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CONSISTENCY OF A STRONG UNIFORMIZATION PRINCIPLE

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Abstract. We prove the consistency of a strong uniformization principle for subsets of the Baire space of cardinality $\aleph_1$.

0. Introduction. There are many consistency results on uniformization principles which can be seen as strong negations of Jensen's principle $\Diamond$. One example is the consistency of

$\boxtimes_1$ there exists an injective sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of elements of ${}^\omega\omega$ which has the 2-uniformization property, that is, if c_α ($\alpha < \omega_1$) are elements of ${}^\omega 2$ then for some $h\colon {}^{\omega>}\omega \to 2$, for every $\alpha < \omega_1$ and every sufficiently large $n < \omega$ we have $h(\eta_\alpha \restriction n) = c_\alpha(n)$.

See, for example, [1, 9] for the consistency of this principle, [5] for negative ZFC results and [1] for connections to abelian groups. We would like to deal with colorings with $\aleph_0$ many colors, but the parallel of $\boxtimes_1$ for this fails (see [6, 1.2(3)]). Weakening the demand in another direction motivates us to formulate

$\boxtimes_2$ there exists an injective sequence $\langle \eta_\alpha : \alpha < \omega_1 \rangle$ of elements of ${}^\omega\omega$ such that for every countable group $G = (G, +_G)$ and every $\{c_\alpha : \alpha < \omega_1\} \subseteq {}^\omega G$ there exist functions $h\colon {}^{\omega>}\omega \to G$ and $\zeta\colon \omega_1 \to \omega_1$ such that for every $\alpha < \omega_1$ and every positive $n < \omega$ we have

$$c_\alpha(n) = h(\eta_\alpha \restriction n) +_G h(\eta_{\zeta(\alpha)} \restriction n).$$

We have in mind an abelian group (thus the additive notation), but this makes sense for any countable group. We omit the restriction "for every large enough n" as we have the function ζ.

In this paper we prove the consistency of $\boxtimes_2$ (Corollary 2.2). We first thought of using non-meagerness of $\{\eta_\alpha : \alpha < \omega_1\}$ but eventually continued

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the ideas from [7, §1]. The main part of the present paper derives the consistency of a uniformization principle from which $\boxtimes_2$ follows (see Definition 1.4 and Theorem 1.5). The proof uses a forcing iteration by finite support; most of the work goes into showing that the iteration satisfies the countable chain condition. As the individual iterands are not absolutely c.c.c., the proof needs to analyze the iteration as a whole.

We have not managed to settle the consistency of the following relative:

$\boxtimes_3$ for every infinite countable group $G = (G, +_G)$, there exist pairwise distinct $\eta_\alpha \in {}^\omega G$ for $\alpha < \omega_1$ such that for every $\{c_\alpha : \alpha < \omega_1\} \subseteq {}^\omega G$ there exist functions $h\colon {}^{\omega>}G \to G$ and $\zeta\colon \omega_1 \to \omega_1$ such that for any $\alpha < \omega_1$ and $n < \omega$ we have

$$c_\alpha(n) = h(\eta_\alpha \restriction (n+1)) +_G \eta_{\zeta(\alpha)}(n).$$

This would give a result on Ext related to a problem on splitters (there are R-modules G such that $\mathrm{Ext}(G, G) = 0$, for R a subring of the rationals; see Göbel–Shelah [2, 3]). More specifically, if $\boxtimes_3$ holds for some such R with one prime we get the consistency of the existence of such G of cardinality $\aleph_1$ and density $\aleph_0$. We intend to deal with this in [4].

1. Consistency of a uniformization principle for $\aleph_1$

Notation 1.1. For finite sequences η and ν, $\eta \trianglelefteq \nu$ means that η is an initial segment of ν, and $\eta \triangleleft \nu$ means that η is a proper initial segment of ν. We let $\ell g(\eta)$ denote the length of η.

Notation 1.2. We let

(1) $\mathscr{F}_{\aleph_0}$ denote the set of pairs (h, ν) for which there exist a non-zero $n < \omega$ and a sequence $\eta \in {}^n\omega$ such that $\nu \in {}^n\omega$ and h is a function from

$$\{\rho : \rho \trianglelefteq \eta \vee \rho \triangleleft \nu\}$$

to ω (so (η, ν) can be reconstructed from $\mathrm{dom}(h)$);

(2) $\mathscr{F}_{*,\aleph_0}$ denote the set of functions from $\mathscr{F}_{\aleph_0}$ to ω.

The "s.i.u." defined in part (1) below is closely related to $\boxtimes_2$ from the introduction (see Theorem 2.1). Note that the main case below is $i_1^* = i_2^* = \aleph_1$.

Definition 1.3.

(1) We say that $(\bar\eta^1, \bar\eta^2)$ satisfies the $\aleph_0$-*strong inside uniformization property* ($\aleph_0$-s.i.u.) when, for some ordinals i_1^* and i_2^*:

(a) $\bar\eta^\ell = \langle \eta_i^\ell : i < i_\ell^* \rangle$ for $\ell \in \{1, 2\}$;

(b) $\eta_i^\ell \in {}^\omega\omega \setminus \{\eta_j^\ell : j < i\}$ for $i < i_\ell^*$ and $\ell = 1, 2$;

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(c) for each sequence $\langle f_i : i < i_1^* \rangle \in {}^{i_1^*}(\mathscr{F}_{*,\aleph_0})$ we can find functions $h\colon {}^{\omega>}\omega \to \omega$ and $g\colon i_1^* \to i_2^*$ satisfying

(∗) for every $i < i_1^*$ and for every non-zero $n < \omega$ the function h *obeys* f_i at $((\eta_i^1 \restriction n), \eta_{g(i)}^2 \restriction n)$, which means that

$$h(\eta_{g(i)}^2 \restriction n) = f_i(h \restriction \{\rho : \rho \trianglelefteq \eta_i^1 \restriction n \vee \rho \triangleleft \eta_{g(i)}^2 \restriction n\}, \eta_{g(i)}^2 \restriction n).$$

(2) We may replace $(\bar{\eta}^1, \bar{\eta}^2)$ by $\bar{\eta}$ if $\bar{\eta}^1 = \bar{\eta}^2 = \bar{\eta}$.

(3) We say that λ has the $\aleph_0$-s.i.u. if some sequence $\bar{\eta} \in {}^{\lambda}({}^{\omega}\omega)$ has the $\aleph_0$-s.i.u.

Definition 1.4. A sequence $\bar{\eta}$ is *universally* $\aleph_0$*-s.i.u.* when, for some ordinal i^*:

(a) $\bar{\eta} = \langle \eta_i : i < i^* \rangle$ where $\eta_i \in {}^{\omega}\omega \setminus \{\eta_j : j < i\}$ for all $i < i^*$;

(b) for all $\bar{\eta}^1 = \langle \eta_i^1 : i < i^* \rangle$ such that $\eta_i^1 \in {}^{\omega}\omega \setminus \{\eta_j^1 : j < i\}$ for $i < i^*$, the pair $(\bar{\eta}^1, \bar{\eta})$ has the $\aleph_0$-s.i.u.

Our main result is the following.

Theorem 1.5. *There is a c.c.c. partial order of cardinality* $2^{\aleph_1}$ *forcing the existence of a universally* $\aleph_0$*-s.i.u. sequence of length* ω_1*.*

The proof is broken into a series of definitions and claims. We fix for this section a regular cardinal $\chi > 2^{2^{\aleph_1}}$, and let λ be $2^{\aleph_1}$. Let $<_\chi^*$ be a strict wellordering of $H(\chi)$.

Definition 1.6. For $\alpha \in [1, \lambda]$, let $\mathfrak{K}_\alpha$ be the family of

$$\mathfrak{q} = \langle (\mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \underset{\sim}{\bar{f}}_\beta, \bar{N}_\beta) : \beta < \alpha \rangle$$

such that

(a) $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta : \beta < \alpha \rangle$ is a finite support iteration;

(b) $\mathbb{Q}_0$ is the set of finite partial functions from ω_1 to ${}^{\omega>}\omega$, ordered by

$$p \leq_{\mathbb{Q}_0} q \;\Leftrightarrow\; (\forall i \in \mathrm{Dom}(p))(i \in \mathrm{Dom}(q) \wedge p(i) \trianglelefteq q(i));$$

(c) $\underset{\sim}{\bar{f}}_0 = \bar{N}_0 = \emptyset$;

(d) for all $\beta \in [1, \alpha)$:

(α) $\underset{\sim}{\bar{f}}_\beta$ is a $\mathbb{P}_\beta$-name for an ω_1-sequence of members of $\mathscr{F}_{*,\aleph_0}^{\mathbf{V}[\mathbb{P}_\beta]}$ (for each $j < \omega_1$ we let $\underset{\sim}{f}_{\beta,j}$ be the induced $\mathbb{P}_\beta$-name for the jth member of the realization of $\underset{\sim}{\bar{f}}_\beta$),

(β) $\underset{\sim}{\bar{\eta}}_\beta^1$ is a $\mathbb{P}_\beta$-name for an ω_1-sequence of pairwise distinct members of ${}^{\omega}\omega$ (for each $j < \omega_1$ we let $\underset{\sim}{\eta}_{\beta,j}^1$ be the induced $\mathbb{P}_\beta$-name for the jth member of the realization of $\underset{\sim}{\bar{\eta}}_\beta^1$),

(γ) $\bar{N}_\beta$ is a $\subseteq$-increasing continuous sequence $\langle N_{\beta,i} : i < \omega_1 \rangle$ such that

- each $N_{\beta,i}$ is a countable elementary submodel of $(\mathscr{H}(\chi), \in, <^*_\chi)$,
- $\mathfrak{q}\restriction\beta, \beta \in N_{\beta,0}$,
- $\bar{N}_\beta\restriction(i+1) \in N_{\beta,i+1}$ for each $i < \omega_1$;

(e) for all $\beta \in (0, \alpha)$, if $\omega_1^{\mathbf{V}}$ is countable in $\mathbf{V}^{\mathbb{P}_\beta}$ then $\underset{\sim}{\mathbb{Q}}_\beta$ is the trivial forcing there; otherwise, in $\mathbf{V}^{\mathbb{P}_\beta}$, the conditions of $\underset{\sim}{\mathbb{Q}}_\beta$ are the triples $p = (h^p, w^p, g^p)$ such that, letting

- $\mathbb{G}_\beta$ be a $\mathbb{V}$-generic filter for $\mathbb{P}_\beta$,
- for each $i < \omega_1$, $\underset{\sim}{\eta}_i$ denote the natural name for the ith element of ${}^\omega\omega$ added by $\mathbb{Q}_0$ and $\zeta_\beta(i)$ denote $N_{\beta,i} \cap \omega_1$,

we have:

(α) h^p is a function with domain a finite subset of ${}^{\omega>}\omega$ closed under initial segments and range contained in ω,
(β) w^p is a finite subset of ω_1,
(γ) g^p is a function with domain w^p and each value $g^p(j)$ in the corresponding set $\{\zeta_\beta(\omega j + n) : 0 < n < \omega\}$,
(δ) for all $n < \omega$ and $j \in w^p$,

$$\underset{\sim}{\eta}^1_{j,\mathbb{G}_\beta}\restriction n \in \mathrm{Dom}(h^p) \;\Leftrightarrow\; \underset{\sim}{\eta}_{g^p(j),\mathbb{G}_\beta}\restriction n \in \mathrm{Dom}(h^p),$$

(ε) for each $j \in w^p$ there exists an $n \in \omega$ such that
 (i) $\underset{\sim}{\eta}^1_{j,\mathbb{G}_\beta}\restriction n, \underset{\sim}{\eta}_{g^p(j),\mathbb{G}_\beta}\restriction n \in \mathrm{Dom}(h^p)$,
 (ii) for all $i \in w^p\setminus\{j\}$, $\underset{\sim}{\eta}_{g^p(i),\mathbb{G}_\beta}\restriction(n+1) \neq \underset{\sim}{\eta}_{g^p(j),\mathbb{G}_\beta}\restriction(n+1)$,
(ζ) for all $j \in w^p$ and $n \in (0, \omega)$, if $\underset{\sim}{\eta}^1_{j,\mathbb{G}_\beta}\restriction n \in \mathrm{Dom}(h^p)$, then h^p obeys $\underset{\sim}{f}_{\beta,j,\mathbb{G}_\beta}$ at $(\underset{\sim}{\eta}^1_{i,\mathbb{G}_\beta}\restriction n,\ \underset{\sim}{\eta}_{g^p(j),\mathbb{G}_\beta}\restriction n)$;

(f) for all $\beta \in (0, \alpha)$ (for which $\omega_1^{\mathbf{V}}$ is uncountable in $\mathbf{V}^{\mathbb{P}_\beta}$) the order on $\underset{\sim}{\mathbb{Q}}_\beta$ in $\mathbf{V}^{\mathbb{P}_\beta}$ is: $p \le q$ iff $h^p \subseteq h^q \wedge w^p \subseteq w^q \wedge g^p \subseteq g^q$.

Notation 1.7. Given a $\mathfrak{q}$ in $\mathfrak{K}_\alpha$ for some ordinal α, we let

$$\langle(\mathbb{P}^{\mathfrak{q}}_\beta, \underset{\sim}{\mathbb{Q}}^{\mathfrak{q}}_\beta, \underset{\sim}{\bar{f}}^{\mathfrak{q}}_\beta, \bar{N}^{\mathfrak{q}}_\beta) : \beta < \alpha^{\mathfrak{q}}\rangle$$

denote the components of $\mathfrak{q}$.

Notation 1.8. Given $\alpha \in [1, \lambda]$ and $\mathfrak{q}$ in $\mathfrak{K}_\alpha$, we let $\mathrm{Lim}(\mathfrak{q})$ denote $\mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is $\mathbb{P}_{\alpha-1} * \underset{\sim}{\mathbb{Q}}_{\alpha-1}$ if α is a successor ordinal and $\bigcup_{\beta<\alpha}\mathbb{P}_\beta$ otherwise. When $\mathfrak{q}$ is clear from the context, we let

- $\zeta_\beta(i)$ (for $\beta \in (0, \alpha)$ and $i < \omega_1$) be $N_{\beta,i} \cap \omega_1$;
- $\underset{\sim}{\eta}_i$ (for $i < \omega$) be the natural $\mathbb{Q}_0$-name for the ith element of ${}^\omega\omega$ added by $\mathbb{Q}_0$ (i.e., the union of the sequences $p(i)$ for p in the $\mathbb{Q}_0$-generic filter);
- $\underset{\sim}{h}_\beta$ (for $\beta \in (0, \alpha)$) be the natural $\mathbb{P}_{\beta+1}$-name for $\bigcup\{h^{p(\beta)} : p \in \underset{\sim}{G}_{\mathbb{P}_\alpha}\}$ (in the case where $\omega_1^{\mathbf{V}}$ is uncountable in $\mathbf{V}^{\mathbb{P}_\beta}$).

The following two claims show that the partial orders $\mathbb{Q}_\alpha$ ($\alpha \in [1, \lambda)$) force instances of the universal $\aleph_0$-s.i.u. The proof of Claim 1.9 is routine.

CLAIM 1.9. *If $\alpha \in [1, \lambda]$ and $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \underset{\sim}{\bar f}_\beta, \bar N_\beta : \beta < \alpha\rangle \in \mathfrak{K}_\alpha$ then every condition in $\mathbb{P}_1$ forces each of the following statements:*

(1) $\forall i\ \forall j\ (i < j < \omega_1) \Rightarrow (\underset{\sim}{\eta}_i, \underset{\sim}{\eta}_j \in {}^\omega\omega \wedge \underset{\sim}{\eta}_i \neq \underset{\sim}{\eta}_j)$.
(2) $\forall n \in \omega\ \forall \beta \in [1, \omega_1)\ \forall \varepsilon < \omega_1$

$$ {}^n\omega = \big\{\eta_j{\restriction}n : j \in \{N_{\beta,\omega\varepsilon+k} \cap \omega_1 : k \in (0, \omega)\}\big\}. $$

CLAIM 1.10. *If*

- $\alpha \in [1, \lambda]$;
- $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\beta, \underset{\sim}{\bar f}_\beta, \bar N_\beta : \beta < \alpha\rangle \in \mathfrak{K}_\alpha$;
- $\beta \in (0, \alpha)$;
- *$\mathbb{G}_\beta \subseteq \mathbb{P}_{\beta+1}$ is a $\mathbb{V}$-generic filter, $\mathbb{G}_1$ is its restriction to $\mathbb{P}_1$ and $\mathbb{G}_\beta$ is its restriction to $\mathbb{P}_\beta$;*
- *$\omega_1^{\mathbf{V}}$ is uncountable in $V[\mathbb{G}_\beta]$,*

then

(1) *$\underset{\sim}{h}_{\beta,\mathbb{G}_{\beta+1}}$ is a function from ${}^{\omega>}\omega$ to ω;*
(2) *in $\mathbf{V}[\mathbb{G}_{\beta+1}]$ the function $\underset{\sim}{h}_{\beta,\mathbb{G}_{\beta+1}}$ witnesses the universal $\aleph_0$-s.i.u. for the sequence $\langle \underset{\sim}{\eta}_{i,\mathbb{G}_1} : i < \omega_1\rangle$ with respect to $\underset{\sim}{\bar f}_{\beta,\mathbb{G}_\beta}$ and $\underset{\sim}{\bar\eta}^1_{\beta,\mathbb{G}_\beta}$.*

Proof. For each $i < \omega_1$, let $\eta_i = \underset{\sim}{\eta}_{i,\mathbb{G}_1}$, $\eta^1_i = \underset{\sim}{\eta}^1_{\beta,i,\mathbf{G}_\beta}$ and $f_i = \underset{\sim}{f}_{\beta,i,\mathbf{G}_\beta}$. We prove the first part first. Trivially $\underset{\sim}{h}_{\beta,\mathbb{G}_{\beta+1}}$ is a partial function from ${}^{\omega>}\omega$ to ω. Let $\nu \in {}^{\omega>}\omega$; we shall prove that, in $V[\mathbb{G}_\beta]$, $\Vdash_{\underset{\sim}{\mathbb{Q}}_{\beta,\mathbb{G}_\beta}} \nu \in \mathrm{Dom}(\underset{\sim}{h}_\beta)$. Working in $V[\mathbb{G}_\beta]$, fix $p \in \underset{\sim}{\mathbb{Q}}_{\beta,\mathbb{G}_\beta}$. We need to find a condition q satisfying $p \leq q$ in $\underset{\sim}{\mathbb{Q}}_{\beta,\mathbb{G}_\beta}$ and such that $\nu \in \mathrm{Dom}(h^q)$. If $\nu \in \mathrm{Dom}(h^p)$ we are done, so suppose otherwise. Let $n^* \geq \ell g(\nu)$ be such that $n^* > \sup\{\ell g(\rho) : \rho \in \mathrm{Dom}(h^p)\}$. By extending ν if necessary we may assume that $\ell g(\nu) = n^*$.

Our condition q will have $w^q = w^p$ and $g^q = g^p$. It remains to define h^q which will extend h^p. For each $i \in w^p$, let $\eta^2_i = \eta_{g^p(i)}$ We let

$$ \mathrm{Dom}(h^q) = \{\rho : \rho \trianglelefteq \nu \vee \rho \in \mathrm{Dom}(h^p) \vee \exists j \in w\ \exists \ell \in \{1, 2\}\ \rho \trianglelefteq \eta^\ell_j{\restriction}n^*\}. $$

If $\rho \in \mathrm{Dom}(h^q) \setminus \mathrm{Dom}(h^p)$ and ρ is not of the form $\eta_{g^p(j)}{\restriction}m$ for some $j \in w^p$ and $m \leq n^*$, then we let $h^q(\rho) = 0$. For the remaining sequences ρ, we define $h^q(\rho)$ by recursion on j, and for each j by recursion on m, letting

$$ h^q(\eta_{g^p(j)}{\restriction}m) = f_j\big(h^q{\restriction}\{\rho' : (\rho' \trianglelefteq \eta_j{\restriction}m) \vee (\rho' \triangleleft \eta_{g^p(j)}{\restriction}m)\}, \eta_{g^p(j)}{\restriction}m\big). $$

By part (e)(ϵ)(ii) of Definition 1.6 there are no conflicts in doing this. This completes the proof of the first part of the claim.

We now prove the second part. By the definition of the order on $\underset{\sim}{\mathbb{Q}}_\beta$, and the first part of the claim, it suffices to prove that, in $\mathbf{V}[\mathbb{G}_\beta]$, for every $i < \omega_1$ the set of $p \in \underset{\sim}{\mathbb{Q}}_{\beta,\mathbb{G}_\beta}$ with $i \in w^p$ is a dense subset of $\underset{\sim}{\mathbb{Q}}_\beta$. Fix

$i < \omega_1$ and $p \in \underset{\sim}{\mathbb{Q}}_{\beta,\mathbb{G}_\beta}$. Again, for each $k \in w^p$, we let $\eta^2_k = \eta_{g^p(k)}$. By part (2) of Claim 1.9, there exists a $j \in \{N_{\beta,\omega i+k} \cap \omega_1 : k \in (0,\omega)\}$ such that $\{\rho : \rho \triangleleft \eta_j \wedge \ell g(\rho) > 0\}$ is disjoint from $\mathrm{Dom}(h^p) \cup \{\rho : \rho \triangleleft \eta^2_k,\ k \in \mathrm{Dom}(g^p)\}$ (it is enough to choose a suitable value for $\eta_j(0)$).

Choose $n^* > 0$ such that $\ell g(\rho) < n^*$ for all $\rho \in \mathrm{Dom}(h^p)$. As in the proof of the first part we can find a function h^* from

$$\mathrm{Dom}(h^p) \cup \{\rho : \rho \trianglelefteq \eta^1_i{\restriction}n^* \vee \exists k \in w^p\ \exists \ell \in \{1,2\}\ \rho \trianglelefteq \eta^\ell_k{\restriction}n^*\}$$

to ω such that $h^p \subseteq h^*$ and h^* obeys f_k at $(\eta^1_k{\restriction}m, \eta_{g^p(k)}{\restriction}m)$ for all $k \in w^p$ and $m \in [1, n^*]$. Next choose $h^{**} \supseteq h^*$ with domain $\mathrm{Dom}(h^*) \cup \{\eta_j{\restriction}m : m \leq n^*\}$, as in the proof of the first part, so that h^{**} obeys f_i at $(\eta^1_i{\restriction}m, \eta_j{\restriction}m)$ for all $m \in [1, n^*]$.

Lastly, we let $g^q = g^p \cup \{(i,j)\}$, $w^q = w^p \cup \{i\}$ and $h^q = h^{**}$. Clearly, $p \leq q$ and $i \in w^q$, so we are done. $\blacksquare_{1.10}$

We make one additional observation about the successor stages of our iterations (Claim 1.12 below).

Definition 1.11. We let $\mathbb{Q}_*$ be the set of $p = (h^p, w^p, g^p)$ such that

- (α) h^p is a function whose domain is a finite subset of ${}^{\omega>}\omega$ closed under initial segments, and whose range is a subset of ω;
- (β) w^p is a finite subset of ω_1;
- (γ) g^p is an increasing function from w^p to ω_1 and such that $\alpha < g^p(\alpha)$ for all $\alpha \in w^p$.

We define an order $\leq_{\mathbb{Q}_*}$ on $\mathbb{Q}_*$ by setting $p \leq_{\mathbb{Q}_*} q$ if and only if

$$h^p \subseteq h^q \wedge w^p \subseteq w^q \wedge g^p \subseteq g^q.$$

The following claim is straightforward.

Claim 1.12. *For each $\beta \leq \lambda$, $\Vdash_{\mathbb{P}_\beta} \underset{\sim}{\mathbb{Q}}_\beta \subseteq \mathbf{V}$. Furthermore, in $V^{\mathbb{P}_\beta}$, for all $p, q \in \underset{\sim}{\mathbb{Q}}_\beta$ we have $p \leq_{\underset{\sim}{\mathbb{Q}}_\beta} q \Leftrightarrow p \leq_{\mathbb{Q}_*} q$.*

We now move to an analysis of the initial segments of our iterations.

Definition 1.13. Let $\mathfrak{K}^+_\alpha$ be the set of $\mathfrak{q} \in \mathfrak{K}_\alpha$ such that for every $\beta < \alpha$ the forcing notion $\mathbb{P}^{\mathfrak{q}}_\beta$ satisfies the c.c.c.

Claim 1.14. *To prove Theorem 1.5 it suffices to prove that for all $\alpha < \lambda$ and all $\mathfrak{q} \in \mathfrak{K}^+_\alpha$ the forcing notion $\mathbb{P}^{\mathfrak{q}}_\alpha$ satisfies the c.c.c.*

Proof. By bookkeeping, as $\lambda^{\aleph_1} = \lambda$, there is $\mathfrak{q} \in \mathfrak{K}_\lambda$ such that

- ($*$) for each $\beta < \lambda$, each $\mathbb{P}_\beta$-name $\underset{\sim}{\bar{f}}$ for a member of ${}^{\omega_1}(\mathscr{F}_{*,\aleph_0})$ and each $\mathbb{P}_\beta$-name $\underset{\sim}{\bar{\eta}}^1$ for a member of ${}^{\omega_1}({}^\omega\omega)$, there exists a $\gamma \in [\beta, \lambda)$ such that $\underset{\sim}{\bar{f}}^{\mathfrak{q}}_\gamma$ and $\underset{\sim}{\bar{\eta}}^1_\gamma$ are the natural reinterpretations of $\underset{\sim}{\bar{f}}$ and $\underset{\sim}{\bar{\eta}}^1$ respectively as $\mathbb{P}_\gamma$-names.

Then one sees by induction that for all $\alpha \in [1,\lambda]$, $\mathrm{Lim}(\mathfrak{q}\restriction\alpha)$ satisfies the c.c.c., noting that the c.c.c. is preserved by finite support iterations. $\blacksquare_{1.14}$

For the rest of the section we fix $\alpha \in [1,\lambda)$ and $\mathfrak{q} \in \mathfrak{K}^+_\alpha$. We aim to show that $\mathbb{P}^{\mathfrak{q}}_\alpha$ satisfies the c.c.c. (we will drop the superscripts $\mathfrak{q}$, however). By the definition of finite support iterations, for each $\beta \leq \lambda$, $\mathbb{P}_\beta$ is the set of finite functions p with domain contained in β such that for each $\gamma \in \mathrm{Dom}(p)$, $p(\gamma)$ is a $\mathbb{P}_\gamma$-name of a member of $\underset{\sim}{\mathbb{Q}}_\gamma$. We define some dense subsets of $\mathbb{P}_\alpha$.

Definition 1.15. Suppose that $\beta \leq \alpha$.

(1) We let D^0_β be the set of $p \in \mathbb{P}_\beta$ such that

- (a) $0 \in \mathrm{Dom}(p)$;
- (b) for each $\gamma \in \mathrm{Dom}(p)$, there exists a set $x \in \mathbf{V}$ such that $p(\gamma) = \check{x}$";
- (c) for all $\gamma \in \mathrm{Dom}(p) \setminus \{0\}$ and $i \in w^{p(\gamma)}$, if $j = g^p(i)$ then $j \in \mathrm{Dom}(p(0))$, and, letting n^* be the length of the largest initial segment of $p(0)(j)$ in $\mathrm{Dom}(h^{p(\gamma)})$,
  - (i) for some $\nu \in {}^{(n^*)}2 \cap \mathrm{Dom}(h^{p(\gamma)})$, $(p\restriction\gamma) \Vdash (\underset{\sim}{\eta}^1_{\gamma,i}\restriction n^*) = \check{\nu}$,
  - (ii) $n^* < \ell g(p(0)(j))$,
  - (iii) $p(0)(j)\restriction(n^*+1)$ is not equal to $p(0)(g^{p(\gamma)}(k))\restriction(n^*+1)$ for any $k \in w^{p(\gamma)}$ with $g^{p(\gamma)}(k) < j$.

(2) We let D^1_β be the set of finite functions p with $\mathrm{Dom}(p) \subseteq \beta$ and

- (a) $0 \in \mathrm{Dom}(p)$ and $p(0) \in \mathbb{Q}_0$;
- (b) for all $\gamma \in \mathrm{Dom}(p) \setminus \{0\}$:
  - $p(\gamma)$ is a triple $(h^{p(\gamma)}, w^{p(\gamma)}, g^{p(\gamma)})$ in $\mathbb{Q}_*$,
  - $\mathrm{Rang}(g^{p(\gamma)}) \subseteq \mathrm{Dom}(p(0))$.

(3) We define the order $\leq_{D^1_\beta}$ on D^1_β by setting $p \leq_{D^1_\beta} q$ if and only if

- (a) $\mathrm{Dom}(p) \subseteq \mathrm{Dom}(q)$;
- (b) $p(0) \leq_{\mathbb{Q}_0} q(0)$;
- (c) $\forall \gamma \in \mathrm{Dom}(p) \setminus \{0\}$ $p(\gamma) \leq_{\mathbb{Q}_*} q(\gamma)$.

(4) We let $D^{0,*}_\beta$ be the set of $p \in D^1_\beta$ such that for all $\gamma \in \mathrm{Dom}(p) \setminus \{0\}$ and all $i \in w^p$, if $j = g^{p(\gamma)}(i)$ then $j \in \mathrm{Dom}(p(0))$, and, letting $n^* = \ell g(p(0)(j))$, we have:

- (a) $p(0)(j) \in \mathrm{Dom}(h^{p(\gamma)})$;
- (b) there is $q \in D^0_\gamma \cap N_{\gamma+1,i+1}$ satisfying $q \leq_{D^1_\beta} p\restriction\gamma$ such that
  - (i) for some $\nu \in {}^{(n^*)}2 \cap \mathrm{Dom}(h^{p(\gamma)})$, $q \Vdash_{\mathbb{P}_\gamma} (\underset{\sim}{\eta}^1_{\gamma,i}\restriction n^*) = \check{\nu}$,
  - (ii) q forces that $h^{p(\gamma)}$ obeys $\underset{\sim}{f}_{\gamma,i}$ at $(\nu\restriction m, p(0)(j)\restriction m)$ for all $m \in (0, n^*]$.

(5) Given $p \in D^{0,*}_\beta$ and $n < \omega$, we let $p^{\langle n \rangle}$ be the following function:

(a) $\mathrm{Dom}(p^{\langle n \rangle}) = \mathrm{Dom}(p)$;
(b) $\forall \gamma \in \mathrm{Dom}(p) \setminus \{0\}\ p^{\langle n \rangle}(\gamma) = p(\gamma)$;
(c) $\mathrm{Dom}(p^{\langle n \rangle}(0)) = \mathrm{Dom}(p(0))$;
(d) $i \in \mathrm{Dom}(p(0)) \Rightarrow (p^{\langle n \rangle}(0))(i) = (p(0)(i))\hat{\ }\langle n + \mathrm{otp}(i \cap \mathrm{Dom}(p(0)))\rangle$.

(6) Given $\beta \leq \alpha$, $p \in D^1_\beta$ and a countable elementary submodel N of

$$(\mathscr{H}(\chi), \in, <^*_\chi),$$

we let $p \upharpoonright N$ denote the element q of D^1_β such that:

(a) $\mathrm{Dom}(q) = \mathrm{Dom}(p) \cap N$;
(b) $q(0) = p(0) \restriction (N \cap \omega_1)$;
(c) for all $\gamma \in \mathrm{Dom}(q) \setminus \{0\}$, $q(\gamma) = (h^{q(\gamma)}, w^{q(\gamma)}, g^{w(\gamma)})$ is defined by:
  (α) $h^{q(\gamma)} = h^{p(\gamma)}$,
  (β) $w^{q(\gamma)} = \{i \in w^{p(\gamma)} : g^{p(\gamma)}(i) \in N\}$,
  (γ) $g^{q(\gamma)} = g^{p(\gamma)} \restriction w^{q(\gamma)}$.

Remark 1.16. (1) Each member of D^0_β has a clear description, but the satisfaction of "$p \in D^0_\beta$" is complicated; it depends on the bookkeeping involved in the definition of $\mathfrak{q}$.

(2) The set D^0_β can be viewed as a subset of D^1_β (it is not literally a subset but we ignore this distinction in what follows, and above). Unlike with D^0_β, membership in D^1_β is simply defined.

(3) The set of $D^{0,*}_\beta$ consists of $p \in D^1_\beta$ which are in some sense close to being in D^0_β, needing only to be strengthened in coordinate 0 (see Claim 1.18 below). Clause 1.15(4)(b) is crucial; having such $q \in N_{\gamma+1,i+1}$ will hold densely often.

Claim 1.17 lays out some of the basic properties of the terms defined in Definition 1.15.

Claim 1.17. *Fix $\beta \leq \alpha$.*

(0) *For all $\gamma < \beta$,*
- $D^0_\gamma = \{p \in D^0_\beta : \mathrm{Dom}(p) \subseteq \gamma\} = \{p \restriction \gamma : p \in D^0_\beta\}$;
- $D^1_\gamma = \{p \in D^1_\beta : \mathrm{Dom}(p) \subseteq \gamma\} = \{p \restriction \gamma : p \in D^1_\beta\}$;
- $D^{0,*}_\gamma = \{p \in D^{0,*}_\beta : \mathrm{Dom}(p) \subseteq \gamma\} = \{p \restriction \gamma : p \in D^{0,*}_\beta\}$;
- $\leq_{D^1_\gamma} = \leq_{D^1_\beta} \restriction D^1_\gamma$.

(1) *D^0_β is a dense subset of $\mathbb{P}_\beta$.*

(2) *If $p \in D^1_\beta$, $v \subseteq \mathrm{Dom}(p)$ and $0 \in v$ then $p \restriction v \in D^1_\beta$.*

(3) *If $\beta \leq \alpha$, $p \in D^{0,*}_\beta$ and $i < \omega_1$ then $p \upharpoonright N_{\beta,i} \in D^{0,*}_\beta$ and $D^1_\beta \models$ "$p \upharpoonright N_{\beta,i} \leq p$".*

(4) *If* $p, q \in D^0_\beta$ *then* $p \le_{\mathbb{P}_\beta} q$ *iff* $p \le_{D^1_\beta} q$.
(5) $\le_{D^1_\beta}$ *is a partial order on* D^1_β.

Proof. Parts (0), (2), (4) and (5) follow immediately from the definitions, and part (1) is routine.

For part (3), let $p' = p \restriction N_{\beta,i}$. Clause (a) of Definition 1.15(4) should be clear; the main issue is clause (b). So assume that $\gamma_1 \in \mathrm{Dom}(p') \setminus \{0\}$ and $h^{p'(\gamma_1)}(i_1) = j_1$, hence $\gamma_1 \in N_{\beta,i} \cap \beta$ and $i_1, j_1 \in N_{\beta,i} \cap \omega_1$. Now as p satisfies clause (b) there is q as there; in particular, $q \in D^0_{\gamma_1} \cap N_{\gamma_1+1,i_1+1}$. But $\gamma_1 \in \mathrm{Dom}(p') \subseteq N_{\beta,i}$ and $i_1 \in N_{\beta,i}$ (as $g^{p'(\gamma_1)}(i_1) = j_i$) and $\langle N_{\gamma_1,\varepsilon} : \varepsilon < \omega_1 \rangle$ is in $N_{\beta,i}$, hence $N_{\gamma_1+1,i_1+1} \in N_{\beta,i}$, recalling Definition 1.6(e), so clearly $q \le_{D^1_\beta} p \restriction \gamma$ implies $q \le_{D^1_\beta} p' \restriction \gamma$. $\blacksquare_{1.17}$

Extending a $p \in D^{0,*}_\beta$ to an element of D^0_β (for some $\beta \le \alpha$) requires only extending the members of $p(0)$ to make them distinct. Claim 1.18 records one way of doing this.

Claim 1.18. *Suppose that* $\beta \le \alpha$ *and* $p \in D^{0,*}_\beta$. *For all but finitely many* $n \in \omega$, *we have* $p \le_{D^1_\beta} p^{\langle n \rangle} \in D^0_\beta$.

Proof. All that is needed is to ensure parts (c)(i) and (c)(ii) of Definition 1.15 of D^0_β. Choosing n larger than every element of the union of the ranges of the functions $p(0)(j)$ ($j \in \mathrm{Dom}(p(0))$) will do this. $\blacksquare_{1.18}$

Recall that our one remaining goal in this section is show that $\mathbb{P}_\alpha$ satisfies the c.c.c.

Definition 1.19. Conditions $p_1, p_2 \in D^{0,*}_\beta$ (for some $\beta \le \alpha$) are a Δ-*system pair* when:

(a) if $0 \in \mathrm{Dom}(p_1) \cap \mathrm{Dom}(p_2)$ then for all $i \in \mathrm{Dom}(p_1) \cap \mathrm{Dom}(p_2(0))$, $p_1(0)(i) = p_2(0)(i)$;
(b) $\mathrm{Dom}(p_1(0)) \cap \mathrm{Dom}(p_2(0))$ is an initial segment of both $\mathrm{Dom}(p_1(0))$ and $\mathrm{Dom}(p_2(0))$;
(c) for all $\gamma \in \mathrm{Dom}(p_1) \cap \mathrm{Dom}(p_2) \setminus \{0\}$,
  (α) $h^{p_1(\gamma)} = h^{p_2(\gamma)}$,
  (β) $w^{p_1} \cap w^{p_2}$ is an initial segment of both w^{p_1} and w^{p_2},
  (γ) for all $i \in w^{p_1} \cap w^{p_2}$, we have $g^{p_1(\gamma)}(i) = g^{p_2(\gamma)}(i)$.

Remark 1.20. If $\beta \le \alpha$ and p_1, p_2 in D^1_β are compatible, then they have a least upper bound in D^1_β, which we call $p_1 + p_2$. If compatible p_1, p_2 are in $D^{0,*}_\beta$, then so is $p_1 + p_2$. A If p_1, p_2 are a Δ-system pair then they are compatible.

Claim 1.21 below is used in the proof of Crucial Claim 1.23. For r and q as in the claim below, it may be that $r(0)(j) = q(0)(k)$ for some j, k not in $\mathrm{Dom}(r(0)) \cap \mathrm{Dom}(q(0))$. In this case $r + q$ is not in $D^0_{\beta_*}$.

Claim 1.21. *Suppose that*

- $\beta_* \leq \beta \leq \alpha$;
- $i < \omega_1$;
- $q, r \in D^{0,*}_{\beta_*}$;
- $r \in N_{\beta,i}$;
- $r \geq q \restriction N_{\beta,i}$.

Then r and q are compatible in D^1_β, and $r + q$ is in $D^{0,*}_{\beta_*}$.

Proof. For each $\gamma \in \mathrm{Dom}(q)$, $N_{\beta,i} \cap \omega_1 = N_{\gamma,(N_{\beta,i}\cap\omega_1)} \cap \omega_1$ is a limit ordinal, so for all $i \in N_{\beta,i} \cap \mathrm{Dom}(g^{q(\gamma)})$, we have $g^{q(\gamma)}(i) \in N_{\beta,i} \cap \omega_1$, by part (e)($\gamma$) of Definition 1.6. Given this, the compatibility of r and q is straightforward (since the only issue comes from part (c)(β) of Definition 1.15(b) of $q \restriction N_{\beta,i}$). $\blacksquare_{1.21}$

Definition 1.22. We say p is (β, δ)-*good* when:

(i) $p \in D^0_{\beta+1}$;
(ii) if $\beta \in \mathrm{Dom}(p) \setminus \{0\}$, $g^{p(\beta)}(i) = j$ and $\delta < j$ then for some n^* the demands in the definition of $D^{0,*}_\beta$ (Definition 1.15(4)) hold.

Crucial Claim 1.23. *For all $\beta \leq \alpha$ and all $p \in D^0_\beta$ there exist $q \in D^{0,*}_\beta$ such that $p \leq_{D^1_\beta} q$.*

Proof. We prove by induction primarily on β_* with $\beta_* + 1 \leq \alpha$ and secondarily on limit $\delta < \omega_1$ that (letting $\beta = \beta_* + 1$):

$\boxplus_{\beta_*,\delta}$ if $p \in D^0_\beta$ is (β_*, δ)-good then for some $q \in D^{0,*}_\beta$ we have

- $p \leq_{D^1_\beta} q$;
- $q(\beta_*) = p(\beta_*)$.

This is enough because our iteration is by finite support and because (whenever $\beta_* + 1 = \beta \leq \alpha$) every $p \in D^0_\beta$ is (β_*, δ)-good for all sufficiently large δ.

The case where $\beta_* = 0$ is trivial. Suppose then that $\beta_* > 0$ and that $\boxplus_{\beta_{**},\delta}$ holds for all $\beta_{**} < \beta$ and limit $\delta < \omega_1$. We now show $\boxplus_{\beta_*,\delta}$ for all limit $\delta < \omega_1$ by induction.

If $\delta = \omega$ then we apply the induction hypothesis for β_* to obtain a $q_0 \in D^{0,*}_{\beta_*}$ above $p \restriction \beta_*$. Then $q_0 \cup (\beta_*, p(\beta_*))$ is as desired, as $\zeta_{\beta_*}(i) > \omega$ for all $i < \omega_1$, so the assumption that p is (β_*, ω)-good implies that the requirements for $q_0 \cup (\beta_*, p(\beta_*))$ being in $D^{0,*}_\beta$ are satisfied in the case $\gamma = \beta_*$.

Fix then a countable limit ordinal δ such that $\boxplus_{\delta'}$ holds for all limit $\delta' < \delta$, and fix a (β_*, δ)-good $p \in D^0_\beta$. If there is no $i \in \mathrm{Dom}(g^{p(\beta_*)})$ with $g^{p(\beta_*)}(i) = \delta$ then p is (β_*, δ')-good for some limit $\delta' < \delta$ and we are done, so suppose otherwise. Let p_0 be p with i removed from $w^{p(\beta_*)}$ (and thus from $\mathrm{Dom}(g^{p(\beta_*)})$). Then p_0 is (β_*, δ')-good for some limit $\delta' < \delta$, so there exists a q_0 as in $\boxplus_{\delta'}$ relative to p_0. By Claim 1.17(1) there is $p_1 \in D^0_\beta$ above q_0, and again we may assume that $p_1(\beta_*) = q_0(\beta_*)$. As $\beta_* < \alpha$, $\mathbb{P}_{\beta_*}$ satisfies the c.c.c., so there exists an $r_0 \in \mathbb{P}_{\beta_*} \cap N_{\beta,i+1}$ above $q_0 \restriction N_{\beta,i+1}$ deciding enough of $\underset{\sim}{f}_{\beta_*,i}$ and $\underset{\sim}{\eta}^1_{\beta_*,i}$, in agreement with p_1, to satisfy Definition 1.15(4) with respect to β_* and i. We can strengthen r_0 inside $N_{\beta,i+1}$ to a condition $r_1 \in D^0_{\beta_*}$ (which is dense) and then again to a condition $r_2 \in D^{0,*}_{\beta_*}$ (by the induction hypothesis for β^*). Now let $q = q_0 + r_2$, which is in $D^{0,*}_{\beta_*}$, by Claim 1.21. Then $q \cup (\beta_*, p(\beta_*))$ is as desired. $\blacksquare_{1.23}$

Conclusion 1.24. *$\mathbb{P}_\alpha$ satisfies the c.c.c.*

Proof. Let p_ε ($\varepsilon < \omega_1$) be elements of $\mathbb{P}_\alpha$. By Claim 1.17(1), we may assume without loss of generality that each p_ε is in D^0_α. Applying Crucial Claim 1.23, choose for each $\varepsilon < \omega_1$ a $q_\varepsilon \in D^{0,*}_\alpha$ such that $q_\varepsilon \geq_{D^1_\alpha} p_\varepsilon$. Use the Δ-system lemma to find $\varepsilon < \zeta < \omega_1$ such that (q_ε, q_ζ) form a Δ-system pair, as in Definition 1.19. By Remark 1.20, q_ε and q_ζ have a common upper bound $q = q_\varepsilon + q_\zeta$ in $D^{0,*}_\alpha$.

By Claim 1.18 there is a $p \in D^0_\alpha$ such that $q \leq_{D^1_\alpha} p$. Then

$$p_\varepsilon \leq_{D^1_\alpha} q_\varepsilon \leq_{D^1_\alpha} q \leq_{D^1_\alpha} p$$

and

$$p_\zeta \leq_{D^1_\alpha} q_\zeta \leq_{D^1_\alpha} q \leq_{D^1_\alpha} p,$$

and hence by Claim 1.17(4) & (5), we have $\mathbb{P}_\alpha \models$ "$p_\varepsilon \leq p \wedge p_\zeta \leq p$", so we are done. $\blacksquare_{1.24}$

2. Conclusion. In this section we show that an $\aleph_0$-s.i.u. sequence witnesses the principle $\boxtimes_2$ from the introduction. We prove this in slightly greater generality, modifying Definition 1.3 by replacing ${}^\omega\omega$ with ${}^\omega\mu$ (for some cardinal μ) and making the obvious changes. For any set X, we let $\mathscr{F}_X = \{(h, \nu_1) :$ for some $n, \nu_0 \in {}^nX$ and $\nu_1 \in {}^{n+1}X$, h is a function from $\{\rho : \rho \triangleleft \nu_0 \vee \rho \triangleleft \nu_1\}$ to $X\}$ and define $\mathscr{F}_{*,X}$ and the X-s.i.u. analogously.

Theorem 2.1. *Let λ_1 and λ_2 be ordinals, and let μ be a cardinal. Suppose that*

(a) *$\eta^\ell_\alpha \in {}^\omega\mu$ for $\alpha < \lambda_\ell$ and $\bar\eta^\ell = \langle \eta^\ell_\alpha : \alpha < \lambda_\ell \rangle$ for $\ell = 1, 2$;*
(b) *$(\bar\eta^1, \bar\eta^2)$ has the μ-s.i.u.;*
(c) *G is a group of cardinality μ.*

Then

$\boxtimes^2_{\bar\eta,G}$ *given $c_\alpha \in {}^\omega G$ ($\alpha < \lambda_1$) we can find functions $h\colon {}^{\omega>}\mu \to G$ and $\zeta\colon \lambda_1 \to \lambda_2$ such that*

$$c_\alpha(n) = h(\eta^1_\alpha \restriction n) \cdot_G h(\eta^2_{\zeta(\alpha)} \restriction n)$$

for all $\alpha < \lambda_1$ and $n \in (0, \omega)$.

Proof. For notational simplicity, we suppose that μ is the set of elements of G. Given $c_\alpha \in {}^\omega\mu$ $(\alpha < \lambda_1)$ we define functions f_α $(\alpha < \lambda_1)$ as follows. If $n < \omega$, $\nu \in {}^n\mu$ and h is a function from

$$\{\rho : \rho \lhd \eta^1_\alpha \restriction (n+1) \vee \rho \lhd \nu\}$$

to μ, we let $f_\alpha(h)$ be the unique $x \in \mu$ such that

$$c_\alpha(n) = h(\eta^1_\alpha \restriction n) \cdot_G x.$$

Since $(\bar\eta^1, \bar\eta^2)$ has the μ-s.i.u., there exist $h : {}^{\omega>}\mu \to \mu$ and $\zeta : \lambda_1 \to \lambda_2$ such that

(∗) for all $\alpha < \lambda_1$ and every non-zero $n < \omega$, h obeys f_i at n, i.e.,

$$h(\eta^2_{\zeta(\alpha)} \restriction n) = f_\alpha(h \restriction \{\rho \lhd \eta^1_\alpha \restriction (n+1) \vee \rho \lhd \eta^2_{\zeta(\alpha)} \restriction n\}).$$

It follows that for all $\alpha < \lambda_1$ and all $n \in (0, \omega)$,

$$c_\alpha(n) = h(\eta^1_\alpha \restriction n) \cdot_G h(\eta^2_{\zeta(\alpha)} \restriction n),$$

as required. $\blacksquare_{2.1}$

COROLLARY 2.2. *If $\aleph_1$ has the $\aleph_0$-s.i.u., then $\boxtimes_2$ holds.*

Similarly, for any pair of cardinals μ, κ we can define $\mathscr{F}_{\mu,\kappa}$ to be the set of pairs (h, ν_1) such that for some $\nu_0, \nu_1 \in {}^{\kappa>}\mu$ of the same length, h is a function from $\{\rho : \rho \trianglelefteq \nu_0 \vee \rho \lhd \nu_1\}$ to ω, and define $\mathscr{F}_{*,\mu,\kappa}$, the (μ, κ)-s.i.u. and being a universal (μ, κ)-s.i.u. sequence accordingly.

The proof of the following result, a modification of the proof of Theorem 1.5, will appear elsewhere.

THEOREM 2.3. *Assume V satisfies $\kappa = \kappa^{<\kappa} = \mu$, $\theta = \kappa^+ < \lambda = \lambda^\theta$, $2^\kappa = \kappa^+ = 2^\kappa$. Then for some κ^+-c.c. and $(<\kappa)$-complete forcing notion $\mathbb{P}$ of cardinality λ we have $\Vdash_{\mathbb{P}}$ "there is a universal κ-s.i.u. sequence $\bar\eta \in {}^\theta({}^\kappa\kappa)$".*

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