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A Bothersome Question

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In diesem Text im Stil eines Interviews versucht der israelische Logiker Saharon Shelah, einem allgemein (nicht unbedingt mathematisch) gebildeten Publikum die Schönheit von höchst abstrakten mathematischen Problemen näher zu bringen, und insbesondere zu erklären, was ihn an der der Arithmetik der unendlichen Kardinalzahlen so fasziniert.

Im darauf folgenden Artikel liefert Martin Goldstern einige historische und technische Hintergründe zu Shelahs Theorie.

Für das nächste Heft ist ein ausführliches Interview mit Shelah geplant.

Q. — Mathematics, mathematics, hasn't everything been discovered already generations ago?

A. — What an educated person knows today generally was discovered hundreds of years ago. Concerning the mathematics of the 20th century, the educated person will in general neither know what has been discovered, nor what the questions are that mathematicians are asking now. This is a pity.

It seems to be like this: the more mathematics has advanced, deepened and become beautiful, and its use has broadened and wonderful theories and basic puzzling questions have been answered, the more it has become unknown and closed to those outside of a select few. Physicists can assume that the reader has heard about black holes, and biologists – about DNA; a mathematician would be lucky if the reader knows something about the calculus of infinitesimals, discovered/invented by Leibnitz and Newton in the 17th century.

Q. — What can be "beautiful" about mathematics?

A. — Michelangelo said that he just discovers the form hidden in the marble and I can identify with him.

Q. — Perhaps you can explain what you hope to discover?

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A. — I'll try to explain without cheating too much. A problem I love and that has been bothering me for many years, and that many have dealt with: what are the arithmetical laws of infinite numbers.

Q. — Infinite numbers??

A. — These are numbers which measure the number of elements in infinite sets. Usually when the question is asked: how many members/elements are there in a set, the notion of the number is already clear, which isn't the case in infinite sets.

Q. — What do you mean, isn't there only one infinite number – infinity!

A. — This has no meaning if we have not defined the infinite numbers. We can know when, in two sets, there is the same number of elements even if we haven't counted them, such as, for instance the number of living people, which equals the number of brains of living people (I will omit the obvious joke about the reader's least favorite politician). In general, two sets, say the Tribe of Simeon and the Tribe of Levi will be considered as having the same number of elements if one can find a "matching" between the Tribe of Simeon and the Tribe of Levi – to each Simeonite will be matched one single Levite and vice versa.

In this way, numbers can be defined, even infinite numbers. Cantor, towards the end of the 19th century, discovered them. He named the smallest infinite number $\aleph_0$; this is the number of elements the set of all finite numbers has.

Q. — What can one do with them?

A. — It turns out that you can naturally define arithmetic operations: addition, multiplication and exponentiation (among all numbers, finite and infinite).

Q. — Can you explain what these operations are? For instance, addition?

A. — In giving 2 disjoint sets (in other words, without a common element), the number of elements in the union will be the sum of the number of elements. It seems that $\aleph_0 = \aleph_0 + 2$, because the number of natural numbers $(0,1,2,3,\dots)$ equals the number of integers greater than -3 (i.e., $-2,-1,0,1,2,3,4,5,\dots$), why? Because one can "match" -2 to 0, -1 to 1, 0 to 2, and in general, n to $n+2$.

Q. — Doesn't this show that everything is nonsense? It negates Aristotle's great rule "the whole is greater than the part", and therefore for all numbers a, b greater than 0, $a+b > a$.

A. — It would be unreasonable to expect that all usual rules of arithmetic will continue to hold; what's so beautiful and surprising is that many of them *do* hold, e.g. the usual equations like $(a+b)c = ac+bc$ and $(a^b)^c = a^{bc}$ (and for the pedantic: for every numbers, a,b, we have $a = b$ or $a < b$ or $b < a$).

Q. — So surely those operations are very awkward and it's impossible to know anything about these numbers.

A. — To the contrary, in a certain way this arithmetic is more transparent. It was discovered that the sum of two numbers where at least one of them in infinite is the maximum of the two, and it is same with products (except that $x \cdot 0 = 0$ even for infinite x):

$$\begin{array}{ll} \kappa + \lambda = \lambda & \text{for all infinite } \lambda > \kappa \\ \kappa \cdot \lambda = \lambda & \text{for all infinite } \lambda > \kappa, \text{ if } \kappa > 0 \end{array}$$

If this were true for finite numbers, it would mean that $7 + 123 = 123$. Wouldn't you prefer to make computations such as this in school?

Q. — If so, it's all too simple, and in effect there is actually only one infinite number that is necessarily $\aleph_0$, which would solve all the problems.

A. — The number of natural numbers is not equal to the number of real numbers (in other words, infinite decimal fractions) or what is equivalent, the number of points in the plane. This can be seen as a particular case of the following: For every number a, "$2^a > a$". Also, for every number a there is a successor, the smallest number bigger than itself, which we will call a^+, so we can define $\aleph_1$ as the successor of $\aleph_0$, the number following $\aleph_0$; $\aleph_2$ as the successor of $\aleph_1$ and we can continue to define $\aleph_n$ for any natural number n.

Q. — And that's it?

A. — No, for example, for every n there exists the first number under which there are $\aleph_n$ numbers and it is called $\aleph_{\aleph_n}$.

Q. — If addition and multiplication are so simple, then probably the power operation is not so complicated.

A. — As you recall, we have two functions which increase the (infinite) number: the successor a^+ and the power 2^a. It's extremely tempting to hope that they are actually one and the same operation, i.e., for every infinite a, $2^a = a^+$; this hypothesis is called the Generalized Continuum Hypothesis. If this hypothesis is correct, then the power operation is very simple indeed, and we would completely understand all laws of arithmetic of infinite numbers.

Q. — Of what use would this be?

A. — Whoever wants to prove general theorems on "large" infinite sets, this hypothesis will be very useful.

Q. — Do mathematicians truly consider this an important problem?

A. — When Hilbert, considered the outstanding mathematician since the beginning of the 20th century, prepared a list of the most important mathematical problems, with 23 questions (this is the best known list of its kind), he chose this as question number 1 (he asked: is $\aleph_1 = 2^{\aleph_0}$, but he meant: find all the arithmetic laws for infinite numbers).

On the other hand, the majority of mathematicians whom I have met aren't particularly interested in it. All mathematicians are in agreement among themselves as to what is correct, but not necessarily about what is important, and what is beautiful and exciting. Mathematics is an Exact Art.

Q. — Presumably, do you hope to prove the Generalized Continuum Hypothesis? Or at least to refute it? Or have you missed the boat?

A. — Too late. Gödel showed that (from the usual axioms of set theory) it's impossible to contradict it. On the other hand, it was proved that it's impossible to prove it. Moreover, except for the obvious monotonicity: $2^{\aleph_0} \leq 2^{\aleph_1} \leq \ldots$ (plus a little more) there aren't actually any additional restrictions.

Q. — If this is so, then there is nothing new to discover?

A. — No, because if you look at products of infinitely many but still "few" large numbers (for instance, just $\aleph_0$ numbers), there is a lot to say. For instance, if $2^{\aleph_0}$ is any $\aleph_n$, then the product of all the $\aleph_n$'s is not large, it is smaller than $\aleph_{\aleph_4}$.

Q. — Isn't there a typographical error here? It seems to me that you are talking about infinite numbers, so why in the dickens does 4 appear here?

A. — That's exactly what I want to find out, and I feel like it's a a basic problem, the key to the mystery of the mathematical laws of infinite numbers.

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