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ON THE NUMBER OF MINIMAL MODELS

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Abstract. Answering a problem of Fuhrken we prove that for every κ, $1 \leq \kappa \leq \aleph_0$, there is a (countable) complete theory T, with no prime model, and exactly κ minimal models (up to isomorphism).

§0. Introduction. We restrict ourselves to countable complete theories T. M is *minimal* if it has no proper elementary submodel, so necessarily M is countable and let $m(T)$ be the number of such models. M is *weakly minimal* if $N < M \Rightarrow N \cong M$ so M is countable, and let $wm(T)$ be the number of such models.

The first to deal with this was Vaught who notices that prime models are weakly minimal. It was, probably, Engeler who raised the question of converse implication. Fuhrken [1] constructed theories with the following qualities:

(i) a theory T every model of which contains a minimal model but the theory has no prime model and $m(T) = 2^{\aleph_0}$.

(ii) a theory some models of which contain and some models of which do not contain a minimal model.

He also noticed that if T has a prime model, then it is either the unique minimal model or $m(T) = 0$, and posed the problem whether a theory having just one minimal model has a prime model. However he stated for some T_0, $m(T_0) = 3$, but as Morely (in ***Mathematical Reviews***) and Marcus [2] note, $m(T_0) = 2^{\aleph_0}$. Marcus [2] also dealt with $m(T)$ and proved that if (a) $M \models T$ is minimal, if it omits each $p \in \Gamma$ (Γ countable) *and* (b) T has no prime models, *then* $m(T) = 2^{\aleph_0}$. Remembering Morley [3] and analysing minimality, it is clear that $m(T) > \aleph_1 \Rightarrow m(T) = 2^{\aleph_0}$.

Problem. Is $m(T) = \aleph_1$ possible?

REMARKS.(1) The language used to construct the example is infinite. This fact is not essential and can be overcome in the following manner: Rename the P_η, Q_η's as R_n $(n < \omega)$. Let

$$|M^*_\kappa| = |M_\kappa| \cup \{\langle a, n, k\rangle : a \in |M_\kappa| \text{ and } M_\kappa \models R_n[a] \Rightarrow k < 2n,$$
$$M_\kappa \models \neg R_n[a] \Rightarrow k < 2n + 1\}.$$

Take F: $F(\langle a, n, k\rangle) = a$ and $F(a) = a$ for $a \in |M_\kappa|$.

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$$E = \{\langle\langle a, n, k\rangle, \langle a, n, k'\rangle\rangle : a, n, k, k'\},$$

$$M^*_\kappa = (|M^*_\kappa|, P, Q, F, G, H, H^{-1}, E),$$

$$T^*_\kappa = \mathrm{Th}(M^*_\kappa)$$

which is a complete theory, in a finite language, with exactly κ minimal models.

(2) The theory constructed below is not finitely axiomatizable. The question whether a finitely axiomatizable theory with the desired properties exists is open.

(3) The result was announced in [4] and notes were distributed in the Fall of 1975.

(4) In fact in our example also $wm(T) = \kappa$, the only change we need for this is in (E) (at the end). There define N^{**} as the submodel of N with universe $|N| - \{b' : b', b$ realize the same type in $N\}$.

(5) In fact we can get more. Clearly $0 \le m(T) \le wm(T) \le 2^{\aleph_0}$; now if $0 \le \kappa^* \le \kappa$, $\kappa, \kappa^* \in \omega \cup \{\aleph_0, 2^{\aleph_0}\}$ then for some T with no prime model $m(T) = \kappa^*$, $wm(T) = \kappa$. For this we should change the T in §1 as follows (we concentrate on the case $\kappa \le \aleph_0$). Let $B = \{\eta \in A :$ for some $l < \kappa^*$, $\nu \in E$, $(\forall m)[l(\nu) \le m < \omega \rightarrow F_\nu(\eta_l) \restriction m \in A]$ and $\eta = F_\nu(\eta_l) \restriction m, m \ge l(\nu)\}$ and we redefine Q by using B, instead of A.

Now in the definition of M we omit the Q_τ's and instead add for $n < \omega$

$$R_n = \{\langle\langle \eta, \nu_1, c_1\rangle, \langle \eta, \nu_2, c_2\rangle\rangle : \langle \eta, \nu_l, c_l\rangle \in Q \quad \text{for } l = 1, 2$$

and one of the following occurs:

(a) $(\eta \restriction n) \notin (A - B)$, $\langle \eta, \nu_1, c_1\rangle \in Q_n$ and $\langle \eta, \nu_2, c_2\rangle \in Q_n$,

(b) $(\eta \restriction n) \in A - B$, and $\langle \eta, \nu_1, c_1\rangle \in Q_n \Leftrightarrow (\eta_1, \nu_2, c_2\rangle \in Q_n\}$. Now N^l_κ $(l < \kappa^*)$ will be minimal and N^l_κ $(\kappa^* \le l < \kappa)$ will be weakly minimal.

§1.

THEOREM. *For every κ, $\aleph_0 \ge \kappa \ge 1$ there is a theory T_κ which has no prime model and has exactly κ minimal models.*

Notation. For any ordinal α, ${}^\alpha 2$ is the set of sequences η of zeroes and ones with a length of $l(\eta) = \alpha$. ${}^{\alpha >}2 = \bigcup_{\alpha > \beta} {}^\beta 2$. $\eta(i)$ will denote the ith element of η. $E_n = \{\eta \in {}^\omega 2 : \forall i \ge n, \ \eta(i) = 0\}$ and $E = \bigcup_{n<\omega} E_n$. There is a natural correspondence between E and ${}^{\omega >}2$, and we shall often not distinguish between them. For each $\nu \in E$ we define a function $F_\nu : {}^\omega 2 \rightarrow {}^\omega 2$ by $(F_\nu(\eta))(i) = \nu(i) + \eta(i)$ mod 2 for $\eta \in {}^\omega 2$, $i \in \omega$. $\lessdot$ is the relation of being an initial segment. For $\eta \in {}^{\omega >}2$, $P_\eta = \{\tau \in {}^\omega 2 : \eta \lessdot \tau\}$. We shall deal with models related to

$$M = ({}^\omega 2, \ldots, F_\nu, \ldots, P_\eta, \ldots)_{\nu, \eta \in {}^{\omega >}2}.$$

It is not difficult to verify that $T = \mathrm{Th}(M)$ satisfies:

(i) It has elimination of quantifiers. (This fact is, essentially, proven in the proof given below.)

(ii) Each of its elements generates an elementary submodel which is minimal, and in this method $2^{\aleph_0}$ nonisomorphic minimal models to T can be obtained. For every $l < \kappa$, choose $\eta_l \in {}^\omega 2$ such that $\nu \in E$, $l < m < \kappa \Rightarrow F_\nu(\eta_l) \ne \eta_m$. Next choose a set $A \subseteq {}^{\omega >}2$ such that:

(i) each $\eta \in A$ has exactly one successor $\eta\hat{}\langle i\rangle$ in A;

(ii) for every $l < \kappa$, $\nu \in {}^{\omega>}2$ for every big enough n, $F_\nu(\eta_l) \restriction n \in A$;

(iii) for any $\tau \in {}^{\omega}2$ if $(\exists n)(\forall i > n)[\tau \restriction i \in A]$ then $\tau = F_\nu(\eta_l)$ for some $\nu \in E$, $l < \kappa$.

Let

$$Q = \{\langle \eta, \nu, c\rangle : \eta \in {}^{\omega}2,\ \nu \in {}^{\omega}2,\ c \in Z \text{ (the integers)},$$
$$(\forall i < \omega)[\eta \restriction (i+1) \in A \Rightarrow \nu(i) = 0]\}.$$

$P = {}^{\omega}2$. Let $G: Q \to P$ be defined by $G(\langle \eta, \nu, c\rangle) = \eta$, $H: Q \to Q$ be defined by $H(\langle \eta, \nu, c\rangle) = \langle \eta, \nu, c+1\rangle$. For any $\tau \in {}^{\omega>}2$, $Q_\tau = \{\langle \eta, \nu, c\rangle \in Q : \tau \lessdot \nu\}$. Define

$$M_\kappa = (P \cup Q, P, Q, G, H, H^{-1}, \ldots, F_\eta, \ldots, P_\eta, \ldots, Q_\eta, \ldots)_{\eta \in {}^{\omega>}2} \quad \text{and}$$
$$M^n_\kappa = (P \cup Q, P, Q, G, H, H^{-1}, \ldots, F_\eta, \ldots, P_\eta, \ldots, Q_\eta, \ldots)_{\eta \in {}^{n\geq}2}.$$

(Technically we assume G, H, H^{-1} are defined on P as the identity.)

$T_\kappa = \mathrm{Th}(M_\kappa)$ is the desired theory, and let $T^n_\kappa = \mathrm{Th}(M^n_\kappa)$. Let N^l_κ be the submodel of M_κ such that

$$P(N^l_\kappa) = \{\eta \in {}^{\omega}2 : \eta = F_\nu(\eta_l),\ \nu \in E\},$$
$$Q(N^l_\kappa) = \{q \in Q(M_\kappa) : G(q) \in P(N^l_\kappa)\}.$$

The proof of the adequacy of T_κ will be established by proving

(A) T_κ has elimination of quantifiers and $N^l_\kappa < M_\kappa$.

(B) Each N^l_κ is minimal.

(C) Each minimal model of T_κ is isomorphic to some N^l_κ.

(D) The models N^l_κ $(l < \kappa)$ are pairwise nonisomorphic.

(E) T_κ has no prime models.

We shall use the following basic fact: Let S be a theory and let $\{\theta_j(x) : j < m\}$ be formulas in the appropriate language such that $S \vdash$ "the family $\{\theta_j(x) : j < m\}$ is a partition" then, in order to eliminate the main quantifier (in S) from a formula $(\exists x)\psi(x, \bar{y})$ it suffices to eliminate it from each of the formulas $(\exists x)(\psi(x, \bar{y}) \wedge \theta_j(x))$, $j < m$.

Remarks. From T^n_κ it follows that for each term $t(x)$ exactly one of the two following possibilities holds:

(i) $P(t(x))$ and for some $\nu \in E$, $t(x) = F_\nu(x)$ or $t(x) = F_\nu(G(x))$ depending on whether $P(x)$ or $Q(x)$.

(ii) $Q(t(x))$ and $Q(x)$ and, for some $c \in Z$, $t(x) = H^c(x)$.

(A) T_κ has elimination of quantifiers.

It suffices to prove that for each n, $T^n_\kappa = \mathrm{Th}(N^n_\kappa)$ has elimination of quantifiers. The elimination is proved using the following facts about T^n_κ.

(1) $\{P, Q\}$ is a partition, $\{P_\eta : \eta \in {}^n2\}$ is a partition of P, $\{Q_\tau : \tau \in {}^n2\}$, $\{P_\tau G : \tau \in {}^n2\}$ are partitions of Q. $[P_\tau G$ is $P_\tau(G(x))]$.

(2) $P_\nu(F_\tau(x)) \leftrightarrow P_{\nu+\tau(\mathrm{mod}\ 2)}(x)$; $F_\nu F_\tau(x) = F_{\nu+\tau(\mathrm{mod}\ 2)}(x)$; $HH^{-1}(x) = H^{-1}H(x) = x$; if $\nu \lessdot \tau$ then $P_\tau(x) \to P_\nu(x)$.

(3) $P(x) \to [H(x) = x \wedge G(x) = x]$,

$$Q(x) \to [Q(H(x)) \wedge P(G(x)) \wedge [G(H^c(x)) = G(x)]$$
$$\wedge [Q_\tau(x) \leftrightarrow Q_\tau(H^c(x))] \wedge H^c(x) \neq x]$$

for any $c \in Z$, $c \neq 0$.

(4) For any $\eta \in {}^{n\geq}2$, $k \in \omega$, $(\forall y_0 \cdots y_{k-1})(\exists x)(P_\eta(x) \wedge \bigwedge_{i<k} x \neq y_i)$. For any η, $\tau \in E$, $k \in \omega$,

$$(\forall y_0 \cdots y_{k-1})(\exists x)[Q_\eta(y_0) \wedge P_\tau(G(y_0)) \to Q_\eta(x) \wedge P_\tau(G(x)) \wedge \bigwedge_{i<k} x \neq y_i].$$

(5) $P(x) \to x = F_\nu(x)$ iff ν is constantly zero.

(6) For any τ_1, τ_2 either $\neg(\exists y)[Q(y) \wedge Q_{\tau_1}(y) \wedge P_{\tau_2}(G(y))]$ or

$$(\forall x)(\exists y)(P_{\tau_2}(x) \to Q(y) \wedge Q_{\tau_1}(y) \wedge G(y) = x)$$

(depending on τ_1 and τ_2, expecially depending on whether for any $i < l(\tau_1), l(\tau_2)$: $\tau_1(i) = 1 \to \tau_2(i+1) \notin A$). By transforming formulas into disjunctive normal form it suffices to eliminate the quantifier from $\exists x_0 \bigwedge_{i<k} \varphi_i$ where the φ_i are either atomic or negations of atomic formulas. Evidently, we can assume x_0 appears in each φ_i. Using the basic fact and (1) it suffices to consider the following two cases:

(I) $\varphi_0 = P(x_0)$.

(II) $\varphi_0 = Q(x_0)$.

Case (I). $\varphi_0 = P(x_0)$.

Using (1) we can assume $\varphi_1 = P_\eta(x_0)$ for some $\eta \in {}^n2$. Using (2) it is easy to see that all the φ_i of the form $P_\nu(t(x_0))$ are decided by φ_1. Hence the only case left to check is when the φ_i $(i > 1)$ are either equalities or inequalities.Using (2) and (3) we can assume that each φ_i $(i > 1)$ is either $x_0 = t_i$ or $x_0 \neq t_i$, for some term t_i. If for some equality $x_0 = t_i$, x_0 does not appear in t_i, then this equation can be used to eliminate x_0. On the other hand, equalities of the form $x_0 \neq t(x_0)$ are, using (2), (3), (5) and the fact that all functions are 1-place, either true or false, independent of x_0. The inequalities $x_0 \neq t_i$ are taken care of by (4).

Case (II). $\varphi_0 = Q(x_0)$.

Using (1) we can assume $\varphi_1 = Q_{\tau_1}(x_0)$ and $\varphi_2 = P_{\tau_2}(G(x_0))$ for some $\tau_1, \tau_2 \in {}^n2$. Using (3) it follows that $\varphi_1 \wedge \varphi_2$ decides all formulas of the form $\pm P_\tau(t(x_0))$ (i.e. the formula or its negation) and $\pm Q_\tau(t(x_0))$, we assume therefore that for $i \geq 3$, φ_i is an equality or inequality, i.e. of the form $\pm(t_i^1(x_0) = t_i^2(y))$. For each $i \geq 3$, φ_i can be either equality or inequality. A formula of the form $\pm(t_i^1(x_0) = t_i^2(x_0))$ is either true or false, independent of x_0 and should, therefore, not be considered. We are, thus, left with formulas of the form $\pm(t_i^1(x_0) = t_i^2(y))$ with $y \neq x_0$.

(a) $t_i^1(x_0) = t_i^2(y)$. If t_i^1 does not include G then the formula is of the form $H^c(x_0) = t_i^2(y)$ or equivalently $x_0 = H^{-c}t_i^2(y)$ which can be used to eliminate x_0. Alternatively, if t_i^1 does include G, the formula is (using (3)) $F_\nu(G(x_0)) = t_i^2(y)$ or equivalently $G(x_0) = F_\nu(t_i^2(y))$, i.e. of the form $G(x_0) = t_i^3(y)$.

(b) $t_i^1(x_0) \neq t_i^2(y))$. If t_i^1 does not include G, then the formula is equivalent to $x_0 \neq t_i^3(y)$. Alternatively, if t_i^1 does include G, it is equivalent to $G(x_0) \neq t_i^3(y)$.

We thus have to show that the main quantifier in

$$\psi = \exists x_0 \Big[Q(x_0) \wedge Q_{\tau_1}(x_0) \wedge P_{\tau_2}(G(x_0)) \wedge \bigwedge_{3 \le i < j_1} G(x_0) = t_i^3(y_i)$$

$$\wedge \bigwedge_{j_1 \le i \le j_2} x_0 \ne t_i^3(y_i) \bigwedge_{j_2 \le i < k} G(x_0) \ne t_i^3(y_i) \Big]$$

can be eliminated. This follows from (6) and (4) (considering separately whether $j_1 = 2$ or not). It is easy to check N_κ^l satisfies (1)–(6) hence, by the elimination of quantifiers $N_\kappa^l < M_\kappa$.

We shall henceforth refer to (1)–(6) as S_κ^n and $\bigcup_n S_\kappa^n = S_\kappa$.

(B) Each N_κ^l is minimal.

Let $N < N_\kappa^l$ and choose any $a \in P(N)$. Then $a = F_\nu(\eta_l)$ for some $\nu \in E$; therefore, $\eta_l = F_\nu(a)$. Hence $\{F_\tau(\eta_l): \tau \in E\} \subset P(N)$; thus $P(N) = P(N_\kappa^l)$. What remains to prove is $Q(N) = Q(N_\kappa^l)$. Take any $b = \langle \eta, \nu, c \rangle \in Q$ such that $\eta \in P(N_\kappa^l)$. By the defining property of A, $(\exists n)(\forall m \ge n)[\eta \restriction m \in A]$. Thus, by the defining property of Q, there is $n = n_\eta$ such that for any $m \ge n$, $\nu(m) = 0$. Define $\tau \in E$ by $\tau = \nu \restriction n$. Since $N_\kappa^l \models \exists x (Q(x) \wedge Q_\tau(x) \wedge G(x) = \eta)$ then this formula is satisfied in N by some $b' = \langle \eta, \nu', c' \rangle \in Q$. It is obvious that $\nu' = \nu$. By a proper choice of $d \in Z$ we get $H^d(b') = b \in Q(N)$.

(C) Each minimal model N of T_κ is isomorphic to some N_κ^l, $l < \kappa$.

Choose $a \in P(N)$. Let $\eta \in {}^\omega 2$ be such that for any $n \in \omega$, $N \models P_{\eta \restriction n}(a)$. We shall show $(\exists n)(\forall m \ge n)[\eta \restriction m \in A]$ thus proving that $\eta = F_\nu(\eta_l)$ for some $l < \kappa$, $\nu \in E$. If it were not so, then $(\forall n)(\exists m \ge n)[\eta \restriction m \notin A]$ choose $b \in Q(N)$ s.t. $G(b) = a$ and let N^{**} be a submodel of N with universe $N - \{H^c(b): c \in Z\}$. Since $N^{**} \models S_k$ (notice that (6) is proved using the above-assumed property of η), $N^{**} < M$ contradicting minimality. It follows that $\eta = F_\nu(\eta_l)$. Using N^{**} as above, clearly

$$(\forall a, b \in Q^N)(G(a) = G(b) \rightarrow (\exists c \in Z) H^c(a) = b).$$

So there is an isomorphism $f: N \rightarrow N_\kappa^l$, $f(F_\tau(a)) = F_\tau F_\nu(\eta_l)$.

(D) The models N_κ^l $(l < \kappa)$ are pairwise nonisomorphic since for $i < j < \kappa$, $\nu \in E$: $\{\tau \in E: P_\tau(\eta_i)\} \ne \{\tau \in E: P_\tau(F_\nu(\eta_j))\}$.

(E) For $\kappa > 1$, T_κ has two nonisomorphic minimal models and therefore no prime model.

For $\kappa = 1$—the unique minimal model is not prime since each $\eta \in P(N_1^0)$ satisfies no formula which decides its type. The proof is as follows: since there is elimination of quantifiers it suffices to show that for each $\tau \in E$ the formula $P_\tau(x)$ does not determine the type of x and this follows from the fact that there is more than one $\eta \in P(N_1^0)$ satisfying $P_\tau(\eta)$.

So there is an embedding $f: N_\kappa^l \rightarrow N$, such that $f(F_\tau(\eta_l)) = F_\tau F_\nu(a)$. Clearly f is elementary, thus N's minimality $N \cong N_\kappa^l$.

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