Canad. J. Math. Vol. **63** (6), 2011 pp. 1416–1435
doi:10.4153/CJM-2011-057-1

MAD Saturated Families and SANE Player

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Abstract. We throw some light on the question: is there a MAD family (a maximal family of infinite subsets of $\mathbb{N}$, the intersection of any two is finite) that is saturated (=completely separable *i.e.,* any $X \subseteq \mathbb{N}$ is included in a finite union of members of the family *or* includes a member (and even continuum many members) of the family). We prove that it is hard to prove the consistency of the negation:

(i) if $2^{\aleph_0} < \aleph_\omega$, then there is such a family;
(ii) if there is no such family, then some situation related to pcf holds whose consistency is large (and if $\mathfrak{a}_* > \aleph_1$ even unknown);
(iii) if, *e.g.,* there is no inner model with measurables, *then* there is such a family.

1 Introduction

We try to throw some light on the following problem.

Problem 1.1 Is there, provably in ZFC, a completely separable MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$; see Definition 1.3(1) and (4).

Erdös–Shelah [5] investigates the ZFC-existence of families $\mathcal{A} \subseteq \mathcal{P}(\omega)$ with separability properties, continuing Hechler [7], which mostly uses MA. Problem 1.1 is [5, Problem A, p. 209]; see Miller [8] and Goldstern–Judah–Shelah [6] on the existence for larger cardinals. It seemed natural to prove the consistency of a negative answer by CS iteration making the continuum $\aleph_2$, but this had not worked out; the results here show this is impossible.

The celebrated matrix-tree theorem of Balcar, Pelant, and Simon [1], Balcar and Simon [2] is related to our starting point. Gruenhut and Shelah try to generalize it, hoping eventually to get applications, *e.g.,* "there is a subgroup of ${}^\omega\mathbb{Z}$ that is reflexive (*i.e.,* canonically isomorphic to the dual of its dual)" and "less" (see [4, Problem D7]), but have had no success so far. We then had tried to use such constructions to answer Problem 1.1 positively, but this does not work. Simon [3] proved (in ZFC), that there is an infinite almost disjoint $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \subseteq \omega$ and $(\exists^\infty A \in \mathcal{A})[B \cap A \text{ infinite}] \Rightarrow (\exists A \in \mathcal{A})(A \subseteq B)$. Shelah and Steprans [10] tried to continue it with dealing with Hilbert spaces.

Here $\mathfrak{s}$ and ideals (formally $J \in \mathrm{OB}$) are central. Originally we had a unified proof using games between the MAD and the SANE players (SANE is naturally the

Received by the editors August 23, 2009; revised June 18, 2010.
Published electronically September 15, 2011.
Research supported by the United States-Israel Binational Science Foundation (Grant No. 2006108). Publication 935.
AMS subject classification: **03E05**, 03E04, 03E17.
Keywords: set theory, MAD families, pcf, the continuum.

opponent of MAD) but with some parameters for the properties. As on the one hand it was claimed this was unreadable and on the other hand we have a direct proof, which was presented (for $\mathfrak{s} < \mathfrak{a}_*$), in the Hebrew University and Rutgers, we use the later one. A minor price is that the proof in Section 2 says to repeat the earlier one with the following changes. The major price is that some information is lost: we use smaller, more complicated cardinal invariants, and there are some points in the proof that we hope will serve other proofs (including covering all cases), so we hope to return to the main problem and relatives elsewhere.

A related problem of Balcar and Simon is: given a MAD family $\mathcal{B}$ we look for such $\mathcal{A}$ refining it, *i.e.*, $(\forall B \in \mathrm{id}^+_{\mathcal{A}})(\exists A \in \mathcal{A})(A \subseteq^* B)$. At present there is no difference between the two problems (see also Theorems 2.1, 3.1, and 3.6).

Conclusion 1.2

(1) If $2^{\aleph_0} < \aleph_\omega$, then there is a saturated MAD family.
(2) Moreover, in (1) for any dense $J_* \subseteq [\omega]^{\aleph_0}$ we can find such a family contained in J_*.

Definition 1.3

(1) We say $\mathcal{A}$ is an AD (family) for B when $\mathcal{A} \subseteq [B]^{\aleph_0}$ is infinite and almost disjoint (*i.e.*, $A_1 \neq A_2 \in \mathcal{A} \Rightarrow A_1 \cap A_2$ finite). We say $\mathcal{A}$ is MAD for B when $\mathcal{A}$ is AD for B and is $\subseteq$-maximal among such $\mathcal{A}$'s. If $B = \omega$, we may omit it.
(2) For $\mathcal{A} \subseteq [\omega]^{\aleph_0}$, $\mathrm{id}_{\mathcal{A}}$ is the ideal generated by $\mathcal{A} \cup [\omega]^{<\aleph_0}$.
(3) A MAD family $\mathcal{A}$ is saturated when: if $B \in \mathrm{id}^+_{\mathcal{A}}$ (see 1.7(3)), then B almost contains some member of $\mathcal{A}$ (equivalently, if $B \in \mathrm{id}^+_{\mathcal{A}}$, then B almost contains continuum many members of A, because if $B \in \mathrm{id}^+_{\mathcal{A}}$, then there is an AD family $\mathcal{B} \subseteq [B]^{\aleph_0} \cap \mathrm{id}^+_{\mathcal{A}}$ of cardinality $2^{\aleph_0}$).

Definition 1.4

(1) Let $\mathfrak{a}$ be the minimal cardinality of a MAD family.
(2) Let $\mathfrak{a}_*$ be the minimal κ such that there is a sequence $\langle A_\alpha : \alpha < \kappa+\omega \rangle$ of pairwise almost disjoint (*i.e.*, with finite intersection) infinite subsets of ω satisfying: there is no infinite set $B \subseteq \omega$ almost disjoint to A_α for $\alpha < \kappa$ but where $B \cap A_{\kappa+n}$ is infinite for infinitely many n-s.

Observation 1.5 We have $\mathfrak{b} \leq \mathfrak{a}_* \leq \mathfrak{a}$.

Remark 1.6

(1) Note that if there is a MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \in \mathrm{id}^+_{\mathcal{A}} \Rightarrow (\exists^{2^{\aleph_0}} A \in \mathcal{A})$ $(B \cap A$ is infinite), then there is a MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ such that $B \in \mathrm{id}^+_{\mathcal{A}} \Rightarrow$ $(\exists^{2^{\aleph_0}} A \in \mathcal{A})(A \subseteq B)$; equivalently $B \in \mathrm{id}^+_A \Rightarrow (\exists A \in \mathcal{A})(A \subseteq B)$. Just list our tasks and fulfil them by dividing each member of $\mathcal{A}$ into two infinite sets to fulfil one task.
(2) So the four variants of "there is $\mathcal{A}$ ..." in 1.3(4), 1.6(1) are equivalent.

Notation 1.7

(1) For $A \subseteq \omega$, let $A^{[\ell]}$ be A if $\ell = 1$ and $\omega \backslash A$ if $\ell = 0$.
(2) For $J \subseteq [\omega]^{\aleph_0}$, let $J^{\perp} = \{B : B \in [\omega]^{\aleph_0}$ and $[A \in J \Rightarrow A \cap B$ finite$]\}$, and also for $\bar{A} = \langle A_s : s \in S \rangle$, let $\bar{A}^{\perp} = \{A_s : s \in S\}^{\perp}$.
(3) $\mathrm{id}_{\mathcal{A}}(B)$ is the ideal of $\mathcal{P}(B)$ generated by

$$(\mathcal{A} \restriction B) \cup [B]^{<\aleph_0} \quad \text{and} \quad \mathrm{id}^{+}_{\mathcal{A}}(B) = [B]^{\aleph_0} \backslash \mathrm{id}_{\mathcal{A}}(B),$$

(on $\mathcal{A} \restriction B$, see (7)); if $B = \omega$, we may omit it.
(4) $A \subseteq^* B$ means that $A \backslash B$ is finite.
(5) If $\mathcal{C} \subseteq \mathcal{P}(B)$ and $\eta \in {}^{\mathcal{C}}2$, then $I_{\mathcal{C},\eta}(B)$ is $\{C \subseteq B : C \subseteq^* A^{[\eta(A)]}$ for every $A \in \mathcal{C}\}$; if $B = \omega$ we may omit it.
(6) In part (5), if ν is a function extending η, then let $I_{\mathcal{C},\nu} = I_{\mathcal{C},\eta}$.
(7) For $\mathcal{A} \subseteq \mathcal{P}(B_2)$ and $B_1 \subseteq B_2$, let $\mathcal{A} \restriction B_1 = \{A \cap B_1 : A \in \mathcal{A}$ and $A \cap B_1$ is infinite$\}$.

Definition 1.8

(1) Let $\mathrm{OB} = \{I \subseteq [\omega]^{\aleph_0} : I \cup [\omega]^{<\aleph_0}$ is an ideal of $\mathcal{P}(\omega)\}$.
(2) For $A \subseteq \omega$, let $\mathrm{ob}(A) = \{B : B \in [\omega]^{\aleph_0}$ and $B \subseteq^* A\}$ so $\mathrm{ob}(\omega) = [\omega]^{\aleph_0}$.
(3) $\eta \perp \nu$ means $\neg(\eta \trianglelefteq \nu) \wedge \neg(\nu \trianglelefteq \eta)$.
(4) We say $\mathcal{A}$ is AD in $J \subseteq [\omega]^{\aleph_0}$ when $\mathcal{A}$ is AD and $\mathcal{A} \subseteq J$.
(5) We say $\mathcal{A}$ is MAD in $J \subseteq [\omega]^{\aleph_0}$ when $\mathcal{A}$ is AD in J and is $\subseteq$-maximal among such $\mathcal{A}$'s.
(6) $J \subseteq [\omega]^{\aleph_0}$ is hereditary when $A \in [\omega]^{\aleph_0} \wedge A \subseteq^* B \in J \Rightarrow A \in J$.
(7) $J \subseteq [\omega]^{\aleph_0}$ is dense when $(\forall B \in [\omega]^{\aleph_0})(\exists A \in J)[A \subseteq B]$.

2 The Simple Case: $\mathfrak{s} < \mathfrak{a}_*$

We here give a proof for the case $\mathfrak{s} < \mathfrak{a}_*$.

Theorem 2.1

(1) *If $\mathfrak{s} < \mathfrak{a}_*$, then there is a saturated MAD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$.*
(2) *Moreover, given a dense $J_* \subseteq [\omega]^{\aleph_0}$ we can demand $\mathcal{A} \subseteq J_*$.*

Proof Stage A: Let $\kappa = \mathfrak{s}$, so $\mathrm{cf}(\kappa) > \aleph_0$. For (1) let $J_* \subseteq [\omega]^{\aleph_0}$ be a dense (and even hereditary) subset of $[\omega]^{\aleph_0}$, *i.e.*, as in part (2) and in both cases without loss of generality every finite union of members of J_* is co-infinite, *i.e.*, $\omega \notin \mathrm{id}_{J_*}$.

Choose a sequence $\langle C^*_\alpha : \alpha < \kappa \rangle$ of subsets of ω exemplifying $\mathfrak{s} = \kappa$, *i.e.*,

$$\neg(\exists B \in [\omega]^{\aleph_0}) \bigwedge_{\alpha} (B \subseteq^* C^*_\alpha \vee B \subseteq^* \omega \backslash C^*_\alpha).$$

For $i < \kappa$ and $\eta \in {}^{i}2$, let $C^*_\eta = C^*_i$. The aim of this notation is to simplify later proofs where we say "repeat the present proof but ...".

Stage B: For $\alpha \leq 2^{\aleph_0}$ let AP_α, the set of α-approximations, be defined by the following conditions:

$\boxplus_1$(a) $\mathcal{T} = \mathcal{T}_t$ is a subtree of ${}^{\kappa>}2$, *i.e.*, closed under initial segments;

(b) let $\mathrm{suc}(\mathcal{T}) = \{\eta \in \mathcal{T} : \ell g(\eta)$ is a successor ordinal$\}$ and[1] $c\ell(\mathcal{T}) = \{\eta \in {}^{\kappa\geq}2$: if $i < \ell g(\eta)$, then $\eta{\restriction}i \in \mathcal{T}\}$;
(c) $1 \leq |\mathcal{T}| \leq \aleph_0 + |\alpha|$;
(d) $\bar{I} = \bar{I}_t = \langle I_\eta : \eta \in c\ell(\mathcal{T})\rangle = \langle I^t_\eta : \eta \in c\ell(\mathcal{T}_t)\rangle$;
(e) $\bar{A} = \bar{A}_t = \langle A_\eta : \eta \in \mathrm{suc}(\mathcal{T})\rangle = \langle A^t_\eta : \eta \in \mathrm{suc}(\mathcal{T}_t)\rangle$;

such that

(f) $A_\eta \in I_\eta \cap J_*$ or[2] $A_\eta = \varnothing$ and $\mathscr{S}_t = \{\eta \in \mathrm{suc}(\mathcal{T}_t) : A_\eta \neq \varnothing\}$
(g) $I_\eta = \{A \in [\omega]^{\aleph_0}$: if $i < \ell g(\eta)$, then $A \subseteq^* (C^*_{\eta\restriction i})^{[\eta(i)]}$, and if $i+1 < \ell g(\eta)$, then $A \cap A_{\eta\restriction(i+1)}$ is finite$\}$; so I_η is well defined also when $\eta \in c\ell(\mathcal{T})$.

We let

(h) $C^t_\eta = C^*_\eta$ (for generalizations);

$\boxplus_2$ $\mathrm{AP} = \bigcup\{\mathrm{AP}_\alpha : \alpha \leq 2^{\aleph_0}\}$;

$\boxplus_3$ $s \leq_{\mathrm{AP}} t$ if and only if (both are from AP and)

(a) $\mathcal{T}_s \subseteq \mathcal{T}_t$;
(b) $\bar{I}_s = \bar{I}_t{\restriction}c\ell(\mathcal{T}_s)$;
(c) $\bar{A}_s = \bar{A}_t{\restriction}\,\mathrm{suc}(\mathcal{T}_s)$.

Stage C: We assert various properties of AP; of course s, t denote members of AP:

$\boxplus_4$(a) $\leq_{\mathrm{AP}}$ partially orders AP;
(b) $\eta \vartriangleleft \nu \in c\ell(\mathcal{T}_t) \Rightarrow I^t_\nu \subseteq I^t_\eta$;
(c) if $\eta \in c\ell(\mathcal{T}_t)$, then $I^t_\eta \in$ OB, *i.e.*, $I^t_\eta \cup [\omega]^{<\aleph_0}$ is an ideal of $\mathcal{P}(\omega)$;
(d) $\langle A^t_\eta : \eta \in \mathscr{S}_t\rangle$ is almost disjoint (so $A^t_\eta \in$ ob(ω) and $\eta \neq \nu \in \mathscr{S}_t \Rightarrow A^t_\eta \cap A^t_\nu$ finite; recall that here we can assume $\mathscr{S}_t = \mathrm{suc}(\mathcal{T}_t)$);
(e) if $\eta \in c\ell(\mathcal{T}_t)$ and $\ell g(\eta) = \kappa$, then $I^t_\eta = \varnothing$;
(f) if $s \leq_{\mathrm{AP}} t$, then $c\ell(\mathcal{T}_s) \subseteq c\ell(\mathcal{T}_t)$ and $\eta \in c\ell(\mathcal{T}_s) \Rightarrow I^s_\eta = I^t_\eta$ (and clause (b) of $\boxplus_3$ follows from clauses (a),(c));
(g) • if $\nu \in c\ell(\mathcal{T}_s)\backslash\mathcal{T}_s$ and $\eta \in \mathscr{S}_s$ and $B \in I^s_\nu$, then $B \cap A_\eta$ is finite
• if $\nu \in \mathcal{T}_s$ and $\eta \in \mathscr{S}_s$ but $\neg(\nu \trianglelefteq \eta)$ and $B \in I^s_\nu$, then $B \cap A_\eta$ is finite.

[Why clause (d)? Let $\eta_0 \neq \eta_1 \in \mathscr{S}_t$; if $\eta_0 \perp \eta_1$, let $\rho = \eta_0 \cap \eta_1$, hence for some $\ell \in \{0,1\}$ we have $\rho\hat{\ }\langle\ell\rangle \trianglelefteq \eta_0, \rho\hat{\ }\langle 1-\ell\rangle \trianglelefteq \eta_1$, so $A_{\eta_k} \in I^t_{\eta_k} \subseteq I^t_{\rho\hat{\ }<k>} \subseteq$ ob$((C^t_\rho)^{[k]})$ for $k = 0,1$, hence $A_{\eta_0} \cap A_{\eta_1} \subseteq^*$ ob$((C^t_\rho)^{[\ell]}) \cap$ ob$((C^t_\rho)^{[1-\ell]}) = \varnothing$. If $\eta_0 \vartriangleleft \eta_1$, note that $A^t_{\eta_1} \in I^t_{\eta_1} \subseteq$ ob$(\omega\backslash A^t_{\eta_0})$ by clause $\boxplus_1(g)$. Also if $\eta_1 \vartriangleleft \eta_0$ similarly, so clause (d) holds indeed.

Why Clause (e)? Recall the choice of $\langle C^*_\alpha : \alpha < \kappa\rangle$ and $\langle C^t_\eta : \eta \in {}^{\kappa>}2\rangle$, hence $\alpha < \kappa \Rightarrow C^t_{\eta\restriction\alpha} = C^*_\alpha$. So if $B \in I^t_\eta$, then $B \in I_{\eta\restriction(\alpha+1)}$, hence $(B \subseteq^* C^*_\alpha \vee B \subseteq^* \omega\backslash C^*_\alpha)$ for every $\alpha < \kappa$, a contradiction to the choice of $\langle C^*_\alpha : \alpha < \kappa\rangle$.]

$\boxplus_5$(a) $\alpha < \beta \leq 2^{\aleph_0} \Rightarrow \mathrm{AP}_\alpha \subseteq \mathrm{AP}_\beta$;
(b) $\mathrm{AP}_0 \neq \varnothing$ (*e.g.*, use t with $\mathcal{T}_t = \{\langle\ \rangle\}$);
(c) if $\langle t_i : i < \delta\rangle$ is $\leq_{\mathrm{AP}}$-increasing, $t_i \in \mathrm{AP}_{\alpha_i}$ for $i < \delta$, $\langle\alpha_i : i < \delta\rangle$ is increasing, δ a limit ordinal and $\alpha_\delta = \bigcup\{\alpha_i : i < \delta\}$, then $t_\delta = \bigcup\{t_i : i < \delta\}$ naturally defined belongs to $\mathrm{AP}_{\alpha_\delta}$ and $i < \delta \Rightarrow t_i \leq_{\mathrm{AP}} t_\delta$;

[1] So $c\ell(\{\langle\ \rangle\}) = \{\langle\ \rangle, \langle 0\rangle, \langle 1\rangle\}$.
[2] the case "$A_\eta = \varnothing$" is not needed in this proof

$\boxplus_6$ let J_t be the ideal on $\mathcal{P}(\omega)$ generated by $\{A^t_\eta : \eta \in \mathscr{S}_t\} \cup [\omega]^{<\aleph_0}$.

For $s \in$ AP and $B \in$ ob(ω) we define:

$(*)_1$ $S_B = S^s_B := S^1_B \cup S^2_B$, where

(a) $S^1_B = S^{s,1}_B := \{\eta \in c\ell(\mathcal{T}_s) : [B \backslash A]^{\aleph_0} \cap I^s_\eta \neq \varnothing$ for every $A \in J_s\}$;

(b) $S^2_B = S^{s,2}_B := \{\eta \in c\ell(\mathcal{T}_s)$: for infinitely many $\nu, \eta \trianglelefteq \nu \in \mathscr{S}_s$ and the set $B \cap A_\nu$ is infinite$\}$;

(c) $S^3_B = S^{3,s}_B := S_B$.

$(*)_2$ $\mathrm{SP}^\iota_B = \mathrm{SP}^{s,\iota}_B := \{\eta \in \mathcal{T}_s : \eta\hat{\ }\langle 0\rangle \in S^{s,\iota}_B$ and $\eta\hat{\ }\langle 1\rangle \in S^{s,\iota}_B\}$ for $\iota = 1, 2, 3$, and $S_B = S^s_B = S^3_B$.

Note that

$(*)_3$ for $\iota = 1, 2, 3$,

(a) S^ι_B is a subtree of $c\ell(\mathcal{T}_s)$;

(b) $\langle\ \rangle \in S_B \Leftrightarrow B \in J^+_s \Leftrightarrow \langle\ \rangle \in S^1_B$;

(c) $\mathrm{SP}^\iota_B \subseteq \mathcal{T}_s$;

(d) if $B \subseteq A$ are from $[\omega]^{\aleph_0}$, then $S^\iota_B \subseteq S^\iota_A$, $\mathrm{SP}^\iota_B \subseteq \mathrm{SP}^\iota_A$.
[Why? For $\iota = 1$, the first statement holds by recalling $\boxplus_4$(b). The second, $\langle\ \rangle \in S^\iota_B \Leftrightarrow B \in J^+_s$, holds as $I^s_{\langle\ \rangle} =$ ob(ω). The third, $\mathrm{SP}^\iota_B \subseteq \mathcal{T}_s$ as by the definition of $c\ell(\mathcal{T}_s)$ we have $\eta\hat{\ }\langle\ell\rangle \in c\ell(\mathcal{T}_s) \Rightarrow \eta \in \mathcal{T}_s$. The fourth is obvious. For $\iota = 2$ this is even easier and for $\iota = 3$ it follows.]

$(*)_4$ If $\eta \in S_B$ and $\nu_0 \triangleleft \nu_1 \triangleleft \cdots \triangleleft \nu_{n-1}$ is a listing of $\{\nu \triangleleft \eta : \nu \in \mathrm{SP}_B\}$ (so this set is finite) and we let $C_s(\eta, B) := \bigcap\{(C^s_{\nu_\ell})^{[\eta(\ell g(\nu_\ell))]} : \ell < n\}$, then $S_{B \cap C_s(\eta,B)} = \{\nu \in S_B : \nu \trianglelefteq \eta$ or $\eta \trianglelefteq \nu\}$.
[Why? Clearly $(\forall A \in I^s_\eta)(A \subseteq^* C_s(\eta, B))$ by the definition of I^s_η (see $\boxplus_1$(g)), but $(\exists A \subseteq I^s_\eta)(|A \cap B| = \aleph_0)$, hence $B \cap C_s(\eta, B) \in$ ob(ω).
As $B \cap C_s(\eta, B) \subseteq B$, clearly $S_{B \cap C_s(\eta,B)} \subseteq S_B$. Also as $\eta \in S_B$ and $(\forall A \in I^s_\eta)(A \subseteq^* C_s(\eta, B))$, clearly $\eta \in S_{B \cap C_s(\eta,B)}$ and moreover $\{\nu \in S_{B \cap C_s(\eta,B)} : \eta \trianglelefteq \nu\} = \{\nu \in S_B : \eta \trianglelefteq \nu\}$ by $\boxplus_4(b)$.
Also, as S_B and $S_{B \cap C_s(\eta,B)}$ are subtrees, clearly $\{\nu : \nu \trianglelefteq \eta\} \subseteq S_B \cap S_{B \cap C_s(\eta,B)}$ and $\eta \trianglelefteq \nu \in c\ell(\mathcal{T}_s) \Rightarrow [\nu \in S_B \Leftrightarrow \nu \in S_{B \cap C_s(\eta,B)}]$.
So to prove the equality it suffices to assume $\alpha < \ell g(\eta), \nu \in S_B, \ell g(\eta \cap \nu) = \alpha, \ell g(\nu) > \alpha$, and $\nu \in S_{B \cap C_s(\eta,B)}$ and get a contradiction. If $\ell < n$ and $\alpha = \ell g(\nu_\ell)$, then $(\forall A \in I^s_\nu)[A \subseteq^* (C^s_{\eta\restriction\alpha})^{[1-\eta(\alpha)]}]$, so it is an easy contradiction. If $\alpha \notin \{\ell g(\nu_\ell) : \ell < n\}$, we can get a contradiction to $\eta\restriction\alpha \notin \mathrm{SP}_B$. So we are done proving $(*)_4$.]

$(*)_5$ (a) For every $\eta \in c\ell(\mathcal{T}_t)$ the set $\{B \in$ ob$(\omega) : \eta \notin S_B\}$ belongs to OB;

(b) if $\iota = 1, 2, 3$ and $B = B_0 \cup \cdots \cup B_n \subseteq \omega$, then $S^\iota_B = S^\iota_{B_0} \cup \cdots \cup S^\iota_{B_n}$;

(c) if $A \in J_t$ and $B_2 = B_1 \backslash A$, then $S^\iota_{B_2} = S^\iota_{B_1}$, $\mathrm{SP}^\iota_{B_2} = \mathrm{SP}^\iota_{B_1}$ for $\iota = 1, 2, 3$;

(d) $S^2_B \subseteq \mathcal{T}_t$ for $B \in$ ob(ω);

(e) if $\eta \triangleleft \nu \in c\ell(\mathcal{T}_t)$, then $\eta \in \mathcal{T}_t$;

(f) if $B \in$ ob(ω) and $s \leq_{\mathrm{AP}} t$, then

- $\bullet_1$ $S^{s,1}_B \supseteq S^{t,1}_B \cap c\ell(\mathcal{T}_s)$;
- $\bullet_2$ $S^{s,2}_B \subseteq S^{t,2}_B \cap \mathcal{T}_s$ (inclusion in different direction? yes!);
- $\bullet_3$ $S^{s,3}_B \supseteq S^{t,3}_B \cap c\ell(\mathcal{T}_s)$, in fact $(S^{t,2}_B \cap c\ell(\mathcal{T}_s)) \backslash S^{s,2}_B \subseteq S^{s,1}_B$;

(clause (f) is not used here).

$(*)_6$ For $\iota = 1, 2, 3$ we have

(a) if $B \subseteq \omega, \ell < 2, \nu \in S^\iota_B \cap \mathcal{T}_t$, and $B \subseteq^* (C^t_\nu)^{[\ell]}$ then $\nu\hat{\ }\langle \ell \rangle \in S^\iota_B$, but $\nu\hat{\ }\langle 1-\ell \rangle \notin S^\iota_B$;

(b) if $B \subseteq \omega$ and $\nu \in S^\iota_B \cap \mathcal{T}_t$, then for some $\ell < 2$ we have $\nu\hat{\ }\langle \ell \rangle \in S^\iota_B$;

(c) $\omega \notin J_t$.

[Why? Read the definitions recalling $(*)_5$(c). For clause (c) recall that $\nu \in \mathscr{S}_s \Rightarrow A_s \in J_s$ and by $\boxplus_1$(f) we have $J_s \subseteq J_*$ and by Stage A, $\omega \notin \mathrm{ob}(J_*)$.]

Stage D:

$\boxplus_7$ If $\alpha < 2^{\aleph_0}, s \in \mathrm{AP}_\alpha$ and $B \in \mathrm{ob}(\omega) \backslash J_s$, then we can find $t \in \mathrm{AP}_{\alpha+1}$ such that $s \leq_{\mathrm{AP}} t$ and B contains A_η for some $\eta \in \mathscr{S}_t \backslash \mathcal{T}_s$.

This is a major point, and we shall prove it in Stage F below.

Stage E: We prove the theorem.

Let $\langle B_\alpha : \alpha < 2^{\aleph_0} \rangle$ list $\mathcal{P}(\omega)$, each appearing $2^{\aleph_0}$ times. By induction on $\alpha \leq 2^{\aleph_0}$ we choose t_α such that

$\circledast$ (a) $t_\alpha \in \mathrm{AP}_\alpha$;

(b) $\beta < \alpha \Rightarrow t_\beta \leq_{\mathrm{AP}} t_\alpha$;

(c) if $\alpha = \beta + 1$, then either $B_\alpha \in J_{t_\beta}$ or B_α contains A_η, for some $\eta \in \mathscr{S}_{t_\alpha} \backslash \mathcal{T}_{t_\beta}$.

For $\alpha = 0$ use $\boxplus_5$(b).
For α limit use $\boxplus_5$(c).
For $\alpha = \beta + 1$ use $\boxplus_7$.

Now let $t \in \mathrm{AP}$ be $\bigcup\{t_\alpha : \alpha < 2^{\aleph_0}\}$ and recalling $\boxplus_4$(d) it is easy to check that $\bar{A}_t$ is a saturated MAD family, enough for Theorem 2.1(1), and recalling that by $\boxplus_1$(f) it is $\subseteq J_*$, also enough for Theorem 2.1(2).

Stage F: The rest of the proof is dedicated to the proof of $\boxplus_7$, so α, s and B are given. The proof is now split into cases.

Case 1: Some $\nu \in S_B$ is such that $\nu \in c\ell(\mathcal{T}_s) \backslash \mathcal{T}_s$.

By $(*)_5$(d) we have $\nu \in S^1_B$. Clearly, as $\nu \in S_B$, there is $B_1 \in [B]^{\aleph_0} \cap I^s_\nu$. Note that $\ell g(\nu) > 0$ as $\langle\ \rangle \in \mathcal{T}_s$ by clause (c) of $\boxplus_1$.

Note that, $A \in I^s_\nu \wedge \eta \in \mathscr{S}_s \Rightarrow A \cap A^s_\eta$ is finite, *e.g.*, by the proof of $\boxplus_4$(d) or better by $\boxplus_4(g)$.

Subcase 1A: Assume $\ell g(\nu)$ is a successor ordinal.

Let $B_2 \subseteq B_1$ be such that $B_2 \in J_*$ and $B_1 \backslash B_2$ are infinite. Now define t as follows: $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu\}; A^t_\rho$ is A^s_ρ if $\rho \in \mathrm{suc}(\mathcal{T}_s)$ and is B_2 if $\rho = \nu$; lastly define I^t_ρ for $\rho \in \mathcal{T}_t$ as in clause (g) of $\boxplus_1$. It is easy to check that t is as required.

Subcase 1B: Assume $\ell g(\nu)$ is a limit ordinal.

Clearly $\ell g(\nu) < \kappa$ by $\boxplus_4$(e), as $I^s_\nu \neq \varnothing$ because $B_1 \in I^s_\nu$. Clearly there is $\ell \in \{0, 1\}$ such that $B'_1 := (C^s_\nu)^{[\ell]} \cap B_1$ is infinite. Let $B_2 \subseteq B'_1$ be such that B_2 and $B'_1 \backslash B_2$ are infinite and $B_2 \in J_*$. We define t by $\mathcal{T}_t = \mathcal{T}_s \cup \{\nu, \nu\hat{\ }\langle \ell \rangle\}, A^t_\rho$ is A^s_ρ if $\rho \in \mathrm{suc}(\mathcal{T}_s)$, and is B_2 if $\rho = \nu\hat{\ }\langle \ell \rangle$ and I^t_ρ for $\rho \in \mathcal{T}_t$ is defined as in clause (g) of $\boxplus_1$.

It is easy to check that t is as required.

Case 2: $\mathrm{SP}_B = \varnothing$ but not case 1.

Let $\nu_B^* := \bigcup\{\eta : \eta \in S_B\}$.

Subcase 2A: $\nu_B^* \in S_B^1$.

As $S_B \subseteq c\ell(\mathcal{T}_s)$ by the definition of S_B, and since we are assuming "not case 1" we have $S_B \subseteq \mathcal{T}_s$; hence $\nu_B^* \in \mathcal{T}_s$, so $\ell g(\nu_B^*) < \kappa$.

We define B_2^* as $B \cap A_{\nu_B^*}$ if $A_{\nu_B^*}$ is well defined and $B_2^* = \varnothing$ otherwise; and for $\ell = 0, 1$, let $B_\ell^* := B \cap (C^*_{\ell g(\nu_b^*)})^{[\ell]} \backslash B_2^*$.

Then we have

$(*)_7$ $\langle B_0^*, B_1^*, B_2^* \rangle$ is a partition of B.

Hence by $(*)_5$(b), for some $\ell = 0, 1, 2$, we have

$(*)_8$ $\nu_B^* \in S^1_{B_\ell^*}$,

$(*)_9$ $\ell \neq 2$,

and

$(*)_{10}$ $\rho := \nu_B^{*\frown}\langle \ell \rangle \in S_{B_\ell^*}$.

[Why? By the definitions noting $\rho \in c\ell(\mathcal{T}_s)$.]

Also as $B_\ell^* \subseteq B$, clearly

$(*)_{11}$ $S_{B_\ell^*} \subseteq S_B$.

But $(*)_{10}$ and $(*)_{11}$ contradict the choice of ν_B^*.

Subcase 2B: $\nu_B^* \notin S_B^1$.

By $(*)_3$(b) and $(*)_6$(c) and the assumption of $\boxplus_7$, we have $\langle\,\rangle \in S_B^1$, and by $(*)_6$(b), clearly $\langle 0 \rangle \in S_B^1$ or $\langle 1 \rangle \in S_B^1$, hence $\nu_B^* \neq \langle\,\rangle$. If $\nu_B^* = \nu^\frown\langle \ell \rangle$, by the definition of ν_B^*, we have $\nu_B^* \in S_B$, contradicting the subcase assumption. Hence, necessarily $\ell g(\nu_B^*)$ is a limit ordinal $\leq \kappa$; call it δ. So $\alpha < \delta \Rightarrow \nu_B^* {\restriction} \alpha \in S_B$ but $\rho \triangleleft \varrho \in c\ell(\mathcal{T}_s) \Rightarrow \rho \in \mathcal{T}_s$, hence $\alpha < \delta \Rightarrow \nu_B^* {\restriction} \alpha \in \mathcal{T}_s$. Now for every $\alpha < \delta$ let $\nu^*_{B,\alpha} := (\nu_B^* {\restriction} \alpha)^\frown\langle 1 - \nu_B^*(\alpha) \rangle$, so clearly $\nu^*_{B,\alpha} \in c\ell(\mathcal{T}_s) \backslash S_B$. Hence as $\nu^*_{B,\alpha} \notin S_B^2$, by the definition of $S_B^2 \subseteq S_B$ the set $\mathcal{A}_\alpha = \{\mathcal{A}_\rho : \rho \in \mathscr{S}_s, \nu^*_{B,\alpha} \trianglelefteq \rho$ and $B \cap A_\rho$ is infinite$\}$ is finite. So we can find $n = n(\alpha) < \omega$ and $A^*_{\alpha,0}, \ldots, A^*_{\alpha,n(\alpha)-1}$ enumerating $\mathcal{A}_\alpha$, but also $\nu^*_{B,\alpha} \notin S_B^1 \subseteq S_B$, hence $\mathrm{ob}(B \backslash \bigcup \mathcal{A}_\alpha) = \mathrm{ob}(B \backslash \bigcup\{A^*_{\alpha,\ell} : \ell < n(\alpha)\}$ is disjoint to $I_{\nu^*_{B,\alpha}} \cap J_s^+$ and by the choice of $\mathcal{A}_\alpha$ and $(*)_4$, $\mathrm{ob}(B \backslash \bigcup \mathcal{A}_\alpha) = [B \backslash (A^*_{\alpha,0} \cup \cdots \cup A^*_{\alpha,n(\alpha)-1})]^{\aleph_0}$ is disjoint to $I^s_{\nu^*_{B,\alpha}}$. Let $A^*_{\alpha,n(\alpha)}$ be $A_{\nu_B^* {\restriction} \alpha}$ when defined and $\varnothing$ otherwise. By the definitions of $I^s_{\nu^*_{B,\alpha}}, I^s_{\nu^*_B {\restriction} \alpha}$ we have, for $\alpha < \delta$,

$\odot_1$(a) $[B \cap (C^s_{\nu_B^* {\restriction} \alpha})^{[1-\nu_B^*(\alpha)]} \backslash (A^*_{\alpha,0} \cup \cdots \cup A^*_{\alpha,n(\alpha)})]^{\aleph_0}$ is disjoint to $I^s_{\nu_B^* {\restriction} \alpha}$;

(b) $A^*_{\alpha,\ell} \in \{A_\rho : \rho \in \mathscr{S}_t$ and $\nu^*_{B,\alpha} \trianglelefteq \rho$ (hence $A_{\alpha,\ell} \subseteq (C^s_{\nu_B^* {\restriction} \alpha})^{[1-\nu_B^*(\alpha)]}$)$\}$ for $\ell < n(\alpha)$ (not needed presently).

[Why clause (b)? By the choice of $\mathcal{A}_\alpha$.]

Let $\mathcal{A}^* = \{B \cap A : A = A^*_{\alpha,k}$ for some $\alpha < \delta, k \leq n(\alpha)$ and $B \cap A$ is infinite$\}$.

So $\mathcal{A}^*$ is a family of pairwise almost disjoint infinite subsets of B, and if $\mathcal{A}^*$ is finite, $B \backslash \bigcup\{A : A \in \mathcal{A}^*\}$ is still infinite because $\mathcal{A}^* \subseteq J_s$ and we are assuming $B \notin J_s$.

Let $\Lambda := \{\nu \in \mathscr{S}_s : \nu_B^* \trianglelefteq \nu, |A_\nu \cap B| = \aleph_0\}$.

$\odot_2$ There is a set B_1 such that

(a) $B_1 \subseteq B$ is infinite;
(b) B_1 is almost disjoint to any $A \in \mathcal{A}^*$;
(c) if Λ is finite then $\nu \in \Lambda \Rightarrow |B_1 \cap A_\nu| < \aleph_0$;
(d) if Λ is infinite then for infinitely many $\nu \in \Lambda$ we have $|B_1 \cap A_\nu| = \aleph_0$.

[Why? First assume Λ is finite, so without loss of generality, it is empty. If $\mathcal{A}^*$ is finite use the paragraph above on $\mathcal{A}^*$. Otherwise as $|\mathcal{A}^*| \leq |\delta| + \aleph_0 \leq \kappa = \mathfrak{s}$ and by the theorem's assumption, $\mathfrak{s} < \mathfrak{a}_* \leq \mathfrak{a}$, and by the definition of $\mathfrak{a}$, it follows that $\odot_2$ holds.
Second, assume that Λ is infinite and choose pairwise distinct $\nu_n \in \Lambda$ for $n < \omega$. Now recall that we are assuming $\mathfrak{s} < \mathfrak{a}_*$, and apply Definition 1.4 of $\mathfrak{a}_*$ to $\mathcal{A}^*$ and $\langle A_{\nu_n} : n < \omega\rangle$ to get an infinite $B_1 \subseteq B$ as required.]

$\odot_3$(a) $B_1 \in J_s^{\perp}$;
(b) if $\neg(\nu_B^* \trianglelefteq \eta)$ and $\eta \in \mathscr{S}_s$, then $B_1 \cap A_\eta$ is finite.
[Why? For clause (a), note that first $B_1 \subseteq B \subseteq \omega$, second B_1 is infinite by clause (a) of $\odot_2$, third $B_s \notin J_s$ is proved by dividing into two cases. If Λ is finite, use clause (b) of $\odot_3$, proved below, and clause (c) of $\odot_2$; and if Λ is infinite, use $\odot_2$(d)).

So let us turn to proving clause (b); we should prove that $\eta \in \mathrm{suc}(\mathcal{T}_s) \wedge \neg(\nu_B^* \trianglelefteq \eta) \Rightarrow B_1 \cap A_\eta$ finite.
If $A_\eta \in \{A^*_{\alpha,n} : \alpha < \delta, n \leq n(\alpha)\}$, then either $A_\eta \cap B$ is finite hence $A_\eta \cap B_1 \subseteq A_\eta \cap B$ is finite, or $A_\eta \cap B$ is infinite, hence $A_\eta \cap B \in \mathcal{A}^*$, so $B_1 \cap (A_\eta \cap B)$ is finite by the choice of B_1. But $B_1 \subseteq B$ hence $B_1 \cap A_\eta$ is finite. So assume $A_\eta \notin \{A^*_{\alpha,m} : \alpha < \delta, n \leq n(\alpha)\}$. By the choice of $A_{\alpha,n(\alpha)}$ for $\alpha < \delta$ necessarily $\neg(\eta \triangleleft \nu_B^*)$. Recall that we are assuming that $\neg(\nu_B^* \trianglelefteq \eta)$. Together, for some $\alpha < \delta$, we have $\alpha = \ell g(\nu_B^* \cap \eta) < \delta$ and $\nu_B^* \restriction \alpha \triangleleft \eta$, and we get a contradiction by the choice of $\mathcal{A}_\alpha = \{A^*_{\alpha,\ell} : \ell < n(\alpha)\}$ and $A^*_{\alpha,n(\alpha)}$.]

We shall now prove by induction on $\alpha \leq \delta$ that $B_1 \in I^s_{\nu_B^* \restriction \alpha}$. For $\alpha = 0$ recall that $I^s_{\nu_B^* \restriction \alpha} = [\omega]^{\aleph_0}$ for α limit $I^s_{\nu_B^* \restriction \alpha} = \bigcap\{I^s_{\nu_B^* \restriction \beta} : \beta < \alpha\}$ and use the induction hypothesis. For $\alpha = \beta + 1$, first note that by $\odot_2$(b) B_1 is almost disjoint to $A_{\nu_B^* \restriction \beta}$ if $\nu_B^* \restriction \beta \in \mathscr{S}_s \subseteq \mathrm{suc}(\mathcal{T}_s)$, and second, B_1 is almost disjoint to $(C^s_{\nu_B^* \restriction \beta})^{[1-\nu_B^*(\beta)]}$. (Otherwise, recalling $\odot_3$(b), we get a contradiction to the assumption $\mathrm{SP}_B = \varnothing$ by $(*)_6$(a) and the induction hypothesis.) Together with the definition of $I^s_{\nu_B^* \restriction \beta}, I^s_{\nu_B^* \restriction \alpha}$ we have $B_1 \in I^s_{\nu_B^* \restriction \alpha}$. Having carried the induction, in particular $B_1 \in I^s_{\nu_B^* \restriction \delta} = I_{\nu_B^*}$. Now recalling first $B_1 \notin J_s$ by $\odot_3(a)$, second $B_1 \subseteq B$ by $\odot_2(a)$, and third, the choice of $\mathcal{A}^*$, J_s. Together they contradict the subcase assumption $\nu_B^* \notin S^1_B$.

Case 3: None of the above.

Without loss of generality:

$\oplus_1$ if $B_1 \subseteq B$ but $B_1 \notin J_s$, then none of the two cases above holds.

We try to choose $\bar{\eta}^n = \langle \eta_\rho : \rho \in {}^n2\rangle$ by induction on n such that:

(a) $\eta_\rho \in \mathrm{SP}_B$;
(b) if $\rho = \varrho\hat{\ }\langle \ell\rangle$ then $\eta_\varrho\hat{\ }\langle \ell\rangle \trianglelefteq \eta_\rho$;
(c) $\{\nu : \nu \triangleleft \eta_\rho$ and $\nu \in \mathrm{SP}_B\} = \{\eta_{\rho\restriction k} : k < \ell g(\rho)\}$.

For $n = 0$, since $\mathrm{SP}_B \neq \varnothing$ as this is not Case 2 (and not Case 1), we can choose $\eta_\rho \in SP_B$ with minimal length. If $n = m+1$ and $\rho \in {}^m2$, by the induction hypothesis,

$\eta_\rho \in \mathrm{SP}_B$, hence $\eta_\rho \in \mathcal{T}_s$, and by the definition of SP_B for $\ell = 0, 1$, the sequence $\eta_\rho{}^\frown\langle\ell\rangle$ belongs to S_B.

First assume $\{\nu \in \mathrm{SP}_B : \eta_\rho{}^\frown\langle\ell\rangle \trianglelefteq \nu\} = \varnothing$. So $B_1 := B \cap C_s(\eta_\rho{}^\frown\langle\ell\rangle, B) \notin J_s$, noting that $C_s(\eta_\rho{}^\frown\langle\ell\rangle, B) = \bigcap\{C^{[\rho(k)]}_{\eta_\rho\restriction k} : k \le m\}$, recalling it is defined in $(*)_4$ from Stage C using $\eta_\rho{}^\frown\langle\ell\rangle \in S_B$; hence $\eta_\rho{}^\frown\langle\ell\rangle \in S_{B_1}$.

Now by $(*)_4$ we know $S_{B_1} = \{\nu \in S_B : \nu \trianglelefteq \eta_\rho{}^\frown\langle\ell\rangle$ or $\eta_\rho{}^\frown\langle\ell\rangle \trianglelefteq \nu\}$, so case 2 or case 1 holds for B_1, a contradiction to $\oplus_1$.

Second, assume that we have $(\exists\eta)(\eta_\rho{}^\frown\langle\ell\rangle \trianglelefteq \eta \in \mathrm{SP}_B)$, so choose such $\eta_{\rho^\frown\langle\ell\rangle}$ of minimal length.

Hence we have carried the inductive choice of $\langle\bar\eta^n : n < \omega\rangle$.

For each $\rho \in {}^\omega 2$ let $\eta_\rho = \bigcup\{\eta_{\rho\restriction n} : n < \omega\}$, so clearly $\eta_\rho \in c\ell(\mathcal{T}_s)$. Also $\langle\eta_\rho : \rho \in {}^\omega 2\rangle$ is without repetitions and each η_ρ belongs to $c\ell(\mathcal{T}_s)$, so as $|\mathcal{T}_s| < 2^{\aleph_0}$ there is $\rho \in {}^\omega 2$ such that $\eta_\rho \notin \mathcal{T}_s$. By clause (c) above we have $\{\varrho : \varrho \triangleleft \eta_\rho$ and $\varrho \in SP_B\} = \{\eta_{\rho\restriction n} : n < \omega\}$.

Note that

$\oplus_2$ $\langle C_s(\eta_{\rho\restriction k}, B) : k < \omega\rangle$ is $\subseteq$-decreasing.

Let $\mathcal{W} = \{\alpha < \ell g(\eta_\rho)$: for some $\nu \in \mathscr{S}_s$ we have $\ell g(\nu \cap \eta_\rho) = \alpha$ and $A_\nu \cap B$ is infinite$\}$.

First, assume $\mathcal{W}$ is an unbounded subset of $\ell g(\eta_\rho)$. In this case we choose $\alpha_n \in \mathcal{W}$ such that $\alpha_{n+1} > \alpha_n \ge \ell g(\eta_{\rho\restriction n})$ for $n < \omega$, and we choose $\nu_n \in \mathscr{S}_s$ such that $\ell g(\nu_n \cap \eta_\rho) = \alpha_n$ and $A_{\nu_n} \cap B$ is infinite. So we can choose an infinite $B_0 \subseteq B$ such that $n < \omega$ implies $B_0 \backslash \bigcup\{A_{\nu_{n\restriction k}} : k < n\} \subseteq^* C_s(\eta_\rho\restriction\alpha_n, B)$ and $(B_0 \bigcap A_{\nu_n} \in \mathrm{ob}(\omega))$.

So,

$\oplus_3$ $B_0 \subseteq B, B_0 \notin J_s$;

$\oplus_4$ the set SP_{B_0} is empty.

[Why? By $(*)_4$, for each $n < \omega$ we have $S_{B_0} \subseteq \{\nu : \nu \trianglelefteq \eta_{\rho\restriction n} \vee \eta_{\rho\restriction n} \trianglelefteq \nu\}$, hence $S_{B_0} \cap \mathcal{T}_s \subseteq \{\nu : \nu \triangleleft \eta_\rho$ or $\eta_\rho \trianglelefteq \nu\}$, but $\eta_\rho \notin \mathcal{T}_s$, so $\mathrm{SP}_{B_0} = \varnothing$.]

$\oplus_5$ S_{B_0} is not empty.

[Why? By $(*)_3$(b).]

By $\oplus_4$ and $\oplus_5$ for the set B_0, Case 1 or Case 2 holds, so we get a contradiction to $\oplus_1$.

Second, assume $\sup(\mathcal{W}) < \ell g(\eta_\rho)$, so we can choose $n(*) < \omega$ such that $\sup(\mathcal{W}) < \ell g(\eta_{\rho\restriction n(*)})$. Now $\nu \in \mathscr{S}_s \wedge \eta_{\rho\restriction n(*)} \trianglelefteq \nu$ implies that $B \cap A^s_\nu$ is finite, as otherwise, recalling $\eta_\rho \in c\ell(\mathcal{T}_s)\backslash\mathcal{T}_s$, necessarily $\alpha = \ell g(\eta_\rho \cap \nu) < \ell g(\eta_\rho)$ and of course $\alpha \ge \ell g(\eta_{\rho\restriction n(*)})$, but see the choice of $n(*)$, so $\eta_{\rho\restriction n(*)} \notin S^2_\beta$. Hence $\eta_{\rho\restriction n(*)} \in S^1_B$, so we can choose an infinite $B_1 \subseteq B$ such that $B_1 \in I^s_{\eta_{\rho\restriction n(*)}}$. Checking by cases, $B_1 \in \mathrm{ob}(\omega)$ is almost disjoint to any $A_\nu, \nu \in \mathscr{S}_s$. Obviously $B_1 \in I^s_{\eta_\rho}$, so Case 1 holds as exemplified by η_ρ, again a contradiction to $\oplus_1$. ■

3 The Other Cases

Theorem 3.1 (1) *If* $\kappa = \mathfrak{s} = \mathfrak{a}_*$ *and* $\mathrm{cf}([\mathfrak{s}]^{\aleph_0}, \subseteq) = \mathfrak{s}$, *then there is a saturated MAD family.*

(2) *If* $\kappa = \mathfrak{s} = \mathfrak{a}_*$ *and* $\mathbf{U}(\kappa) = \kappa$ *and* $J_* \subseteq [\omega]^{\aleph_0}$ *is dense, then there is a saturated MAD family* $\subseteq J_*$.

Definition 3.2 (1) For cardinals $\partial \leq \sigma \leq \theta \leq \lambda$ (also the case $\theta < \sigma$ is OK) let $\mathbf{U}_{\theta,\sigma,\partial}(\lambda) = \operatorname{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq\sigma}$ such that for every $X \in [\lambda]^\theta$, for some $u \in \mathcal{P}$, we have $|X \cap u| \geq \partial\}$. If $\partial = \sigma$, we may omit ∂; if $\sigma = \partial = \aleph_0$, we may omit them both; and if $\sigma = \partial = \aleph_0 \wedge \theta = \lambda$ we may omit θ, σ, ∂. In the case of our theorem, it means: $\mathbf{U}(\kappa) = \operatorname{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\kappa]^{\leq\aleph_0}$ and $(\forall X \in [\kappa]^\kappa)(\exists u \in \mathcal{P})(|X \cap u| \geq \aleph_0)\}$.
(2) If in addition J is an ideal on θ, then let $\mathbf{U}_{\theta,\sigma,J}(\lambda) = \operatorname{Min}\{|\mathcal{P}| : \mathcal{P} \subseteq [\lambda]^{\leq\sigma}$ such that for every function $f : \theta \to \lambda$, for some $u \in \mathcal{P}$, the set $\{i < \theta : f(i) \in u\}$ does not belong to $J\}$.
(3) Let $\Pr(\kappa, \theta, \sigma, \partial)$ mean: $\kappa \geq \theta \geq \sigma \geq \partial$, and we can find $(E, \bar{\mathcal{P}})$ witnessing it (if $\partial = \sigma$, we may omit ∂; if $\sigma = \partial = \aleph_0$, we may omit them; if $\sigma = \partial = \aleph_0 \wedge \theta = \kappa$, we may omit θ, σ, ∂) which means:

(a) $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in E \rangle$
(b) E is a club of κ and $\gamma \in E \Rightarrow |\gamma|$ divide γ
(c) if $u \in \mathcal{P}_\alpha$ then $u \in [\alpha]^{\leq\sigma}$ has no last member
(d) $\bullet_1$ $\bar{\mathcal{P}}$ is $\subseteq$-increasing,
    $\bullet_2$ $|\mathcal{P}_\alpha| < \kappa$,
(e) if $w \subseteq \kappa$ is bounded and $\operatorname{otp}(w) = \theta$ and $\sup(w) \in \operatorname{acc}(E)$, then for some u, j we have (so $\theta > \partial$):
    $\bullet_1$ $|u \cap w| \geq \partial$,
    $\bullet_2$ $j \in \operatorname{acc}(E)$,
    $\bullet_3$ $u \in \mathcal{P}_j$,
    $\bullet_4$ $|w \cap j| < \theta$, *i.e.*, $j < \sup(w)$;
(f) if $i \in \{0\} \cup E$ and $j = \min(E \backslash (i+1))$, $w \subseteq [i, j)$, $\operatorname{otp}(w) = \theta$, then for some set u;
    $\bullet_1$ $u \in \mathcal{P}_j$ and $u \subseteq (i, j)$,
    $\bullet_2$ $|u \cap w| \geq \partial$.

Explanation 3.3 The proof of Theorem 3.1 is based on the proof of Theorem 2.1. The difference is that in the proof of $\odot_2$ of subcase 2B of stage F, if $\ell g(\nu^*_B) = \kappa$, it does not follow that we have $|\mathcal{A}^*| < \mathfrak{a}_*$, so we have to do something else when $|\mathcal{A}^*| = \mathfrak{a}_* = \mathfrak{s}$. By the assumption $\mathbf{U}(\kappa) = \kappa$ there is a sequence $\langle u_\alpha : \omega \leq \alpha < \kappa \rangle$ of members of $[\kappa]^{\aleph_0}$ such that $u_\alpha \subseteq \alpha$ and for every $X \in [\kappa]^\kappa$ for some $\alpha, u_\alpha \cap X$ is infinite. Now if *e.g.*, $\ell g(\nu) = \alpha \geq \omega$, we can use u_α and apply 3.5 below to an appropriate $\bar{B}_\nu$ getting $\mathcal{P}_\nu$. We then add it to the family $\{C^*_\alpha : \alpha < \kappa\}$, witnessing $\mathfrak{s} = \kappa$ by the family $\mathcal{P}_\nu$, as in Observation 3.5. So now we really need to use C^s_ν rather than C^*_α.

Observation 3.4 If $\Pr(\kappa, \theta, \sigma, \partial)$ is witnessed by $(E, \bar{\mathcal{P}})$, then we can find $(E', \bar{\mathcal{P}}')$ as in 3.2(3), but

(d)$'\bullet_2$ if $j > \sup(j \cap E')$, then $|\mathcal{P}'_j| \leq j$,
(e) as above but require just $\sup(w) \in E$.

Proof Use any club $E' \subseteq \mathrm{acc}(E)$ of κ such that $\delta \in E' \Rightarrow |\mathcal{P}_\delta| \leq |\min(E' \backslash (\delta+1))|$ and $\delta \in \mathrm{nacc}(E') \Rightarrow \mathrm{cf}(\delta) \neq \mathrm{cf}(\theta)$, and let $\mathcal{P}'_\gamma$ be $\mathcal{P}_\gamma$ if $\gamma \in \mathrm{acc}(E')$ and let $\mathcal{P}'_\gamma$ be $\bigcup\{\mathcal{P}_\beta : \beta \in E \cap \gamma\}$ if $\gamma \in \mathrm{nacc}(E')$. ■

Observation 3.5 Assume $\bar{B}^* = \langle B^*_n : n < \omega \rangle$ satisfies $B^*_n \in [\omega]^{\aleph_0}, B^*_{n+1} \subseteq B^*_n$, and $|B^*_n \backslash B^*_{n+1}| = \aleph_0$ for infinitely many n's. Then we can find $\mathcal{P}$ such that

$(*)$(a) $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ is of cardinality $\mathfrak{b}$;

(b) if $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an AD family, $B \subseteq \omega$ and $(\exists^\infty n)(B \cap B^*_n \setminus B^*_{n+1})$ or just for some sequence $\langle (n_i, A_i) : i < \omega \rangle$ we have $n_i < n_{i+1}, A_i \in \mathscr{A} \setminus \{A_j : j < i\}$ and $(B_{n_i} \setminus B_{n_i+1}) \cap A_i$ is infinite for every i, then for some countable (infinite) $\mathcal{P}' \subseteq \mathcal{P}$ there exists for $2^{\aleph_0}$ functions $\eta \in {}^{\mathcal{P}'}2$ such that for some $\mathrm{id}_{\mathcal{A}}$-positive set $A \subseteq^* B$ we have: $A \subseteq^* C^{[\eta(C)]}$ for every $C \in \mathcal{P}'$ and $A \subseteq^* B^*_n$ for every n.

Proof of Observation 3.5 Let $\mathcal{B} = \{\bar{B} : \bar{B} = \langle B_n : n < \omega \rangle$, where $B_n \subseteq \omega$ is infinite, $B_n \supseteq B_{n+1}$, and $B_n \backslash B_{n+1}$ is infinite for infinitely many $n < \omega\}$, *i.e.,* the set of $\bar{B}$ satisfying the demands on $\bar{B}^*$.

For $\bar{B} \in \mathcal{B}$ and $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ let $\mathrm{pos}(\bar{B}, \mathcal{A}) = \{B \subseteq \omega : B$ as in $(*)$(b)$\}$. So Observation 3.5 says that for every $\bar{B} \in \mathcal{B}$ there is $\mathcal{P} \subseteq [\omega]^{\aleph_0}$ of cardinality $\mathfrak{b}$ such that if $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ is an AD family and $B \in \mathrm{pos}(\bar{B}, \mathcal{A})$, then there is a countable infinite $\mathcal{P}' \subseteq \mathcal{P}$ as required in $(*)$(b) of Observation 3.5

Consider the statement:

$\boxplus$ if $\bar{B} \in \mathcal{B}$, then we can find $\mathbf{B}$ such that

(a) $\mathbf{B} = \langle \bar{B}_\delta : \delta \in S^{\mathfrak{b}}_{\aleph_0} \rangle$, recalling $S^{\mathfrak{b}}_{\aleph_0} = \{\delta < \mathfrak{b} : \mathrm{cf}(\delta) = \aleph_0\}$;

(b) $\delta \in S^{\mathfrak{b}}_{\aleph_0} \Rightarrow \bar{B}_\delta \in \mathcal{B}$;

(c) if $\mathcal{A}$ is an AD family and $B \in \mathrm{pos}(\bar{B}, \mathcal{A})$, then for some club E of $\mathfrak{b}$, for every $\delta \in E \cap S^{\mathfrak{b}}_{\aleph_0}$ we have $(\exists^\infty n)[B \cap (B_{\delta,n} \backslash B_{\delta,n+1}) \in \mathrm{id}^+_{\mathcal{A}}]$;

(d) if $\delta_1 < \delta_2$ are from $S^{\mathfrak{b}}_{\aleph_0}$, then for some $n < \omega$ the set $B_{\delta_1,n} \cap B_{\delta_2,n}$ is finite.

Why is this statement enough? Because it allows us to find a subset $\mathcal{B}'$ of $\mathcal{B}$ of cardinality $\mathfrak{b}$ such that $\bar{B}^* \in \mathcal{B}'$ and for every $\bar{B} \in \mathcal{B}'$ for some $\mathbf{B} = \langle \bar{B}_\delta : \delta \in S^{\mathfrak{b}}_{\aleph_0} \rangle$ as in $\boxplus$, we have $\delta \in S^{\mathfrak{b}}_{\aleph_0} \Rightarrow \bar{B}_\delta \in \mathcal{B}'$. Now $\mathcal{P}$, the closure by Boolean operations of $\{B_n : \bar{B} \in \mathcal{B}'$ and $n < \omega\}$ is as required.

Why? Let $\bar{B} \in \mathcal{B}'$ (*e.g.,* $\bar{B}^*$) and an AD family $\mathcal{A} \subseteq [\omega]^{\aleph_0}$ and assume $B \in \mathrm{pos}(\bar{B}, \mathcal{A})$ is given.

We choose by induction on $n < \omega$ a sequence $\langle \bar{B}_\eta, m_\eta : \eta \in {}^n2 \rangle$ such that

- $\bar{B}_\eta \in \mathcal{B}'$ moreover $(\exists^\infty n)(B_{\eta,n} \backslash B_{\eta,n+1} \in \mathrm{id}^+_{\mathcal{A}})$ for $\eta \in {}^n2$;
- $\bar{B}_\eta = \bar{B}$ if $\eta = \langle \, \rangle$ so $n = 0$;
- $B \in \mathrm{pos}(\bar{B}_\eta, \mathcal{A})$, m_η and $\bigcap\{B_{\eta\restriction\ell, m_{\eta\restriction\ell}}\} \in \mathrm{pos}(\bar{B}_\eta, \mathscr{A})$ if $\eta \in {}^n2$;
- if $\nu\char`\^\langle 0 \rangle, \nu\char`\^\langle 1 \rangle \in {}^n2$, then the set $B_{\nu\char`\^\langle 0 \rangle, m_{\nu\char`\^\langle 0 \rangle}} \cap B_{\nu\char`\^\langle 1 \rangle, m_{\nu\char`\^\langle 1 \rangle}}$ is finite.

For $n = 0$ this is trivial and for $n = m+1$ we use $\boxplus$(c), *i.e.,* the construction of $\mathcal{B}'$. For every $n < \omega, \varrho \in {}^n2$, let $B_\varrho = \bigcap\{B_{\eta\restriction k, m_{\eta\restriction k}} : k \leq n\}$. So $B_\varrho \in \mathrm{id}^+_{\mathcal{A}}$ and $m < \ell g(\varrho) \Rightarrow B_\varrho \subseteq B_{\varrho\restriction m}$ and if $\varrho_1 \neq \varrho_2 \in {}^n2$ then we have $B_{\varrho_1} \cap B_{\varrho_2}$ is finite. Obviously $[\varrho \in {}^\omega 2 \Rightarrow (\forall n < \omega)(\exists k < \omega)(B_{\varrho\restriction n} \backslash B_{\varrho\restriction k} \in \mathrm{id}^+_{\mathcal{A}})]$, and for each $\varrho \in {}^\omega 2$ there is $C_\varrho \in \mathrm{id}^+_{\mathcal{A}}$ such that $C_\varrho \subseteq^* B_{\varrho\restriction n}$ for $n < \omega$.

[Why? We try by induction on $k < \omega$ to choose $A_{\varrho,k}, A'_{\varrho,k} \in \mathrm{ob}(\omega)$ such that

$A'_{\varrho,k} \in \mathcal{A}, A_{\varrho,k} \subseteq A'_{\varrho,k}$ and $m < k \Rightarrow A'_{\varrho,k} \neq A'_{\varrho,m}$ and $A_{\varrho,k} \subseteq^* B_{\varrho\restriction k}$. Now first, if we succeed, then we can find $C \in \mathrm{ob}(\omega)$ such that for every $n < \omega$ we have that $C \cap A_{\varrho,n}$ is infinite and $C \setminus \bigcup\{A'_{\varrho,m} : m < n\} \subseteq B_{\varrho\restriction k_n}$. If there is an infinite $C' \subseteq C$ almost disjoint to every member of $\mathcal{A}$, then $C_\varrho = C'$ is as required. If there is no such C', then we can find pairwise distinct $A''_n \in \mathcal{A} \setminus \{A'_{\varrho,m} : m < \omega\}$ such that $C \cap A''_n$ is infinite for every $n < \omega$. Clearly $A''_n \cap C \subseteq^* B_{\varrho\restriction m}$ for every $n, m < \omega$, and there is an infinite $C_\varrho \subseteq C$ such that $C_\varrho \subseteq^* B_{\varrho\restriction m}$ and $C_\varrho \cap A''_n$ is infinite for every $n, m < \omega$, so C_ϱ is as required.
Second, if $k < \omega$ and we cannot choose $A_{\varrho,k}$, then we can choose $C_\varrho \in \mathrm{ob}(\omega)$ such that $n < \omega \Rightarrow C_\varrho \subseteq^* B_{\varrho\restriction n}$ and $C_\varrho \cap A_{\varrho,m} = \varnothing$ for $m < k$, and C_ϱ is as required, so we are done.]

So $\mathcal{P}' = \{B_{\eta\restriction k,m} : k, m < \omega\}$ is as required.
So proving $\boxplus$ is enough.

Why does this statement hold? Let $\bar{f} = \langle f_\alpha : \alpha < \mathfrak{b}\rangle$ be a sequence of members of ${}^\omega\omega$ witnessing $\mathfrak{b}$, such that without loss of generality $f_\alpha \in {}^\omega\omega$ is increasing and $\alpha < \beta < \mathfrak{b} \Rightarrow f_\alpha <_{J^{\mathrm{bd}}_\omega} f_\beta$.

For $\alpha < \mathfrak{b}$ let $C_\alpha := \bigcup\{B_n \cap [0, f_\alpha(n)) : n < \omega\}$, so clearly

$(*)_1$(a) $\alpha < \beta \Rightarrow C_\alpha \subseteq^* C_\beta$,
(b) $\alpha < \mathfrak{b} \wedge n < \omega \Rightarrow C_\alpha \subseteq^* B_n$.

We choose $\alpha_\varepsilon = \alpha(\varepsilon) < \mathfrak{b}$ by induction on $\varepsilon < \mathfrak{b}$, increasing with ε as follows: for $\varepsilon = 0$ let $\alpha_\varepsilon = \min\{\alpha < \mathfrak{b} : C_\alpha \text{ is infinite}\}$, for $\varepsilon = \zeta + 1$ let $\alpha_\varepsilon = \min\{\alpha < \mathfrak{b} : \alpha > \alpha_\zeta \text{ and } C_\alpha \setminus C_{\alpha(\zeta)} \text{ is infinite}\}$, and for ε limit let $\alpha_\varepsilon = \bigcup\{\alpha_\zeta : \zeta < \varepsilon\}$. By the choice of $\bar{f}$, every α_ε is well defined; see the proof of $\oplus_\alpha$ below.

So $\langle \alpha_\varepsilon : \varepsilon < \mathfrak{b}\rangle$ is increasing and continuous with limit $\mathfrak{b}$. For each $\delta \in S^{\mathfrak{b}}_{\aleph_0}$ let $\langle \varepsilon(\delta, n) : n < \omega\rangle$ be increasing with limit δ and, lastly, let

$$\bar{B}_\delta = \left\langle C_{\alpha(\delta)} \setminus \bigcup_{m \le n} C_{\alpha(\varepsilon(\delta,m))} : n < \omega \right\rangle,$$

so $B_{\delta,n} = C_{\alpha(\delta)} \setminus \bigcup_{m \le n} C_{\alpha(\varepsilon(\delta,m))}$, hence $B_{\delta,n+1} \subseteq B_{\delta,n}$ and $B_{\delta,n} \setminus B_{\delta,n+1}$ is infinite by the choice of $\alpha_{\varepsilon(\delta,n)+1}$. Clearly $\bar{B}_\delta \in \mathcal{B}$ (which also follows from the proof below).

Why is $\langle \bar{B}_\delta : \delta \in S^{\mathfrak{b}}_{\aleph_0}\rangle$ as required in $\boxplus$? Clauses (a) and (b) are obvious, and clause (d) is easy; as, if $\delta_1 < \delta_2$, then for some n we have $\delta_1 < \alpha(\varepsilon(\delta_2, n))$, hence

$$B_{\delta_1,n} \cap B_{\delta_2,n} \subseteq B_{\alpha(\varepsilon(\delta_1,n))} \cap (B_{\alpha(\delta_2)} \setminus C_{\alpha(\varepsilon(\delta_2,n))}) \subseteq^* B_{\alpha(\delta_1)} \cap (B_{\alpha(\delta_2)} \setminus B_{\alpha(\delta_1)}) = \varnothing.$$

Lastly, to check clause (c) of $\boxplus$ let $\mathcal{A}$ be an AD family and $B \subseteq \omega$ be such that

$(*)_2$ $u = u_B := \{n < \omega : B \cap B_n \setminus B_{n+1} \notin \mathrm{id}_\mathcal{A}\}$ is infinite, or just $u = \{n_i : i < \omega\}$ where $\langle (n_i, A_i) : i < \omega\rangle$ is as in $(*)$(b) of the observation.

It is enough to prove that for every $\alpha < \mathfrak{b}$:

$\oplus_\alpha$ there is $\beta \in (\alpha, \mathfrak{b})$ such that $B \cap C_\beta \setminus C_\alpha \in \mathrm{id}^+_\mathcal{A}$.

[Why is it enough? Because then for some club E of $\mathfrak{b}$, for every $\delta \in E \cap S^{\mathfrak{b}}_{\aleph_0}$ we would have $(\forall \varepsilon < \delta)(\alpha_\varepsilon < \delta)$ and $(\forall \alpha < \delta)(\exists \beta)(\alpha < \beta < \delta \wedge C_\beta \setminus C_\alpha \in \mathrm{id}^+_\mathcal{A})$, hence $(\exists^\infty n)((C_{\alpha(\varepsilon(\delta,n+1))} \setminus C_{\alpha(\varepsilon(\delta,n))}) \in \mathrm{id}^+_\mathcal{A})$, which means $(\exists^\infty n)(B_{\delta,n} \setminus B_{\delta,n+1}) \in \mathrm{id}^+_\mathcal{A})$ as required.]

So let us prove $\oplus_\alpha$. If $\oplus_\alpha$ fails, for every $\beta \in (\alpha, \mathfrak{b})$ there are $n = n(\beta)$ and $A_{\beta,0}, \ldots, A_{n(\beta)-1} \in \mathcal{A}$ such that $B \cap C_\beta \backslash C_\alpha \subseteq^* A_{\beta,0} \cup \cdots \cup A_{\beta,n(\beta)-1}$. Without loss of generality, $n(\beta)$ is minimal, hence by $(*)_1$ the sequence $\langle n(\beta) : \beta \in [\alpha, \mathfrak{b})\rangle$ is non-decreasing, but $\mathfrak{b} = \mathrm{cf}(\mathfrak{b}) > \aleph_0$, hence, for some $\alpha_* \in [\alpha, \mathfrak{b})$, the sequence $\langle n(\beta) : \beta \in [\alpha_*, \mathfrak{b})\rangle$ is constant, so let $n(\alpha_*) = n_*$.

As $\mathcal{A}$ is AD and $B \cap C_{\alpha_*} \backslash C_\alpha \subseteq^* A_{\alpha_*,0} \cup \cdots \cup A_{\alpha_*,n_*-1}$ and $\beta \in (\alpha_*, \mathfrak{b}) \Rightarrow B \cap C_{\alpha_*} \backslash C_\alpha \subseteq B \cap C_\beta \backslash C_\alpha \subseteq A_{\beta,0} \cup \cdots \cup A_{\beta,n_*-1}$, using "$\mathcal{A}$ is almost disjoint" and the minimality of $n_{\alpha_*} = n_*$ it follows that $\{A_{\alpha_*,\ell} : \ell < n_*\} \subseteq \{A_{\beta,\ell} : \ell < n_*\}$, hence they are equal.

So,

$\odot$ $\beta \in (\alpha, \mathfrak{b}) \Rightarrow B \cap C_\beta \backslash C_\alpha \subseteq^* A_{\alpha_*,0} \cup \cdots \cup A_{\alpha_*,n_*-1}$.

For each $n \in w = \{n_i : A_i \notin \{A_{\alpha_*,\ell} : \ell < n_*\}\}$, as $B \cap B_n \backslash B_{n+1} \in \mathrm{id}^+_{\mathcal{A}}$ and $A_{\alpha_*,0}, \ldots, A_{\alpha_*,n_*-1}$ are from $\mathrm{id}_{\mathcal{A}}$, clearly there is

$$k_n \in (B \cap B_n \backslash B_{n+1} \backslash C_\alpha) \backslash A_{\alpha_*,0} \backslash \cdots \backslash A_{\alpha_*,n_*-1} \backslash \{k_0, \ldots, k_{n-1}\}.$$

By the choice of $\bar{f}$ there is $\beta \in (\alpha_*, \mathfrak{b})$ such that $u_1 := \{n \in w : k_n < f_\beta(n)\}$ is infinite. As f_β is increasing, clearly $n \in u_1 \Rightarrow k_n < f_\beta(n) \Rightarrow k_n \in C_\beta \backslash C_\alpha$. So $\{k_n : n \in u_1\} \in [\omega]^{\aleph_0}$ is infinite and is a subset of $B \cap C_\beta \backslash C_\alpha \backslash A_{\alpha_*,0}, \ldots, A_{\alpha_*,n_*-1}$, which is a contradiction, so $\oplus_\alpha$ indeed holds, and we are done. ■

Proof of Theorem 3.1 We prove part (2), and part (1) follows. We imitate the proof of Theorem 2.1.

Stage A: Let $\kappa = \mathfrak{s}$. Let $\mathcal{P} \subseteq [\kappa]^{\aleph_0}$ witness $\mathbf{U}(\kappa) = \kappa$. For transparency we assume $\omega \in \mathcal{P}$ and $u \in \mathcal{P} \Rightarrow \mathrm{otp}(u) = \omega$. this holds without loss of generality as $\mathfrak{b} \leq \mathfrak{a}_* = \mathfrak{s} = \kappa$.

[Why? It is enough to show that for every countable $u \subseteq \kappa$ there is a family $\mathcal{P}_u$ of cardinality $\leq \mathfrak{b}$ of subsets of u each of order type ω such that every infinite subset of u has an infinite intersection with some member of $\mathcal{P}$. Without loss of generality, u is a countable ordinal α, and we prove this by induction on α. For α a successor ordinal or not divisible by ω^2 this is trivial, so let $\langle \alpha_n : n < \omega\rangle$ be an increasing sequence of limit ordinals with limit α, but $\alpha_0 = 0$. Let $\langle \beta_{n,k} : k < \omega\rangle$ list $[\alpha_n, \alpha_{n+1})$ with no repetitions, let $\langle f_\epsilon \in {}^\omega\omega : \epsilon < \mathfrak{b}\rangle$ exemplify $\mathfrak{b}$, each f_ϵ increasing, and let

$$\mathcal{P}_\alpha = \bigcup \{\mathcal{P}_\beta : \beta < \alpha\} \cup \{\{\beta_{n,k} : n < \omega, k < f_\epsilon(n)\} : \varepsilon < \mathfrak{b}\}.$$

Clearly $\mathcal{P}_\alpha$ has the right form and cardinality.

Lastly, assume $v \subseteq u$ is infinite. If for some $\gamma < \alpha$, $u \cap \gamma$ is infinite, use the choice of $\mathcal{P}_\gamma$. Otherwise let $f \in {}^\omega\omega$ be defined by $f(n) = \min\{k : (\exists m)[n \leq m \wedge \beta_{m,k} \in v]\}$, and use $\epsilon < \mathfrak{b}$ large enough.]

Let $\langle u_\alpha : \alpha < \kappa\rangle$ list $\mathcal{P}$, possibly with repetitions. Without loss of generality $n \leq \omega \Rightarrow u_n = \omega$ and $\alpha \geq \omega \Rightarrow u_\alpha \subseteq \alpha$. For $\alpha < \kappa$ let $\langle \gamma(\alpha, k) : k < \omega\rangle$ list u_α in increasing order and $\gamma_{\alpha,k} = \gamma(\alpha, k)$.

Let $\langle \mathcal{U}_\alpha : \alpha < \kappa\rangle$ be a partition of κ into sets each of cardinality κ such that $\min(\mathcal{U}_{1+\alpha}) \geq \sup(u_\alpha) + 1$ and $\omega \subseteq \mathcal{U}_0$. Let $\langle C^*_\alpha : \alpha \in \mathcal{U}_0\rangle$ list a subset of $\mathcal{P}(\omega)$

witnessing $\mathfrak{s} = \kappa$, and, as in Stage A of the proof of Theorem 2.1, the set $J_* \subseteq \mathrm{ob}(\omega)$ is dense, and $\omega \notin \mathrm{id}_{J_*}$.

If $\bar{B}$ is as in the assumption of Observation 3.5 and $\alpha \in (0, \kappa)$, let $\mathcal{P}_{\bar{B}}$ be as in the conclusion of Observation 3.5, and for $\alpha < \kappa$ let $\langle C^*_{\bar{B},\alpha,i} : i \in \mathcal{U}_\alpha \rangle$ list $\mathcal{P}_{\bar{B}}$.

Stage B: We proceed as in the proof of 2.1, but we use $C^s_\rho(\rho \in \mathcal{T}_s)$, which may really depend on s, and where $C^t_\rho, \bar{B}^t_{\nu,\beta}$ are defined in clauses $\boxplus_1$(e),(g),(h),(i), and (j) below (so the $\boxplus$(e),(g), and (h) from Theorem2.1 are replaced), and depend just on $\mathcal{T}_t, \bar{A}_t$ and $\bar{I}_t$, too[3], where:

$\boxplus_1$ (a)–(d) and (f) as in Theorem 2.1 of course and

(e) $\bullet_1$ as before, *i.e.*, $\bar{A} = \langle A^t_\eta : \eta \in \mathrm{suc}(\mathcal{T}_t) \rangle$,

$\bullet_2$ $\bar{C} = \bar{C}_t = \langle C^t_\eta : \eta \in \Lambda_t \rangle$, where $\Lambda_t = \{\eta : \eta \in {}^i 2$ and $i \in \mathcal{U}_0$ or $\alpha > 0, i \in \mathcal{U}_\alpha$ and $\eta{\restriction}(\sup(u_\alpha)) \in \mathcal{T}_t$ or (for Theorem 3.6) $\eta \in \mathcal{T}_t\}$,

$\bullet_3$ $A_\eta = \varnothing$ if $\eta \in \mathcal{T} \setminus \mathrm{suc}(\mathcal{T})$

(g) as in Theorem 2.1 but replacing $C^*_{\eta\restriction i}$ by $C^t_{\eta\restriction i}$,

(h) if $i \in \mathcal{U}_0$ and $\nu \in \mathcal{T}_t \cap {}^i 2$ then $C^s_\nu = C^*_i$,

(i) if $\beta \in (0, \kappa)$ and $\nu \in {}^{\sup(u_\beta)}2$ and both $\langle C^t_{\nu\restriction i} : i \in u_\beta \rangle$ and $\langle A^t_{\nu\restriction i} : i \in u_\beta \rangle$ are well defined then we let $\bar{B}^t_{\nu,\beta} = \langle B^t_{\nu,\beta,n} : n < \omega \rangle$ be defined by

$$B^t_{\nu,\beta,n} = \bigcap \{ (C^t_{\nu\restriction\gamma(\beta,k)})^{[\nu(\gamma(\beta,k))]} \setminus A^t_{\nu\restriction\gamma(\beta,k)} : k < n \},$$

(j) if $\beta \in (0, \kappa), i \in \mathcal{U}_\beta$ so $i \geq \sup(u_\beta)$ and $\rho \in {}^i 2$ and $\bar{B}^t_{\rho\restriction \sup(u_\beta),\beta}$ is well defined, then, recalling stage A, $C^t_\rho = C^*_{\bar{B}^t_{\rho\restriction \sup(u_\beta),\beta},\beta,i}$.

Note that $\mathcal{T}_t, \overline{A}_t$ determine t, *i.e.*, $\overline{I_t}, \Lambda_t, \overline{C_t}$, and $\langle \overline{B^t}_{\nu,\beta} : \nu, \beta$ as above$\rangle$.

Stage C: As in Theorem 2.1 we just add:

$\boxplus_4$ (h) if $s \leq_{\mathrm{AP}} t$ and $\bar{B}^s_{\nu,\beta}$ is well defined then $\bar{B}^t_{\nu,\beta}$ is well defined and equal to it;

(i) if $s \leq_{\mathrm{AP}} t$ and C^s_ν is well defined then C^t_ν is well defined and equal to it, so $\Lambda_s \subseteq \Lambda_t$;

(j) C^s_ν is well defined when $\nu \in c\ell(\mathcal{T}_s) \cap {}^{\kappa>}2$.

In the proof of $\boxplus_4$(e) use the choice of $\langle C^s_\nu : \nu \in {}^i 2, i \in \mathcal{U}_0 \rangle$, *i.e.*, of $\langle C^*_\alpha : \alpha \in \mathcal{U}_0 \rangle$ in Stage A.

Stages D and E: As in Theorem 2.1.

Stage F: The only difference is in the proof of $\odot_2$ in subcase (2B). Recall:

Case 2: $\mathrm{SP}_B = \varnothing$, but not Case 1, *i.e.*, $S_B \subseteq \mathcal{T}_s$, recall $B \subseteq \omega, B \notin J_s$.

Subcase 2B: $\nu^*_B \notin S_B$, where $\nu^*_B = \bigcup \{\eta : \eta \in S_B\}$.

$\odot_2$ there is a set B_1 such that

(a) $B_1 \subseteq B$ is infinite;

(b) B_1 is almost disjoint to every $A \in \mathcal{A}^*$

(c) if Λ is finite then $\nu \in \Lambda \Rightarrow |B_1 \cap A_\nu| < \aleph_0$;

[3] also here we require $\eta \in \mathrm{suc}(\mathcal{T}_s) \Rightarrow A_\eta \neq \varnothing$

(d) if Λ is infinite then for infinitely many $\nu \in \Lambda$ we have $|B_1 \cap A_\nu| = \aleph_0$.

The rest of the proof is proving $\odot_2$. If $|\mathcal{A}^*| < \kappa$ then $\mathcal{A}^*$ has cardinality $< \kappa = \mathfrak{s}$ hence by the theorem's assumption $|\mathcal{A}^*| < \mathfrak{s} = \mathfrak{a}_*$; so $\odot_2$ follows as in the proof of 2.1. So we can assume $|\mathcal{A}^*| = \kappa$, but $|\mathcal{A}^*| \le \aleph_0 + |\ell g(\nu^*_B)|$, hence necessarily $\ell g(\nu^*_B) = \kappa$ follows.

Let

$$\begin{aligned}\mathcal{W} := \{\alpha < \kappa : \quad & \text{for some } \ell \le n(\alpha) \text{ we have } A^*_{\alpha,\ell} \cap B \in \mathcal{A}^* \\ & (\text{equivalently } A^*_{\alpha,\ell} \cap B \text{ is infinite}) \text{ but} \\ & A^*_{\alpha,\ell} \notin \{A^*_{\alpha_1,\ell_1} : \alpha_1 < \alpha \text{ and } \ell_1 \le n(\alpha_1)\}\}.\end{aligned}$$

For $\alpha \in \mathcal{W}$ choose $\ell(\alpha) \le n(\alpha)$ such that $B \cap A^*_{\alpha,\ell(\alpha)}$ is infinite and $A^*_{\alpha,\ell(\alpha)} \notin \{A^*_{\alpha_1,\ell_1} : \alpha_1 < \alpha \text{ and } \ell_1 \le n(\alpha_1)\}$. In fact by $\odot_1$(b) the last condition follows. As $n(\alpha) < \omega$ for $\alpha < \kappa$, clearly $|\mathcal{W}| = \kappa$, because $|\mathcal{A}^*| = \kappa$, hence by the choice of $\mathcal{P}$ there is $u_* \in \mathcal{P}$ such that $|\mathcal{W} \cap u_*|$ is infinite; let $\alpha(*) \in [\omega, \kappa)$ be such that $u_{\alpha(*)} = u_*$ and let $\nu = \nu^*_B \restriction \sup(u_*)$; recall that $\mathrm{otp}(u_*) = \omega$. Note that

$\odot_{2.1}$ $k < \omega \Rightarrow B^s_{\nu,\alpha(*),k+1} \subseteq B^s_{\nu,\alpha(*),k} \subseteq \omega$,

by their choice in $\boxplus_1(i)$.

Recall also that $\langle \gamma_{\alpha(*),k} : k < \omega \rangle$ list u_* in increasing order and so:

$\odot_{2.2}$ $v := \{k < \omega : \gamma_{\alpha(*),k} \in \mathcal{W}\}$ is infinite;

$\odot_{2.3}$ $[B^s_{\nu,\alpha(*),k}]^{\aleph_0} \supseteq I^s_{\nu\restriction\gamma(\alpha(*),k)}$ for $k < \omega$.

[Why? As for $k(1) < k$, $(C^s_{\nu\restriction\gamma(\alpha(*),k(1))})^{[\nu(\gamma(\alpha)(*),k(1))]}$ and $\omega \backslash A^*_{\nu\restriction\gamma(\alpha(*),k(1))}$ belong to $\{X \subseteq \omega : [X]^{\aleph_0} \supseteq I^s_{\nu\restriction\gamma(\alpha(*),k)}\}$ hence by the definition of $B^s_{\nu,\alpha(*),k}$ in $\boxplus_1$(i) it satisfies $\odot_{2.3}$.]

$\odot_{2.4}$ $k \in v \Rightarrow B \cap B^s_{\nu,\alpha(*),k} \backslash B^s_{\nu,\alpha(*),k+1}$ is infinite.

[Why? For $k \in v$ let $\beta = \gamma(\alpha(*),k)$, $n = n(\beta)$, and $\ell = \ell(\beta)$. On the one hand

$$[B \cap A^*_{\beta,\ell}]^{\aleph_0} \subseteq [A^*_{\beta,\ell}]^{\aleph_0} \subseteq I_{\nu^*_B\restriction\beta}.$$

On the other hand $A^*_{\beta,\ell}$ is trivially disjoint to $(C^s_{\nu^*_B\restriction\beta})^{[\nu^*_B(\beta)]} \backslash A^s_{\nu^*_B\restriction\beta}$ if $A^*_{\beta,\ell} = A^*_{\beta,n}$, and is almost disjoint to $(C^s_{\nu^*_B\restriction\beta})^{[\nu(\beta)]}$ otherwise; *i.e.*, as

$$\left[\left(C^s_{\nu^*_B\restriction\beta}\right)^{[1-\nu(\beta)]}\right]^{\aleph_0} \supseteq I_{(\nu^*_\beta\restriction\beta)^\frown\langle 1-\nu(\beta)\rangle} \supseteq \{A^*_{\alpha,\ell}\}.$$

Hence $(C^s_{\nu^*_B\restriction\beta})^{[\nu^*_B(\beta)]} \backslash A^*_{\beta,\ell}$ is almost disjoint to $B \cap A^*_{\beta,\ell}$, an infinite set from $I_{\nu^*_B\restriction\beta}$ and hence by $\odot_{2.3}$ from $[B^s_{\nu,\alpha(*),k}]^{\aleph_0}$. So

$$B \cap B^s_{\nu,\alpha(*),k} \backslash B^s_{\nu,\alpha(*),k+1} = B \cap B^s_{\nu,\alpha(*),k} \backslash \left(B^s_{\nu,\alpha(*),k} \cap \left(C^s_{\nu^*_B\restriction\beta}\right)^{[\nu^*_B(\beta)]} \backslash A^*_{\beta,k}\right)$$

almost contains this infinite set and hence is infinite as promised.]

So by the choice of $\mathcal{P}_{\bar{B}^s_{\nu,\alpha(*)}}$, *i.e.*, Observation 3.5 and clauses (i) and (j) of $\boxplus_1$ for some $\beta \in \mathcal{U}_{\alpha(*)}$, so $\beta \ge \alpha(*) \ge \ell g(\nu)$, we have $B \backslash (C^s_{\nu^*_B\restriction\beta})^{[\ell]} \notin J_s$ for $\ell = 0, 1$. Hence

$B_1 := B \backslash (C^s_{\nu^*_B \restriction \beta})^{[\nu^*_B(\beta)]} \notin J_s$. Recalling that for $\beta \in \mathcal{U}_\alpha, \alpha \neq 0$ and $\rho \in {}^\beta 2$, the set C^s_ρ depends just on $\ell g(\rho)$ and $\rho \restriction \sup(u_\alpha)$ (and our s).

Now consider B_1 instead of B. Clearly S_{B_1} is a subset of S_B and $\nu^*_B \restriction (\beta+1)$ is not in it, but $B_1 \in \mathrm{ob}(B)$, hence $S_{B_1} \subseteq S_B$, and hence S_{B_1} is $\subseteq \{\nu^*_B \restriction \gamma : \gamma \leq \beta\}$ and B_1 falls under subcase (2A) as $\beta < \kappa = \ell g(\nu^*_B)$. ■

Theorem 3.6 *When* $\mathfrak{a}_* < \kappa = \mathfrak{s}, J_* \subseteq \mathrm{ob}(\omega)$ *is dense and* $\mathrm{Pr}(\kappa, \mathfrak{a})$*, there is a saturated MAD family* $\mathcal{A} \subseteq J_*$*; see Definition* 3.2(3).

Proof of Theorem 3.6 We imitate the proofs of Theorems 2.1 and 3.1. Note that $\mathfrak{b} \leq \mathfrak{a}_* < \mathfrak{s}$.

Stage A: Similarly to Stage A of the proof of Theorem 3.1, let $(E, \bar{\mathcal{P}}^*)$ be as in Definition 3.2(3) and Observation 3.4. As $\mathfrak{b} < \kappa$, without loss of generality $u \in \mathcal{P}^*_\alpha \Rightarrow \mathrm{otp}(u) = \omega$ for $\alpha \in [\omega, \kappa)$. As we can replace E by any appropriate club E' of κ contained in $\mathrm{acc}(E)$ (see Observation 3.4) there, without loss of generality $\mathrm{otp}(E) = \mathrm{cf}(\kappa)$, $\min(E) \geq \omega$ and $\gamma \in E \Rightarrow \gamma + 1 + \mathfrak{b} < \min(E \backslash (\gamma+1))$. Let $\langle \gamma^*_i : i < \mathrm{cf}(\kappa)\rangle$ list E in increasing order.

Let $\langle u_\gamma : \gamma < \kappa\rangle$ be such that $\langle u_\gamma : \gamma^*_i \leq \gamma < \gamma^*_{i+1}\rangle$ list $\mathcal{P}_{\gamma^*_{i+1}}$ (which includes $\mathcal{P}_{\gamma^*_i}$) and $u_j = \omega$ for $j < \gamma^*_0$.

Let $\langle \mathcal{U}_\alpha : \alpha < \kappa\rangle$ be a partition of $\{2i+1 : i < \kappa\}$ such that $\min(\mathcal{U}_{1+\alpha}) \geq \alpha + \omega, |\mathcal{U}_{1+\alpha}| = \mathfrak{b}, |\mathcal{U}_0| = \kappa, 1 \leq \alpha < \gamma^*_i \Rightarrow \mathcal{U}_\alpha \subseteq \gamma^*_i$.

Let $\langle C^*_i : i \in \mathcal{U}_0\rangle$ list a family of subsets of ω witnessing $\mathfrak{s} = \kappa$. Also J_* is as in the proof of Theorem 2.1.

Let $\mathcal{P}_{\bar{B}}, \langle C^*_{\bar{B},\alpha,i} : i \in \mathcal{U}_\alpha\rangle$ be as in Theorem 3.1, Stage A.

Stage B: As in 3.1, *i.e.,* the case $\mathfrak{s} = \mathfrak{a}_*$, but we change $\boxplus_1$(f):

$\boxplus_1$(f) • $A_\eta \in I_\eta \cap J_*$ or $A_\eta = \varnothing$ and
• $\mathscr{S}_t := \{\eta \in \mathcal{T}_t : A_\eta \neq \varnothing\} \subseteq \{\eta \in \mathcal{T}_t : \ell g(\eta) = \gamma^*_i + 1$ for some $i < \kappa\}$.

Stage C: As in the proof of Theorem 3.1.

Stage D: Here there is a minor change: we replace $\boxplus_7$ in Theorems 2.1 and 3.1 by $\boxplus_7, \boxplus_8$, and $\boxplus_9$ below:

$\boxplus_7$ if $\alpha < 2^{\aleph_0}, s \in \mathrm{AP}_\alpha$ and $B \in J^+_s$, then there are a limit ordinal $\xi \in \kappa \backslash E$ and $t \in \mathrm{AP}_{\alpha+1}$ such that $s \leq_{\mathrm{AP}} t$ and $|S^t_B \cap {}^\xi 2| = 2^{\aleph_0}$; we may add $\mathscr{S}_t = \mathscr{S}_s$.

This is proved in Stage F.

To clarify why this is acceptable, recall $\boxplus_1$(f) and note that:

$(*)$ if $s \leq_{\mathrm{AP}} t, B \in \mathrm{ob}(\omega), \eta \in S^s_B \backslash \mathcal{T}_s$ and $\eta \notin \mathcal{T}_t$, then $\eta \in S^t_B$.

Now we need:

$\boxplus_8$ if $\xi \in \kappa \backslash E$ is a limit ordinal, $\alpha < 2^{\aleph_0}, t \in \mathrm{AP}_\alpha, B \in \mathrm{ob}(\omega)$ and $|S^t_B \cap {}^\xi 2| = 2^{\aleph_0}$, $\zeta = \min(E \backslash \xi)$, then for every t_1 and $\alpha + \zeta \leq \beta < 2^{\aleph_0}$ such that $t \leq_{\mathrm{AP}} t_1 \in \mathrm{AP}_\beta$ there is a t_2 satisfying $t_1 \leq_{\mathrm{AP}} t_2 \in \mathrm{AP}_{\beta+1}$ and $(\exists \eta \in \mathrm{suc}(\mathcal{T}_{t_2}))[\eta \notin \mathcal{T}_{t_1} \wedge A^{t_2}_\eta \in \mathrm{ob}(\omega) \wedge A^{t_2}_\eta \subseteq B]$.

The proof of $\boxplus_8$ is like the proof of Case 1 in Stage F in the proof of Theorem 2.1, but we elaborate. We are given β, ξ, ζ, and t_1 such that $t \leq_{\mathrm{AP}} t_1 \in \mathrm{AP}_\beta$; now we choose $\rho \in S^t_B \cap {}^{\xi}2 \backslash \mathcal{T}_{t_1}$ which exists since $|S^t_B \cap {}^{\xi}2| = 2^{\aleph_0} > |\mathcal{T}_{t_1}|$. Recalling $(*)$, necessarily $\rho \in S^{t_1,1}_B$. Choose B_1 such that $B_1 \subseteq B, B_1 \in I^{t_1}_\rho$.

Note that for every $\varepsilon \in [\xi, \zeta + 1)$, either $C^{t_1}_\varrho$ is well defined for every $\varrho \in {}^{\varepsilon}2$ such that $\rho \trianglelefteq \varrho$ and its value is the same for all such ϱ (when ε is odd), or $C^{t_1}_\varrho$ for $\rho \trianglelefteq \varrho \in {}^{\varepsilon}2$ is not well defined (when ε is even). So $\mathcal{B} = \{C^{t_1}_\varrho : \rho \triangleleft \varrho \in {}^{\zeta+1\geq}2$ and $C^{t_1}_\varrho$ is well defined$\}$ is a family of $\leq |\zeta| < \kappa = \mathfrak{s}$ subsets of B_1. Hence there is an infinite $B_2 \subseteq B_1$ such that $\rho \trianglelefteq \varrho \in {}^{\zeta>}2 \wedge (C^t_\varrho$ well defined$) \Rightarrow B_2 \subseteq^* C^t_\varrho \vee B_2 \subseteq^* \omega \backslash C^t_\varrho$, and without loss of generality, $B_2 \in J_*$.

We choose η such that $\rho \triangleleft \eta \in {}^{\zeta+1}2$: $[\ell g(\rho) \leq \gamma < \zeta + 1 \wedge (C^t_{\eta \restriction \gamma}$ is well defined$) \wedge B_2 \subseteq^* (C^t_{\eta\restriction\gamma})^{[\ell]} \wedge \rho \in \{0, 1\} \Rightarrow \eta(\gamma) = \ell]$. Let us define $t_2 \in \mathrm{AP}_{\beta+\zeta+2} := \mathrm{AP}_{\beta+1}$ (as $\alpha + \zeta + 1 \leq \beta$ and $|\alpha_1| = |\alpha_2| \Rightarrow \mathrm{AP}_{\alpha_1} = \mathrm{AP}_{\alpha_2}$) as follows:

(a) $\mathcal{T}_{t_2} := \mathcal{T}_{t_1} \cup \{\varrho : \varrho \trianglelefteq \eta\}$;
(b) $A^{t_2}_\varrho$ is $A^{t_1}_\varrho$ if well defined, is B_2 if $\varrho = \eta$, and is $\varnothing$ if $\eta \in \mathrm{suc}(\mathcal{T}_{t_2})$ but $A^{t_2}_\varrho$ is not already defined;
(c) $C^{t_2}_\varrho$ is $C^{t_1}_\varrho$ if $\varrho \in \mathcal{T}_{t_1}$, and we choose $C^{t_2}_{\eta\restriction\varepsilon}$ by induction on $\varepsilon \in [\xi, \zeta + 2]$ as follows: if it is determined by $\boxplus_1$ we have no choice otherwise let it be $\omega^{[\eta(\varepsilon)]}$.

The other objects of t_2 are determined by those we have chosen. So $\boxplus_8$ holds indeed.

$\boxplus_9$ If $s \in \mathrm{AP}_\alpha$ and $\rho \in c\ell(\mathcal{T}_s)$, then for some t, we have $s \leq_{\mathrm{AP}} t \in \mathrm{AP}_{\alpha+3}$, $\mathcal{T}_s \subseteq \mathcal{T}_t \subseteq \mathcal{T}_s \cup \{\rho, \rho\hat{\ }\langle 0\rangle, \rho\hat{\ }\langle 1\rangle\}$, and $I^s_\rho \neq \varnothing \Rightarrow \rho \in \mathcal{T}_t$ and $I^s_\rho \neq \varnothing \wedge \ell < 2 \wedge I^t_{\rho\hat{\ }<\ell>} \neq \varnothing \Rightarrow \rho\hat{\ }\langle \ell\rangle \in \mathcal{T}_t$.
[Why? It is easier than $\boxplus_8$.]

Stage E: This is similar to Theorem 2.1 with the changes necessitated by the change in Stage D.

Stage F: We prove $\boxplus_7$, and the proof splits into cases.

Case 1: Some $\nu \in S_B$ is such that $\nu \in c\ell(\mathcal{T}_s) \backslash \mathcal{T}_s$.

Let $B_1 \in \mathrm{ob}(B) \cap I^s_\nu$. Such B_1 exists as $\nu \in S_B$ but $\nu \notin S^{s,2}_B$ as $\nu \notin \mathcal{T}_s$.

Let $C_{\nu,n} \in \mathrm{ob}(\omega)$ for $n < \omega$ be such that $\bigcap\{C^{[\varrho(n)]}_{\nu,n} : n < \ell g(\varrho)\} \cap B_1$ is infinite for every $\varrho \in {}^{\omega>}2$.

We choose $\mathcal{T}_t = T_s \cup \{\nu\hat{\ }\rho : \rho \in {}^{\omega>}2\}$. For $\rho \in {}^{\omega>}2$, we choose $C^t_{\nu\hat{\ }\rho}$ by induction on $\ell g(\rho)$: if $\ell g(\nu\hat{\ }\rho) = \ell g(\nu) + n$ is even and $n \in \{2m, 2m + 1\}$, then $C^t_{\nu\hat{\ }\rho} = C_{\nu,m}$, n odd, (see Stage a), and we act as in the proof of $\boxplus_8$. Lastly, let $A^t_{\nu\hat{\ }\rho} = \varnothing$.

Easily,

$(*)$ if $s \leq_{\mathrm{AP}} s_1, |\mathcal{T}_{s_1}| < 2^{\aleph_0}$, then $|S^{s_1}_B \cap {}^{\ell g(\nu)+\omega}2| = 2^{\aleph_0}$.

So we are done with Case 1.

Case 2: $SP^s_B = \varnothing$, but not Case 1; let $\nu^*_B = \bigcup\{\eta : \eta \in S_B\}$.

Subcase 2A: $\nu^*_B \in S_B$.

This is as in the proof Theorem 2.1, ending in contradiction.

Subcase 2B: $\nu^*_B \notin S_B$.

With the exception of $\odot_2$, which we will be elaborating on, this is as in the proofs of Theorems 2.1 and 3.1 but in the end replacing "Subcase 1B" by "Case 1". Recall from Stage 2B as in the proof of Theorem 2.1, $\delta := \ell g(\nu^*_B)$ is a limit ordinal and

$\nu^*_{B,\alpha} = (\nu^*_B{\restriction}\alpha)^\frown\langle 1-\nu^*_B(\alpha)\rangle$ for $\alpha < \delta$ and $\langle\langle A^*_{\alpha,n} : n \le n(\alpha)\rangle : \alpha < \delta\rangle$ and $\mathcal{A}^* := \{B \cap A : A = A^*_{\alpha,k}$ for some $\alpha < \delta, k \le n(\alpha)$ and $B \cap A$ is infinite$\}$, $\Lambda = \{\nu \in \mathscr{S}_s : \nu^*_B \trianglelefteq \nu$ and $A_\nu \cap B$ is infinite$\}$ are as in Stage (2B) of the proof of Theorem 2.1. We have to prove:

$\odot_2$ there is a set B_1 such that:

- (a) $B_1 \subseteq B$ is infinite;
- (b) B_1 is almost disjoint to any $A \in \mathcal{A}^*$;
- (c) if Λ is finite, then $\nu \in \Lambda \Rightarrow |B_1 \cap A_\nu| < \aleph_0$;
- (d) if Λ is infinite, then for infinitely many $\nu \in \Lambda$ we have $|B_1 \cap A_\nu| = \aleph_0$.

Why does $\odot_2$ hold? If $|\mathcal{A}^*| < \mathfrak{a}_*$, then as in the proof of Theorem 2.1, the statement of $\odot_2$ follows. So we can assume $|\mathcal{A}^*| \ge \mathfrak{a}_*$, let

$$\mathcal{W} := \{\alpha < \kappa : \quad \text{for some } \ell \le n(\alpha) \text{ we have } A^*_{\alpha,\ell} \cap B \in \mathcal{A}^* \\ \text{(equivalently } A^*_{\alpha,\ell} \cap B \text{ is infinite)}\},$$

and let

$$\mathcal{W}' = \{\alpha \in W : |\alpha \cap \mathcal{W}| < \mathfrak{a}_*\}.$$

Subcase 2B(α): $\sup(\mathcal{W}') \in \operatorname{acc}(E)$.

For $\alpha \in \mathcal{W}$ choose $\ell(\alpha) \le n(\alpha)$ such that $B \cap A^*_{\alpha,\ell(\alpha)}$ is infinite, hence $A^*_{\alpha,\ell} \notin \{A^*_{\alpha_1,\ell_1} : \alpha_1 < \alpha, \ell_1 \le n(\alpha_1)\}$. As $n(\alpha) < \omega$ for $\alpha < \kappa$, clearly $|\mathcal{W}| = |\mathcal{A}^*| \ge \mathfrak{a}_*$, hence $\operatorname{otp}(\mathcal{W}') = \mathfrak{a}_*$.

So by Definition 3.2(3), *i.e.,* the choice of $(E, \bar{\mathcal{P}})$, there is a pair (u_*, γ^*_j) as in clause (e) there. So $u_* \in \mathcal{P}_{\gamma^*_j}$, hence $u_* = u_{\alpha(*)}$ for some $\alpha(*) \in [\gamma^*_j, \gamma^*_{j+1}), \gamma^*_{j+1} < \sup(\mathcal{W}')$, and $\mathcal{W} \cap \sup(E \cap \gamma^*_j) = \mathcal{W}' \cap \sup(E \cap \gamma^*_j)$ has cardinality $< \mathfrak{a}_*$. Let $\nu = \nu^*_B{\restriction}\sup(u_*)$, and recall $\operatorname{otp}(u_*) = \omega$.

Recall also that $\langle \gamma_{\alpha(*),n} : n < \omega\rangle$ list u_* in increasing order and so $v := \{n < \omega : \gamma_{\alpha(*),n} \in \mathcal{W}\}$ is infinite. Clearly $n \in v \Rightarrow B^s_{\nu,\alpha(*),n} \backslash B^s_{\nu,\alpha(*),n+1}$ is infinite as in the proof of 3.1. So by the choice of $\mathcal{P}_{\bar{B}^s_\nu,\alpha(*)}$, *i.e.,* Theorem 3.5 and clauses (i) and (j) of $\boxplus_1$, for some $\beta \in \mathcal{U}_{\alpha(*)}$, so $\beta \ge \ell g(\nu)$, we have $B\backslash(C^s_{\nu^*_B{\restriction}\beta})^{[\ell]} \notin J_s$ for $\ell = 0, 1$. Hence $B_1 := B\backslash(C^s_{\nu^*_B})^{[\nu^*_B(\beta)]} \notin J_s$, recalling that for $\beta \in \mathcal{U}_\alpha, \alpha \ne 0$ and $\rho \in {}^\beta 2$ the set C^s_ρ depends just on $\ell g(\rho)$ and $\rho{\restriction}\sup(u_\alpha)$ (and our s).

We finish as in the proof of Theorem 3.1.

Subcase 2B(β): $\sup(\mathcal{W}') \in (\gamma^*_i, \gamma^*_j]$ where $j = i+1$ so $\gamma^*_i, \gamma^*_j \in E$; let $\gamma^* = \sup(\mathcal{W}')$.

Apply Definition 3.2(3)(f) to $\gamma^*_i, \gamma^*_j, \mathcal{W}'\backslash\gamma^*_j$ we get $u = u_* \in \mathcal{P}_{\gamma^*_j}$, so $u = u_{\alpha(*)} \subseteq [\gamma^*_i, \gamma^*_j)$ for some $\alpha(*) \in (\gamma^*_i, \gamma^*_j)$.

Let $\beta = \sup(u_*), \nu = \nu^*_\beta{\restriction}\beta$, so by $\odot_{2.1}$ and $\odot_{2.4}$ in stage 2B of the proof of Theorem 3.1, and by the choice of $\mathcal{P}_{\bar{B}_{\nu,\alpha(*)}}$, there is a $\mathcal{Q} \subseteq \mathcal{P}_{\bar{B}^s_{\nu,\alpha(*)}}$ of cardinality $\aleph_0$ and $\Lambda \subseteq {}^{\mathcal{Q}}2$ of cardinality $2^{\aleph_0}$ such that for every $\rho \in \Lambda$ there is $B_\rho \in \operatorname{ob}(B) \cap J^+_s$ such that

$$A \in \mathcal{Q} \Rightarrow B_\rho \subseteq^* A^{[\rho(A)]},\ n < \omega \Rightarrow B_{\eta,n} := B_n \backslash \bigcup\{C^{[\nu^*_B(\gamma_{\alpha(*),k})]}_{\nu^*_B{\restriction}\gamma_{\alpha(*),k}} : k < n\} \in I^s_{\nu^*_B{\restriction}\gamma_{\alpha(*),n}}.$$

Clearly for some $v \subseteq \mathcal{U}_{\alpha(*)} \subseteq (\gamma^*_i, \gamma^*_j)$ of cardinality $\aleph_0, \nu \triangleleft \rho \in \mathcal{T}_s \wedge \ell g(\rho) \ge \sup(v) \Rightarrow \{C^s_{\rho{\restriction}\varepsilon} : \varepsilon \in v\} = \mathcal{Q}$.

For $\eta \in \Lambda$ analyzing S_{B_η} and recalling $\gamma^*_{j+1} < \mathfrak{s}$ clearly $S_{B_\eta} \cap \{\nu \in \mathcal{T}_s : \ell g(\nu) \leq \sup(u_*)\}$ is $\{\nu^*_B \restriction \gamma : \gamma < \sup(u_*)\}$ and $\sup(u_*) > \gamma^* + 1$, so there is no ρ such that A^s_ρ is non-empty, $\rho \in \mathcal{T}_s$ and $\sup(u_*) \leq \ell g(\rho) < \gamma^*_j$, so $S_{B_\eta} \cap \{\nu \in \mathcal{T}_s : \ell g(s) < \gamma^*_j\}$ does not depend on $\langle A^s_\eta : \eta \in \operatorname{suc}(\mathcal{T}_s), \ell g(\eta) \geq \gamma^*_j \rangle$ so we can finish easily as in case 1.

Case 3: See the proof of Theorem 3.1. ■

4 Further Discussion

The cardinal invariant $\mathfrak{s}$ plays a major role here, so the claims depend on how $\mathfrak{s}$ and $\mathfrak{a}_*$ are compared; when $\mathfrak{s} = \mathfrak{a}_*$ it is not clear whether the further hypothesis of Theorem 3.1(2) may fail. If $\mathfrak{s} > \mathfrak{a}_* > \aleph_1$, it is not clear if the hypothesis of Theorem 3.6 may fail. Recall 2.1, dealing with $\mathfrak{s} < \mathfrak{a}_*$, the first case is proved in ZFC, but the others need pcf assumptions.

All this does not exclude the case $\mathfrak{s} = \aleph_{\omega+1}, \mathfrak{a}_* = \aleph_1$, hence $\mathfrak{b} = \aleph_1$, as in [9]. Fulfilling the promise from §0 and the abstract:

Claim 4.1 (1) *If there is no inner model with a measurable cardinal (and even the non-existence of much stronger statements), then there is a saturated MAD family, $\mathcal{A}$.*

(2) *Also if $\mathfrak{s} < \aleph_\omega$, there is one.*

(3) *Moreover, if $J_* \subseteq \operatorname{ob}(\omega)$ is dense, then we can demand $\mathcal{A} \subseteq J_*$.*

Proof These follow from Theorems 2.1, 3.1, and 3.6 (using well known results). ■

We now remark on some further possibilities.

Definition 4.2 (1) We say $\mathscr{S} \subseteq \operatorname{ob}(\omega)$ is $\mathfrak{s}$-free when:

(a) for every $A \in \operatorname{ob}(\omega)$ there is $B \in \operatorname{ob}(A)$ such that B induces an ultrafilter on $\mathscr{S}$; *i.e.,* $C \in \mathscr{S} \Rightarrow A \subseteq^* C \vee A \subseteq^* (\omega \backslash C)$.

(1A) We say $\mathscr{S} \subseteq \operatorname{ob}(\omega)$ is $\mathfrak{s}$-free in I when $I \in \mathrm{OB}$ and for every $A \in I$ there is $B \in \operatorname{ob}(A)$ inducing an ultrafilter on S.

(2) We say $\mathscr{S} \subseteq \operatorname{ob}(\omega)$ is $\mathfrak{s}$-richly free when clause (a) holds and

(b) if $A \in \operatorname{ob}(\omega)$ and the set $\{D \cap \mathscr{S} : D$ an ultrafilter on ω containing $\operatorname{ob}(A)\}$ is infinite, then it has cardinality continuum.

(3) We say $\mathscr{S} \subseteq \operatorname{ob}(\omega)$ is $\mathfrak{s}$-anti-free if no $B \in \operatorname{ob}(\omega)$ induces an ultrafilter on $\mathscr{S}$.

(4) Let $\mathfrak{S}$ be $\{\kappa$: there is a $\subseteq$-increasing sequence $\langle \mathscr{S}_i : i < \kappa \rangle$ of $\mathfrak{s}$-richly-free families such that $\bigcup\{\mathscr{S}_i : i < \kappa\}$ is not $\mathfrak{s}$-free$\}$.

(5) Recall $\mathfrak{s} = \min\{|\mathscr{S}| : \mathscr{S} \subseteq \operatorname{ob}(\omega)$ and no $B \in \operatorname{ob}(\omega)$ induces an ultrafilter on $\mathscr{S}\}$.

(6) We say $\operatorname{ch}_{\dim}(\mathcal{B}) < \kappa$ when $(\exists \eta \in {}^{\mathcal{B}}2)(I = I_{\mathcal{B},\eta}) \Rightarrow \operatorname{cf}(I, \supseteq) < \kappa$, recalling 1.7(5).

Observation 4.3 (1) If $\mathscr{S}$ is $\mathfrak{s}$-free and $\mathscr{S}' \subseteq \mathscr{S}$, then $\mathscr{S}'$ is $\mathfrak{s}$-free.

(2) If $\mathscr{S} \subseteq \operatorname{ob}(\omega)$ and $|\mathscr{S}| < \mathfrak{s}$, then $\mathscr{S}$ is $\mathfrak{s}$-free.

(3) If $\mathscr{S}_n \subseteq \operatorname{ob}(\omega)$ is $\mathfrak{s}$-free for $n < \omega$, then $\bigcup\{\mathscr{S}_n : n < \omega\}$ is $\mathfrak{s}$-free.

(4) $\mathfrak{s} \in \mathfrak{S}$.
(5) $\kappa \in \mathfrak{S}$ if and only if $\mathrm{cf}(\kappa) \in \mathfrak{S}$.
(6) $\kappa \in \mathfrak{S} \Rightarrow \aleph_1 \leq \kappa \leq 2^{\aleph_0}$.
(7) In the definition of $\mathfrak{S}$ we can add "$\bigcup\{\mathscr{S}_i : i < \kappa\}$ is $\mathfrak{s}$-anti-free".
(8) $\mathrm{cf}(\mathfrak{s}) > \aleph_0$, in fact $\kappa \in \mathfrak{S} \Rightarrow \mathrm{cf}(\kappa) > \aleph_0$.

Definition 4.4 (1) We say $A \in \mathrm{ob}(\omega)$ obeys $f \in {}^{\omega}\omega$ when, for every $n_1 < n_2$, from A we have $f(n_1) < n_2$.
(2) Let $\bar{f} = \langle f_\alpha : \alpha < \delta\rangle$ be a sequence of members of ${}^{\omega}\omega$. We say $\bar{A} = \langle A_\alpha : \alpha \in u\rangle$ obeys $\bar{f}$ when $u \subseteq \delta$ and A obeys f_α for $\alpha \in A$.
(3) $\mathfrak{a}_{\bar{f}} = \mathrm{Min}\{|u|$: there are $B \in \mathrm{ob}(\omega)$ and $\bar{A} = \langle A_\alpha : \alpha \in u\rangle$ obeying $\bar{f}$ such that $\{A_\alpha \cap B : \alpha \in u\}$ is a MAD of $B\}$.

Remark 4.5 (1) Also note that in Theorems 2.1, 3.1, and 3.6 we can replace $\mathfrak{s}$ by a smaller (or equal) cardinal invariant $\mathfrak{s}_{\mathrm{tree}}$, the tree splitting number.
(2) Let $\mathfrak{s}_{\mathrm{tree}}$ be the minimal κ such that there is a sequence $\bar{C} = \langle C_\eta : \eta \in {}^{\kappa>}2\rangle$ such that $C_\eta \in \mathrm{ob}(\omega)$ for $\eta \in {}^{\kappa>}2$, and there is no $\eta \in {}^{\kappa}2$ and $A \in \mathrm{ob}(\omega)$ such that $\epsilon < k \Rightarrow A \subseteq^* C_\eta^{[\eta(\epsilon]}$. Note that the minimal κ for which there is such sequence $\langle C_\eta : \eta \in {}^{\kappa>}2\rangle$ has uncountable cofinality.
(3) Also in Theorem 2.1 we may weaken $\mathfrak{s} < \mathfrak{a}_*$ to $\mathfrak{s} < \mathfrak{a} \wedge \mathfrak{s} \leq \mathfrak{a}_*$.

Acknowledgment The author thanks Alice Leonhardt for the excellent typing and Shimoni Garti and the referee for helpful corrections.

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