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SATURATED NULL AND MEAGER IDEAL

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Abstract. We prove that the meager ideal and the null ideal could both be somewhere $\aleph_1$-saturated.

1. Introduction

In [2], starting with a measurable cardinal, Komjáth constructed a model of Zermelo–Fraenkel with choice (ZFC) in which there is a nonmeager set of reals which cannot be partitioned into uncountably many nonmeager sets. In [3], starting with a measurable cardinal, Shelah constructed a model of ZFC in which there is a nonnull set of reals which cannot be partitioned into uncountably many nonnull sets. Our main result is that the two consistency results can be combined.

Theorem 1.1. *Suppose that there is a measurable cardinal. Then there is a countable chain condition (ccc) forcing $\mathbb{P}$ such that in $V^{\mathbb{P}}$, there is a set $X \subseteq \mathbb{R}$ such that X is neither null nor meager, X cannot be partitioned into uncountably many nonnull sets, and X cannot be partitioned into uncountably many nonmeager sets.*

Let us briefly point out why other boolean combinations are also possible. Ulam showed that if there is an $\aleph_1$-saturated sigma ideal $\mathcal{I}$ on some set X such that $\mathcal{I}$ contains every countable set, then there is a weakly inaccessible cardinal below $|X|$. It follows that, under the continuum hypothesis, every nonmeager (resp., nonnull) set of reals can be partitioned into uncountably many nonmeager (resp., nonnull) sets.

Suppose that X is a nonmeager set of reals that cannot be partitioned into uncountably many nonmeager sets. Let $\mathbb{P}$ be the forcing for adding $\aleph_1$ Cohen reals. Then in $V^{\mathbb{P}}$, X continues to be nonmeager and it is easy to check that it still cannot be partitioned into uncountably many nonmeager sets. Also, in $V^{\mathbb{P}}$, the real line can be covered by $\aleph_1$ null sets. It follows that every nonnull set in $V^{\mathbb{P}}$ can be partitioned into uncountably many nonnull sets.

Similarly, if X is a nonnull set of reals that cannot be partitioned into uncountably many nonnull sets, then adding $\aleph_1$ random reals gives us a model where X remains nonnull, it cannot be partitioned into uncountably many nonnull sets, and every nonmeager set can be partitioned into uncountably many nonmeager sets.

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Notation 1.2. A subset $W \subseteq 2^\omega$ is fat if for every clopen set C, either $W \cap C = \emptyset$ or $\mu(W \cap C) > 0$. A subtree $T \subseteq {}^{<\omega}2$ is fat if $[T] = \{x \in 2^\omega : (\forall n < \omega)(x \upharpoonright n \in T)\}$ is fat. For a clopen subset $C \subseteq 2^\omega$, define $\mathrm{supp}(C)$ to be the smallest (finite) set F such that $(\forall x, y \in 2^\omega)((x \upharpoonright F = y \upharpoonright F) \implies (x \in C \iff y \in C))$. Random denotes the random real forcing. Note that $\{[T] : T \subseteq {}^{<\omega}2$ is a fat tree$\}$ is dense in Random. Cohen denotes Cohen forcing. Its conditions are members of ${}^{<\omega}\omega$ ordered by end extension. In forcing, we use the convention that a larger condition is the stronger one; so $p \geq q$ means that p extends q. If $\mathbb{P}, \mathbb{Q}$ are forcing notions and $\mathbb{Q} \subseteq \mathbb{P}$, we write $\mathbb{Q} \lessdot \mathbb{P}$ if every maximal antichain in $\mathbb{Q}$ is also a maximal antichain in $\mathbb{P}$. For $x, y \in \omega^\omega$, define $x \oplus y \in \omega^\omega$ by $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n+1) = y(n)$.

2. Eventually different forcing

Suppose that $\bar{Y} = \langle y_i : i < \theta \rangle$, where each $y_i \in \omega^\omega$. Define a forcing notion $\mathbb{E} = \mathbb{E}(\bar{Y})$ as follows. $p \in \mathbb{E}$ iff $p = (\sigma_p, F_p) = (\sigma, F)$, where $\sigma \in {}^{<\omega}\omega$ and $F \in [\theta]^{<\aleph_0}$. For $p, q \in \mathbb{E}$, $p \leq q$ iff $\sigma_p \preceq \sigma_q$, $F_p \subseteq F_q$ and for every $k \in [|\sigma_p|, |\sigma_q|)$, for every $i \in F_p$, $\sigma_q(k) \neq y_i(k)$. It is easy to see that $\mathbb{E}$ is a sigma-centered forcing that makes the set $\{y_i : i < \theta\}$ meager since it adds the real $\tau_{\mathbb{E}} = \bigcup\{\sigma_p : p \in G_{\mathbb{E}}\}$ which satisfies $(\forall i < \theta)(\forall^\infty k)(y_i(k) \neq \tau_{\mathbb{E}}(k))$. The following lemma is well known. We include a short proof for completeness.

Lemma 2.1. *Let $\bar{Y}$, $\mathbb{E} = \mathbb{E}(\bar{Y})$, and $\tau_{\mathbb{E}}$ be as above. Let $\mathring{x} \in 2^\omega \cap V^{\mathbb{E}}$. Then there is a Borel function $B : \omega^\omega \to 2^\omega$ such that $\Vdash_{\mathbb{E}} B(\tau_{\mathbb{E}}) = \mathring{x}$.*

Proof of Lemma 2.1. For each $n < \omega$ and $i < 2$, choose $\mathcal{A}_{i,n} \subseteq \mathbb{E}$ such that for every $p \in \mathcal{A}_{i,n}$, $p \Vdash \mathring{x}(n) = i$ and $\mathcal{A}_{0,n} \cup \mathcal{A}_{1,n}$ is a maximal antichain in $\mathbb{E}$. Define $B : \omega^\omega \to 2^\omega$ as follows. Given $z \in \omega^\omega$ and $n < \omega$, look for unique $i < 2$ and $(\sigma, F) \in \mathcal{A}_{i,n}$ such that $\sigma \subseteq z$ and for every $k \in [|\sigma|, \omega)$ and $\gamma \in F$, $z(k) \neq y_\gamma(k)$, and define $B(z)(n) = i$. If there are no such unique i and (σ, F), define $B(z)(n) = 0$. Note that if $(\sigma_0, F_0), (\sigma_1, F_1) \in \mathcal{A}_{0,n} \cup \mathcal{A}_{1,n}$ are incompatible, then either σ_0 and σ_1 are incomparable or (say) $\sigma_0 \prec \sigma_1$ and for some $k \in [|\sigma_0|, |\sigma_1|)$ and $\gamma \in F_0$, we have $\sigma_1(k) \neq y_\gamma(k)$. Hence $\Vdash_{\mathbb{E}} B(\tau_{\mathbb{E}}) = \mathring{x}$. $\square$

Note that if $Y = \emptyset$, then $\mathbb{E}(Y)$ is Cohen forcing. We can think of $\mathbb{E}(Y)$ as adding a "partial Cohen" real with memory Y which becomes decreasingly Cohen-like with increasing memory.

3. Background ideas

Let us describe some of the ideas that led to the model witnessing Theorem 1.1. By a result of Solovay, we must start with a measurable cardinal κ. Let $\mathcal{I}$ be a witnessing normal prime ideal. We are going to construct a ccc forcing $\mathbb{P}$ that adds two sets of reals $X = \{x_\alpha : \alpha < \kappa\}$ and $Y = \{y_\alpha : \alpha < \kappa\}$ such that (A) and (B) below hold. Let $\mathcal{J} = \{W \subseteq \kappa : (\exists W' \in \mathcal{I})(W \subseteq W')\}$ be the ideal generated by $\mathcal{I}$ in $V^{\mathbb{P}}$. Since $\mathbb{P}$ is ccc, $\mathcal{J}$ is an $\aleph_1$-saturated κ-additive ideal on κ. We would like to have for every $W \subseteq \kappa$

(A) $$W \in \mathcal{J} \iff \{x_\alpha : \alpha \in W\} \text{ is meager,}$$

(B) $$W \in \mathcal{J} \iff \{y_\alpha : \alpha \in W\} \text{ is null,}$$

This would suffice for Theorem 1.1 since if N is a dense G_δ null subset of $\mathbb{R}$, then the set $(N \cap X) \cup ((\mathbb{R} \setminus N) \cap Y)$ is both nonmeager and nonnull, and it cannot be partitioned into uncountably many nonmeager or nonnull sets.

In [2], Komjáth starts by adding κ Cohen reals $X = \{x_\alpha : \alpha < \kappa\}$. So every meager subset of X is currently countable. Using a finite support product, he then makes every subset of X of the form $\{x_\alpha : \alpha \in W\}$ (where $W \in \mathcal{I}$) meager. He finally invokes the properties of product forcing to show that X remains nonmeager in the final model. Note that the analogous construction fails for the null ideal: If we start by adding a set Y of κ random reals and then, using a finite support iteration (for ccc), add null sets containing some subsets of Y, then we inevitably add Cohen reals at stages of cofinality ω which make all of Y null. To get around this difficulty, Shelah [3] proceeds as follows. Let $\langle X_\alpha : \alpha < \lambda\rangle$ be a list where each member of $\mathcal{I}$ occurs $\lambda = 2^\kappa$ times. First add λ Cohen reals $\langle c_\alpha : \alpha < \lambda\rangle$. Each c_α codes a null G_δ-set N_α in a natural way. We now do a finite support iteration of length κ adding a "partial random" y_ξ at stage $\xi < \kappa$ whose memory is $V_1 = V[\langle c_\alpha : \xi \notin X_\alpha\rangle][\langle y_\eta : \eta < \xi\rangle]$. This means that y_ξ is Random^{V_1}-generic. The expectation is that if $\xi \in X_\alpha$, then $y_\xi \in N_\alpha$ (although showing this requires some work) and that $Y = \{y_\xi : \xi < \kappa\}$ would be the desired set of reals in the final model.

To combine these two constructions via a single forcing, we first reverse Komjáth construction as follows. Let $\langle X_\alpha : \alpha < \lambda\rangle$ be the list mentioned above. First add λ Cohen reals $\langle c_\alpha : \alpha < \lambda\rangle$. Each c_α codes an F_σ-meager set—namely, $M_\alpha = \{y \in \omega^\omega : (\forall^\infty k)(y(k) \neq c_\alpha(k))\}$. Now do a finite support iteration of length κ adding a partial Cohen real x_ξ at stage $\xi < \kappa$ with memory $C_\xi = \{c_\alpha : \xi \in X_\alpha\}$. This means that x_ξ is $\mathbb{E}(C_\xi)$-generic. Note that if $\xi \in X_\alpha$, then $x_\xi \in M_\alpha$. It is not difficult to check that $X = \{x_\xi : \xi < \kappa\}$ is a nonmeager set on which the meager ideal is $\aleph_1$-saturated.

The next section begins by describing iterations $\bar{\mathbb{P}}_\lambda = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \lambda + \kappa\rangle$ for $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ (where $\lambda_0 = 2^\kappa$) which combine partial Cohen and partial random reals. The reason behind considering $\bar{\mathbb{P}}_\lambda$ for various λ's and not just for $\lambda = \lambda_0$ will become clear during the proof of Lemma 7.9, where we use automorphisms of $\mathbb{P}_{\lambda+\kappa}$ for $\lambda > \lambda_0$ to construct certain finitely additive measures on $\mathcal{P}(\omega) \cap V^{\mathbb{P}_{\lambda_0+\xi}}$ for $\xi < \kappa$.

4. Forcing

Suppose κ is measurable and $\mathcal{I}$ is a normal prime ideal on κ. Put $\lambda_0 = 2^\kappa$. For $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, define the following.

(1) $\langle X_\alpha : \alpha < \lambda_0^{+\omega}\rangle$ is a sequence of members of $\mathcal{I}$.
(2) For every $n < \omega$ and $X \in \mathcal{I}$, $|\{\alpha < \lambda_0^{+n} : X_\alpha = X\}| = \lambda_0^{+n}$.
(3) For $\xi < \kappa$, $C^\lambda_{\lambda+\xi} = C_{\lambda+\xi} = \{\alpha < \lambda : \xi \in X_\alpha\}$. This is the memory of the partial Cohen real to be added at stage $\lambda + \xi$ (see item (7)).
(4) For $\xi < \kappa$, $A^\lambda_{\lambda+\xi} = A_{\lambda+\xi} = \{\alpha < \lambda : \xi \notin X_\alpha\} \cup [\lambda, \lambda + \xi)$. This is the memory of the partial random real to be added at stage $\lambda + \xi$ (see item (7)).
(5) $\bar{\mathbb{P}}_\lambda = \langle \mathbb{P}_{\lambda,\alpha}, \mathbb{Q}_{\lambda,\alpha} : \alpha < \lambda + \kappa\rangle$ is a finite support iteration with limit $\mathbb{P}_{\lambda,\lambda+\kappa}$. In the contexts where the value of λ is constant, we drop the λ in the subscript and just write $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$.
(6) For $\alpha < \lambda$, $\mathbb{Q}_\alpha = \mathsf{Cohen}$ with generic real $\tau_\alpha \in \omega^\omega$.

(7) For $\xi < \kappa$, $\mathbb{Q}_{\lambda+\xi} = \mathbb{Q}^1_{\lambda+\xi} \times \mathbb{Q}^2_{\lambda+\xi}$, where $\mathbb{Q}^1_{\lambda+\xi} = (\mathsf{Random})^{V[\langle \tau_i : i \in A_{\lambda+\xi}\rangle]}$ with generic partial random $\tau^1_{\lambda+\xi} \in 2^\omega$ and $\mathbb{Q}^2_{\lambda+\xi} = \mathbb{E}(\langle \tau_\alpha : \alpha \in C_{\lambda+\xi}\rangle)$ with generic $\tau^2_{\lambda+\xi} \in \omega^\omega$. Let $\tau_{\lambda+\xi} = \tau^1_{\lambda+\xi} \oplus \tau^2_{\lambda+\xi}$.

(8) Define $\mathbb{P} = \mathbb{P}_{\lambda_0, \lambda_0+\kappa}$.

The model for Theorem 1.1 will be $V^{\mathbb{P}}$. The verification of this will conclude with the proof of Lemma 7.9. In the remainder of this section, we establish some basic facts about these iterations.

The following claim is easily proved by induction on $\xi \leq \kappa$ using Lemma 2.1 and the standard properties of Cohen and random forcings.

Claim 4.1. For every $\xi \leq \kappa$, $\mathring{x} \in 2^\omega \cap V^{\mathbb{P}_{\lambda+\xi}}$, there are a Borel function $B : \omega^\omega \to 2^\omega$ and $\langle (n_k, \gamma_k) : k < \omega\rangle$ such that every $\gamma_k < \lambda + \xi$ and $n_k < \omega$, and that $\Vdash_{\mathbb{P}} \mathring{x} = B(\langle \tau_{\gamma_k}(n_k) : k < \omega\rangle)$.

Definition 4.2. Let $\mathbb{P}'_{\lambda,\lambda+\kappa} = \mathbb{P}'_{\lambda+\kappa}$ be the set of conditions $p \in \mathbb{P}_{\lambda+\kappa}$ satisfying the following requirements.

(a) For each $\alpha \in \lambda \cap \mathrm{dom}(p)$, $p(\alpha) = \sigma_\alpha \in {}^{<\omega}\omega$. In this case, define $\mathrm{supp}(p(\alpha)) = \emptyset$.

(b) For every $\alpha \in \mathrm{dom}(p) \cap [\lambda, \lambda+\kappa)$, letting $p(\alpha) = (p(\alpha)(1), p(\alpha)(2))$, we have the following.

  (i) There exist $\langle (n_k, \gamma_k) : k < \omega\rangle$, $\rho \in {}^{<\omega}2$, and a Borel function B such that for every k, $n_k < \omega$, $\gamma_k \in A_\alpha$, the range of B consists of fat trees in ${}^{<\omega}2$ and $\Vdash_{\mathbb{P}_\alpha} p(\alpha)(1) = [B(\langle \tau_{\gamma_k}(n_k) : k < \omega\rangle)]$ is a subset of $[\rho]$ of relative measure more than $1 - 2^{-(n-j+10)}$, where $n = |\mathrm{dom}(p) \cap [\lambda, \lambda+\kappa)|$ and $j = |\mathrm{dom}(p) \cap [\lambda, \alpha)|$. Recall that for $X \subseteq Y \subseteq 2^\omega$, the relative measure of X in Y is $\mu(X)/\mu(Y)$.

  (ii) $\Vdash_{\mathbb{P}_\alpha} p(\alpha)(2) = (\nu, F)$, where $F \in [C_\alpha]^{<\aleph_0}$, $\nu \in {}^{<\omega}\omega$, $F \subseteq \mathrm{dom}(p)$ and where, for each $\beta \in F$, $|\sigma_\beta| \geq |\nu|$.

  (iii) In this case, define $\mathrm{supp}(p(\alpha)) = \{\gamma_k : k < \omega\} \cup F$.

(c) Define $\mathrm{supp}(p) = \mathrm{dom}(p) \cup \bigcup\{\mathrm{supp}(p(\alpha)) : \alpha \in \mathrm{dom}(p)\}$.

For $\xi < \kappa$, $\mathbb{P}'_{\lambda,\lambda+\xi} = \mathbb{P}'_{\lambda+\xi} \subseteq \mathbb{P}_{\lambda+\xi}$ is defined analogously.

Using Claim 4.1 and the Lebesgue density theorem, it is easily checked that $\mathbb{P}'_{\lambda+\xi}$ is dense in $\mathbb{P}_{\lambda+\xi}$ for every $\xi \leq \kappa$.

Suppose that $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ and $\xi_\star \leq \kappa$. Let $h : \lambda + \xi_\star \to \lambda + \xi_\star$ be a bijection satisfying the following.

(1) $h \restriction [\lambda, \lambda + \xi_\star)$ is the identity.

(2) For every $\xi < \xi_\star$ and $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi}$ iff $h(\alpha) \in A_{\lambda+\xi}$ (equivalently, $\alpha \in C_{\lambda+\xi}$ iff $h(\alpha) \in C_{\lambda+\xi}$).

Define $\hat{h} : \mathbb{P}'_{\lambda+\xi_\star} \to \mathbb{P}'_{\lambda+\xi_\star}$ as follows. For $p \in \mathbb{P}'_{\lambda+\xi_\star}$, put $\hat{h}(p) = p'$, where

(a) $\mathrm{dom}(p') = \{h(\alpha) : \alpha \in \mathrm{dom}(p)\}$;

(b) for $\alpha \in \mathrm{dom}(p) \cap \lambda$, $p'(h(\alpha)) = p(\alpha)$; and

(c) for $\alpha \in \mathrm{dom}(p) \cap [\lambda, \lambda + \xi_\star)$, put $p'(\alpha)(1) = B(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega\rangle)$, where B, $\langle (n_k, \gamma_k) : k < \omega\rangle$ are as in Definition 4.2(b)(i) for the αth coordinate of p and $p'(\alpha)(2) = (\nu, h[F])$, where $(\nu, F) = p(\alpha)(2)$.

Claim 4.3. $\hat{h}$ is an automorphism of $\mathbb{P}'_{\lambda+\xi_\star}$.

Proof. By induction on $\xi_\star$. $\square$

Definition 4.4. For $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ and $A \subseteq \lambda+\kappa$, define $\mathbb{P}'_{\lambda,A} = \mathbb{P}'_A = \{p \in \mathbb{P}'_{\lambda+\kappa} : \mathrm{supp}(p) \subseteq A\}$.

The following lemma describes a sufficient condition on $A \subseteq \lambda + \xi$ for ensuring that $\mathbb{P}'_A \lessdot \mathbb{P}_{\lambda+\xi}$. It is used in the proofs of Corollary 4.6 and Claim 7.11.

Lemma 4.5. *Let $\xi_\star \leq \kappa$, let $A \subseteq \lambda + \xi_\star$, and let $[\lambda, \lambda+\xi_\star) \subseteq A$. Suppose that for every countable $B \subseteq \lambda$, there is a bijection $h : \lambda+\xi_\star \to \lambda+\xi_\star$ such that*

(a) *$h \restriction ((B \cap A) \cup [\lambda, \lambda+\xi_\star))$ is the identity;*
(b) *for every $\xi < \xi_\star$ and $\alpha < \lambda$, $\alpha \in A_{\lambda+\xi}$ iff $h(\alpha) \in A_{\lambda+\xi}$;*
(c) *$h[B] \subseteq A$; and*
(d) *$h[B \cap A_{\lambda+\xi_\star}] \subseteq A \cap A_{\lambda+\xi_\star}$.*

Then $\mathbb{P}'_A \lessdot \mathbb{P}'_{\lambda+\xi_\star}$.

Proof of Lemma 4.5. By induction on $\xi_\star$. If $\xi_\star = 0$ or the limit, this is clear. So assume $\xi_\star = \xi + 1$ and put $\alpha = \lambda + \xi$. By inductive hypothesis, $\mathbb{P}'_{A\cap\alpha} \lessdot \mathbb{P}'_\alpha$, so it suffices to check the following: If $\{p_n : n < \omega\} \subseteq \mathbb{P}'_A$, $p \in \mathbb{P}'_{A\cap\alpha}$, and $p \Vdash_{\mathbb{P}'_{A\cap\alpha}} \{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_{A\cap\alpha}}\}$ is predense in $(\mathsf{Random})^{V[\langle\tau_\beta:\beta\in A\cap A_\alpha\rangle]} \times \mathbb{E}(\langle\tau_\beta : \beta \in A \cap C_\alpha\rangle)$, then $p \Vdash_{\mathbb{P}'_\alpha} \{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_\alpha}\}$ is predense in $(\mathsf{Random})^{V[\langle\tau_\beta:\beta\in A_\alpha\rangle]} \times \mathbb{E}(\langle\tau_\beta : \beta \in C_\alpha\rangle)$.

Suppose that this fails for some $\{p_n : n < \omega\} \subseteq \mathbb{P}'_A$ and $p \in \mathbb{P}'_{A\cap\alpha}$. Choose $q \in \mathbb{P}'_\alpha$, ν, F, D, $\langle(n_k,\gamma_k) : k < \omega\rangle$ such that

- $q \geq p$;
- $\nu \in {}^{<\omega}\omega$, $F \in [C_\alpha]^{<\aleph_0}$, D is a Borel function on ω^ω whose range consists of fat trees, each $\gamma_k \in A_\alpha$; and
- $q \Vdash_{\mathbb{P}'_\alpha} r = [D(\langle\tau_{\gamma_k}(n_k) : k < \omega\rangle)] \wedge (r,(\nu,F))$ is incompatible with every member of $\{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in \mathbb{G}_{\mathbb{P}'_\alpha}\}$.

Let W be the union of the following sets: $\mathrm{dom}(q)$, $\mathrm{supp}(q)$, $\mathrm{supp}(p)$, $\bigcup\{\mathrm{dom}(p_n) : n < \omega\}$, $\bigcup\{\mathrm{supp}(p_n) : n < \omega\}$, and $\{\gamma_k : k < \omega\} \cup F$. Using the hypothesis on A, we can find a bijection $h : \alpha \to \alpha$ such that

- $h \restriction ((B \cap A) \cup [\lambda,\alpha))$ is the identity,
- $(\forall\eta < \xi)(\forall\beta < \lambda)(\beta \in A_{\lambda+\eta} \iff h(\beta) \in A_{\lambda+\eta})$,
- $h[B] \subseteq A$, and
- $h[B \cap A_\alpha] \subseteq A \cap A_\alpha$.

So $\hat{h}$ is an automorphism of $\mathbb{P}'_\alpha$. As $h[B] \subseteq A$, $\hat{h}(q) \in \mathbb{P}'_{A\cap\alpha}$. Since $h \restriction (B \cap A)$ is the identity, it follows that $\hat{h}(p) = p$ and, for every $n < \omega$, $\hat{h}(p_n) = p_n$. Since $\{\gamma_k : k < \omega\} \cup F \subseteq W$ and $h[B \cap A_\alpha] \subseteq A \cap A_\alpha$, we have that $\Vdash_{\mathbb{P}'_\alpha} r' = \hat{h}(r) = [D(\langle\tau_{h(\gamma_k)}(n_k) : k < \omega\rangle)] \in (\mathsf{Random})^{V[\langle\tau_\beta:\beta\in A\cap A_\alpha\rangle]}$ and $\Vdash_{\mathbb{P}'_\alpha} (\nu, h[F]) \in \mathbb{E}[\langle\tau_\beta : \beta \in A \cap C_\alpha\rangle]$. It follows that $\hat{h}(q) \Vdash_{\mathbb{P}'_\alpha} (r',(\nu,h[F]))$ is incompatible with every condition in $\{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_\alpha}\}$. Since $P'_{A\cap\alpha} \lessdot P'_\alpha$, we also get that $\hat{h}(q) \Vdash_{\mathbb{P}'_{A\cap\alpha}} (r',(\nu,h[F]))$ is incompatible with every condition in $\{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_{A\cap\alpha}}\}$. But since $p = \hat{h}(p) \leq \hat{h}(q)$ and $p \Vdash_{\mathbb{P}'_{A\cap\alpha}} \{p_n(\alpha) : n < \omega, p_n \restriction \alpha \in G_{\mathbb{P}'_{A\cap\alpha}}\}$ is predense in $(\mathsf{Random})^{V[\langle\tau_\beta:\beta\in A\cap A_\alpha\rangle]} \times \mathbb{E}(\langle\tau_\beta : \beta \in A \cap C_\alpha\rangle)$, we get a contradiction. $\square$

Corollary 4.6. *For every $\xi_\star < \kappa$, $\mathbb{P}'_{A_{\lambda+\xi_\star}} \lessdot \mathbb{P}'_{\lambda+\xi_\star}$.*

Proof of Corollary 4.6. Let $B \subseteq \lambda$ be countable. By Lemma 4.5 clauses (a)–(d), it suffices to construct a bijection $h : \lambda+\xi_\star \to \lambda+\xi_\star$ such that $h \restriction ((B \cap A_{\lambda+\xi_\star}) \cup$

$[\lambda, \lambda+\xi_\star))$ is the identity, $(\forall \xi < \xi_\star)(\forall \alpha < \lambda)(\alpha \in A_{\lambda+\xi} \iff h(\alpha) \in A_{\lambda+\xi})$, and $h[B] \subseteq A_{\lambda+\xi_\star}$. For each $x \subseteq \xi_\star$, let $W_x = \{\alpha < \lambda : X_\alpha \cap \xi_\star = x\}$. Let $W_{x,0} = \{\alpha \in W_x : \xi_\star \notin X_\alpha\}$ and $W_{x,1} = \{\alpha \in W_x : \xi_\star \in X_\alpha\}$, so $W_x = W_{x,0} \sqcup W_{x,1}$. Note that for every $x \subseteq \xi_\star$, $|W_{x,0}| = |W_{x,1}| = \lambda$. So for each $x \subseteq \xi_\star$, we can choose a bijection $h_x : W_x \to W_x$ such that $h_x[W_{x,0} \cap B] \subseteq W_{x,1}$ and that $h_x \restriction (W_{x,1} \cap B)$ is the identity. Put $h \restriction \lambda = \bigcup\{h_x : x \subseteq \xi_\star\}$. $\square$

5. Meager ideal

Recall that $\mathbb{P} = \mathbb{P}_{\lambda_0, \lambda_0+\kappa}$. Throughout this section and the next, we fix $\lambda = \lambda_0 = 2^\kappa$. In $V^{\mathbb{P}}$, let $\mathcal{J} = \{Y \subseteq \kappa : (\exists X \in \mathcal{I})(Y \subseteq X)\}$ be the ideal generated by $\mathcal{I}$. Since $\mathbb{P}$ is ccc, $\mathcal{J}$ is an $\aleph_1$-saturated κ-additive ideal over κ. The next lemma says that the meager ideal restricted to $\{\tau^2_{\lambda+\xi} : \xi < \kappa\}$ is isomorphic to $\mathcal{J}$ and is, therefore, $\aleph_1$-saturated. Its proof will conclude at the end of Section 7.

Lemma 5.1. *In $V^{\mathbb{P}}$, for every $Y \subseteq \kappa$, $\{\tau^2_{\lambda+\xi} : \xi \in Y\}$ is meager iff $Y \in \mathcal{J}$.*

Proof of Lemma 5.1. Suppose that $\mathring{Y} \in \mathcal{J}$. Since $\mathbb{P}$ is ccc, we can find $X \in \mathcal{I}$ such that $\Vdash \mathring{Y} \subseteq X$. Choose $\alpha < \lambda$ such that $X = X_\alpha$. Note that $\Vdash (\forall \xi \in X_\alpha)(\forall^\infty k)(\tau^2_{\lambda+\xi}(k) \neq \tau_\alpha(k))$. Hence $\{\tau^2_{\lambda+\xi} : \xi \in \mathring{Y}\}$ is meager in $V^{\mathbb{P}}$. $\square$

Next suppose that $\mathring{Y} \notin \mathcal{J}$. Toward a contradiction, wlog, suppose that $p \in \mathbb{P}'$ forces $\{\tau^2_{\lambda+\xi} : \xi \in \mathring{Y}\}$ to be nowhere dense in ω^ω. Let $\mathring{T} \subseteq {}^{<\omega}\omega$ be a nowhere dense subtree such that $p \Vdash \{\tau^2_{\lambda+\xi} : \xi \in \mathring{Y}\} \subseteq [\mathring{T}]$. For each $\sigma \in {}^{<\omega}\omega$, let $\mathcal{A}_\sigma$ be a maximal antichain of conditions in $\mathbb{P}'$ deciding $\sigma \in \mathring{T}$. Put $W = \bigcup\{\text{supp}(p) : p \in \mathcal{A}_\sigma, p \in \mathcal{A}_\sigma\}$.

Choose $\xi < \kappa$ and $p' \in \mathbb{P}'$ such that $p' \geq p$, $\xi \notin \bigcup\{X_\alpha : \alpha \in W \cap \lambda\}$, $\lambda + \xi > \sup(W)$ and $p' \Vdash \xi \in \mathring{Y}$, and hence $p' \Vdash \tau^2_{\lambda+\xi} \in [\mathring{T}]$. By extending p', we can assume that $\lambda + \xi \in \text{dom}(p')$. Let $q \in \mathbb{P}'$ be such that $\text{dom}(q) = \text{dom}(p') \cap (\lambda+\xi+1)$, $q \restriction (\lambda+\xi) = p' \restriction (\lambda+\xi)$, $q(\lambda+\xi)(1) = 2^\omega$, and $q(\lambda+\xi)(2) = (\emptyset, \emptyset)$. Since $\mathring{T} \in V^{\mathbb{P}_{\lambda+\xi}}$, $q \Vdash_{\mathbb{P}_{\lambda+\xi+1}} \tau^2_{\lambda+\xi} \in [\mathring{T}]$.

Put $\text{dom}(q) \cap \lambda = \{\alpha_j : j < m_\star\} \sqcup \{\beta_j : j < r_\star\}$, where $\{\beta_j : j < r_\star\} = \{\beta \in \text{dom}(q) \cap \lambda : \xi \notin X_\beta\}$ and the α_j's and β_j's are increasing with j. Note that $W \cap \{\alpha_j : j < m_\star\} = \emptyset$. Put $\text{dom}(q) \cap [\lambda, \lambda+\xi) = \{\lambda + \xi_j : j < n_\star\}$, where the ξ_j's are increasing with j. For $j < r_\star$, let $q(\beta_j) = \eta_j$. For $j < n_\star$, let $q(\lambda+\xi_j)(2) = (\nu_j, F_j)$, and let $\rho_j \in {}^{<\omega}2$ be such that $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} q(\lambda+\xi_j)(1)$ is a fat subset of $[\rho_j]$ of relative measure more than $1 - 2^{-(n_\star - j + 10)}$. By extending q, we can also assume that

- $q(\lambda+\xi)(2) = (\nu_\star, F_\star)$, where $\nu_\star \in {}^{l_\star}\omega$;
- $F_\star = \{\alpha_j : j < m_\star\}$ is nonempty;
- for every $j < m_\star$, $q(\alpha_j) = \sigma_j \in {}^{l_\star}\omega$, so $|\sigma_j| = |\nu_\star|$; and
- for each $j < n_\star$, $\nu_j \in {}^{l_\star}\omega$.

To produce such a q, first extend each $q(\alpha)$ for $\alpha \in \text{dom}(q) \cap \lambda$ such that they all have the same sufficiently large length $l_\star$. Let $K \subseteq \omega$ be the finite set of values these $q(\alpha)$'s take. Next for each $j < n_\star$, extend each ν_j to a member of ${}^{l_\star}\omega$ with new values from $\omega \setminus K$. Finally extend $q(\lambda+\xi)(2)$ to $(\nu_\star, F_\star)$, where $\nu_\star \in {}^{l_\star}\omega$ and $F_\star = \{\alpha_j : j < m_\star\}$. This is permissible because $\xi \in X_{\alpha_j}$ for every $j < m_\star$.

For $\alpha < \lambda$, define $\mathring{S}_{\nu_\star, \alpha} = \{\nu \in {}^{<\omega}\omega : (\nu_\star \preceq \nu) \wedge (\forall k \in [|\nu_\star|, |\nu|))(\nu(k) \neq \tau_\alpha(k))\}$.

Claim 5.2. $q \restriction (\lambda+\xi) \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{T} \supseteq \bigcap_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_j}$.

Proof of Claim 5.2. Suppose not. Choose $q \restriction (\lambda+\xi) \le q_1 \in \mathbb{P}'_{\lambda+\xi}$, $\nu_\star \preceq \nu_1 \in {}^{<\omega}\omega$ such that $q_1 \Vdash_{\mathbb{P}_{\lambda+\xi}} \nu_1 \in \bigcap_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_j} \wedge \nu_1 \notin \mathring{T}$. Let $q_2 \ge q_1$, $q_2 \in \mathbb{P}'_{\lambda+\xi+1}$ be such that $q_2 \restriction (\lambda+\xi) = q_1 \restriction (\lambda+\xi)$ and $q_2(\lambda+\xi)(2) = (\nu_1, F_\star)$. Then $q_2 \ge q$ and $q_2 \Vdash_{\mathbb{P}_{\lambda+\xi+1}} \nu_1 \subseteq \tau^2_{\lambda+\xi}$. Hence $q_2 \Vdash_{\mathbb{P}_{\lambda+\xi+1}} \tau^2_{\lambda+\xi} \notin \mathring{T}$. This is a contradiction. $\square$

Choose $\langle \alpha_{i,j} : i<\lambda, j<m_\star\rangle$ such that the following hold.

- For every $i<\lambda$ and $j<m_\star$, $\alpha_{i,j} \in \lambda \setminus (W \cup \mathrm{dom}(q))$.
- For every $i_1, i_2 < \lambda$ and $j_1, j_2 < m_\star$, $\alpha_{i_1,j_1} = \alpha_{i_2,j_2}$ iff $(i_1,j_1) = (i_2,j_2)$.
- For every $i<\lambda$ and $j<m_\star$, $X_{\alpha_{i,j}} = X_{\alpha_j}$.

For $i<\lambda$, the map $h_i : \lambda+\xi \to \lambda+\xi$ defined by

$$h_i(\alpha) = \begin{cases} \alpha_{i,j} & \text{if } j<m_\star \text{ and } \alpha=\alpha_j, \\ \alpha_j & \text{if } j<m_\star \text{ and } \alpha = \alpha_{i,j}, \\ \alpha & \text{otherwise} \end{cases}$$

induces an automorphism $\hat{h}_i$ of $\mathbb{P}'_{\lambda+\xi}$ that fixes $\mathring{T}$. Let $q_i = \hat{h}_i(q \restriction (\lambda+\xi))$. Then for each $i<\lambda$, we have the following.

(1) $\mathrm{dom}(q_i) = \{\alpha_{i,j} : j<m_\star\} \sqcup \{\beta_j : j<r_\star\} \sqcup \{\lambda+\xi_j : j<n_\star\}$.
(2) For every $j<m_\star$, $q_i(\alpha_{i,j}) = q(\alpha_j) = \sigma_j \in {}^{l_\star}\omega$.
(3) For every $j<r_\star$, $q_i(\beta_j) = q(\beta_j) = \eta_j$.
(4) For every $j<n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} q_i(\lambda+\xi_j)(1)$ is a fat subset of $[\rho_j]$ of fractional measure more than $1-2^{-(n_\star-j+10)}$.
(5) For every $j<n_\star$, $q_i(\lambda+\xi_j)(2) = (\nu_j, F_{i,j})$, where $\nu_j \in {}^{l_\star}\omega$ and $F_{i,j} = h_i[F_j]$.
(6) $q_i \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{T} \supseteq \bigcap_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_{i,j}}$.

Since λ is uncountable, by a Δ-system argument, we can further assume that for some $\langle F^\star_j : j<n_\star\rangle$, for every $j<n_\star$, $\langle F_{i,j} : i<\omega\rangle$ forms a Δ-system with root $F^\star_j$.

For $i<\omega$, define $g(i) : [l_\star, l_\star+i) \to \omega$ such that for every $k \in \mathrm{dom}(g(i))$, $g(i)(k) = i$.

Definition 5.3. For each $i<\omega$, define $q^\star_i \in \mathbb{P}'_{\lambda+\xi}$ by $\mathrm{dom}(q^\star_i) = \mathrm{dom}(q_i)$ and

$$q^\star_i(\alpha) = \begin{cases} \sigma_j \cup g(i) & \text{if } j<m_\star \text{ and } \alpha = \alpha_{i,j}, \\ q_i(\alpha) & \text{otherwise.} \end{cases}$$

Let $\bar{q}^\star = \langle q^\star_i : i<\omega\rangle$.

The next claim provides a sufficient condition to complete the proof of Lemma 5.1.

Claim 5.4. Suppose that there exists a $q_\star \in \mathbb{P}$ such that

$$q_\star \Vdash (\exists^\infty i)(q^\star_i \in G_{\mathbb{P}}).$$

Then $q_\star \Vdash [\mathring{T}]$ has a nonempty interior.

Proof of Claim 5.4. Let G be $\mathbb{P}$-generic over V with $q_\star \in G$. Suppose that $\nu_\star \preceq \nu \in {}^{<\omega}\omega$. Choose $i<\omega$ such that $(\forall k \in \mathrm{dom}(\nu))(\nu(k) < i)$ and $q^\star_i \in G$. Since $q^\star_i(\alpha_{i,j}) = \sigma_j \cup g(i)$ for every $j<m_\star$, it follows that $\nu \in \bigcap_{j<m_\star} \mathring{S}_{\nu_\star,\alpha_{i,j}}$. By Claim 5.2, it follows that $\nu \in \mathring{T}$. Hence $q_\star \Vdash [\nu_\star] \subseteq [\mathring{T}]$. $\square$

So to complete the proof of Lemma 5.1, it is sufficient to construct $q_\star \in \mathbb{P}$, satisfying the hypothesis of Claim 5.4. This will be done in Section 7.

6. Null ideal

Definition 6.1. For each $n < \omega$, let $\langle C^n_k : k < \omega \rangle$ be a one-to-one listing of all clopen subsets of 2^ω of measure 2^{-n}. For $\alpha < \lambda$, define $\mathring{N}_\alpha = \bigcap_k \bigcup_{n > k} C^n_{\tau_\alpha(n)}$. So $\mathring{N}_\alpha$ is a null G_δ-set coded by τ_α.

The next claim says that the null ideal restricted to $\{\tau^1_{\lambda+\xi} : \xi < \kappa\}$ is isomorphic to $\mathcal{J}$ and is, therefore, $\aleph_1$-saturated. Its proof will be completed at the end of Section 7.

Lemma 6.2. *In $V^{\mathbb{P}}$, for every $Y \subseteq \kappa$, $\{\tau^1_{\lambda+\xi} : \xi \in Y\}$ is null iff $Y \in \mathcal{J}$.*

Proof of Lemma 6.2. Toward a contradiction, suppose that $p \Vdash \mathring{Y} \notin \mathcal{J} \wedge \{\tau^1_{\lambda+\xi} : \xi \in \mathring{Y}\}$ is null. Let $\mathring{N}$ be a null Borel set in $V^{\mathbb{P}}$ such that $p \Vdash \mathring{N} \supseteq \{\tau^1_{\lambda+\xi} : \xi \in \mathring{Y}\}$. Choose a Borel function B coded in V, and choose $\langle (n_k, \gamma_k) : k < \omega \rangle$ such that for every $k < \omega$, $\gamma_k < \lambda + \kappa$, $n_k < \omega$, and $\Vdash B(\langle \tau_{\gamma_k}(n_k) : k < \omega \rangle) = \mathring{N}$. Let $A = \bigcup \{X_{\gamma_k} : k < \omega \wedge \gamma_k < \lambda\}$. Then $A \in \mathcal{I}$. Choose $q \geq p$ and $\xi < \kappa$ such that $q \Vdash \xi \in \mathring{Y} \setminus A$ and $\lambda + \xi > \sup(\{\gamma_k : k < \omega\})$. Since $\mathring{N}$ is coded in $V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi} \rangle]$ (as $\{\gamma_k : k < \omega\} \subseteq A_{\lambda+\xi}$), it follows that $q \Vdash \tau^1_{\lambda+\xi} \notin \mathring{N}$. This is a contradiction.

Next suppose that $\mathring{Y} \in \mathcal{J}$. Since $\mathbb{P}$ is ccc, we can find $X \in \mathcal{I}$ such that $\Vdash \mathring{Y} \subseteq X$. We would like to show that $\{\tau^1_{\lambda+\xi} : \xi \in X\}$ is null. Choose $\alpha < \lambda$ such that $X = X_\alpha$. It is clearly enough to show that for every $\xi \in X_\alpha$, $\Vdash_{\mathbb{P}_{\lambda+\xi+1}} \tau^1_{\lambda+\xi} \in \mathring{N}_\alpha$. Suppose that this fails, and fix $\xi \in X_\alpha$, $p \in \mathbb{P}'_{\lambda+\xi+1}$, and $k_\star < \omega$ such that $p \Vdash (\forall k \geq k_\star)(\tau^1_{\lambda+\xi} \notin C^k_{\tau_\alpha(k)})$. We can assume that $\alpha \in \mathrm{dom}(p)$ and $p(\alpha) = \sigma_\star \in {}^{l_\star}\omega$ for some $l_\star > k_\star$. Choose a Borel function B and $\langle (n_j, \gamma_j) : j < \omega \rangle$ such that $\gamma_j \in A_{\lambda+\xi}$, that the range of B consists of fat trees and $\Vdash_{\mathbb{P}_{\lambda+\xi}} B(\langle \tau_{\gamma_j}(n_j) : j < \omega \rangle) = \mathring{T}$, and that $[\mathring{T}] = p(\lambda+\xi)(1)$. It follows that $p \restriction (\lambda+\xi) \Vdash_{\mathbb{P}_{\lambda+\xi}} (\forall k \geq k_\star)([\mathring{T}] \cap C^k_{\tau_\alpha(k)} = \emptyset)$. Let $W = \{\gamma_j : j < \omega\}$, and note that $\alpha \notin W$.

Put $\mathrm{dom}(p) \cap \lambda = \{\alpha\} \sqcup \{\beta_j : j < r_\star\}$ and $\mathrm{dom}(p) \cap [\lambda, \lambda+\xi) = \{\lambda+\xi_j : j < n_\star\}$, where β_j and ξ_j are increasing with j. For $j < r_\star$, let $p(\beta_j) = \eta_j$. For $j < n_\star$, let $p(\lambda+\xi_j)(2) = (\nu_j, F_j)$, and let $\rho_j \in {}^{<\omega}2$ be such that $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p(\lambda+\xi_j)(1)$ is a fat subset of $[\rho_j]$ of relative measure more than $1 - 2^{-(n_\star - j + 10)}$. By possibly extending p, we can assume that for every $j < n_\star$, $\nu_j \in {}^{l_\star}\omega$. Choose $\langle \alpha_i : i < \lambda \rangle$ such that the following hold.

- For all $i < j < \lambda$, $\alpha_i < \alpha_j < \lambda$.
- $X_{\alpha_i} = X_\alpha$ (so $\alpha_i \notin W$).
- $\alpha_i \notin \mathrm{supp}(p)$.

For $i < \lambda$, the map $h_i : \lambda+\xi \to \lambda+\xi$ defined by

$$h_i(\gamma) = \begin{cases} \alpha & \text{if } \gamma = \alpha_i, \\ \alpha_i & \text{if } \gamma = \alpha, \\ \gamma & \text{otherwise} \end{cases}$$

induces an automorphism $\hat{h}_i$ of $\mathbb{P}'_{\lambda+\xi}$ that fixes $\mathring{T}$. Let $p_i = \hat{h}_i(p \restriction (\lambda+\xi))$. Then for each $i < \lambda$, we have the following.

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(1) $\mathrm{dom}(p_i) = \{\alpha_i\} \sqcup \{\beta_j : j < r_\star\} \sqcup \{\lambda + \xi_j : j < n_\star\}$.
(2) $p_i(\alpha_i) = p(\alpha) = \sigma_\star \in {}^{l_\star}\omega$.
(3) For every $j < r_\star$, $p_i(\beta_j) = p(\beta_j) = \eta_j$.
(4) For every $j < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_i(\lambda+\xi_j)(1)$ is a fat subset of $[\rho_j]$ of fractional measure more than $1 - 2^{-(n_\star - j + 10)}$.
(5) For every $j < n_\star$, $p_i(\lambda+\xi_j)(2) = (\nu_j, F_{i,j})$, where $\nu_j \in {}^{l_\star}\omega$ $F_{i,j} = h_i[F_j]$.
(6) $p_i \Vdash_{\mathbb{P}_{\lambda+\xi}} (\forall k \geq k_\star)([\mathring{T}] \cap C^k_{\tau_{\alpha_i}(k)} = \emptyset)$.

As before, by thinning out we can assume that for some $\langle F^\star_j : j < n_\star\rangle$, for every $j < n_\star$, $\langle F_{i,j} : i < \omega\rangle$ forms a Δ-system with root $F^\star_j$. $\square$

For each $i < \omega$, we will extend p_i on the α_ith coordinate to get p'_i as follows.

Definition 6.3. For each $n < \omega$, let $K_n = \{k < \omega : \mathrm{supp}(C^{l_\star}_k) \subseteq n\}$. Note that for all $n \geq l_\star$, $|K_n| = \binom{2^n}{2^{n-l_\star}}$. Define $\bar{k} = \langle k_n : n < \omega\rangle$ by $k_0 = 0$, $k_{n+1} - k_n = \binom{2^n}{2^{n-l_\star}}$. Let $f : \omega \to \omega$ be such that $f[[k_n, k_{n+1})] = K_n$. For each $i < \omega$, $\gamma \in \mathrm{dom}(p_i)$, define

$$p'_i(\gamma) = \begin{cases} p_i(\gamma) & \text{if } \gamma \neq \alpha_i, \\ \sigma_\star \cup \{(l_\star, f(i))\} & \text{if } \gamma = \alpha_i. \end{cases}$$

Lemma 6.4. *Suppose that $K < \omega$, that $F \subseteq [\lambda, \lambda+\kappa)$ is finite, that $\langle \rho_\theta : \theta \in F\rangle$ is a sequence in ${}^{<\omega}2$, that $\langle a_\theta : \theta \in F\rangle$ is a sequence in $(1/2, 1)$, and that $\langle q_j : j < K\rangle$ is a sequence of conditions in $\mathbb{P}'$ such that for every $j < K$, $\mathrm{dom}(q_j) = F$, for each $\theta \in F$, $\Vdash_{\mathbb{P}_\theta} q_j(\theta)(1)$ is a subset of $[\rho_\theta]$ of relative measure $\geq a_\theta$, and $q_j(\theta)(2)$ is the empty condition. Then there exists a $q^\star \in \mathbb{P}'$ with $\mathrm{dom}(q^\star) = F$ such that for every $\theta \in F$, $\Vdash_{\mathbb{P}_\theta} q^\star(\theta)(1)$ is a fat subset of $[\rho_\theta]$ of relative measure $\geq 2a_\theta - 1$ and $q^\star(\theta)(2)$ is the empty condition and*

$$q^\star \Vdash_{\mathbb{P}} |\{j < K : q_j \in G_{\mathbb{P}}\}| \geq K 2^{-|F|} \prod_{\theta \in F} a_\theta.$$

Proof of Lemma 6.4. By induction on $|F|$. Suppose that $F = \{\theta\}$. Work in $V^{\mathbb{P}_\theta}$. Define $\phi = \sum_{j<K} 1_{q_j(\theta)(1)}$, where $1_{q_j(\theta)(1)}$ is the characteristic function of $q_j(\theta)(1)$. Put $A = \{x \in [\rho_\theta] : \phi(x) \geq \frac{K a_\theta}{2}\}$. It suffices to show that $\mu(A) > \mu([\rho_\theta])(2a_\theta - 1)$. We have

$$K a_\theta \mu([\rho_\theta]) \leq \int \phi d\mu = \int_A \phi d\mu + \int_{2^\omega \setminus A} \phi d\mu \leq K\mu(A) + (\mu([\rho_\theta]) - \mu(A))\frac{K a_\theta}{2}.$$

Solving gives $\dfrac{\mu(A)}{\mu([\rho_\theta])} \geq \dfrac{a_\theta}{2 - a_\theta} > 2a_\theta - 1$.

Now suppose that $|F| \geq 2$ and that β is the largest member of F. Let $F' = F \setminus \{\beta\}$, $q'_j = q \restriction F'$. Choose $q' \in \mathbb{P}'$ with domain F' such that for every $\theta \in F'$, $\Vdash_{\mathbb{P}_\theta} q'(\theta)(1)$ is a subset of $[\rho_\theta]$ of relative measure $\geq 2a_\theta - 1$, $q'(\theta)(2)$ is the empty condition, and $q' \Vdash_{\mathbb{P}} |\{j < K : q'_j \in G_{\mathbb{P}}\}| \geq K2^{-|F'|} \prod_{\theta \in F'} a_\theta$. Let $\mathring{W} = \{j < K : q'_j \in G_{\mathbb{P}}\}$. Let $\{W_i : i < N\}$ list all subsets of K of size $\geq K2^{-|F'|} \prod_{\theta \in F'} a_\theta$. Choose a maximal antichain $\{r_i : i < N\}$ in $\mathbb{P}'_\beta$ above q' such that each $r_i \Vdash_{\mathbb{P}} \mathring{W} = W_i$. Work in $V^{\mathbb{P}_\beta}$. For each $i < N$, arguing as above, we can get a condition $s_i \in \mathbb{Q}^1_\beta$ such that $r_i \Vdash_{\mathbb{P}_\beta} \mu(s_i) \geq 2a_\beta - 1$ and $s_i \Vdash_{\mathbb{Q}_\beta} |\{j \in W_i : q_j(\beta)(1) \in G_{\mathbb{Q}^1_\beta}\}| \geq \dfrac{|W_i| a_\beta}{2}$. Choose $q^\star$ such that $q^\star(\theta) = q'(\theta)$ if $\theta \in F'$ and for each $i < N$, $r_i \Vdash_{\mathbb{P}_\beta} q^\star(\beta)(1) = s_i$. $\square$

For $i < \omega$, let p''_i be defined by $\mathrm{dom}(p''_i) = \{\lambda+\xi_j : j < n_\star\}$, and for every $j < n_\star$, $p''_i(\lambda + \xi_j)(1) = p'_i(\lambda + \xi_j)(1)$ and $p''_i(\lambda + \xi_j)(2)$ is the empty condition. Note that $p''_i \in \mathbb{P}'_{\lambda+\xi}$. For each $n < \omega$, apply Lemma 6.4 to the sequence $\langle p''_i : i \in [k_n, k_{n+1})\rangle$ to obtain $q^\star_n$ such that the following hold.

Definition 6.5.

(a) $q^\star_n \in \mathbb{P}'_{\lambda+\xi}$ and $\mathrm{dom}(q^\star_n) = \{\lambda + \xi_j : j < n_\star\}$.

(b) For every $j < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} q^\star_n(\lambda + \xi_j)$ is a subset of $[\rho_j]$ of relative measure $\geq 2(1 - 2^{-(n_\star - j+10)}) - 1 = 1 - 2^{-(n_\star - j+9)}$.

(c) $q^\star_n \Vdash_{\mathbb{P}_{\lambda+\xi}} |\{i \in [k_n, k_{n+1}) : p''_i \in \mathbb{G}_{\mathbb{P}_{\lambda+\xi}}\}| \geq (k_{n+1} - k_n)2^{-n_\star} \prod_{j<n_\star}(1 - 2^{-(n_\star - j+10)}) > (k_{n+1} - k_n)4^{-n_\star}$.

Definition 6.6. For each $n < \omega$ and $i \in [k_n, k_{n+1})$, define $p^\star_i \in \mathbb{P}'_{\lambda+\xi}$ by $\mathrm{dom}(p^\star_i) = \mathrm{dom}(p_i)$ and

$$p^\star_i(\alpha) = \begin{cases} \sigma_\star \cup \{(l_\star, f(i))\} \cup \{(k, i) : k \in [l_\star+1, i+l_\star+1)\} & \text{if } \alpha = \alpha_i, \\ p'_i(\beta_j) = \eta_j & \text{if } j < r_\star \text{ and } \alpha = \beta_j, \\ (p'_i(\alpha)(1) \cap q^\star_n(\alpha)(1), (\nu_j, F_{i,j})) & \text{if } j < n_\star \text{ and } \alpha = \lambda+\xi_j. \end{cases}$$

Let $\bar{p}^\star = \langle p^\star_i : i < \omega\rangle$.

Note that for every $j < n_\star$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p^\star_i(\lambda + \xi_j)(1)$ is a subset of $[\rho_j]$ of relative measure more than $1 - 2^{-(n_\star - j+8)}$. The next claim provides a sufficient condition to complete the proof of Lemma 6.2.

Claim 6.7. Suppose that for some $p_\star \in \mathbb{P}$ and $\varepsilon > 0$,

$$p_\star \Vdash (\exists^\infty n) \frac{|\{i \in [k_n, k_{n+1}) : p^\star_i \in G_{\mathbb{P}}\}|}{k_{n+1} - k_n} \geq \varepsilon.$$

Then $p_\star \Vdash [\mathring{T}]$ is finite.

Proof of Claim 6.7. For $n < \omega$, let $\mathring{W}_n = \{i \in [k_n, k_{n+1}) : p^\star_i \in G_{\mathbb{P}}\}$, and let $\mathring{a}_n = |\mathring{T} \cap {}^n2|$. Note that $p_\star \Vdash (\forall i \in \mathring{W}_n)(\forall \sigma \in \mathring{T} \cap {}^n2)(C^{l_\star}_{f(i)} \cap [\sigma] = \emptyset)$ because $l_\star > k_\star$, $p^\star_i(\alpha_i)(l_\star) = f(i)$ and $p^\star_i \Vdash (\forall k \geq k_\star)(C^k_{\tau_{\alpha_i}(k)} \cap [\mathring{T}] = \emptyset)$. It follows that $|\mathring{W}_n| \leq \binom{2^n - \mathring{a}_n}{2^{n-l_\star}}$. Hence

$$\frac{|\mathring{W}_n|}{k_{n+1} - k_n} \leq \frac{\binom{2^n - \mathring{a}_n}{2^{n-l_\star}}}{\binom{2^n}{2^{n-l_\star}}} = \prod_{j=1}^{\mathring{a}_n} \left(1 - \frac{2^{n-l_\star}}{2^n - \mathring{a}_n + j}\right) \leq \left(1 - \frac{2^{n-l_\star}}{2^n}\right)^{\mathring{a}_n}.$$

Therefore

$$\frac{|\mathring{W}_n|}{k_{n+1} - k_n} \leq (1 - 2^{-l_\star})^{\mathring{a}_n}.$$

As $\mathring{a}_n$ is increasing with n, it follows that $p_\star$ forces $\lim_n \mathring{a}_n < \infty$, and hence forces $[\mathring{T}]$ to be finite. □

To complete the proof of Theorem 1.1, it suffices to construct conditions $q_\star, p_\star$ satisfying the hypotheses of Claims 5.4 and 6.7. Let us try to illustrate the main difficulty in doing this for $\bar{p}^\star$.

Let

$$\mathring{A} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p^\star_k \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\},$$

and for each $\xi < \kappa$, let

$$\mathring{B}_\xi = \{i < \omega : p_i^\star \restriction (\lambda + \xi) \in G_{\mathbb{P}}\}.$$

Put $\mathrm{dom}(p_\star) = \{\beta_j : j < r_\star\} \sqcup \{\lambda + \xi_j : j < n_\star\}$ and $p_\star \restriction \{\beta_j : j < r_\star\} = p_i^\star \restriction \{\beta_j : j < r_\star\}$ (this does not depend on $i < \omega$). For $j < n_\star$, define $p_\star(\lambda+\xi_j)(2) = (\nu_j, F_j^\star)$.

Note that $p_\star \restriction \lambda \Vdash \mathring{A}$ is infinite. It is clearly necessary to choose the random coordinates $p_\star(\lambda+\xi_j)(1)$ for $j < n_\star$ such that $p_\star \Vdash \mathring{A} \cap \mathring{B}_{\xi_{n_\star - 1}+1}$ is infinite. Suppose that we have constructed $p_\star \restriction (\lambda + \xi_j)$ such that $p_\star \restriction (\lambda + \xi_j) \Vdash \mathring{A} \cap \mathring{B}_{\xi_j}$ is infinite, and that we would like to choose $p_\star(\lambda+\xi_j)(1) \in \mathsf{Random}^{V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_j}\rangle]}$ (recall that $p_\star(\lambda + \xi_j)(2) = (\nu_j, F_j^\star)$) such that $p_\star \restriction (\lambda + \xi_{j+1}) \Vdash \mathring{A} \cap \mathring{B}_{\xi_{j+1}}$ is infinite. The problem is that we do not have access to $\mathring{B}_{\xi_j} \in V^{\mathbb{P}_{\lambda+\xi_j}}$ in $V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_j}\rangle]$, and hence it is unclear how to proceed.

To get around this difficulty, we will construct an auxiliary finitely additive measure $\mathring{\mathfrak{m}}$ on $\mathcal{P}(\omega) \cap V^{\mathbb{P}}$ which carries enough information about the partial randoms appearing at stages $\{\lambda + \xi_j : j < n_\star\}$ to allow us to choose appropriate $p_\star(\lambda + \xi_j)(1)$'s. Definition 7.7 lists a sufficient set of requirements on $\mathring{\mathfrak{m}}$ for this. The construction of $\mathring{\mathfrak{m}}$ in Lemma 7.9 is inductive and uses Lemma 7.3 to code enough information about the partial randoms to allow the inductive step to proceed. The class of blueprints in Definition 7.4 is general enough to allow a Löwenheim–Skolem type argument (Claim 7.10) in the proof of Lemma 7.9.

7. Measures and blueprints

An algebra $\mathcal{A}$ is a family of subsets of ω that contains all finite subsets of ω and is closed under complementation and finite union. A finitely additive measure on an algebra $\mathcal{A}$ is a function $\mathfrak{m} : \mathcal{A} \to [0, 1]$ that satisfies the following.

- For every finite $F \subseteq \omega$, $\mathfrak{m}(F) = 0$.
- $\mathfrak{m}(\omega) = 1$.
- If $A_1, A_2 \in \mathcal{A}$, and $A_1 \cap A_2 = \emptyset$, then $\mathfrak{m}(A_1 \cup A_2) = \mathfrak{m}(A_1) + \mathfrak{m}(A_2)$.

Suppose that $\mathfrak{m} : \mathcal{P}(\omega) \to [0, 1]$ is a finitely additive measure and that $f : \omega \to [0, 1]$. Following Lebesgue, define

$$\int f d\mathfrak{m} = \lim_{n\to\infty} \sum_{k=0}^{2^n} \frac{k a_k}{2^n},$$

where $a_k = \mathfrak{m}(\{n < \omega : k/2^n \le f(n) < (k + 1)/2^n\})$.

The following is a standard application of the Hahn–Banach theorem.

Lemma 7.1. *Suppose that $\mathfrak{m} : \mathcal{A} \to [0, 1]$ is a finitely additive measure on an algebra $\mathcal{A}$, and that $X \subseteq \omega$. Let $a \in [0, 1]$ be such that for every $A, B \in \mathcal{A}$, if $A \subseteq X \subseteq B$, then $\mathfrak{m}(A) \le a \le \mathfrak{m}(B)$. Then there exists a finitely additive measure $\mathfrak{m}' : \mathcal{P}(\omega) \to [0, 1]$ that extends $\mathfrak{m}$ and $\mathfrak{m}'(X) = a$.*

The proofs of the next two lemmas can be found in [1].

Lemma 7.2. *Suppose that $\mathfrak{m} : \mathcal{P}(\omega) \to [0, 1]$ is a finitely additive measure. For $i \in \{1, 2\}$, let $\mathbb{R}_i$ be a forcing notion, and let $\mathring{\mathfrak{m}}_i \in V^{\mathbb{R}_i}$ be such that $\Vdash_{\mathbb{R}_i} \mathring{\mathfrak{m}}_i : \mathcal{P}(\omega) \to [0, 1]$ is a finitely additive measure extending $\mathfrak{m}$. Then there exists a $\mathring{\mathfrak{m}}_3 \in V^{\mathbb{R}_1 \times \mathbb{R}_2}$ such that $\Vdash_{\mathbb{R}_1 \times \mathbb{R}_2} \mathring{\mathfrak{m}}_3 : \mathcal{P}(\omega) \to [0, 1]$ is a finitely additive measure extending both $\mathring{\mathfrak{m}}_1$ and $\mathring{\mathfrak{m}}_2$.*

Lemma 7.3. *Suppose that* $\mathfrak{m} : \mathcal{P}(\omega) \to [0, 1]$ *is a finitely additive measure. Let* $\mathbb{B} = \mathsf{Random}$*,* $r \in \mathbb{B}$*. Define* $\mathring{\mathfrak{m}}_r \in V^{\mathbb{B}}$ *as follows. For* $\mathring{X} \in \mathcal{P}(\omega) \cap V^{\mathbb{B}}$*, define*

$$\mathring{\mathfrak{m}}_r(\mathring{X}) = \sup\left\{\inf\left\{\int \frac{\mu(q \cap [[n \in \mathring{X}]]_{\mathbb{B}})}{\mu(q)} d\mathfrak{m} : q \geq p\right\} : p \geq r, p \in G_{\mathbb{B}}\right\}.$$

Then the following hold.

(1) $r \Vdash \mathring{\mathfrak{m}}_r : \mathcal{P}(\omega) \to [0, 1]$ *is a finitely additive measure extending* $\mathfrak{m}$.

(2) *If* $\mathring{X} \in \mathcal{P}(\omega) \cap V^{\mathbb{B}}$ *and* $a > 0$ *satisfy, for every* $n < \omega$, $\dfrac{\mu(r \cap [[n \in \mathring{X}]]_{\mathbb{B}})}{\mu(r)} \geq a$, *then there exists an* $s \geq r$ *such that* $s \Vdash \mathring{\mathfrak{m}}_r(\mathring{X}) \geq a$.

The next definition introduces blueprints. Their role is clarified in Claim 7.8. Note the return of variable $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ here.

Definition 7.4. For $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, let $\mathcal{T}_\lambda$ be the set of tuples

$$t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) = (\bar{\alpha}^t, m^t, \bar{\sigma}^t, \bar{\beta}^t, r^t, \bar{\eta}^t, \bar{\xi}^t, n^t, \bar{\rho}^t, \bar{\nu}^t, \bar{F}^t, l^t, \bar{\varepsilon}^t),$$

where

(i) $l, m, n, r < \omega$;
(ii) $\bar{\alpha} = \langle \alpha_{i,j} : i < \omega, j < m \rangle$, where each $\alpha_{i,j} < \lambda$;
(iii) for every $i_1, i_2 < \omega$ and $j_1, j_2 < m$, $\alpha_{i_1,j_1} = \alpha_{i_2,j_2}$ iff $(i_1, j_1) = (i_2, j_2)$;
(iv) $\bar{\sigma} = \langle \sigma_{i,j} : i < \omega, j < m \rangle$, where each $\sigma_{i,j} \in {}^{<\omega}\omega$;
(v) $\bar{\beta} = \langle \beta_j : j < r \rangle$ is a sequence of pairwise distinct members of $\lambda \setminus \{\alpha_{i,j} : i < \omega, j < m\}$;
(vi) $\bar{\eta} = \langle \eta_j : j < r \rangle$, where each $\eta_j \in {}^{<\omega}\omega$;
(vii) $\bar{\xi} = \langle \xi_j : j < n \rangle$ is an increasing sequence in κ;
(viii) $\bar{\rho} = \langle \rho_j : j < n \rangle$, where each $\rho_j \in {}^{<\omega}2$;
(ix) $\bar{\nu} = \langle \nu_j : j < n \rangle$, where each $\nu_j \in {}^{l}\omega$;
(x) $\bar{F} = \langle F_{i,j} : i < \omega, j < n \rangle$, where each $F_{i,j} \in [C_{\lambda+\xi_j}]^{<\aleph_0}$, and for every $j < n$, $\langle F_{i,j} : i < \omega \rangle$ forms a Δ-system with root F_j; and
(xi) $\bar{\varepsilon} = \langle \varepsilon_j : j < n \rangle$, where $\varepsilon_{n-1} \in (0, 2^{-8})$ and $2\varepsilon_j \leq \varepsilon_{j+1}$ for every $j < n - 1$.

We call members of $\mathcal{T}_\lambda$ blueprints. They are intended to code information about certain sequences of conditions in $\mathbb{P}'_\lambda$ that look like $\bar{q}^\star$ and $\bar{p}^\star$ from Definitions 5.3 and 6.6 in the following sense.

Definition 7.5. Suppose that $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$ and that $\bar{p} = \langle p_i : i < \omega \rangle$ is a sequence in $\mathbb{P}'_\lambda$. We say that $\bar{p}$ is of type t if the following hold.

(a) For every $i < \omega$, $\mathrm{dom}(p_i) = \{\alpha_{i,j} : j < m\} \sqcup \{\beta_j : j < r\} \sqcup \{\lambda + \xi_j : j < n\}$.
(b) For every $i < \omega$ and $j < m$, $p_i(\alpha_{i,j}) = \sigma_{i,j}$.
(c) For every $i < \omega$ and $j < r$, $p_i(\beta_j) = \eta_j$.
(c) For every $i < \omega$ and $j < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_i(\lambda+\xi_j)(1)$ is a subset of $[\rho_j]$ of relative measure more than $1 - \varepsilon_j$.
(d) For every $i < \omega$ and $j < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_i(\lambda + \xi_j)(2) = (\nu_j, F_{i,j})$.

Definition 7.6. Let $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, and let $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$.

(1) We say that t is q-like for every $i < \omega$ and $j < m$, $|\sigma_{i,j}| = l + i$, and $(\forall k \in [l, l + i))(\sigma_{i,j}(k) = i)$.

(2) We say that t is p-like if for every $n < \omega$, $i \in [k_n, k_{n+1})$, and $j < m$, $|\sigma_{i,j}| = l + 1 + i$, $\langle \sigma_{i,j}(l) : i \in [k_n, k_{n+1}) \rangle$ are pairwise distinct and $(\forall k \in [l+1, l+1+i))(\sigma_{i,j}(k) = i)$, where $\langle k_n : n < \omega \rangle$ is as in Definition 6.3.

Note that if $\bar{q}^\star$ is of type t, then t is q-like, and if $\bar{p}^\star$ is of type t, then t is p-like.

For $t \in \mathcal{T}_\lambda$, $\xi < \kappa$, we write $t \restriction \xi$ for the blueprint which is obtained by restricting the sequence $\bar{\xi}^t$ to ordinals below ξ and modifying $\bar{\rho}^t, \bar{\nu}^t, \bar{F}^t, \bar{\varepsilon}^t$, and n^t accordingly. The next definition relates finitely additive measures in $V^{\mathbb{P}_{\lambda+\kappa}}$ and blueprints in $\mathcal{T}_\lambda$.

Definition 7.7. Suppose that $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$, $\bar{k} = \langle k_n : n < \omega \rangle$ is an increasing sequence in ω with $k_0 = 0$, $\xi_{n-1} < \xi \le \kappa$, and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$. We say that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$ if the following hold.

(1) $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}} : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure.
(2) For every $j < n$, letting $V_j = V[\langle \tau_\alpha : \alpha \in A_{\lambda+\xi_j} \rangle]$, we have $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V_j) \in V_j$.
(3) For every $\bar{p} = \langle p_i : i < \omega \rangle$ of type t, there exists a $p_{\bar{p}} \in \mathbb{P}'_{\lambda+\xi}$ such that the following hold.
   (a) $\mathrm{dom}(p_{\bar{p}}) = \{\beta_j : j < r\} \sqcup \{\lambda + \xi_j : j < n\}$.
   (b) For every $j < r$, $p_{\bar{p}}(\beta_j) = \eta_j$.
   (c) For every $X \in \mathcal{P}(\omega) \cap V$ that satisfies $(\forall n < \omega)(|X \cap [k_n, k_{n+1})| \le 1)$, we have
$$p_{\bar{p}} \restriction \lambda \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(X) = 0.$$
   (d) $p_{\bar{p}} \restriction \lambda \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{A}_{\bar{p},\bar{k}}) = 1$, where
$$\mathring{A}_{\bar{p},\bar{k}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\}.$$
   (e) For every $j < n$, $\Vdash_{\mathbb{P}_{\lambda+\xi_j}} p_{\bar{p}}(\lambda + \xi_j)(1) \subseteq [\rho_j]$ and $p_{\bar{p}}(\lambda + \xi_j)(2) = (\nu_j, F_j)$.
   (f) For every $j < n$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{Y}_{\bar{p},\bar{k},j}) = 1$, where $i \in \mathring{Y}_{\bar{p},\bar{k},j}$ iff letting $N < \omega$ be such that $i \in [k_N, k_{N+1})$, we have $p_i(\lambda + \xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}$ and $|\{i' \in [k_N, k_{N+1}) : p_{i'}(\lambda + \xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}\}| \ge k_{N+1} - k_N - m^t$.
   (g) For every $j < n$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{X}_{\bar{p},j}) \ge 1 - 2\varepsilon_j > 0$, where
$$\mathring{X}_{\bar{p},j} = \{i < \omega : p_i \restriction [\lambda, \lambda + \xi_j + 1) \in \mathbb{G}_{\mathbb{P}}\}.$$

The next claim provides a sufficient condition for the existence of $q_\star$ and $p_\star$ satisfying the hypotheses of Claims 5.4 and 6.7, respectively.

Claim 7.8. Suppose that for every $t \in \mathcal{T}_\lambda$, if t is either q-like or p-like, then there are $\xi^t_{n-1} < \xi < \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$, where $\bar{k}$ is as in Definition 6.3. Then there exist $q_\star$ and $p_\star$ satisfying the hypotheses of Claims 5.4 and 6.7, respectively.

Proof of Claim 7.8. Choose $t \in \mathcal{T}_\lambda$ such that $\bar{q}^\star$ from Definition 5.3 is of type t. Choose $\xi_{n_\star-1} < \xi < \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$. Let $q_\star = p_{\bar{q}^\star}$ be as in clause (3) of Definition 7.7. Let $\mathring{X}_{\bar{q}^\star, n_\star-1} = \{i < \omega : q^\star_i \restriction [\lambda, \lambda + \xi) \in G_{\mathbb{P}}\}$, and let
$$\mathring{A}_{\bar{q}^\star,\bar{k}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(q^\star_k \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\}.$$

Then $q_\star$ forces $\mathring{\mathfrak{m}}(\mathring{X}_{\bar{q}^\star,n_\star-1}) > 0$ and $\mathring{\mathfrak{m}}(\mathring{A}_{\bar{q}^\star,\bar{k}}) = 1$ and hence forces $\mathring{A}_{\bar{q}^\star,\bar{k}} \cap \mathring{X}_{\bar{q}^\star,n_\star-1}$ to be infinite. It follows that $q_\star \Vdash (\exists^\infty i)(q_i^\star \in G_{\mathbb{P}})$. Hence $q_\star$ satisfies the hypothesis of Claim 5.4.

Next choose $t \in \mathcal{T}_\lambda$ such that $\bar{p}^\star$ from Definition 6.6 is of type t. Choose $\xi_{n_\star-1} < \xi < \kappa$ and $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$. Let $p_\star = p_{\bar{p}^\star}$ be as in clause (3) of Definition 7.7.

Let $\mathring{X}_{\bar{p}^\star,n_\star-1} = \{i < \omega : p_i^\star \restriction [\lambda, \lambda+\xi) \in G_{\mathbb{P}}\}$. For $j < n_\star$, let $\mathring{Y}_{\bar{p}^\star,\bar{k},j}$ be defined by $i \in \mathring{Y}_{\bar{p}^\star,\bar{k},j}$ iff $p_i^\star(\lambda+\xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}$, and for some $N < \omega$, $i \in [k_N, k_{N+1})$ and $|\{i' \in [k_N, k_{N+1}) : p_{i'}^\star(\lambda+\xi_j)(2) \in G_{\mathbb{Q}^2_{\lambda+\xi_j}}\}| \geq k_{N+1} - k_N - 1$ (recalling that $m^t = 1$ for the blueprint of $\bar{p}^\star$). Finally let

$$\mathring{A}_{\bar{p}^\star,\bar{k}} = \{i < \omega : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k^\star \restriction \lambda \in \mathbb{G}_{\mathbb{P}}))\}.$$

Then $p_\star$ forces $\mathring{\mathfrak{m}}(\mathring{A}_{\bar{p}^\star,\bar{k}}) = 1$, $\mathring{\mathfrak{m}}(\mathring{X}_{\bar{p}^\star,n_\star-1}) > 0$, and for every $j < n_\star$, $\mathring{\mathfrak{m}}(\mathring{Y}_{\bar{p}^\star,\bar{k},j}) = 1$. Hence it also forces

$$\mathring{A}_{\bar{p}^\star,\bar{k}} \cap \mathring{X}_{\bar{p}^\star,n_\star-1} \cap \bigcap_{j<n_\star} \mathring{Y}_{\bar{p}^\star,\bar{k},j}$$

to be infinite. Let i be a member of this set, and fix n such that $i \in [k_n, k_{n+1})$. The set $\{i' \in [k_n, k_{n+1}) : p_{i'}^\star \notin G_{\mathbb{P}}\}$ has size at most $n_\star + (k_{n+1} - k_n)(1 - 4^{-n_\star})$. The first contribution comes from Definition 7.7(3)(f) (noting that $m^t = 1$), and the second comes from the partial random coordinates (see Definitions 6.6 and 6.5(c)). It follows that

$$p_\star \Vdash (\exists^\infty n) \frac{|\{i \in [k_n, k_{n+1}) : p_i^\star \in G_{\mathbb{P}}\}|}{k_{n+1} - k_n} \geq 4^{-(n_\star+1)}.$$

Hence $p_\star$ satisfies the hypothesis of Claim 6.7. □

The following lemma finishes the proof of Theorem 1.1.

Lemma 7.9. *Suppose that $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$, that $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$, $\xi_{n-1} < \xi < \kappa$, and that $\bar{k} = \langle k_n : n < \omega \rangle$ is as in Definition 6.3. Assume that t is either q-like or p-like. Then there exists an $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$.*

Proof of Lemma 7.9. The proof is by induction on $n = n^t = |\bar{\xi}|$. □

Suppose that $n = 0$. Fix $\xi < \kappa$. Since $n = 0$, there is a unique $\bar{p}$ of type t. Put $p_{\bar{p}} = \{(\beta_j, \eta_j) : j < r\}$. Define $\mathring{X}_{\bar{p}} = \{i : (\exists n < \omega)(i \in [k_n, k_{n+1}) \wedge (\forall k \in [k_n, k_{n+1}))(p_k \in G_{\mathbb{P}_{\lambda+\xi}}))\}$. Let $\mathcal{W} = \{X : X \in \mathcal{P}(\omega) \cap V \wedge (\forall n < \omega)(|X \cap [k_n, k_{n+1})| \leq 1)\}$. Since $\lim_n(k_{n+1} - k_n) = \infty$, it follows that for every finite $\mathcal{F} \subseteq \mathcal{W}$, $p_{\bar{p}} \Vdash_{\mathbb{P}_\lambda} \mathring{X}_{\bar{p}} \setminus \bigcup \mathcal{F}$ is infinite. Hence we can choose $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi}}$ such that $\Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}} : \mathcal{P}(\omega) \to [0, 1]$ is a finitely additive measure and, for every $X \in \mathcal{F}$, $p_{\bar{p}} \Vdash_{\mathbb{P}_{\lambda+\xi}} \mathring{\mathfrak{m}}(\mathring{X}_{\bar{p}} \setminus X) = 1$. It follows that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$.

Next fix $\lambda_0 \leq \lambda < \lambda_0^{+\omega}$ and $t = (\bar{\alpha}, m, \bar{\sigma}, \bar{\beta}, r, \bar{\eta}, \bar{\xi}, n+1, \bar{\rho}, \bar{\nu}, \bar{F}, l, \bar{\varepsilon}) \in \mathcal{T}_\lambda$ such that t is either q-like or p-like. It suffices to construct $\mathring{\mathfrak{m}} \in V^{\mathbb{P}_{\lambda+\xi_n+1}}$ such that $\mathring{\mathfrak{m}}$ satisfies $(t, \bar{k})$. Let $\mathcal{T}'_{\lambda^+} = \{t' \in \mathcal{T}_{\lambda^+} : t' = (\bar{\alpha}^{t'}, m, \bar{\sigma}, \bar{\beta}^{t'}, r, \bar{\eta}, \bar{\xi} \restriction n, n, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle F_{i,j} : i < \omega, j < n \rangle, l, \bar{\varepsilon} \restriction n)\}$. By inductive assumption, for every $t' \in \mathcal{T}'_{\lambda^+}$, there exists an $\mathring{\mathfrak{m}}^{t'} \in V^{\mathbb{P}_{\lambda^+, \lambda^+ + \xi_n}}$ such that $\mathring{\mathfrak{m}}^{t'}$ satisfies $(t', \bar{k})$. Fix such a map $t' \mapsto \mathring{\mathfrak{m}}^{t'}$ on $\mathcal{T}'_{\lambda^+}$.

Claim 7.10. There exists an $\mathring{\mathfrak{m}} \in V^{\mathbb{P}'_{\lambda,\lambda+\xi_n}}$ that satisfies $(t \restriction \xi_{n+1}, \bar{k})$, where $t \restriction \xi_{n+1} = (\alpha, \sigma, \bar{\beta}, r, \bar{\xi} \restriction n, n, \bar{\sigma}, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle F_{i,j} : i < \omega, j < n\rangle, l, \bar{\varepsilon} \restriction n)$ and $\Vdash_{\mathbb{P}'_{\lambda,\lambda+\xi_n}} \mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda,A}}) \in V^{\mathbb{P}'_{\lambda,A}}$, where $A = A^{\lambda}_{\lambda+\xi_n}$.

Proof of Claim 7.10. Let χ be sufficiently large. Choose M_0, M_1 elementary submodels of $(\mathcal{H}_\chi, \in, <_\chi)$ such that $M_0 \in M_1$, $|M_0| = |M_1| = \lambda$, and for $l \in \{0,1\}$, $\bar{\mathbb{P}}_{\lambda^+}$, $\mathcal{T}'_{\lambda^+}$, and the map $t' \mapsto \mathring{\mathfrak{m}}^{t'}$ are in M_l, $\lambda + 1 \subseteq M_l$, and ${}^{\leq\kappa}M_l \subseteq M_l$. Note that if $B_j \in \{\lambda \cap A^{\lambda}_{\lambda+\xi_j}, \lambda \setminus A^{\lambda}_{\lambda+\xi_j}\}$ for $j < n+1$, then $|\bigcap_{j<n+1} B_j| = \lambda$. Also, if $D_j \in \{\lambda^+ \cap A^{\lambda^+}_{\lambda^++\xi_j}, \lambda^+ \setminus A^{\lambda^+}_{\lambda^++\xi_j}\}$ for $j < n+1$, then $|M_0 \cap \bigcap_{j<n+1} D_j| = \lambda$ and $|(M_1 \setminus M_0) \cap \bigcap_{j<n+1} D_j| = \lambda$. So we can choose a bijection $h : \lambda + \xi_n \to M_1 \cap (\lambda^+ + \xi_n)$ such that the following hold.

(i) For every $\xi < \xi_n$, $h(\lambda + \xi) = \lambda^+ + \xi$.

(ii) For every $j < n$ and $\alpha < \lambda$, $\alpha \in A^{\lambda}_{\lambda+\xi_j}$ iff $h(\alpha) \in A^{\lambda^+}_{\lambda^++\xi_j}$; hence also $\alpha \in C^{\lambda}_{\lambda+\xi_j}$ iff $h(\alpha) \in C^{\lambda^+}_{\lambda^++\xi_j}$.

(iii) For every $\alpha < \lambda$, $\alpha \in A^{\lambda}_{\lambda+\xi_n}$ iff $h(\alpha) \in M_0$.

Let $t' = (\langle h(\alpha_{i,j}) : i < \omega, j < m\rangle, m, \bar{\sigma}, \langle h(\beta_j) : j < r\rangle, r, \bar{\eta}, \bar{\xi} \restriction n, n, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle h[F_{i,j}] : i < \omega, j < n\rangle, l, \bar{\varepsilon} \restriction n)$. As ${}^{\omega}M_1 \subseteq M_1$, $t' \in M_1$. Hence also $\mathring{\mathfrak{m}}^{t'} \in M_1$.

Define $\hat{h} : \mathbb{P}'_{\lambda,\lambda+\xi_n} \to \mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}$ as follows: $\hat{h}(p) = p'$, where $\mathrm{dom}(p') = \{h(\alpha) : \alpha \in \mathrm{dom}(p)\}$. If $\alpha \in \mathrm{dom}(p) \cap \lambda$, then $p'(h(\alpha)) = p(\alpha)$. If $\alpha \in \mathrm{dom}(p) \cap [\lambda, \lambda + \xi_n)$, then $p'(\alpha)(1) = B(\langle \tau_{h(\gamma_k)}(n_k) : k < \omega\rangle)$, where B, $\langle (n_k, \gamma_k) : k < \omega\rangle$ are as in Definition 4.2(b)(i) for coordinate α and $p'(\alpha)(2) = (\nu, h[F])$, where $(\nu, F) = p(\alpha)(2)$.

Subclaim 7.11. *The following hold.*

(1) $\hat{h} : \mathbb{P}'_{\lambda,\lambda+\xi_n} \to \mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}$ *is an isomorphism.*

(2) $\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_0} \lessdot \mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1} \lessdot \mathbb{P}'_{\lambda^+,\lambda^++\xi_n}$.

(3) *For $j < n$, put $A_j = A^{\lambda^+}_{\lambda^++\xi_j} \cap M_1$. Then* $\Vdash_{\mathbb{P}'_{\lambda^+,\lambda^++\xi_j}} \mathring{\mathfrak{m}}^{t'} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+,A_j}}) \in V^{\mathbb{P}'_{\lambda^+,A_j}}$.

(4) *For $l \in \{0,1\}$,* $\Vdash_{\mathbb{P}'_{\lambda^+,\lambda^++\xi_n}} \mathring{\mathfrak{m}}^{t'} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_l}}) \in V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_l}}$.

□

Proof of Subclaim 7.11. (1) and (4) should be clear. For (2), use Lemma 4.5. For (3), use the fact that $\mathring{m}^{t'}$ satisfies $(t', \bar{k})$.

Choose $\mathring{\mathfrak{m}}' \in V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}}$ such that

$$\Vdash_{\mathbb{P}'_{\lambda^+,\lambda^++\xi_n}} \mathring{\mathfrak{m}}' = \mathring{\mathfrak{m}}^{t'} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda^+,(\lambda^++\xi_n)\cap M_1}})$$

and define $\mathring{\mathfrak{m}} \in V^{\mathbb{P}'_{\lambda,\lambda+\xi_n}}$ by $\hat{h}(\mathring{\mathfrak{m}}) = \mathring{\mathfrak{m}}'$.

By Subclaim 7.11, $\mathring{\mathfrak{m}}$ satisfies $(t \restriction \xi_{n+1}, \bar{k})$, where $t \restriction \xi_{n+1} = (\alpha, \sigma, \bar{\beta}, r, \bar{\xi} \restriction n, n, \bar{\sigma}, \bar{\rho} \restriction n, \bar{\nu} \restriction n, \langle F_{i,j} : i < \omega, j < n\rangle, l, \bar{\varepsilon} \restriction n)$ and, moreover, $\Vdash_{\mathbb{P}'_{\lambda,\lambda+\xi_n}} \mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V^{\mathbb{P}'_{\lambda,A}}) \in V^{\mathbb{P}'_{\lambda,A}}$, where $A = A^{\lambda}_{\lambda+\xi_n}$. This completes the proof of Claim 7.10. □

To complete the proof of Lemma 7.9, we would like to extend $\mathring{\mathfrak{m}}$ to $\mathring{\mathfrak{m}}_1 \in V^{\mathbb{P}_{\lambda+\xi_{n+1}}}$ such that $\mathring{\mathfrak{m}}_1$ satisfies $(t, \bar{k})$. We do this in two steps.

Let $q = \langle(\beta_j, \eta_j) : j < r\rangle$. Note that for every $X \in \mathcal{P}(\omega) \cap V$, if $(\forall n < \omega)(|X \cap [k_n, k_{n+1})| \leq 1)$, then $q \Vdash_{\mathbb{P}_{\lambda+\xi_n}} \mathring{m}(X) = 0$.

Claim 7.12. q forces the following to hold in $V^{\mathbb{P}_{\lambda+\xi_n}}$: Letting $\mathbb{Q} = \mathbb{Q}^2_{\lambda+\xi_n}$, there exists a $\mathbb{Q}$-name $\mathring{\mathfrak{m}}_2$ such that $\Vdash_{\mathbb{Q}} \mathring{\mathfrak{m}}_2 : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure that extends $\mathring{\mathfrak{m}}$ and $(\nu_n, F_n) \Vdash_{\mathbb{Q}} \mathring{\mathfrak{m}}_2(\mathring{Y}) = 1$, where $i \in \mathring{Y}$ iff for some $N < \omega$, $i \in [k_N, k_{N+1})$, $(\nu_n, F_{i,n}) \in G_{\mathbb{Q}}$, and

$$|\{i' \in [k_N, k_{N+1}) : (\nu_n, F_{i',n}) \in G_{\mathbb{Q}}\}| \geq k_{N+1} - k_N - m.$$

Proof of Claim 7.12. Work in $V_1 = V^{\mathbb{P}_{\lambda+\xi_n}}$ above q. By Lemma 7.1, it suffices to show that for every $A \in \mathcal{P}(\omega) \cap V_1$ satisfying $\mathring{\mathfrak{m}}(A) > 0$, $(\nu_n, F_n) \Vdash_{\mathbb{Q}} A \cap \mathring{Y} \neq \emptyset$. Toward a contradiction, suppose that this fails. Choose $(\nu, F) \in \mathbb{Q}$ and $A \in V_1$ such that $(\nu_n, F_n) \leq (\nu, F)$, $\mathring{\mathfrak{m}}(A) > 0$, and $(\nu, F) \Vdash_{\mathbb{Q}} A \cap \mathring{Y} = \emptyset$. We can assume that $|\nu| > |\nu_n| = l$. Choose the $q_1 \in \mathbb{P}'_{\lambda+\xi_n}$, $q_1 \geq q$ that forces this.

First suppose t is q-like. Then, for every $i < \omega$ and $j < m$, $|\sigma_{i,j}| = l + i$ and $(\forall k \in [l, l+i))(\sigma_{i,j}(k) = i)$. Let H be $\mathbb{P}_{\lambda+\xi_n}$-generic over V with $q_1 \in H$. Work in $V[H]$. Since $\mathring{m}(A) > 0$, A is infinite. Choose $N < \omega$ and $i \in [k_N, k_{N+1}) \cap A$ such that $k_N > |\nu|$, $(\forall k \in \mathrm{dom}(\nu))(k_N > \nu(k))$, and for every $i' \in [k_N, k_{N+1})$, $F_{i',n} \setminus F_n \subseteq \{\alpha_{i',j} : j < m\}$. It follows that $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n})$ extends $(\nu_n, F_{i',n})$ for every $i' \in [k_N, k_{N+1})$, and hence that $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n}) \Vdash_{\mathbb{Q}} i \in \mathring{Y} \cap A$. This is a contradiction.

Next suppose t is p-like. Then, for every $N < \omega$, $i \in [k_N, k_{N+1})$, and $j < m$, $|\sigma_{i,j}| = l + 1 + i$, $\langle \sigma_{i,j}(l) : i \in [k_N, k_{N+1})\rangle$ are pairwise distinct and $(\forall k \in [l+1, l+1+i))(\sigma_{i,j}(k) = i)$. Let $X = \{i < \omega : (\exists n < \omega)(\exists j < m)(i \in [k_n, k_{n+1}) \wedge \nu(l) = \sigma_{i,j}(l))\}$. Then, for every $n < \omega$, $|X \cap [k_n, k_{n+1})| \leq m$, and hence $q \Vdash_{\mathbb{P}'_{\lambda+\xi_n}} \mathring{m}(X) = 0$. Let H be $\mathbb{P}_{\lambda+\xi_n}$-generic over V with $q_1 \in H$. Work in $V[H]$. Since $\mathring{m}(A \setminus X) > 0$, $A \setminus X$ is infinite. Choose $N < \omega$ and $i \in [k_N, k_{N+1}) \cap (A \setminus X)$ such that $k_N > |\nu|$, $(\forall k \in \mathrm{dom}(\nu))(k_N > \nu(k))$ and for every $i \in [k_N, k_{N+1})$, $F_{i,n} \setminus F_n \subseteq \{\alpha_{i,j} : j < m\}$. It follows that the set of $i' \in [k_N, k_{N+1})$ for which $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n})$ does not extend $(\nu_n, F_{i',n})$ has size at most m, and hence $(\nu, F \cup \bigcup_{k \in [k_N, k_{N+1})} F_{k,n}) \Vdash_{\mathbb{Q}} i \in \mathring{Y} \cap A$. This is a contradiction. $\square$

Claim 7.13. The following holds in $V^{\mathbb{P}_{\lambda+\xi_n}}$: Let $\mathbb{B} = \mathbb{Q}^1_{\lambda+\xi_n}$. There exist $s \in \mathbb{B}$ and a $\mathbb{B}$-name $\mathring{\mathfrak{m}}_3$ such that $s \geq [\rho_n]$, $\Vdash_{\mathbb{B}} \mathring{\mathfrak{m}}_3 : \mathcal{P}(\omega) \to [0,1]$ is a finitely additive measure extending $\mathring{\mathfrak{m}}$, and that $s \Vdash_{\mathbb{B}} \mathring{\mathfrak{m}}_3(\{i < \omega : p_i(\lambda + \xi_n) \in G_{\mathbb{B}}\}) \geq 1 - \varepsilon_n$.

Proof of Claim 7.13. Put $V_n = V^{\mathbb{P}'_{\lambda, A_{\lambda+\xi_n}}}$ so that $\mathbb{B} = (\mathsf{Random})^{V_n}$. Working in V_n, apply Lemma 7.3 to $\mathring{\mathfrak{m}} \restriction (\mathcal{P}(\omega) \cap V_n)$, with $r = [\rho_n]$, to obtain the extension $\mathring{\mathfrak{m}}_r \in (V_n)^{\mathbb{B}}$ as defined there. By Lemma 7.3(2), we can choose $s \in \mathbb{B}$, $s \geq [\rho_n]$ such that $s \Vdash_{\mathbb{B}} \mathring{\mathfrak{m}}_r(\{i < \omega : p_i(\lambda + \xi_n) \in G_{\mathbb{B}}\}) \geq 1 - \varepsilon_n$.

Since $\mathbb{P}'_{\lambda, A^{\lambda}_{\lambda+\xi_n}} \lessdot \mathbb{P}'_{\lambda+\xi_n}$, we can write $V^{\mathbb{P}'_{\lambda+\xi_n}} = (V_n)^{\mathbb{R}}$ for some $\mathbb{R} \in V_n$. By Lemma 7.2, it follows that $\mathring{m}_r \in (V_n)^{\mathbb{B}}$ and $\mathring{m} \in (V_n)^{\mathbb{Q}}$ have a common extension $\mathring{m}_3 \in (V_n)^{\mathbb{Q} \times \mathbb{B}} = V^{\mathbb{P}_{\lambda+\xi_n} \star \mathbb{Q}^1_{\lambda+\xi_n}}$. So s and $\mathring{m}_3$ are as required.

Since $\mathbb{P}_{\lambda+\xi_n+1} = \mathbb{P}_{\lambda+\xi_n} \star (\mathbb{Q}^1_{\lambda+\xi_n} \times \mathbb{Q}^2_{\lambda+\xi_n})$, using Lemma 7.2 again, we can find a common extension $\mathring{\mathfrak{m}}_1 \in V^{\mathbb{P}_{\lambda+\xi_n+1}}$ of $\mathring{\mathfrak{m}}_2$ and $\mathring{\mathfrak{m}}_3$.

Let us check that $\mathring{\mathfrak{m}}_1$ satisfies $(t, \bar{k})$. So fix $\bar{p} = \langle p_j : j < \omega\rangle$ of type t, and construct $p_{\bar{p}}$ as follows. Put $\bar{q} = \langle p_j \restriction (\lambda + \xi_n) : j < \omega\rangle$. Since $\mathring{\mathfrak{m}}$ satisfies $t \restriction \xi_{n+1}$, we can find $p_{\bar{q}} \in \mathbb{P}'_{\lambda+\xi_n}$, satisfying clauses (3)(a)–(f) in Definition 7.7 for $\bar{q}$.

Define $p_{\bar{p}}$ by $p_{\bar{p}} \restriction (\lambda + \xi_n) = p_{\bar{q}}$, $p_{\bar{p}}(\lambda + \xi_n)(1) = s$ and $p_{\bar{p}}(\lambda + \xi_n)(2) = (\nu_n, F_n)$.

For $j \leq n$, put $\mathring{X}_{\bar{p},j} = \{i < \omega : p_i \restriction [\lambda, \lambda + \xi_j + 1) \in G_{\mathbb{P}}\}$. Clause (3)(f) in Definition 7.7 follows from Claim 7.12. For clause (3)(g), we need to check that $p_{\bar{p}} \Vdash \mathring{m}_1(\mathring{X}_{\bar{p},n}) \geq 1 - 2\varepsilon_n$. Since $p_{\bar{q}} \Vdash \mathring{m}(\mathring{X}_{\bar{p},n-1}) \geq 1 - 2\varepsilon_{n-1}$, $\varepsilon_n \geq 2\varepsilon_{n-1}$, and $p_{\bar{p}} \Vdash \mathring{m}_1(\{i < \omega : p_i \restriction \{\lambda + \xi_j\} \in G_{\mathbb{P}}\}) \geq 1 - \varepsilon_n$ (using Claims 7.12 and 7.13), it follows that $p_{\bar{p}} \Vdash \mathring{m}_1(\mathring{X}_{\bar{p},n}) \geq 1 - 2\varepsilon_{n-1} - \varepsilon_n \geq 1 - 2\varepsilon_n$. Hence $\mathring{m}_1$ satisfies $(t, \bar{k})$. This completes the proof of Lemma 7.9, and therefore of Theorem 1.1. $\square$

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