ extending $\theta_{\mathbf{p}}(\bar{x}_{\mathbf{p}})$. Put $\bar{x}_{\mathbf{q}} := \bar{x}_{\mathbf{p}} \cup \{x_{t,n}\}$ and let $\theta_{\mathbf{q}}(\bar{x}_{\mathbf{q}})$ be a complete formula isolating this type.

(4) is also similar and is left to the reader. \square

The following Proposition follows immediately from the density conditions described above.

Proposition 3.17 *Let G be a \mathbb{Q} -generic filter. Then, in $V[G]$, a rich, \mathcal{U}_G -colored atomic model of T exists, where $\mathcal{U}_G = \{k \in \omega : k \in \mathcal{U}_{\mathbf{p}} \text{ for some } \mathbf{p} \in G\}$.*

Proof There is a congruence \sim_G defined on $X = \{x_{t,n} : t \in \omega_1, n \in \omega\}$ defined by $x_{t,n} \sim_G x_{s,m}$ if and only if $\theta_{\mathbf{p}} \vdash x_{t,n} = x_{s,m}$ for some $\mathbf{p} \in G$. Let M_G be the model of T with universe X/\sim_G and relations $M_G \models \varphi(a_1, \dots, a_k)$ if and only if there are $(x_{t_1, n_1}, \dots, x_{t_k, n_k}) \in X^k$ such that $[x_{t_i, n_i}] = a_i$ for each i and $\theta_{\mathbf{p}} \vdash \varphi(x_{t_1, n_1}, \dots, x_{t_k, n_k})$ for some $\mathbf{p} \in G$. Since $(\mathbb{Q}, \leq_{\mathbb{Q}})$ has c.c.c., M_G has size \aleph_1 . As notation, for each $t \in \omega_1$, let $M_{\leq t}$ be the substructure of M_G with universe $\{[x_{s,m}] : s \leq t, m \in \omega\}$. Then $M^* \leq M_0$ and $M_{\leq s} \leq M_{\leq t} \leq M_G$ whenever $s \leq t < \omega_1$. The definition of a striated type implies that $\text{tp}([x_{t,0}]/A^*)$ is omitted in $M_{<t}$, hence the set $\{[x_{t,0}] : t \in \omega_1\}$ witnesses that (M_G, b^*) is rich. Also, define $c_G := \bigcup \{c_{\mathbf{p}} : \mathbf{p} \in G\}$. Using the fact that each $\mathbf{p} \in \mathbb{Q}$ is fully decided, check that c_G is a \mathcal{U}_G -coloring of (M_G, b^*) . \square

Note that in the Conclusion below, such a $G \in V$ always exists, since \mathcal{B} is countable. Recall the finite fragment ZFC^* of ZFC given by Proposition 2.9.

Conclusion 3.18 *Suppose \mathcal{B} is a countable, transitive model of ZFC^* , with $\{M^*, T, L\} \subseteq \mathcal{B}$, and let $G \in V$, $G \subseteq \mathbb{Q}$ be any filter meeting every dense $D \subseteq \mathbb{Q}$ with $D \in \mathcal{B}$. Then: Let $\mathcal{U}_G = \{k \in \omega : k \in \mathcal{U}_{\mathbf{p}} \text{ for some } \mathbf{p} \in G\}$. Then:*

1. $\mathcal{U}_G \in V$; and
2. In V , there is a \mathcal{U}_G -colored, rich atomic model (N, \bar{b}^*) of T .

Proof That $\mathcal{U}_G \in V$ is immediate, since both \mathcal{B} and G are. As for (2), as G meets every dense set in \mathcal{B} , $\mathcal{B}[G]$ is a countable, transitive model of ZFC^* , and by applying

³ In the proof of (3), e would be a realization of $\varphi(y, b_{s_i, n_i} : i < m)$ in M_k , if one existed.

Proposition 3.17,

$$\mathcal{B}[G] \models \text{‘There is a rich, } \mathcal{U}_G\text{-colored } (M_G, \bar{b}^*) \text{ of size } \aleph_1 \text{’}$$

Let $L' = L \cup \{c, R\} \cup \{c_m : m \in M^*\}$. Working in $\mathcal{B}[G]$, expand M_G to an L' -structure M' , interpreting each c_m by m , interpreting the unary function $c^{M'}$ as $c_G = \bigcup \{c_{\mathbf{p}} : \mathbf{p} \in G\}$, and the unary predicate $R^{M'} = \{[x_t, 0] : t \in \omega_1\}$.

Now, for each $d, d' \in M'$ and $k \in \omega$, the relation $\text{tp}_{M'}(d/A_k) = \text{tp}_{M'}(d'/A_k)$ is definable by an $L'_{\omega_1, \omega}$ -formula. Thus, the binary function $\text{spl} : (M')^2 \rightarrow (\omega + 1)$ is also $L'_{\omega_1, \omega}$ -definable, hence, using the coloring c , there is an $L'_{\omega_1, \omega}$ -sentence Ψ stating that ‘ c induces a \mathcal{U}_G -coloring.’ Finally, using the Q -quantifier to state that R is uncountable, there is an $L'_{\omega_1, \omega}$ -sentence $\Phi \in \mathcal{B}[G]$ stating that the $L(\bar{b}^*)$ -reduct of a given L' -structure is a rich, atomic model of T , that is \mathcal{U}_G -colored via c . We finish by applying Proposition 2.9 to M' and Φ . \square

3.3 Mass production

In this subsection we define a forcing $(\mathbb{P}, \leq_{\mathbb{P}})$ such that a \mathbb{P} -generic filter G produces a perfect set $\{G_{\eta} : \eta \in 2^{\omega}\}$ of \mathbb{Q} -generic filters such that the associated subsets $\{\mathcal{U}_{G_{\eta}} : \eta \in 2^{\omega}\}$ of ω are almost disjoint. Although the application there is very different, the argument in this subsection is similar to one appearing in [9].

We begin with one easy density argument concerning the partial order $(\mathbb{Q}, \leq_{\mathbb{Q}})$. Fundamentally, it allows us to ‘stall’ the construction for any fixed, finite length of time.

Lemma 3.19 *For every $\mathbf{p} \in \mathbb{Q}$ and every $k^* > k_{\mathbf{p}}$, there is $\mathbf{q} \geq_{\mathbb{Q}} \mathbf{p}$ such that $\bar{x}_{\mathbf{q}} = \bar{x}_{\mathbf{p}}$, (hence $c_{\mathbf{q}} = c_{\mathbf{p}}$); but $k_{\mathbf{q}} = k^*$ and $\mathcal{U}_{\mathbf{q}} = \mathcal{U}_{\mathbf{p}}$, i.e., $\mathcal{U}_{\mathbf{q}} \cap [k_{\mathbf{p}}, k^*) = \emptyset$.*

Proof Simply define \mathbf{q} as above and then verify that $\mathbf{q} \in \mathbb{Q}$. \square

Definition 3.20 For $n \in \omega$, let

$$\mathbb{P}_n = \{(k, \bar{p}) : k \in \omega, \bar{p} = \langle p_{\eta} : \eta \in 2^n \rangle, \text{ where each } p_v \in \mathbb{Q} \text{ and every } k_{p_v} = k\}$$

As notation, for $\mathbf{p} \in \mathbb{P}_n$, we let $k(\mathbf{p})$ denote the (integer) first coordinate of \mathbf{p} . For each $\ell < k(\mathbf{p})$, define the *trace of ℓ* , $\text{tr}_{\ell}(\mathbf{p}) = \{v \in 2^n : \ell \in \mathcal{U}_{p_v}\}$.

Let $\mathbb{P} = \bigcup_{n \in \omega} \mathbb{P}_n$. As notation, for $\mathbf{p} \in \mathbb{P}$, $n(\mathbf{p})$ is the unique n for which $\mathbf{p} \in \mathbb{P}_n$.

Definition 3.21 Define an order $\leq_{\mathbb{P}}$ on \mathbb{P} by $\mathbf{p} \leq_{\mathbb{P}} \mathbf{q}$ if and only if

1. $n(\mathbf{p}) \leq n(\mathbf{q})$, $k(\mathbf{p}) \leq k(\mathbf{q})$;
2. $p_v \leq_{\mathbb{Q}} q_{\mu}$ for all pairs $v \in 2^{n(\mathbf{p})}$, $\mu \in 2^{n(\mathbf{q})}$ satisfying $v \leq \mu$; and
3. For all $\ell \in [k(\mathbf{p}), k(\mathbf{q})]$, the set $\{\mu \upharpoonright_{n(\mathbf{p})} : \mu \in \text{tr}_{\ell}(\mathbf{q})\}$ is either empty or is a singleton.

It is easily checked that $(\mathbb{P}, \leq_{\mathbb{P}})$ is a partial order, hence a notion of forcing. The following Lemma describes the dense subsets of \mathbb{P} .

Lemma 3.22 1. For each n and k , $\{\mathbf{p} \in \mathbb{P} : n(\mathbf{p}) \geq n\}$ and $\{\mathbf{p} \in \mathbb{P} : k(\mathbf{p}) \geq k\}$ are dense;

2. Suppose D is a dense, open subset of \mathbb{Q} . Then for every n and every $\mathbf{p} \in \mathbb{P}_n$, there is $\mathbf{q} \in \mathbb{P}_n$ such that $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ and, for every $v \in 2^n$, $\mathbf{q}_v \in D$.

Proof Arguing by induction, it suffices to prove that for any given $\mathbf{p} \in \mathbb{P}$, there is $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ with $n(\mathbf{q}) = n(\mathbf{p}) + 1$ and an $\mathbf{r} \geq_{\mathbb{P}} \mathbf{p}$ with $k(\mathbf{r}) > k(\mathbf{p})$. Fix $\mathbf{p} \in \mathbb{P}$. Say $\mathbf{p} \in \mathbb{P}_n$ and $\mathbf{p} = (k, \bar{p})$. To construct \mathbf{q} , for each $v \in 2^n$, define $q_{v0} = q_{v1} = p_v$. Let $\bar{q} := \langle q_\mu : \mu \in 2^{n+1} \rangle$ and $\mathbf{q} = (k, \bar{q})$. Then $\mathbf{q} \in \mathbb{P}_{n+1}$ and $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ (note that Clause (3) in the definition of $\leq_{\mathbb{P}}$ is vacuously satisfied since $k(\mathbf{p}) = k(\mathbf{q})$).

To construct \mathbf{r} , simply apply Lemma 3.19 to each p_v to produce an extension $r_v \geq_{\mathbb{Q}} p_v$ with $k_{r_v} = k + 1$, but $\mathcal{U}_{r_v} = \mathcal{U}_{p_v}$. Then let $\bar{r} := \langle r_v : v \in 2^n \rangle$ and $\mathbf{r} = (k + 1, \bar{r})$. Then $\mathbf{r} \geq_{\mathbb{P}} \mathbf{p}$ as required.

(2) Fix such a D and n . As we are working exclusively in \mathbb{P}_n and because 2^n is a fixed finite set, it suffices to prove that for any chosen $v \in 2^n$,

For every $\mathbf{p} \in \mathbb{P}_n$ there is $\mathbf{q} \in \mathbb{P}_n$ with $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$ and $q_v \in D$.

To verify this, fix $v \in 2^n$ and $\mathbf{p} \in \mathbb{P}_n$. Concentrating on p_v , as D is dense, choose $q_v \in D \cap \mathbb{Q}$ with $q_v \geq_{\mathbb{Q}} p_v$. Let $k^* := k_{q_v}$. Next, for each $\delta \in 2^n$ with $\delta \neq v$, apply Lemma 3.19 to p_δ , obtaining some $q_\delta \in \mathbb{Q}$ satisfying $q_\delta \geq_{\mathbb{Q}} p_\delta$, $k_{q_\delta} = k^*$, but $\mathcal{U}_{q_\delta} = \mathcal{U}_{p_\delta}$. Now, collect all of this data into a condition $\mathbf{q} \in \mathbb{P}_n$ defined by $k(\mathbf{q}) = k^*$ and $\bar{q} = \langle q_\gamma : \gamma \in 2^n \rangle$, where each q_γ is as above. To see that $\mathbf{q} \geq_{\mathbb{P}} \mathbf{p}$, Clause (3) is verified by noting that for every $\ell \in [k(\mathbf{p}), k^*]$, $\text{tr}_\ell(\mathbf{q})$ is either empty, or equals $\{v\}$, depending on whether or not $\ell \in \mathcal{U}_{q_v}$. \square

Notation 3.23 Suppose $\mathcal{B} \models ZFC^*$ and let $G^* \subseteq \mathbb{P}$, $G^* \in V$ be a filter meeting every dense subset $D^* \subseteq \mathbb{P}$ with $D^* \in \mathcal{B}$. For each n and $v \in 2^n$, let

$$G_v := \{\mathbf{p} \in \mathbb{Q} : \text{for some } \mathbf{p}^* = (k, \bar{p}) \in G^*, \mathbf{p} = \mathbf{p}_v^*\}$$

Then, for each $\eta \in 2^\omega$, let

$$G_\eta := \bigcup \{G_{\eta|n} : n \in \omega\} \text{ and } \mathcal{U}_\eta := \{\ell \in \omega : \ell \in \mathcal{U}_\mathbf{q} \text{ for some } \mathbf{q} \in G_\eta\}$$

Proposition 3.24 In the notation of 3.23:

1. For every $\eta \in 2^\omega$, $G_\eta \subseteq \mathbb{Q}$ is a filter meeting every dense $D \subseteq \mathbb{Q}$ with $D \in \mathcal{B}$;
2. The sets $\{\mathcal{U}_\eta : \eta \in 2^\omega\}$ are an almost disjoint family of infinite subsets of ω .

Proof (1) follows immediately from Lemma 3.22(2).

(2) Choose distinct $\eta, \eta' \in 2^\omega$. Choose n_0 such that $\eta|n \neq \eta'|n$ whenever $n \geq n_0$. By Lemma 3.22(1), choose $\mathbf{p}^* \in G^*$ with $n(\mathbf{p}^*) \geq n_0$. We show that $\mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$ is finite by establishing that if $\ell \in \mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$, then $\ell \leq k(\mathbf{p}^*)$.

To establish this, choose $\ell \in \mathcal{U}_\eta \cap \mathcal{U}_{\eta'}$. By unpacking the definitions, choose $\mathbf{q}^*, \mathbf{r}^* \in G^*$ such that, letting $\mu := \eta|n(\mathbf{q}^*)$ and $\mu' := \eta'|n(\mathbf{r}^*)$, we have $\ell \in \mathcal{U}_{\mathbf{q}_\mu^*} \cap \mathcal{U}_{\mathbf{r}_{\mu'}^*}$. As G^* is a filter, choose $\mathbf{s}^* \in G^*$ with $\mathbf{s}^* \geq_{\mathbb{P}} \mathbf{p}^*, \mathbf{q}^*, \mathbf{r}^*$. As notation, let $\delta := \eta|n(\mathbf{s}^*)$ and $\delta' := \eta'|n(\mathbf{s}^*)$.

Claim: $\ell \in \mathcal{U}_{\aleph_\delta^*} \cap \mathcal{U}_{\aleph_{\delta'}^*}$.

Proof As $\ell \in \mathcal{U}_{\aleph_\mu^*}$, $\ell < k(\mathbf{q}^*)$. From $\mathbf{q}^* \leq_{\mathbb{P}} \mathbf{s}^*$ we conclude $k(\mathbf{q}^*) \leq k(\mathbf{s}^*)$, so $\ell < k(\mathbf{s}^*)$ as well. From $\mathbf{q}^* \leq_{\mathbb{P}} \mathbf{s}^*$ and $\mu \leq \delta$ we obtain $\mathbf{q}_\mu^* \leq_{\mathbb{Q}} \mathbf{s}_\delta^*$. But then, as $\ell \in \mathcal{U}_{\aleph_\mu^*}$, it follows that $\ell \in \mathcal{U}_{\aleph_\delta^*}$. That $\ell \in \mathcal{U}_{\aleph_{\delta'}^*}$ is analogous, using \mathbf{r}^* in place of \mathbf{q}^* .

Finally, assume by way of contradiction that $\ell \geq k(\mathbf{p}^*)$. The Claim implies that $\{\delta, \delta'\} \subseteq \text{tr}_\ell(\mathbf{s}^*)$. As $\ell \in [k(\mathbf{p}^*), k(\mathbf{s}^*)]$, Clause (3) of $\mathbf{p}^* \leq_{\mathbb{P}} \mathbf{s}^*$ implies that $\delta|n(\mathbf{p}^*) = \delta'|n(\mathbf{p}^*)$. But, as $\eta|n(\mathbf{p}^*) = \delta|n(\mathbf{p}^*)$ and $\eta'|n(\mathbf{p}^*) = \delta'|n(\mathbf{p}^*)$, we contradict our choice of \mathbf{p}^* . \square

We close this section with the proof of Proposition 3.1, which we restate for convenience.

Conclusion 3.25 *There is a family $\{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of 2^{\aleph_0} rich, atomic models of T , each of size \aleph_1 , that are pairwise non-isomorphic over \bar{b}^* .*

Proof Choose any countable, transitive model \mathcal{B} of ZFC^* and choose any $G^* \in V$, $G^* \subseteq \mathbb{P}$, G^* meets every dense subset $D^* \in \mathcal{B}$ (as \mathcal{B} is countable, such a G^* exists). For each $\eta \in 2^\omega$, choose G_η and \mathcal{U}_η as in Proposition 3.24, and apply Conclusion 3.18 to get a rich \mathcal{U}_η -colored (N_η, \bar{b}^*) in V . That this family is pairwise non-isomorphic over \bar{b}^* follows immediately from Corollary 3.6, since the sets $\{\mathcal{U}_\eta : \eta \in 2^\omega\}$ are almost disjoint. \square

4 The proof of Theorem 1.4

Assume that the class \mathbf{At}_T is not pcl-small, as witnessed by an (uncountable) model N^* containing a finite tuple \bar{a}^* . Fix a countable, elementary substructure $M^* \preceq N^*$ that contains \bar{a}^* . To aid notation, let $D^* := \text{pcl}_{N^*}(\bar{a}^*)$. We now split into cases, depending on the relationship between the cardinals 2^{\aleph_0} and 2^{\aleph_1} .

Case 1. $2^{\aleph_0} < 2^{\aleph_1}$.

In this case, expand the language of T to $L(D^*)$, adding a new constant symbol for each $d \in D^*$. Then, the natural expansion $N_{D^*}^*$ of N^* to an $L(D^*)$ -structure is a model of the infinitary $L(D^*)$ -sentence Φ that entails $\text{Th}(N_{D^*}^*)$ and ensures that every finite tuple is L -atomic with respect to T . As $N_{D^*}^*$ is a model of Φ that realizes uncountably many types over the empty set (after fixing D^* !), it follows from [6], Theorem 45 of Keisler that there are 2^{\aleph_1} pairwise non- $L(D^*)$ -isomorphic models Φ , each of size \aleph_1 . As $2^{\aleph_0} < 2^{\aleph_1}$, it follows that there is a subfamily of 2^{\aleph_1} pairwise non- L -isomorphic reducts to the original language L . As each of these models are L -atomic, we conclude that \mathbf{At}_T has 2^{\aleph_1} non-isomorphic models of size \aleph_1 .

Case 2. $2^{\aleph_0} = 2^{\aleph_1}$.

Choose \bar{b}^* from M^* as in Proposition 2.10 and apply Conclusion 3.25 to get a set $\mathcal{F}^* = \{(N_\eta, \bar{b}^*) : \eta \in 2^\omega\}$ of atomic models, each of size \aleph_1 , that are pairwise non-isomorphic over \bar{b}^* . Let $\mathcal{F} = \{N_\eta : \eta \in 2^\omega\}$ be the set of reducts of elements from \mathcal{F}^* . By our cardinal hypothesis, \mathcal{F} has size 2^{\aleph_1} . The relation of L -isomorphism is an equivalence relation on \mathcal{F} , and each L -isomorphism equivalence class has size

at most \aleph_1 (since $\aleph_1^{<\omega} = \aleph_1$). As $\aleph_1 < 2^{\aleph_1}$ we conclude that \mathcal{F} has a subset of size 2^{\aleph_1} of pairwise non-isomorphic atomic models of T , each of size \aleph_1 . \square

References

1. Baldwin, J.T.: *Categoricity*. Number 51 in University Lecture Notes. American Mathematical Society, Providence, USA (2009)
2. Baldwin, J., Larson, P.: Iterated elementary embeddings and the model theory of infinitary logic. *Ann. Pure Appl. Logic* **167**(3), 309–334 (2016)
3. Baldwin, J., Laskowski, M.C., Shelah, S.: Constructing many uncountable atomic models in \aleph_1 . *J. Symb. Log.* **81**(3), 1142–1162 (2016)
4. Baldwin, J., Laskowski, M.C.: Henkin constructions of models of size continuum. *Bull. Symb. Log.* (to appear)
5. Baldwin, J., Laskowski, M.C., Shelah, S.: An active chain implies many atomic models in \aleph_1 (preprint)
6. Keisler, H.J.: *Model Theory for Infinitary Logic*. Logic with Countable Conjunctions and Finite Quantifiers. Studies in Logic and the Foundations of Mathematics, vol. 62. North-Holland Publishing Co., Amsterdam (1971)
7. Keisler, H.J.: Logic with the quantifier there exist uncountably many. *Ann. Math. Log.* **1**, 1–93 (1970)
8. Shelah, S.: *Classification Theory and the Number of Nonisomorphic Models*. North-Holland, Amsterdam (1978)
9. Shelah, S.: How special are Cohen and random forcings, i.e. Boolean algebras of the family of subsets of reals modulo meagre or null. *Isr. J. Math.* **88**, 159–174 (1994)
10. Ulrich, D., Rast, R., Laskowski, M.C.: Borel complexity and potential canonical Scott sentences. *Fundam. Math. Fundam. Math.* **239**(2), 101–147 (2017)