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A Calculation of Injective Dimension over Valuation Domains.

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This paper takes up a problem which was posed in a paper by S. Bazzoni [B], about the injective dimension of certain direct sums of divisible modules over a valuation domain. We refer the reader to that paper for the motivation for the problem. We shall make use of the same notation as in [B], which we now proceed to review.

Let R be a valuation domain of global dimension $n+1$, where $n \geqslant 2$. Let $\{L_\alpha : \alpha \in \Lambda\}$ be a family of archimedean ideals of R, where Λ is a set of cardinality $\geqslant \aleph_{n-2}$. For each α let I_α be the injective envelope of R/L_α. Let $I = \prod_{\alpha \in \Lambda} I_\alpha$, and for each $1 \leqslant k \leqslant n$, let D_{n-k} be the submodule of I consisting of those elements having support of cardinality $< \aleph_{n-k}$, i.e. for all $y \in I$, y belongs to D_{n-k} if and only if the cardinality of

$$\{\alpha \in \Lambda : y(\alpha) \neq 0\}$$

is strictly less than $\aleph_{n-k}$.

Bazzoni proves in [B] that the injective dimension of D_{n-k} is at most k. She also shows that the injective dimension of D_{n-1} is exactly 1 and that it is consistent with ZFC that the injective dimension of

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D_{n-2} is exactly 2. It is the main purpose of this paper to prove this latter result in ZFC. In fact we prove:

THEOREM. *The injective dimension of D_{n-k} is $\geqslant 2$ if $2 \leqslant k \leqslant n$.*

Before proving the theorem we prove some lemmas. The first of these is a combinatorial fact. (Compare [Sh; § 6].)

LEMMA 1. *Let $\varkappa$ be a regular cardinal. There exists a family $\{w_\nu^\alpha: \alpha < \varkappa^+, \nu < \varkappa\}$ of subsets of $\varkappa^+$ satisfying for all $\alpha < \varkappa^+$:*

(1) $\alpha = \bigcup\limits_{\nu<\varkappa} w_\nu^\alpha$;

(2) *for all* $\nu < \mu < \varkappa$, $w_\nu^\alpha \subseteq w_\mu^\alpha$;

(3) *for all* $\nu < \varkappa$ *and all* $\beta < \alpha$, $\beta \in w_\nu^\alpha \Rightarrow w_\nu^\beta = w_\nu^\alpha \cap B$;

(4) *for all* $\nu < \varkappa$, *the cardinality of* w_ν^α *is* $< \varkappa$.

PROOF. We shall define the w_ν^α for all ν by induction on α. Let $w_\nu^0 = \emptyset$ for all ν. Now suppose that w_ν^β has been defined for all $\beta < \alpha$. If α is a successor ordinal, say $\alpha = \gamma + 1$, then let $w_\nu^\alpha = w_\nu^\gamma \cup \{\gamma\}$ for all ν. It is easy to see that (1)-(4) hold for α if they hold for γ.

If α is a limit ordinal, let $\lambda =$ the cofinality of α, and let $\eta: \lambda \to \alpha$ be a strictly increasing function such that the supremum of its range is α. Define a function $f: \lambda \to \varkappa$ by the rule:

$$f(\mu) = \text{the least } \nu < \varkappa \text{ such that}$$
$$\text{for all } \tau < \sigma \leqslant \mu,\ \eta(\tau) \in w_\nu^{\eta(\sigma)}.$$

It is easy to see that f is well-defined because of (1) and (2) and because $\varkappa$ is regular and $\geqslant \lambda > |\mu|$. Now for each $\nu < \varkappa$ let

$$w_\nu^\alpha = \bigcup \{w_\nu^{\eta(\mu)}: \mu < \nu \text{ and } f(\mu) \leqslant \nu\}.$$

Conditions (2) and (4) are easily verified. To see that (1) holds, suppose $\gamma < \alpha$ and choose μ such that $\eta(\mu) > \gamma$. Then $\gamma \in w_\tau^{\eta(\mu)}$ for some τ, so if $\nu > \max\{\tau, \mu, f(\mu)\}$, then $\gamma \in w_\tau^\alpha$: To prove (3), let us fix α and ν and let $Y = \{\mu < \nu: f(\mu) \leqslant \nu\}$. Thus

$$w_\nu^\alpha = \bigcup_{\mu \in Y} w_\nu^{\eta(\mu)}.$$

Notice first that if $\tau < \mu$ and $\mu \in Y$, then $\eta(\tau) \in w_\nu^{\eta(\mu)}$; so by induction $w_\nu^{\eta(\tau)} = w_\nu^{\eta(\mu)} \cap \eta(\tau)$. Now if $\beta \in w_\nu^\alpha$ then $\beta \in w_\nu^{\eta(\mu)}$ for some $\mu \in Y$; in this case it is easy to see, using the previous observation, that $\beta \in w_\nu^{\eta(\tau)}$ for any $\tau \in Y$ such that $\beta < \eta(\tau)$. Clearly

$$w_\nu^\beta = w_\nu^{\eta(\mu)} \cap \beta \subseteq w_\nu^\alpha \cap B ,$$

so we are left with proving the opposite inclusion. Suppose $\gamma \in w_\nu^\alpha \cap \beta$; then $\gamma \in w_\nu^{\eta(\tau)}$ for some $\tau \in Y$. As above, $\gamma \in w_\nu^{\eta(\sigma)}$ for any $\sigma \in Y$ such that $\gamma < \eta(\sigma)$, so without loss of generality $\beta < \eta(\tau)$. But then $\gamma \in w_\nu^{\eta(\tau)} \cap \cap \beta = w_\nu^\beta$, since $\beta \in w_\nu^{\eta(\tau)}$.

The second lemma will be used to show that for certain submodules $K' \supseteq K$ of I_α, the quotient K'/K has sufficiently large cardinality. (K and K' will have the form $\{u \in I_\alpha : ru = 0\}$ for an appropriate r.) Here $\mathcal{P}(\gamma)$ is the set of all subsets of γ.

LEMMA 2. *Let $\{r_\nu : \nu < \gamma\}$ be a sequence of elements of R, and let N be a pure-injective module such that for all $\mu < \gamma$ there exists an element $a_\mu \in N$ such that $r_\mu a_\mu = 0$ and $r_{\mu+1} a_\mu \neq 0$. Then for each $S \in \mathcal{P}(\gamma)$ there exist an element x_S of N such that*

(∗) *for all $\beta < \gamma$ and all S, $T \in \mathcal{P}(\gamma)$*, if *$S \cap \beta = T \cap \beta$, then $r_{\beta+1}(x_S - - x_T) = 0$ if and only if $S \cap (\beta + 1) = T \cap (\beta + 1)$.*

PROOF. The idea of the construction is that x_S should «be» $\sum_{\mu \in S} a_\mu$. The actual construction is by induction on γ. If γ is finite and $S \subseteq \gamma$, let $x_S = \sum_{\mu \in S} a_\mu$. (We let $x_\phi = 0$.) Now suppose that for all $\delta < \gamma$ and all $S \subseteq \delta$ we have defined x_S so that (∗) holds. We consider two cases.

Case 1: $\gamma = \delta + 1$ for some δ. We let $x_S = x_{S \cap \delta}$ if $\delta \notin S$, and we let $x_S = x_{S \cap \delta} + a_\delta$ if $\delta \in S$. It is easy to check, using the inductive hypothesis, that (∗) holds.

Case 2: $\gamma = \lambda$, a limit ordinal. Here we use the fact that since N is pure-injective it is algebraically compact: see, for example, [FS; p. 215]. For any $S \subseteq \lambda$ we let x_S be a solution of the set of equations

$$\{r_{\beta+1}(x - x_{S \cap (\beta+1)}) = 0 : \beta < \lambda\}$$

in the single unknown x. (The elements $x_{S \cap (\beta+1)}$ of N have been defined by induction.) This system of equations is finitely solvable in N: indeed, for any finite subset F of λ, if $\delta > \sup(F)$, then $x_{S\cap\delta}$ is a solution of

$$\{_{\beta+1}(x - x_{S\cap(\beta+1)}) = 0 : \beta \in F\}.$$

Hence by algebraic compactness there is a global solution, x_S. It remains to check that $(*)$ is satisfied. So suppose that S and T are subsets of λ, and $\beta < \lambda$ such that $S \cap \beta = T \cap \beta$. We have:

$$x_S - x_T = (x_S - x_{S\cap(\beta+1)}) + (x_{S\cap(\beta+1)} - x_{T\cap(\beta+1)}) + (x_{T\cap(\beta+1)} - x_T)$$

so $r_{\beta+1}(x_S - x_T) = 0 + r_{\beta+1}(x_{S\cap(\beta+1)} - x_{T\cap(\beta+1)}) + 0$; hence we are done by induction. □

The third lemma will guarantee us the existence of the elements a_μ in Lemma 2 provided that $r_{\mu+1} \notin r_\mu R$. (Of course, over a valuation domain, injective = pure-injective + divisible.)

LEMMA 3. *Suppose L is an archimedean ideal and N is a* divisible *module containing R/L. Suppose also that r, s, t are elements of R such that t is a non-unit and $r = st$. Then there exists $a \in N$ such that $ra = 0$ and $sa \neq 0$.*

PROOF. We shall let $\bar{b}$ denote the coset, $b + L$ of $b \in R$ in $R/L \subseteq N$. Since L is archimedean there is an element $b \in L \setminus tL$. If $bt^{-1} \in R$, let $a \in N$ such that $sa = bt^{-1} + L$. Then $ra = \bar{b} = 0$, but $sa \neq 0$ since $bt^{-1} \notin L$ (because $b \notin tL$). If $tb^{-1} \in R$, let $a \in N$ such that $s(tb^{-1})a = \bar{1}$. Then $ra = \bar{b} = 0$, but $sa \neq 0$ since $tb^{-1}(sa) = \bar{1}$. □

We are now ready to give the:

PROOF OF THE THEOREM. Let $D = D_{n-k}$. As Bazzoni observes, we can assume that $|\Lambda| = \aleph_{n-k}$ since we can replace D by the direct summand of D consisting of elements whose support lies in a fixed subset of Λ of size $\aleph_{n-k}$. It suffices to prove that $\mathrm{Ext}^1(J, D) \neq 0$ for some ideal J of R, for then $\mathrm{Ext}^2(R/J, D) \neq 0$ (cf. [FS; VI.5.2]). For this it suffices to prove that the canonical map: $\mathrm{Hom}(J, I) \to \mathrm{Hom}(J, I/D)$ is not surjective. In fact we shall show that this map is not surjective whenever J is an ideal of R which is not generated by a set of size $\aleph_{n-k}$ but is generated by a set of size $\aleph_{n-k+1}$; there is such an ideal because gl. dim $R > n - k + 2$ (cf. [0] or [FS; IV.2.3].)

Let $\{j_{\alpha+1} : \alpha < \aleph_{n-k+1}\}$ be a set of generators of J ordered so that for all $\beta < \alpha$, $j_{\beta+1} \in Rj_{\alpha+1}$ and $j_{\alpha+1} \notin Rj_{\beta+1}$. Thus for every pair of ordinals $\beta < \alpha$ we have a non-unit r^{α}_{β} of R such that $r^{\alpha}_{\beta} j_{\alpha+1} = j_{\beta+1}$. Moreover, for all $\beta < \gamma < \alpha$ we have $r^{\alpha}_{\beta} = r^{\gamma}_{\beta} r^{\alpha}_{\gamma}$.

Let $\varkappa = \aleph_{n-k}$. We may as well suppose that $\Lambda = \varkappa$. So defining $f\colon J \to I/D$ amounts to choosing, for each $\nu < \varkappa$, elements $x^{\alpha}_{\nu} \in I_{\nu}$ ($\alpha < \varkappa^{+} = \aleph_{n-k+1}$) so that for all $\beta < \alpha$, $|\{\nu < \varkappa\colon r^{\alpha}_{\beta} x^{\alpha}_{\nu} \neq x^{\beta}_{\nu}\}| < \varkappa$; for then we can define $f(j_{\alpha+1}) = x^{\alpha} + D$, where $x^{\alpha} = \langle x^{\alpha}_{\nu}\colon \nu < \varkappa \rangle \in I$. We are going to use the sets w^{α}_{ν} ($\alpha < \varkappa^{+}$, $\nu < \varkappa$) constructed in Lemma 1 in order to define the x^{α}_{ν}'s; in fact, we shall construct them so that $r^{\alpha}_{\beta} x^{\alpha}_{\nu} = x^{\beta}_{\nu}$ if $\beta \in w^{\alpha}_{\nu}$. Then f will be defined because, by (1) of Lemma 1, for any $\beta < \alpha$ there exists $\mu < \varkappa$ so that $\beta \in w^{\alpha}_{\mu}$, and hence by (2), the set of ν such that $r^{\alpha}_{\beta} x^{\alpha}_{\nu} \neq x^{\beta}_{\nu}$ is contained in μ, and thus has cardinality less than $\varkappa$.

In order to make f not liftable to a homomorphism into I we shall also require that the x^{α}_{ν} be chosen so that if $\sup(w^{\alpha}_{\nu}) + \varkappa < \beta < \alpha$, then $r^{\alpha}_{\beta} x^{\alpha}_{\nu} \neq x^{\beta}_{\nu}$. (The sum is ordinal addition.) Indeed, if there were a $g\colon J \to I$ which lifted f, then we would have $g(j_{\alpha}) = y^{\alpha}$ for some $y^{\alpha} \in I$ such that $y^{\alpha} = x^{\alpha} + d^{\alpha}$ for some $d^{\alpha} \in D$, for all $\alpha < \varkappa^{+}$. For each $\mu < \varkappa$, let

$$Y_{\mu} \overset{\text{def}}{=} \{\alpha < \varkappa^{+}\colon \mu \notin \operatorname{supp}(d^{\alpha})\};$$

then for some $\nu < \varkappa$, Y_{ν} is a stationary subset of $\varkappa^{+}$ since $\bigcup Y_{\mu} = \varkappa^{+}$ (cf. [J; Lemma 7.4]). Now by (4), $\sup(w^{\alpha}_{\nu}) < \alpha$ if $\operatorname{cf}(\alpha) = \varkappa$, so by Fodor's Lemma ([J; p. 59]) there is a stationary subset Y' of Y_{ν} and an ordinal γ such that for all $\alpha \in Y'$ $\sup(w^{\alpha}_{\nu}) = \gamma$. Hence there are elements β, α of Y' such that $\gamma + \varkappa < \beta < \alpha$. But then $y^{\alpha}(\nu) = x^{\alpha}_{\nu}$ and $y^{\beta}(\nu) = x^{\beta}_{\nu}$, and by construction $r^{\alpha}_{\beta} x^{\alpha}_{\nu} \neq x^{\beta}_{\nu}$, which means that g is not a homomorphism.

Thus it remains only to construct for each ν the elements x^{α}_{ν} of I_{ν} so that for all $\beta < \alpha < \varkappa^{+}$:

(i) $r^{\alpha} x^{\alpha}_{\nu} = x^{\beta}_{\nu}$ if $\beta \in w^{\alpha}_{\nu}$;

(ii) $r^{\alpha}_{\beta} x^{\alpha}_{\nu} \neq x^{\beta}_{\nu}$ if $\beta > \sup(w^{\alpha}_{\nu}) + \varkappa$.

We shall do this for each fixed ν by induction on α. Let $x^{0}_{\nu} = \bar{1}$. Suppose now that x^{β}_{ν} has been defined for all $\beta < \alpha$ so that (i) and (ii) hold where defined. In order to satisfy (i) it is enough to choose x^{α}_{ν} to be a solution, z, of the system of equations

$$(\dagger) \qquad \{r^{\alpha}_{\beta} z = x^{\beta}_{\nu}\colon \beta \in w^{\alpha}_{\nu}\}.$$

Since I is pure-injective, it suffices to show that this system is finitely solvable in I_ν. If F is a finite subset of w_ν^α and $\sigma = \max(F)$, we claim that any z such that $r_\sigma^\alpha z = x_\nu^\sigma$ will be a solution of

$$\{r_\beta^\alpha z = x_\nu^\beta : \beta \in F\}.$$

In fact, if $\beta \in F$ and $\beta < \sigma$, then since $\sigma, \beta \in w_\nu^\alpha$, (3) implies that $\beta \in w_\nu^\sigma$, so $r_\beta^\sigma x_\nu^\sigma = x_\nu^\beta$ and hence $r_\beta^\alpha z = r_\beta^\sigma r_\sigma^\alpha z = r_\beta^\sigma x_\nu^\sigma = x_\nu^\beta$.

Now consider (ii). Let $\delta = \sup(w_\nu^\alpha)$. Let z be a fixed solution of (†). Then (i) will hold if x_ν^α is of the form $z + u$ where $r_\delta^\alpha u = 0$. Let $\beta = \delta + \varkappa + 1$. It suffices to choose u so that $r_\delta^\alpha u = 0$ and for each γ such that $\beta \leqslant \gamma < \alpha$, $r_\beta^\alpha u \neq r_\beta^\gamma x_\nu^\gamma - r_\beta^\alpha z$. (We let $r_\beta^\beta = 1$.) For then, since $r_\beta^\alpha = r_\beta^\gamma r_\gamma^\alpha$, we have that $r_\gamma^\alpha(z + u) \neq x_\nu^\gamma$. But Lemma 2 (with $r_\nu = r_{\delta+\nu}^\alpha$ for $\nu < \varkappa$) in conjunction with Lemma 3 implies that the quotient group

$$\{u \in I_\nu : r_\delta^\alpha u = 0\}/\{u \in I_\nu : r_\beta^\alpha u = 0\}$$

has cardinality $\geqslant 2^\varkappa$. Thus there certainly is a u with the desired properties. This completes the inductive step of the construction, and hence completes the proof of the theorem. □

COROLLARY. *If gI*, $\dim(R) \geqslant 3$, *and for each* $n \in \omega$, I_n *is an injective nodule containing* R/L_n *for some archimedean ideal* L_n *of* R, *then the injective dimension of* $\bigoplus_{n\in\omega} I_n$ *is* $\geqslant 2$. □

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