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Author(s): Uri Avraham and Saharon Shelah
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FORCING WITH STABLE POSETS

URI AVRAHAM AND SAHARON SHELAH

Abstract. The class of stable posets is defined and investigated. We give a forcing construction of a universe of set theory which satisfies a weak form of Martin's Axiom and $2^{\aleph_0} > \aleph_1$ and yet some propositions which follow from CH hold in this universe.

§0. Introduction. We present an axiom which is like Martin's Axiom but deals with a more restricted class of posets than c.c.c. posets dealt by Martin's Axiom. After proving its consistency in §1 we compare this axiom with MA (Martin's Axiom, see [So, Te]) in §2 and §3. Being weaker than MA this axiom gives an intermediate world between that of MA and $V = L$ and thus increases the flexibility in proving consistency results. The axiom was formulated by Shelah who showed the possibility of iterating stable posets and proved 3.1 which shows our axiom permits nonisomorphic $\aleph_1$-dense subsets of the reals to exist. 3.2, 3.5, 3.6 are due to Avraham; 3.2 is an extension of a result of Solovay previously proved for Cohen forcing conditions for adding a real. A. Miller, F. Tall and S. Ben-David made some helpful suggestions.

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§1.

1.1 DEFINITION. For a cardinal κ, a partial order $(P, \leq)$ is *κ-stable* iff for every subset $A \subseteq P$ of cardinality less then κ there is $A^* \subseteq P$ of cardinality less then κ such that any $p \in P$ has an extension $p \leq p' \in P$ and $p^* \in A^*$ such that p' and p^* are compatible exactly with the same elements of A (i.e. for $a \in A$, a and p^* are compatible $\Leftrightarrow$ a and p' are compatible). We say that A and A^* as above "satisfy the requirement of stability". In case $\kappa = \aleph_1$ we say stable instead of $\aleph_1$-stable.

1.2 LEMMA. *If $(P, \leq)$ is κ-stable, satisfies the κ-c.c. (κ a regular cardinal) and if $(\mathbf{Q}, \leq)$ is a name in $V^{RO(P)}$ satisfying with boolean value* 1 *in $V^{RO(P)}$ the κ-stability, then $P * \mathbf{Q}$ is κ-stable.*

PROOF. Let $\{(p_i, q_i), i < \lambda < \kappa\}$ be λ elements of $P * \mathbf{Q}$ (hence $p_i \in P$ and q_i is a name, $\|q_i \in \mathbf{Q}\|^{RO(P)} = 1$). To prove the stability we must find an appropriate set for them. Define $A(q_i) = p_i$, $i < \lambda$. Then A is a name in $V^{RO(P)}$ of a subset of $\mathbf{Q}$ of cardinality $\leq \lambda < \kappa$. As κ remains a cardinal in $V^{RO(P)}$ we can find there, by the κ-stability of $\mathbf{Q}$, a set A^* of cardinality $\lambda_0 < \kappa$ as given by the definition of stability. Write $A^* = \{a_i: i < \lambda_0\}$, where a_i is with boolean value 1 the ith member of A^*.

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Using the κ-c.c. of P we can find $H \subseteq P$ of cardinality less then κ such that for any (p_i, q_i, a_α), $i < \lambda$, $\alpha < \lambda_0$, H contains a maximal antichain from the elements of $\{p \in P \mid p_i \leq p \text{ and } p \Vdash q_i \text{ and } a_\alpha \text{ are compatible in } \mathbf{Q}\}$.

Using the κ-stability of P we get $H \subseteq H^* \subseteq P$, such that H and H^* satisfy the requirement of stability and H^* is of cardinality less then κ.

Taking all the pairs (h^*, a^*), $h^* \in H^*$, $a^* \in A^*$, we claim to arrive at a subset of $P * \mathbf{Q}$ of cardinality $< \kappa$ satisfying the stability requirement for $\{(p_i, q_i), i < \lambda\}$. Indeed, let $(p, q) \in P * \mathbf{Q}$. $p \Vdash^P q$ has an extension and an element in A^* the two of which are compatible with the same elements of A. Hence we can find $p' \geq p$, q' and a_α, $\alpha < \lambda_0$, such that $p' \Vdash q \leq q'$ & q' and a_α are compatible with the same elements of A. By the stability of P an extension $p' \leq p''$ exists having with some $p^* \in H^*$ the same compatibility character over H. Now we will show that (p'', q') and (p^*, a_α) are compatible with exactly the same elements of $\{(p_i, q_i): i < \lambda\}$. This will end the proof as $(p'', q') \geq (p, q)$.

Suppose that (p^*, a_α) and (p_i, q_i) are compatible for some $i < \lambda$. This means that some element in P above p_i and above p^* forces "q_i and a_α are compatible in $\mathbf{Q}$". H was defined so that we can find $h \in H$, $h \geq p_i$, h is compatible with p^* and forces q_i and a_α to be compatible, but as h is compatible with p^*, h is also compatible with p'' so that we have an element above p'' and p_i forcing the compatibility of q_i and a_α, hence this element forces q_i and q' to be compatible (as it forces $q_i \in A$). Hence (p'', q') and (p_i, q_i) are compatible. A similar argument is good for the second direction: if (p'', q'), (p_i, q_i) are compatible then so are (p^*, a_α) and (p_i, q_i). □

Now we have to deal with the limit stages and show that when taking the direct limit the κ-stability holds at a limit stage if it holds at all stages before. So let $(P_\alpha, \leq)$, $\alpha < \delta$, be κ-stable posets such that $\alpha < \beta \Rightarrow P_\alpha \subseteq P_\beta$ and there is a projection $h_{\beta\alpha}: P_\beta \to P_\alpha$ i.e. $h_{\beta\alpha}$ is an order preserving map of P_β onto P_α such that:

(I) $h_{\beta\alpha}(p) = p$ for $p \in P_\alpha$,

(II) $h_{\beta\alpha}(q) \leq q$ for $q \in P_\beta$ and

(III) if $h_{\beta\alpha}(q) < p$ for $p \in P_\alpha$ then for some $q' > q$, $h_{\beta\alpha}(q') = p$. The projections commute. In this case we have

1.3 Lemma. *If* $\forall\alpha < \delta$, $(P_\alpha, \leq)$ *is* κ-*stable then* $\bigcup_{\alpha<\delta} P_\alpha$ *is* κ-*stable.*

Proof. Let $A \subseteq \bigcup_{\alpha<\delta} P_\alpha$ be given, $|A| < \kappa$; we must find $A^* \supseteq A$ to satisfy the requirement of stability. For every $a \in A$ there is a minimal index α such that $a \in P_\alpha$, call such as index a *minimal index*; if α is a supremum of a subset of minimal indexes call α a *relevant index*, then only less than κ indexes are relevant. Hence we can find $A' \supseteq A$, $|A'| < \kappa$ such that if $\xi < \delta$ si a relevant index then the projection of any $a \in A$ in P_ξ is in A'.

Using the κ-stability of each P_α, $\alpha < \delta$, and the regularity of κ, we can find $A^* \subseteq \bigcup_{\alpha<\delta} P_\alpha$, $|A^*| < \kappa$, such that for every relevant index α, $A' \cap P_\alpha$ and $A^* \cap P_\alpha$ satisfy the requirement of stability. We want to show that A and A^* are as required by the definition of stability. So, let $p \in \bigcup_{\alpha<\delta} P_\alpha$ be given. If for some relevant α, $p \in P_\alpha$, then taking such α we can find in P_α, $p' \geq p$ and $p^* \in A^*$ compatible with the same elements of $A' \cap P_\alpha$. It follows that they are compatible with the same elements of A. (For $a \in A$ the projection of a in P_α, a', is in A', using (III) in the definition of projection we see that p' and a are compatible iff p' and a' iff p^* and a' iff p^* and a are compatible.) If for no relevant α, $p \in P_\alpha$, we let $\alpha < \delta$ be the maximal relevant index and set $q \in P_\alpha$ to be the projection of p in P_α. We can find $q' \geq q$ and q^* in P_α, $q^* \in A^*$, compatible with the same element of A. By (III) we find $p' \geq p$ such that the projection of p' in P_α is q', It follows that p' and q^* are compatible with the same elements of A. □

Starting from a world with a regular cardinal $\lambda > \aleph_1$ satisfying $\forall l < \lambda(2^l \leq \lambda)$, $\kappa<\lambda$, κ regular, we can iterate just as in [So, Te] λ times c.c.c. posets κ-stable and of cardinality $< \lambda$ such that in the end a c.c.c. κ-stable poset of cardinality λ is obtained and in its boolean valued world $2^{\aleph_0} = \lambda$ and MA holds for c.c.c. κ-stable posets, i.e. the following proposition holds: If $(P, \leq)$ is a c.c.c. κ-stable poset and $D_\alpha, \alpha < \eta < 2^{\aleph_0}$, are dense subsets of P then there is a filter G over P intersecting every D_α. We use the argument, using c.c.c., that a κ-stable c.c.c. poset of power $< \lambda$ in the extension was κ-stable in the intermediate worlds in which it appeared, and thus was forced at cofinally many stages. For the axiom to apply to c.c.c. κ-stable posets of arbitrary cardinality, a Lowenheim-Skolem argument is needed.

§**2. The consequences of** MA **for stable posets.** Martin's Axiom for c.c.c. stable posets is weaker than the full Martin's Axiom. In this section we investigate what consequences of MA hold also for the restricted version. We assume $2^{\aleph_0} > \aleph_1$ and MA for c.c.c. stable posets for the rest of §2.

2.1 *Souslin Hypothesis.* Every tree all of whose levels are countable is stable, hence there are no Souslin trees (and every Aronszajn tree is special), see [B, M, R].

2.2. Every ladder system on ω_1 can be uniformized, see [D, S2].

2.3. Hence, there is a nonmetrizable normal Moore space, see [D, S2].

2.4. Every subset of the reals of cardinality less than the continuum is of measure zero. (The appropriate poset to do this is even countable, see [M, S].)

2.5. The intersection of less than $2^{\aleph_0}$ dense open sets of reals is dense. (The poset for adding a Cohen real is countable.)

2.6. It follows from 2.5 that no $\kappa \leq 2^{\aleph_0}$ is a real-valued measurable cardinal, see [M, S].

2.7. $2^{\aleph_0} = 2^{\aleph_1}$. Otherwise, $2^{\aleph_0} < 2^{\aleph_1}$ contradicts 2.2, see [D, S1].

2.8 *Questions.* (a) Does $2^\mu = 2^{\aleph_0}$ for all $\mu < 2^{\aleph_0}$?

(b) Is $2^{\aleph_0}$ regular?

§**3.** Now we give some proofs showing that if we start from a world V where the G.C.H. holds and forces with a c.c.c. stable poset of conditions then some properties of V still hold in the extension. This shows that restriction to stable posets gives an intermediate situation between MA and G.C.H. Let V' be an extension of V via c.c.c. stable poset. Define two subsets of the reals to be *far apart* if there is no isomorphsim between uncountable subsets of them. (Note that with the help of $2^{\aleph_0} = \aleph_1$ we can construct in V two subsets A, B of the reals which are far apart, dense and each has $\aleph_1$ points between any two of its points. See [D, M] and [Si].)

3.1. *Any far apart $\aleph_1$-dense subsets of the reals in V remain far apart in V'.*

Hence if C.H. holds in V we have two $\aleph_1$-dense nonisomorphic subsets of the reals in V'. (This is to be contrasted with Baumgartner's result [B], see also [A, S].)

PROOF. Let P be a c.c.c. stable poset. We will show that in $V^{RO(P)}$ A and B are still far apart. *If not* there is in $V^{RO(P)}$ an uncountable subset $E \subseteq A$ and an isomorphism $f\colon E \to B$. Find $F \subseteq E$ countable and dense in E. As P satisfies the c.c.c. we can find in V a countable $F' \subseteq A$ such that for every real e, if $\|e \in F\|^{RO(P)} > 0$ then $e \in F'$. For any $e \in F'$ pick a maximal antichain from $\{p \in P \mid p \Vdash f(e) = b$ for some $b \in B\}$. Let P_0 be a countable subset of P which includes all those chosen antichains and find P_1, a countable subset of P as given by stability. In V there are $\aleph_1$ reals a in A with $\|a \in E\| > 0$. For any such a find $p_a \in P$ forcing $a \in E$ and $f(a) = b_a$ for some $b_a \in B$. By stability find $p'_a \geq p_a$ and $p^*_a \in P_1$ such that p'_a and p^*_a are compatible with the same elements of P_0. As we do this for uncountably many a's and P_1 is countable, we can find an uncountable $E^* \subseteq A$ and $p^* \in P_1$ such that $p^*_a = p^*$ for any $a \in E^*$. We claim that the mapping $a \to b_a$ is an isomorphism defined on the uncountable subset E^* of A and this is a contradiction. To see the claim let $a_1, a_2 \in E^*$, $a_1 < a_2$, $p_{a_i} \Vdash a_i \in E$ and $f(a_i) = b_{a_i}$, $i = 1, 2$. Find an extension of p'_{a_1} forcing $e \in F$ for some $a_1 < e < a_2$ ($p_{a_1} \Vdash a_1 \in E$, and by diluting we can assume that every member of E has, at its right, members of F as near as we wish).

By the definition of P_0, as $e \in F'$, we see that p'_{a_1} is compatible with some $p \in P_0$ which forces $f(e) = \xi$ for some $\xi \in B$, hence $b_{a_1} < \xi$. By stability p^* is compatible with p and so p'_{a_2} is compatible with p and as $e < a_2$ it follows that $\xi < b_{a_2}$, hence $b_{a_1} < b_{a_2}$. □

Using the ideas of 3.1 for dealing with stable posets we show

3.2. *In V', R^V (the reals of V) is not of first category.*

Thus, since R^V is of cardinality $\aleph_1$, and $MA_{\aleph_1}$ implies that every set of reals of cardinality $\aleph_1$ has first category [M, S], $MA_{\aleph_1}$ fails in V'. As R^V is a group it does not have the Baire property in V'. We shall prove that in V', R^V intersects every G_δ dense subset of the reals X and that this intersection even contains an "old" G_δ dense subset of R^V. In V' let $X = \bigcap_{k<\omega}\bigcup_{n<\omega} A_{n,k}$ where $A_{n,k}$ is an open rational interval, such that for any k, $\bigcup_{n<\omega} A_{n,k}$ is dense. V' has been obtained by forcing with a c.c.c. stable poset P. Suppose $\phi \Vdash \forall k \bigcup_{n<\omega} A_{n,k}$ is dense and $\forall n,k$, $A_{n,k}$ is an open rational interval. In V define $B = \{x \in R^V \mid \phi \Vdash x \in X\}$. We want to show that $B \neq \phi$ and even that B is a G_δ dense subset of R^V.

Find a countable $A \subseteq P$ such that for any n, $k < \omega$ and $r_0, r_1 \in Q$ (Q = the rationals) there is in A a maximal antichain from those conditions forcing $A_{n,k} = (r_0, r_1)$ ((r_0, r_1) is the open interval). Let $A^* \subseteq P$ be the countable subset given by the stability. Define in V

$$H = \bigcap_{k<\omega} \bigcap_{q \in A^*} \bigcup_{n<\omega} \bigcup_{r_0, r_1 \in Q} \{x \mid x \in (r_0, r_1) \text{ and } q \nVdash A_{n,k} \neq (r_0, r_1)\}.$$

As A^* is countable H is clearly dense G_δ in V. All we need to show is that $x \in H \to \phi \Vdash x \in X$. Indeed if $x \in H$, let $p \in P$ be any condition and $k < \omega$. We will find an extension of p forcing $x \in \bigcup_{n<\omega} A_{n,k}$. There are $p' \geq p$ and $p^* \in A^*$, p' and p^* are compatible with the same elements of A. As $p^* \in A^*$ and $x \in H$ there are $n < \omega$ and $r_0, r_1 \in Q$ such that $p^* \nVdash A_{n,k} \neq (r_0, r_1)$ and $x \in (r_0, r_1)$. Recalling the definition of A we see that P^* is compatible with some $t \in A$ such that $t \Vdash A_{n,k} = (r_0, r_1)$ but then p' is compatible with t, i.e., we have an extension of p' forcing $x \in A_{n,k}$. □

Likewise if we assume that in V the intersection of fewer than κ G_δ dense sets contains a G_δ set then after a κ-stable forcing extension every G_δ dense set intersects R^V on an old G_δ dense set.

3.3. *If CH holds in V, P does not hold in V'.* P is the assertion that if $A_\alpha \subseteq \omega$, $\alpha < \omega_1$, are $\aleph_1$ subsets of ω the intersection of every finite number of them is infinite *then* there is an infinite subset $s \subseteq \omega$ such that $s - A_\alpha$ is finite for any $\alpha < \omega_1$. MA + $2^{\aleph_0} > \aleph_1$ implies P, see [M, S] and [K, T].

We prove something stronger: if in V we have $A_\alpha \subseteq \omega$, $\alpha < \omega_1$, such that the intersection of every finite subset of them is infinite but for any uncountable $X \subseteq \omega_1$, $\bigcap_{\alpha\in X} A_\alpha$ is finite then in V' this property remains and so no infinite $s \subseteq \omega$ is almost contained in uncountably many A_α's. (With CH one can build such A_α's.) Suppose, in order to get a contradiction, that $\phi \Vdash \bigcap_{\alpha\in X} A_\alpha = A$ is infinite and X is uncountable. In V find a countable $P_0 \subseteq P$ such that for every $n \in \omega$ there is in P_0 a maximal antichain from those conditions forcing $n \in A$. Let $P_1 \subseteq P$ be a countable set as given by stability. In V find an uncountable $X' \subseteq \omega_1$ such that for $\alpha \in X'$ there is a condition p_α forcing $\alpha \in X$. Then we can find $X'' \subseteq X'$ uncountable, $p'_\alpha \geq p_\alpha$ for $\alpha \in X''$ and $p \in P_1$ such that p'_α and p are compatible with the same members of P_0. Now, $\bigcap_{\alpha\in X''} A_\alpha = F$ a finite subset; p is compatible with some $q \in P_0$ such that $q \Vdash n \in \bigcap_{\alpha\in X} A_\alpha$, for some $n \notin F$, hence every p'_α, $\alpha \in X''$, is compatible with that q and so $n \in \bigcap_{\alpha\in X''} A_\alpha$, contradiction. □

F. Tall remarks that 3.3 follows from 3.2; Rothberger [R1] proved that $P \Rightarrow$ every set of reals of power $\aleph_1$ is of first category. A. Miller observed that, by 3.2, for every ultrafilter U in V there is no infinite X in V' almost contained in every element of U.

On the other hand, S. Ben-David and the first author observed that one can construct a filter such that for no infinite $s \subseteq \omega$ s is almost contained in every one of its elements but adding a Cohen real creates some such infinite s. ${}^{\omega}2$ is the binary tree; its branches give $2^{\aleph_0}$ almost disjoint subsets of ω and extend to a maximal family of almost disjoint infinite subsets of ω; $\{b_i \mid i < 2^{\aleph_0}\}$. Now $A_i = \omega - b_i$ has the finite intersection property, no infinite set is almost contained in every A_i (by maximality). If $r \subseteq {}^{\omega}2$ is the generic branch then r is almost contained in every A_i, for suppose $p \Vdash r \cap b_i$ is infinite, then we can construct an old branch intersecting b_i infinitely many times.

A. Miller remarked further that assuming R^V is of second category (3.2) and of power $\aleph_1$, a theorem of Rothberger implies.

3.4. *In V' the union of $\aleph_1$ sets of Lebesgue measure zero is not necessarily of measure zero.* (See [R2].)

3.5. It is possible to combine the full power of Martin's Axiom for posets of cardinality $\aleph_1$ for example and the restricted Martin's Axiom for $\aleph_2$-stable posets. To do this, force MA + $2^{\aleph_0} = \aleph_2$ and then continue iteration with $\aleph_2$-stable c.c.c. posets ω_3 times so as to obtain $2^{\aleph_0} = \aleph_3$ and Martin's Axiom for $\aleph_2$-stable posets. As any poset of cardinality $\aleph_1$ is $\aleph_2$-stable Martin's Axiom holds for all posets of cardinality $\aleph_1$, hence, for example, P does hold for any family of cardinality $\aleph_1$ but P does not hold for family of cardinality $\aleph_2$.

3.6. At first one might think that the reason why our model is close to L (we started the forcing from L) is that every real is included in a world obtained by adding a Cohen generic real. However the arguments of Devlin and Shelah [D, S1] and the fact that in our world every coloring of a ladder system on ω_1 is uniformizable, give us a real which is not defined by a Cohen generic. We believe the following picture will be easily understood by anyone who reads [D, S1]. In L take a ladder system $\langle \eta_\delta \mid \delta \in \Omega \rangle$ and color it $\langle h_\delta \mid \delta \in \Omega \rangle$ such that there is no uniformization for that coloring in L even on a stationary subset of ω_1. Define $F: 2^{\omega_1} \to 2$ as follows for $f: \delta \to 2$. $F(f) = 0$ iff $f(\eta_\delta(m)) = 0$ from some m onward. Let $f: \omega_1 \to 2$ be a uniformization for $\langle h_\delta \mid \delta \in \Omega \rangle$. We define by induction on $\omega \times \omega$ functions $f_{n,k}$, $g_{n,k}$: $\omega_1 \to 2$ the following way: $g_{1,n}(\omega \cdot \alpha) = f(\alpha)$ for $n < \omega$, $\alpha < \omega_1$; $f_{1,n}$ satisfies $F(f_{1,n} \upharpoonright \alpha) = g_{1,n}(\alpha)$ for every limit α. Suppose $g_{i,n}$, $f_{i,n}$, $i < k$, $n < \omega$, have been defined. Then for limit α, $\langle g_{k,n}(\alpha) \mid n < \omega \rangle$ codes in an absolute way $\langle g_{k-1,n} \upharpoonright \alpha + \omega, f_{k-1,n} \upharpoonright \alpha + \omega \mid n < \omega \rangle$ and then $f_{k,n}$, $n < \omega$, satisfies $F(f_{k,n} \upharpoonright \alpha) = g_{k,n}(\alpha)$ for all limit α. If the real a gives the information of $\langle g_{i,k}(\omega), f_{i,k}(\omega) \mid i, k < \omega \rangle$ then in $L[a]$ we can reconstruct $f_{k,n}$ and $g_{k,n}$, hence in $L[a]$ there is a uniformization of $\langle h_\delta \mid \delta \in \Omega \rangle$ and if a is obtained via a countable set of forcing conditions then in L we have a uniformization on a stationary subset which is impossible.

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HEBREW UNIVERSITY
JERUSALEM, ISRAEL