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Universal graphs omitting finitely many finite graphs

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ABSTRACT

If $\mathcal{F}$ is a family of graphs, then a graph is $\mathcal{F}$-free, if it contains no induced subgraph isomorphic to an element of $\mathcal{F}$. If $\mathcal{F}$ is a finite set of finite graphs, λ is an infinite cardinal, we let $\mathrm{CF}(\mathcal{F}, \lambda)$ be the minimal number of $\mathcal{F}$-free graphs of size λ such that each $\mathcal{F}$-free graph of size λ embeds into some of them. We show that if $2^{<\lambda} = \lambda$, then $\mathrm{CF}(\mathcal{F}, \lambda) \leq \mathfrak{c}$ (continuum), there are examples such that $\mathrm{CF}(\mathcal{F}, \lambda)$ is finite but can be arbitrarily large, and give an example such that $\mathrm{CF}(\mathcal{F}, \lambda) \geq \mathfrak{c}$ for any infinite cardinal λ.

1. Introduction

It was R. Rado [7,8] who first constructed a countable graph containing all countable graphs as an induced subgraph. This became known as a *universal countable graph*. In general, if $\mathcal{K}$ is a class of graphs, λ is an infinite cardinal, we let $\mathcal{K}_\lambda$ be the set of graphs in $\mathcal{K}$ of cardinality λ and call $G \in \mathcal{K}_\lambda$ a *universal graph in* $\mathcal{K}_\lambda$ if all graphs in $\mathcal{K}_\lambda$ can strongly (i.e., isomorphically) be embedded into G. There is a huge literature on universal countable graphs omitting a subgraph, see e.g., [2].

We are interested in the case when $\mathcal{K}$ consists of those graphs, which has no induced subgraph (isomorphic to a graph) in $\mathcal{F}$, where $\mathcal{F}$ is a finite set of finite graphs ($\mathcal{F}$-free graphs). For example, if $\mathcal{F}$ consists of a single finite path, then there is a universal graph in every infinite cardinal [4]. If $\mathcal{F}$ consists of a finite clique, $2^{<\lambda} = \lambda$, then an argument similar to Rado's shows that there is a universal graph of cardinality λ. If there is no universal graph in some $\mathcal{K}_\lambda$, it still can happen that there is a small subset L of $\mathcal{K}_\lambda$ such that each $G \in \mathcal{K}_\lambda$ strongly embeds into some $H \in L$. Let $\mathrm{CF}(\mathcal{F}, \lambda)$ denote the minimum size $|L|$ such that an L as above exists for $\mathcal{F}$-free graphs of size λ. (This notion was introduced in [5] and investigated there and in [6].)

Here we prove that $\mathrm{CF}(\mathcal{F}, \lambda) \leq \mathfrak{c}$ for any finite set of graphs $\mathcal{F}$ and any cardinal λ with $2^{<\lambda} = \lambda$ (Theorem 1). In some cases, under the same conditions, there is a universal graph, namely if $\mathcal{F}$ contains only connected graphs, or contains only complements of connected graphs, in particular, if $|\mathcal{F}| = 1$ (Theorem 2).

We give examples $\mathcal{F}_k$ ($k = 2, 3, \ldots$) with $\mathrm{CF}(\mathcal{F}_k, \lambda)$ finite but arbitrarily large (Theorem 3), and we construct a set $\mathcal{F}$ such that $\mathrm{CF}(\mathcal{F}, \lambda) \geq \mathfrak{c}$ for any infinite cardinal λ (Theorem 4).

Notation. Definitions. $\mathfrak{c}$ denotes the cardinality continuum. If f is a function, A a set, then $f[A] = \{f(x) : x \in A\}$. If S is a set, κ a cardinal, then $[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$. A *graph* is a pair (V, X) where $X \subseteq [V]^2$. If (V, X), (V', X') are graphs a *weak embedding* is $f : V \to V'$ where for $x \neq y \in V$ $\{x, y\} \in X$ implies $\{f(x), f(y)\} \in X'$. A *strong embedding* is $f : V \to V'$ where $\{x, y\} \in X$ iff $\{f(x), f(y)\} \in X'$.

If $\mathcal{K}$ is a class of graphs, λ an infinite cardinal, then $\mathrm{CF}(\mathcal{K}_\lambda)$ is $\min |L|$ where $L \subseteq \mathcal{K}_\lambda$, such that each $G \in \mathcal{K}_\lambda$ strongly embeds into some element of L. If $\mathcal{F}$ is a set of graphs, $\mathrm{CF}(\mathcal{F}, \lambda)$ is $\mathrm{CF}(\mathcal{K}_\lambda)$ where $\mathcal{K}$ is the set of $\mathcal{F}$-free graphs.

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2. Results

Theorem 1. *If $\mathcal{F}$ is a set of finite graphs, λ is an infinite cardinal, $2^{<\lambda} = \lambda$, then* $\mathrm{CF}(\mathcal{F}, \lambda) \le \mathfrak{c}$.

Proof. Let T be the theory in the language of graphs stating that there is no induced copy of any $F \in \mathcal{F}$. There are continuum many complete extensions of T, $\{T_\alpha : \alpha < \mathfrak{c}\}$. For each T_α there is a universal model (V_α, X_α) of T_α of cardinality λ by the existence theorem of saturated and special models (see [1], Proposition 5.1.8 and Theorem 5.1.16). Then $\{(V_\alpha, X_\alpha) : \alpha < \mathfrak{c}\}$ witness that $\mathrm{CF}(\mathcal{F}, \lambda) \le \mathfrak{c}$. □

Theorem 2. *If $\mathcal{F}$ is any family of finite graphs, $\lambda > \omega$ is a cardinal, $2^{<\lambda} = \lambda$, then there exists a strongly universal $\mathcal{F}$-free graph of cardinality λ, assuming either*

(a) *each element of $\mathcal{F}$ is connected, or*
(b) *the complement of each element of $\mathcal{F}$ is connected, or*
(c) $|\mathcal{F}| = 1$.

Proof. (a) Let $\{(V_\alpha, X_\alpha) : \alpha < \mathfrak{c}\}$ be the set of graphs as in the proof of Theorem 1 and let (V, X) be the vertex disjoint union of the (V_α, X_α)'s.

The graph (V, X) clearly isomorphically embeds each $\mathcal{F}$-free graph of cardinality λ, and omits each $F \in \mathcal{F}$, as F is connected.

(b) If $\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$, by (a) there is a universal graph U for $\overline{\mathcal{F}}$. Then $\overline{U}$ is universal for $\mathcal{F}$.

(c) follows from (a) or (b) as for a finite graph X either X or $\overline{X}$ is connected. □

Let $\mathcal{F}_k$ be the set of all nonisomorphic graphs on $2k$ vertices, each containing a 1-factor (i.e., k independent edges). Clearly, $|\mathcal{F}_k| \le 2^{\binom{2k}{2}-k} < 2^{2k^2}$. Z. Füredi noticed that almost every random graph on $2k$ vertices has a 1-factor, and does not have automorphisms, consequently

$$|\mathcal{F}_k| \sim \frac{2^{\binom{2k}{2}}}{(2k)!}.$$

Notice that a graph X is $\mathcal{F}_k$-free iff there are no k independent edges in X.

Theorem 3 ($2^{<\lambda} = \lambda$).

(a) $\mathrm{CF}(\mathcal{F}_2, \lambda) = 2$.
(b)

$$2^{\lfloor\sqrt{k}\rfloor} < p(k) \le \mathrm{CF}(\mathcal{F}_{k+1}, \lambda) \le 2^{2k^2}(k+1)^{2^{2k}} \quad (k \ge 2).$$

Here $p(k)$ is the number of partitions of k, i.e., the number of sequences $\langle a_1, \ldots, a_r\rangle$ of positive natural numbers such that $a_1 \ge \cdots \ge a_r$ and $a_1 + \cdots + a_r = k$. $2^{\lfloor\sqrt{k}\rfloor} < p(k)$ is an easy lower estimate and in fact the

$$p(k) \sim \frac{1}{4k\sqrt{3}} e^{\pi\sqrt{2k/3}}$$

asymptotics holds (Hardy–Ramanujan, [3]).

Proof. (a) Two of the $\mathcal{F}_2$-free graphs are X_0, the triangle, and X_1, the path of length 2 (i.e., the path containing two edges). However, inspection shows that if some graph Y isomorphically embeds both X_0 and X_1, then Y contains two independent edges.

For the other direction, if $X_0(\lambda)$ is a triangle plus λ isolated vertices and $X_1(\lambda)$ is a λ-star plus λ isolated vertices, then every $\mathcal{F}_2$-free graph of size λ embeds either into $X_0(\lambda)$ or into $X_1(\lambda)$.

(b) If $a_1 + \cdots + a_r = k$, $a_1 \ge a_2 \ge \cdots \ge a_r \ge 1$, then let $X(a_1, \ldots, a_r)$ be the vertex disjoint union of $C_{2a_1+1}, \ldots, C_{2a_r+1}$. It is easy to see that in $X(a_1, \ldots, a_r)$ the maximal number of independent edges is k, and this also holds if a vertex is removed. We can construct $p(k)$ different graphs this way.

Claim 1. *If X contains no more than k independent edges and X is an induced subgraph of Y which is isomorphic to some $X(a_1, \ldots, a_r)$, then Y contains no edges outside X (just vertices).*

Proof. Let V, W denote the vertex set of Y, X, respectively. As X already contains k independent edges, no edge goes between two vertices of $V - W$. Assume that $a \in W$, $b \in V - W$ and $\{a, b\}$ is an edge of Y. Then there are k independent edges in $Y|(W - \{a\})$, adding $\{a, b\}$ gives $k + 1$ independent edges, a contradiction. □

By Claim 1, if for $\overline{a} = \langle a_1, \ldots, a_r\rangle$, $Y(\overline{a})$ is an $\mathcal{F}_{k+1}$-free graph of size λ embedding $X(\overline{a})$, then $Y(\overline{a})$ is $X(\overline{a})$ plus λ isolated vertices, consequently if $\overline{a} \ne \overline{a}'$, then $Y(\overline{a}) \ne Y(\overline{a}')$. This gives $\mathrm{CF}(\mathcal{F}_{k+1}, \lambda) \ge p(k)$.

For the upper bound, let (V, X) be an $\mathcal{F}_k$-free graph. Choose $A \subseteq V$, $|A| = 2k$, that contains a maximal number of independent edges. There are at most $2^{\binom{2k}{2}} \leq 2^{2k^2}$ graphs on A. By maximality, there is no edge inside $V - A$, so all edges not in A go between A and $V - A$.

Define $H(B) = \{x \in V - A : \Gamma(x) = B\}$ for $B \subseteq A$. If, for some B, $|H(B)| > k$, we add λ more vertices to $H(B)$, and so obtain the set $H^*(B)$, all whose points are joined exactly to B. This gives the enlarged graph X^*.

Claim 2. *There are no $k+1$ independent edges in X^*.*

Proof. Let I be a set of $k+1$ independent edges in X^*. For a fixed $B \subseteq A$, let $\{\{x_i, y_i\} : i < r\}$ be the elements of I with $x_i \in B$, $y_i \in H^*(B)$. Notice that the x_i's are distinct and so are the y_i's. We replace the y_i's with distinct $z_i \in H(B)$, clearly we can do that. Performing this operation for all $B \subseteq A$, we obtain $k+1$ independent edges in X, a contradiction. □

By Claim 2, the graphs of the form X^* embed all graphs omitting $\mathcal{F}_{k+1}$. We have to calculate the number of them: there are 2^{2k^2} graphs on A, for each $B \subseteq A$, $|H^*(B)|$ is either at most k or λ, that gives $k+1$ possibility, the number of different B's is 2^{2k}, in total this gives $2^{2k^2}(k+1)^{2^{2k}}$ possibilities. □

Theorem 4. *There is a finite set $\mathcal{F}$ of finite graphs such that $\mathrm{CF}(\mathcal{F}, \lambda) \geq \mathfrak{c}$ holds for every infinite cardinal λ.*

Proof. We start with a Claim.

Claim 1. *There exist graphs G_0, G_1, and G_2 with distinct specific vertices $x(G_i)$ and $y(G_i)$, such that*

(A) *G_i is connected $(i < 3)$;*
(B) *the triangles of G_i cover the edges of G_i $(i < 3)$;*
(C) *if f is an isomorphism of G_i into an induced subgraph of G_j, then $i = j$, $f(x(G_i)) = x(G_i)$, and $f(y(G_i)) = y(G_i)$.*

Proof. Let T_0, T_1, T_2 be trees with specified neighboring vertices $x(T_i)$, $y(T_i)$, whose degrees are (8,3), (7,4), and (6,5), respectively, and with all other vertices being of degree one. Then erect a triangle on each edge, to obtain G_i from T_i. That is, if $\{a, b\}$ is an edge of T_i, then add a new vertex $p_{a,b}$ and the edges $\{a, p_{a,b}\}$ and $\{b, p_{a,b}\}$ to G_i. Further, set $x(G_i) = x(T_i)$, $y(G_i) = y(T_i)$. Notice that T_i has 13 vertices and 12 edges, all vertices but $x(T_i)$ and $y(T_i)$ have degree 1. G_i has 25 vertices, $d(x(G_i)) = 2d(x(T_i))$ and $d(y(G_i)) = 2d(y(T_i))$, the other vertices have degree 2.

(A) and (B) hold clearly, (C) follows from degree considerations. □

If $u \subseteq \omega - \{0, 1\}$, then we obtain the graph H_u as follows. The vertex set P^u is the disjoint union $P^u_0 \cup P^u_1 \cup P^u_2 \cup P^u_3 \cup \cdots$ with $|P^u_i| = 25$, each P^u_i inducing some $G_{j(i)}$, with x^u_i, y^u_i being the special vertices $x(G_{j(i)})$ and $y(G_{j(i)})$. Further, $j(0) = j(1) = 0$, $j(i) = 1$ if $i \in u$ and $j(i) = 2$ if $2 \leq i \in \omega - u$. Finally, the 'crossing' edges are of the type $\{x^u_i, y^u_j\}$ where $j = i+1$ or $i+2$.

Claim 2. *If T is a triangle in H_u, then $T \subseteq P^u_i$ for some $i < \omega$.*

Proof. Assume indirectly that T is a triangle of H_u such that $T \not\subseteq P^u_i$ for $i < \omega$. Set $B = \{i < \omega : T \cap P^u_i \neq \emptyset\}$.

Case 1. $|B| = 3$.

Let $B = \{i, j, k\}$ with $i < j < k$. The only edges between P^u_i, P^u_j, and P^u_k can be $\{x^u_i, y^u_j\}$, $\{x^u_i, y^u_k\}$, and $\{x^u_j, y^u_k\}$ but they do not contain a triangle.

Case 2. $|B| = 2$.

Set $B = \{i, j\}$ with $i < j$. There is at most one edge between P^u_i and P^u_j which contradicts the fact that either $|T \cap P^u_i| = 2$ or $|T \cap P^u_j| = 2$.

We obtain that $|B| = 1$, as claimed. □

Claim 3. *If $K \subseteq P^u$ induces G_j for some $j < 3$, then $K = P^u_i$ for some i.*

Proof. By the way G_j is constructed, $K = T_0 \cup \cdots \cup T_k$ where each T_m induces a triangle and $T_m \cap (T_0 \cup T_1 \cup \cdots \cup T_{m-1}) \neq \emptyset$. Pick i such that $T_0 \cap P^u_i \neq \emptyset$. By Claim 2, $T_0 \subseteq P^u_i$ and one obtains by induction on m that $T_m \subseteq P^u_i$, finally we have that $K \subseteq P^u_i$. By the above properties of the G_j's, we must have $K = P^u_i$. □

Let $\mathcal{F}$ be the set of finite graphs (V, X) such that either

(a) $V = A \cup B$, $A \cap B \neq \emptyset$, $X|A \simeq G_i$, $X|B \simeq G_j$ for some i, j, or
(b) $V = A \cup B \cup C$, A, B, C are pairwise disjoint, $X|A \simeq X|B \simeq X|C \simeq G_0$, or
(c) V partitions as $V = A_0 \cup A_1 \cup A_2 \cup A_3$, $X|A_i \simeq G_{j_i}$ $(i < 4)$, with the corresponding vertices x_i, y_i, and $\{x_0, y_2\}, \{x_0, y_3\}, \{x_1, y_2\}, \{x_1, y_3\} \in X$.

Claim 4. *Each H_u is $\mathcal{F}$-free.*

Proof. By Claim 3 and the way H_u is defined. □

Notice that each H_u remains $\mathcal{F}$-free if an arbitrary number of isolated vertices are added.

Assume now that λ is an infinite cardinal, $\{(V_i, E_i) : i \in I\}$ are $\mathcal{F}$-free graphs of cardinality λ, $|I| < \mathfrak{c}$ and each H_u embeds into some (V_i, E_i).

There is some $i_0 \in I$ such that (V_{i_0}, E_{i_0}) embeds infinitely many of the H_u's.

In (V_{i_0}, E_{i_0}), by (a) and (b), there are at most (actually, exactly) 2 copies of G_0. There are, therefore, $u \neq u'$, with the embeddings $f : P^u \to V_{i_0}$, $f' : P^{u'} \to V_{i_0}$, such that $Q_i = f[P_i^u] = f'[P_i^{u'}]$ $(i < 2)$, $f(x_i^u) = f'(x_i^{u'}) = x_i$, $f(y_i^u) = f'(y_i^{u'}) = y_i$ $(i < 2)$.

Claim 5. $f[P_n^u] = f'[P_n^{u'}]$ *$(n < \omega)$.*

Proof. We prove the statement by induction on n. By assumption, we know this for $n = 0, 1$. Assume we have it for $0, 1, \ldots, n$. Let Q_i be the common value of $f[P_i^u]$ and $f'[P_i^{u'}]$ for $i \leq n$. Further, let x_i, y_i be the uniquely determined common values of $f(x_i^u) = f'(x_i^{u'})$ and $f(y) = f'(y_i^{u'})$. Then, as (V_{i_0}, E_{i_0}) is $\mathcal{F}$-free, $Q_{n+1} = f[P_{n+1}^u] = f'[P_{n+1}^{u'}]$ is that uniquely determined set that Q_{n+1} induces a copy of some G_j $(j < 3)$ such that x_{n-1} and x_n are joined to the y vertex of $E_{i_0}|Q_{n+1}$. This follows from conditions (a) and (c) above. □

From Claim 5 we obtain that $H_u|P_n^u \simeq H_{u'}|P_n^{u'}$ for $n < \omega$, which implies $u = u'$ a contradiction. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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