Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 948-961

SUFFICIENCY CONDITIONS FOR THE EXISTENCE OF TRANSVERSALS

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1. Introduction. A *transversal* of a family of non-empty sets $\mathscr{F} = \langle F_\nu : \nu \in I \rangle$ is a 1-1 map

$$\varphi : I \to S(\mathscr{F}) = \bigcup_{\nu \in I} F_\nu$$

such that $\varphi(\nu) \in F_\nu$ $(\nu \in I)$. A number of problems in combinatorial mathematics reduce to the question of whether or not a certain family of sets has a transversal. An up-to-date account of this theory is to be found in the book by Mirsky [**9**]. The best known result of this kind is the following theorem.

THEOREM. *If $\mathscr{F} = \langle F_\nu : \nu \in I \rangle$ is either a finite family or an arbitrary family of finite sets, then $\mathscr{F}$ has a transversal if and only if*

$$(1.1) \qquad \left| \bigcup_{\nu \in J} F_\nu \right| \geqslant |J|$$

holds for all finite sets $J \subset I$.

This was proved for finite $\mathscr{F}$ by P. Hall [**7**] (and in an equivalent graph theoretical formulation by J. König [**8**]) and for an arbitrary family of finite sets by M. Hall [**6**]. We shall refer to (1.1) as Hall's condition. If $\mathscr{F}$ is an infinite family with infinite sets, then the problem of finding necessary and sufficient conditions for the existence of a transversal assumes a different complexity and remains unsolved. Rado and Jung [**12**] observed that if $\mathscr{F}$ has just one infinite member, say F_{ν_0}, then there is a transversal if and only if (1.1) holds and

$$F_{\nu_0} \not\subset \bigcup_{J \in \mathscr{C}} \bigcup_{\nu \in J} F_\nu$$

where $\mathscr{C}$ is the set of critical subsets of I, i.e., $J \in \mathscr{C}$ if and only if J is a finite subset of I for which equality holds in (1.1). Brualdi and Scrimger [**3**] and Folkman [**5**] considered the more general problem of a family containing an arbitrary finite number of infinite sets. More recently, Nash-Williams [**10**] conjectured a condition which is both necessary and sufficient for an arbitrary *countable* family of sets to have a transversal, and this was proved by Damerell and Milner [**4**]. The conditions given by these authors are not so easily stated and the reader is referred to the original papers.

Received February 27, 1973 and in revised form, May 3, 1973. The research of the first named author was supported by NRC Grant No. A-5198.

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That there can be no entirely elementary set of conditions which are necessary and sufficient for an arbitrary family of sets to have a transversal may perhaps be illustrated by considering the two families

$$\mathscr{F}_1 = \langle \alpha + 1 : \omega \leqq \alpha < \omega_1 \rangle \quad \text{and} \quad \mathscr{F}_2 = \langle \alpha : \omega \leqq \alpha < \omega_1 \rangle.$$

Here ω denotes the first infinite ordinal, ω_1 the first uncountable ordinal and an ordinal $\alpha = \{\beta : \beta < \alpha\}$ is regarded as the set of all smaller ordinals. Clearly $\mathscr{F}_1$ has a transversal since $\alpha \in \alpha + 1$. However, $\mathscr{F}_2$ has no transversal. For, if $\varphi(\alpha) \in \alpha$ $(\omega \leqq \alpha < \omega_1)$, then by a theorem of Alexandroff and Urysohn [**1**] on regressive functions, there is some $\gamma < \omega_1$ such that $\varphi(\alpha) = \gamma$ for uncountably many $\alpha < \omega_1$. The family $\mathscr{F}_2$ gives a partial answer to [**9**, Problem 3, p. 220].) It is difficult to imagine any criterion involving inequalities between cardinals of sets which will be delicate enough to distinguish between the families $\mathscr{F}_1$ and $\mathscr{F}_2$.

In view of the difficulty just mentioned it seems of interest therefore to have conditions which, though not necessary, are at least sufficient to ensure the existence of a transversal in a family having infinite members. In this connection Professor L. Mirsky asked if the following condition (which is a kind of dual of the finiteness condition in M. Hall's theorem) is sufficient for the existence of a transversal: *each member of* $\mathscr{F}$ *is infinite and each element* $x \in S(\mathscr{F})$ *belongs to only a finite number of sets* $F \in \mathscr{F}$.

If $\mathscr{F} = \langle F_\nu : \nu \in I \rangle$ is a family, we write $F \in \mathscr{F}$ if $F = F_\nu$ for some $\nu \in I$. The cardinality of the family is $|\mathscr{F}| = |I|$. For any set A, put $\mathscr{F}(A) = \langle F_\nu : \nu \in I,\ A \cap F_\nu \neq \emptyset \rangle$ and write $\mathscr{F}(x)$ instead of $\mathscr{F}(\{x\})$. Mirsky's question is answered affirmatively by the following theorem.

THEOREM 1. *If the family of nonempty sets* $\mathscr{F}$ *satisfies*

(1.2) $\quad |F| \geqq |\mathscr{F}(x)| \quad$ *for all* $F \in \mathscr{F}$ *and* $x \in S(\mathscr{F})$,

then $\mathscr{F}$ *has a transversal.*

Dr. C. J. Knight conjectured that the following, more local type of condition, is also sufficient for a transversal. We write $\mathscr{F} \in \mathscr{K}$ if and only if the members of $\mathscr{F}$ are nonempty and

(1.3) $\quad |F| \geqq |\mathscr{F}(F)| \quad (F \in \mathscr{F}).$

The main result proved in this paper settles Knight's conjecture.

THEOREM 2. *If* $\mathscr{F} \in \mathscr{K}$, *then* $\mathscr{F}$ *has a transversal.*

A common weakening of the conditions (1.2) and (1.3) is the condition

(1.4) $\quad |F| \geqq |\mathscr{F}(x)| \quad (x \in S(\mathscr{F}),\ F \in \mathscr{F}(x)$ i.e., $x \in F \in \mathscr{F}).$

We write $\mathscr{F} \in \mathscr{L}$ if the members of $\mathscr{F}$ are nonempty and (1.4) is satisfied. Thus a strengthening of both Theorems 1 and 2 is

THEOREM 3. *If $\mathscr{F} \in \mathscr{L}$, then $\mathscr{F}$ has a transversal.*

Suppose $\mathscr{F} = \langle F_\nu : \nu \in I\rangle \in \mathscr{L}$. Let J be a finite set, $J \subset I$, and let $\mathscr{F}'$ be the sub-family $\langle F_\nu : \nu \in J\rangle$. For $p \in \{1, 2, \ldots, |J|\}$, put

$$n_p = |\{\nu \in J : |F_\nu| = p\}|, \quad m_p = |\{x \in S(\mathscr{F}') : |\mathscr{F}'(x)| = p\}|.$$

Considering the number of pairs (x, F) with $x \in F \in \mathscr{F}'$, $|F| \leqq p$, we obtain by (1.4) the inequality

$$n_1 + 2n_2 + \ldots + pn_p \leqq m_1 + 2m_2 + \ldots + pm_p \quad (1 \leqq p \leqq |J|).$$

It follows that

$$n_1 + n_2 + \ldots + n_p \leqq m_1 + m_2 + \ldots + m_p \quad (1 \leqq p \leqq |J|),$$

and hence (1.1) holds. It follows from this that $\mathscr{L}$ is a sufficient condition for a family of finite sets to have a transversal. The conditions $\mathscr{L}$ and $\mathscr{K}$ are easily seen to be equivalent if all the members of $\mathscr{F}$ are infinite sets and therefore, $\mathscr{L}$ is also sufficient (by Theorem 2) for a family of infinite sets to have a transversal. In an early version of this paper we left Theorem 3 as an open question since we could not prove the special case

(1.5) *if $\mathscr{F}$ is a countable family of countable sets and $\mathscr{F} \in \mathscr{L}$, then $\mathscr{F}$ has a transversal.*

In fact, (1.5) and Theorem 2 implies the complete result stated as Theorem 3 (see § 6). Shelah [**13**] has since proved (1.5) and a simpler proof of this result is given in [**2**]. In § 7 we prove an even stronger result (Theorem 4).

Theorem 3 has an interesting formulation in terms of bipartite graphs. A bipartite graph is a triple $\Gamma = \langle X, \Delta, Y\rangle$ with vertex set $X \cup Y$ (X, Y disjoint sets) and edge set $\Delta \subset \{\{x, y\} : x \in X, y \in Y\}$. Let $v(z) = |\{u \in X \cup Y : \{u, z\} \in \Delta\}|$ ($z \in X \cup Y$) be the valency function of Γ. Then Theorem 3 is equivalent to the following statement: *If $\Gamma = \langle X, \Delta, Y\rangle$ is a bipartite graph such that $v(x) > 0$ for $x \in X$ and $v(x) \geqq v(y)$ whenever $x \in X$, $y \in Y$ and $\{x, y\} \in \Delta$, then there is a matching from X into Y, i.e. there is a* 1-1 *function $\varphi : X \to Y$ such that $\{x, \varphi(x)\} \in \Delta$ ($x \in X$).*

2. Notation. Capital letters denote sets and the cardinal power of A is $|A|$. Small Latin and Greek letters denote ordinal numbers unless stated otherwise. As usual, an ordinal α is the set $\{\beta : \beta < \alpha\}$ of all smaller ordinals. A cardinal number is an initial ordinal, i.e., α is a cardinal if $\beta < \alpha \Rightarrow |\beta| < |\alpha|$. The letters κ, λ, μ always denote infinite cardinals. κ^+ is the successor cardinal of κ.

$\mathscr{F}$ will always denote the family of non-empty sets $\langle F_\nu : \nu \in I\rangle$ with index set I. We write $|\mathscr{F}| = |I|$ and $S(\mathscr{F}) = \bigcup_{\nu \in I} F_\nu$. We shall abuse the usual terminology of sets by applying it to families of sets, but this should not lead to any confusion. Thus, we write $A \in \mathscr{F}$ if $A = F_\nu$ for some $\nu \in I$. We write

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$A, B \in \mathscr{F}, A \neq B$ to mean that A, B are *different* members of $\mathscr{F}$, i.e., $A = F_\mu$, $B = F_\nu$ and $\mu \neq \nu$ (even though we may have $A = B$ in the usual set theoretical sense). $\mathscr{F}' = \langle F_\nu : \nu \in I' \rangle$ is a *subfamily* of $\mathscr{F}$, and we write $\mathscr{F}' \subset \mathscr{F}$, if $I' \subset I$; in this case we also write $\mathscr{F} - \mathscr{F}' = \langle F_\nu : \nu \in I - I' \rangle$. We write $\mathscr{F}'' \subset\subset \mathscr{F}$ if $\mathscr{F}'' = \langle G_\nu : \nu \in I \rangle$ and $G_\nu \subset F_\nu (\nu \in I)$. The family $\mathscr{F}' = \langle F_\nu : \nu \in I' \rangle$ is *disjoint* from $\mathscr{F}$ if $I \cap I' = \emptyset$; it is *strongly disjoint* from $\mathscr{F}$ if it is disjoint and in addition $S(\mathscr{F}) \cap S(\mathscr{F}') = \emptyset$. If $\mathscr{F}, \mathscr{F}'$ are disjoint, then $\mathscr{F} \cup \mathscr{F}' = \langle F_\nu : \nu \in I \cup I' \rangle$.

A *transversal* of $\mathscr{F}$ is an 1-1 function $\varphi : I \to S(\mathscr{F})$ such that $\varphi(\nu) \in F_\nu$ $(\nu \in I)$. Let Trans $(\mathscr{F})$ be the set of all transversals of $\mathscr{F}$. If $\varphi \in$ Trans $(\mathscr{F})$, $\psi \in$ Trans $(\mathscr{F}')$, then range $(\varphi) = \{\varphi(\nu) : \nu \in I\}$ and φ, ψ are said to be *disjoint* if range $(\varphi) \cap$ range $(\psi) = \emptyset$. Thus, if $\mathscr{F}, \mathscr{F}'$ are disjoint families and φ, φ' are disjoint transversals of $\mathscr{F}$ and $\mathscr{F}'$ respectively, then $\varphi \cup \varphi' \in$ Trans $(\mathscr{F} \cup \mathscr{F}')$.

For $A \subset S(\mathscr{F})$, let $\mathscr{F}(A)$ denote the subfamily of $\mathscr{F}$,

$$\mathscr{F}(A) = \langle F_\nu : \nu \in I, A \cap F_\nu \neq \emptyset \rangle.$$

In particular, for a singleton we write $\mathscr{F}(x)$ instead of $\mathscr{F}(\{x\})$. $\mathscr{F}$ has property $\mathscr{K}$, $\mathscr{F} \in \mathscr{K}$, if and only if

$$(2.1) \quad |F| \geqq |\mathscr{F}(F)| \quad (F \in \mathscr{F}),$$

and $\mathscr{F} \in \mathscr{L}$ if and only if

$$|F| \geqq |\mathscr{F}(x)| \quad (x \in S(\mathscr{F}),\ F \in \mathscr{F}(x) \text{ i.e., } x \in F \in \mathscr{F}).$$

If λ is an infinite cardinal we write

$$\mathscr{F}^\lambda = \langle F_\nu : \nu \in I, |F_\nu| = \lambda \rangle.$$

$\mathscr{F}^{<\lambda}$, $\mathscr{F}^{\leqq\lambda}$, $\mathscr{F}^{>\lambda}$, $\mathscr{F}^{\geqq\lambda}$ are similarly defined. For $x \in S(\mathscr{F})$ put

$$\rho_{\mathscr{F}}(x) = \inf \{|F| : F \in \mathscr{F}(x)\}.$$

Thus $\mathscr{F} \in \mathscr{L}$ if and only if $\rho_{\mathscr{F}}(x) \geqq |\mathscr{F}(x)|$ $(x \in S(\mathscr{F}))$. We usually write $S = S(\mathscr{F})$, and then

$$S^\lambda = \{x \in S : \rho_{\mathscr{F}}(x) = \lambda\}.$$

$S^{<\lambda}$, $S^{\leqq\lambda}$ are similarly defined.

A λ-*component* of $\mathscr{F}$ is a minimal non-empty subfamily $\mathscr{H} \subset \mathscr{F}^{\leqq\lambda}$ such that

$$A \in \mathscr{H}, B \in \mathscr{F}^{\leqq\lambda}(A) \Rightarrow B \in \mathscr{H}.$$

Let $\mathscr{F}^{\leqq\lambda} = \langle F_\nu : \nu \in I_\lambda \rangle$. Consider the graph $\mathscr{G}_\lambda$ on the index set I_λ in which $\{\rho, \sigma\}$ is an edge if and only if $\rho, \sigma \in I_\lambda$, $\rho \neq \sigma$ and $F_\rho \cap F_\sigma \neq \emptyset$. Then $\mathscr{H} = \langle F_\nu : \nu \in J \rangle$ is a λ-component of $\mathscr{F}$ exactly when J is the vertex set of a connected component of the graph $\mathscr{G}_\lambda$. Two different λ-components of $\mathscr{F}$ are strongly disjoint subfamilies of $\mathscr{F}$. A *large* λ-*component* of $\mathscr{F}$ is a minimal non-empty subfamily $\mathscr{H} \subset \mathscr{F}$ such that

$$A \in \mathscr{H}, A \cap B \cap S^{\leqq\lambda} \neq \emptyset \Rightarrow B \in \mathscr{H}.$$

Thus every set $A \in \mathscr{F}$ is a member of a large λ-component of $\mathscr{F}$; two large λ-components are disjoint subfamilies of $\mathscr{F}$ but they are not in general strongly disjoint.

If $\mathscr{F} \in \mathscr{L}$, then for any $\lambda \geqq \omega$, the valency of a vertex ν in the graph $\mathscr{G}_\lambda$ described above is at most λ and hence the vertex set of a connected component has cardinality at most λ, i.e. if $\mathscr{H}$ is a λ-component of $\mathscr{F}$, then $|\mathscr{H}| \leqq \lambda$.

Suppose $\mathscr{F}$ is a family of sets such that (2.1) holds and

(2.2) $|A \cap S^{\leqq\lambda}| \leqq \lambda$ for $A \in \mathscr{F}$.

Now (2.1) implies that each element $x \in S^{\leqq\lambda}$ is a member of at most λ different sets of the family $\mathscr{F}$. Therefore, by (2.2), there are at most $\lambda^2 = \lambda$ different sets $B \in \mathscr{F}$ such that $A \cap B \cap S^{\leqq\lambda} \neq \emptyset$. This implies that every large λ-component of $\mathscr{F}$ also has cardinality at most λ.

The cofinality of the cardinal λ, is the least cardinal $\mu = \text{cf}(\lambda)$ such that λ can be expressed as the union of μ subsets each of cardinal less than λ. λ is *regular* if $\text{cf}(\lambda) = \lambda$ and *singular* if $\text{cf}(\lambda) < \lambda$.

A set of ordinals $C \subset \lambda$ is *stationary* in λ if for every regressive function $f : C \to \lambda$ (i.e., $f(\gamma) < \gamma$ for $\gamma \in C - \{0\}$), there is γ_0 such that

$$|\{\gamma \in C : f(\gamma) = \gamma_0\}| = \lambda.$$

We use the well-known result (e.g. [**11**]) that if $\lambda > \omega$ is regular then the set $C = \{\gamma < \lambda : \gamma \text{ is a limit ordinal}\}$ is stationary in λ. A set $C \subset \lambda$ is *cofinal* in λ if for every $x \in \lambda$ there is $y \in C$ such that $x \leqq y$.

3. Elementary lemmas and proof of Theorem 1. We need the following well-known fact.

LEMMA 1. *If* $|\mathscr{F}| \leqq \lambda \leqq |F|$ $(F \in \mathscr{F})$, *then there are sets* $g(F) \subset F$ $(F \in \mathscr{F})$ *such that* $|g(F)| = \lambda$ *and* $g(F_1) \cap g(F_2) = \emptyset$ *for* $F_1, F_2 \in \mathscr{F}$ *and* $F_1 \neq F_2$.

Proof. We may assume that $\mathscr{F} = \langle F_\nu : \nu < \alpha \rangle$, $\alpha \leqq \lambda$. Let $\langle \nu_\rho : \rho < \lambda \rangle$ be any sequence of ordinals such that $\nu_\rho < \alpha (\rho < \lambda)$ and $|\{\rho < \lambda : \nu_\rho = \nu\}| = \lambda$ $(\nu < \alpha)$. Now by transfinite induction we can choose elements $x_\rho \in F_{\nu_\rho} - \{x_\sigma : \sigma < \rho\}$ and the lemma holds with $g(F_\nu) = \{x_\rho : \rho < \lambda \text{ and } \nu_\rho = \nu\}$ $(\nu < \alpha)$.

Since a family of non-empty pairwise disjoint sets obviously has a transversal, we have the following corollary.

COROLLARY 1. *If* $|F| \geqq \lambda \geqq |\mathscr{F}|$ $(F \in \mathscr{F})$, *then* Trans $(\mathscr{F}) \neq \emptyset$.

LEMMA 2. *If* $\mathscr{F} \in \mathscr{K}$ *and* $|\mathscr{F}| \leqq \aleph_0$, *then* Trans $(\mathscr{F}) \neq \emptyset$.

Remark. The condition $\mathscr{F} \in \mathscr{K}$ can be replaced by the weaker hypothesis $\mathscr{F} \in \mathscr{L}$, but the proof is much more difficult in this case (see [**1**; **13**]).

Proof of Lemma 2. We may assume that $\mathscr{F} = \langle F_i : i < \tau \rangle$, where $\tau \leqq \omega$. Let $n < \tau$ and suppose that elements $\varphi(i) \in F_i$ have been chosen for $i < n$. Since $F_n \in \mathscr{F}(F_n)$ and $\mathscr{F} \in \mathscr{K}$, we have that

$$|\{i < n : F_i \in \mathscr{F}(F_n)\}| < |F_n|$$

and hence there is $\varphi(n) \in F_n - \{\varphi(i) : i < n\}$. This defines a transversal φ of $\mathscr{F}$ by induction.

LEMMA. 3. *Let* $\mathscr{F} \in \mathscr{K}$. *If either* (i) $|F| \leqq \aleph_0$ *for all* $F \in \mathscr{F}$ *or* (ii) $|F| = \lambda$ *for all* $F \in \mathscr{F}$, *then* Trans $(\mathscr{F}) \neq \emptyset$.

Proof. If (i) holds put $\mu = \omega$; if (ii) holds put $\mu = \lambda$. Then $\mathscr{F}$ is the union of its μ-components $\mathscr{G}_i$ $(i \in J)$ which are pairwise strongly disjoint. Since $|\mathscr{G}_i| \leqq \mu$ and $\mathscr{G}_i \in \mathscr{K}$ it follows, from Lemma 2 in the case $\mu = \omega$ and from Corollary 1 in the case $\mu > \omega$, that Trans $(\mathscr{G}_i) \neq \emptyset$. Lemma 3 follows since the $\mathscr{G}_i$ are strongly disjoint.

Proof of Theorem 1. The hypothesis implies that there is a cardinal number m such that $|F| \geqq m \geqq |\mathscr{F}(x)|$ for all $F \in \mathscr{F}$ and $x \in S(\mathscr{F})$. Let F' be any subset of F of power m $(F \in \mathscr{F})$. Then it will be enough to show that the family $\mathscr{F}' = \langle F' : F \in \mathscr{F} \rangle \subset \subset \mathscr{F}$ has a transversal. If m is finite then Trans $(\mathscr{F}') \neq \emptyset$ by Hall's theorem. If m is infinite, then for $F' \in \mathscr{F}'$ and $x \in F'$ we have

$$|\mathscr{F}'(x)| \leqq |\mathscr{F}(x)| \leqq m = |F'|$$

i.e., $\mathscr{F}' \in \mathscr{K}$. Therefore, since the members of $\mathscr{F}'$ all have the same cardinality, it follows from Lemma 3 (ii) that Trans $(\mathscr{F}') \neq \emptyset$.

4. A strengthening of $\mathscr{K}$. It will be convenient to consider the following strengthening of condition $\mathscr{K}$. We write $\mathscr{F} \in \mathscr{K}^+$ if and only if the following three conditions are satisfied:

(i) $\mathscr{F} \in \mathscr{K}$,

(ii) $A \in \mathscr{F}^{>\mu} \Rightarrow |A \cap S^{\leqq \mu}| < \mu$,

(iii) $\lambda > \omega, A \in \mathscr{F}^{\lambda}, A \cap S^{<\lambda} \neq \emptyset \Rightarrow A \subset S^{<\lambda}$.

It follows from (ii) and (iii) that if $A \in \mathscr{F}^{\lambda}$ and $A \cap S^{<\lambda} \neq \emptyset$, then λ is a limit cardinal.

LEMMA 4. *Let* $\mathscr{F} \in \mathscr{K}^+, A \in \mathscr{F}^{\lambda}, A \cap S^{<\lambda} \neq \emptyset$. *Then* cf $(\lambda) = \omega$.

Proof. The hypothesis implies that λ is a limit cardinal. Suppose that $\text{cf}(\lambda) = \kappa > \omega$. Let $\langle \lambda_\alpha : \alpha < \kappa \rangle$ be a closed increasing sequence of ordinals with $\lambda = \lim_{\alpha<\kappa} \lambda_\alpha$. By (ii), for each limit ordinal $\alpha < \kappa$ there is an ordinal $f(\alpha) < \alpha$ such that

$$|A \cap S^{\leqq \lambda_\alpha}| \leqq \lambda_{f(\alpha)}.$$

The set of limit ordinals $\alpha < \kappa$ is a stationary subset of κ. Hence there is $\beta < \kappa$ such that $f(\alpha) = \beta$ on some cofinal set $U \subset \kappa$. Since U is cofinal in κ, it follows that

$$|A \cap S^{\leqq\lambda_\alpha}| \leqq \lambda_\beta \quad \text{for all } \alpha < \kappa.$$

By (iii), and the fact that the sets $S^{\leqq\lambda_\alpha}$ increase with α, we have

$$A \subset S^{<\lambda} = \bigcup_{\alpha<\kappa} S^{\leqslant\lambda_\alpha}.$$

This gives the contradiction $|A| \leqq \lambda_\beta^+ < \lambda$.

Before stating the next lemma, we remind the reader that $\mathscr{K} \subset \mathscr{L}$.

LEMMA 5. *Let $\mathscr{J} \in \{\mathscr{K}, \mathscr{L}\}$, $\mathscr{F} \in \mathscr{J}$. Then there is $\mathscr{F}_1 \subset\subset \mathscr{F}$ so that*
(i) $\mathscr{F}_1^{\leqq\omega} \in \mathscr{J}$,
(ii) $\mathscr{F}_1^{>\omega} \in \mathscr{K}^+$.
(iii) *$\mathscr{F}_1^{\leqq\omega}$ and $\mathscr{F}_1^{>\omega}$ are strongly disjoint.*

Proof. We shall define sets $g(F) \subset F$ for $F \in \mathscr{F}$ by induction on the cardinality of F. For $F \in \mathscr{F}^{\leqq\omega}$ put $g(F) = F$. Now let $\lambda > \omega$ and assume that $g(F)$ is defined for $F \in \mathscr{F}^{<\lambda}$. Let $A \in \mathscr{F}^{\lambda}$. Then we define $g(A)$ as follows.

For $\omega \leqq \mu < \lambda$, put $A(\mu) = \{x \in A : x \in g(B)$ for some $B \in \mathscr{F}^{\leqq\mu}\}$, and for $\mu \geqq \lambda$ put $A(\mu) = A$. Then $A(\mu) \subset A(\kappa)$ for $\mu \leqq \kappa$. Put

$$C(\mu) = A(\mu) - \bigcup_{\omega\leqslant\kappa<\mu} A(\kappa).$$

Since $|A(\lambda)| = \lambda$, there is a smallest cardinal, say λ_0, such that $\omega \leqq \lambda_0 \leqq \lambda$ and $|A(\lambda_0)| \geqq \lambda_0$.

Case 1. If $|C(\lambda_0)| \geqq \lambda_0$, let $g(A)$ be any λ_0-subset of $C(\lambda_0)$.

Case 2. If $|C(\lambda_0)| < \lambda_0$, put

$$g(A) = \bigcup_{\omega<\kappa<\lambda_0} A(\kappa) - A(\omega).$$

Notice that if Case 2 holds, then $\lambda_0 > \omega$ (since $C(\omega) = A(\omega)$) and so $|A(\omega)| < \omega$ and hence $|g(A)| = \lambda_0$. Thus, in either case, $|g(A)| = \lambda_0$ and

(4.1) $\quad g(A) \subset A(\lambda_0)$.

The family $\mathscr{F}_1 = \langle g(A) : A \in \mathscr{F}\rangle$ has the required properties.

To prove this we first show that

(4.2) $\quad A \in \mathscr{F}, x \in S_1 \cap A, \rho_1(x) \leqq \mu \Rightarrow x \in A(\mu)$,

where $S_1 = S(\mathscr{F}_1)$ and $\rho_1 = \rho_{\mathscr{F}_1}$. From the hypothesis that $\rho_1(x) \leqq \mu$, it follows that there is some $F \in \mathscr{F}$ such that $x \in g(F)$ and $|g(F)| \leqq \mu$.

(i)′ If $|F| \leqq \mu$, then $x \in A(\mu)$ by the definition of $A(\mu)$.

(ii)′ If $|F| > \mu$, then $g(F) \subset F(\mu)$ by (4.1) and hence there is $B \in \mathscr{F}^{\leqq\mu}$ such that $x \in g(B)$. This again implies that $x \in A(\mu)$, and (4.2) follows. We now verify that $\mathscr{F}_1$ has the required properties. Let $C \in \mathscr{F}_1^{\leqq\omega}$, $x \in C$. There is $A \in \mathscr{F}$ such that $C = g(A)$ and $x \in A$. If $|A| \leqq \omega$, then $C = A$ and we have

(a) $|C| = |A| \geqq |\mathscr{F}(A)| \geqq |\mathscr{F}_1(C)|$ if $\mathscr{J} = \mathscr{K}$ and

(b) $|C| = |A| \geqq |\mathscr{F}(x)| \geqq |\mathscr{F}_1(x)|$ if $\mathscr{J} = \mathscr{L}$. Suppose $|A| > \omega$. Then $|C| = \omega$ and $C \subset A(\omega)$. Hence there is $B \in \mathscr{F}^{\leqq\omega}$ such that $x \in g(B) = B$. Then, since $\mathscr{K} \subset \mathscr{L}$, $\omega = |C| \geqq |B| \geqq |\mathscr{F}(x)| \geqq |\mathscr{F}_1(x)|$ and also

$$|\mathscr{F}_1(C)| = \Big|\bigcup_{x\in C} \mathscr{F}_1(x)\Big| \leqslant \omega = |C|.$$

This proves (i).

Let $\lambda > \omega$, $C \in \mathscr{F}_1^{\lambda}$, $x \in C$. There is $A \in \mathscr{F}$ such that $C = g(A) \subset A(\lambda) \subset S^{\leqq\lambda}$. Thus $\rho_{\mathscr{F}}(x) \leqq \lambda$ and so x is a member of at most λ sets $B \in \mathscr{F}$ and hence at most λ sets $g(B) \in \mathscr{F}_1$. It follows that $|\mathscr{F}_1(C)| \leqq \lambda^2 = |C|$ and hence $\mathscr{F}_1^{>\omega} \in \mathscr{K}$.

Now suppose $C \in \mathscr{F}_1^{>\mu}$. There is $\lambda > \mu$ such that $C = g(A)$, $A \in \mathscr{F}^{\lambda}$. Since $|C| > \mu$, it follows from the definition of $g(A)$ that $|A(\mu)| < \mu$. Therefore, by (4.2),

$$|C \cap S_1^{\leqq\mu}| \leqq |A(\mu)| < \mu.$$

Now let $\lambda > \omega$, $C \in \mathscr{F}_1^{\lambda}$, $C \cap S_1^{<\lambda} \neq \emptyset$. There is $A \in \mathscr{F}^{\kappa}$ such that $C = g(A)$ and $\kappa \geqq \lambda$. Now $C \subset A(\lambda)$ and from the definition of $g(A)$, either

(a) $g(A) \cap A(\mu) = \emptyset$ for $\omega \leqq \mu < \lambda$ or

(b) $g(A) \subset \bigcup_{\omega<\mu<\lambda} A(\mu)$.

Now (a) is false by (4.2) and the assumption that $C \cap S_1^{<\lambda} \neq \emptyset$. So (b) holds. But if $x \in A(\mu) \cap S_1$, then $\rho_1(x) \leqq \mu$ by the definition of $A(\mu)$. Hence $g(A) \subset S_1^{<\lambda}$. This proves (ii).

Finally, suppose $C \in \mathscr{F}_1^{>\omega}$. Then $C = g(A)$ for some $A \in \mathscr{F}^{>\omega}$ and from the definition of $g(A)$, we have $C \cap A(\omega) = \emptyset$. Therefore, by (4.2), $\rho_1(x) > \omega$ for all $x \in C$. This proves that $\mathscr{F}_1^{\leqq\omega}$ and $\mathscr{F}_1^{>\omega}$ and strongly disjoint.

5. Proof of Theorem 2. We shall prove the result by induction on

$$\mu(\mathscr{F}) = \inf\{\mu : |F| \leqq \mu \text{ for all } F \in \mathscr{F}\}.$$

By Lemma 3 (i) the theorem is true if $\mu(\mathscr{F}) = \omega$. Now assume that $\lambda > \omega$ and that

(5.1) $\mathscr{F}' \in \mathscr{K}, \mu(\mathscr{F}') < \lambda \Rightarrow \text{Trans}(\mathscr{F}') \neq \emptyset.$

Let $\mathscr{F} \in \mathscr{K}$, $\mu(\mathscr{F}) = \lambda$. We have to prove that Trans $(\mathscr{F}) \neq \emptyset$. Since $\mathscr{F}_1 \subset\subset \mathscr{F}$ and Trans $(\mathscr{F}_1) \neq \emptyset \Rightarrow$ Trans $(\mathscr{F}) \neq \emptyset$, we may assume by Lemma 5 that $\mathscr{F}_1 \in \mathscr{K}^+$ (and that $\mathscr{F}_1^{\leqq\omega} = \emptyset$, but we do not use this fact). We shall consider separately the three cases (1) λ a successor cardinal, (2) λ a regular limit cardinal, (3) λ a singular limit cardinal.

Case 1. $\lambda = \mu^+$: Since $\mathscr{F} \in \mathscr{K}^+$, it follows from Lemma 4, that $\mathscr{F}^{\lambda}$ and $\mathscr{F}^{<\lambda}$ are strongly disjoint families (since cf $(\lambda) > \omega$). Now $\mathscr{F}^{<\lambda} = \mathscr{F}^{\leqq\mu}$ has a transversal by (5.1) and $\mathscr{F}^{\lambda}$ has a transversal by Lemma 3 (ii). Hence $\mathscr{F} = \mathscr{F}^{<\lambda} \cup \mathscr{F}^{\lambda}$ also has a transversal.

Case 2. λ *a regular limit cardinal:* By Lemma 4, since cf $(\lambda) > \omega$, the families $\mathscr{F}^{\lambda}$ and $\mathscr{F}^{<\lambda}$ are strongly disjoint. Now $\mathscr{F}^{\lambda}$ has a transversal by Lemma 3 (ii) and so is enough to show that $\mathscr{F}^{<\lambda}$ has a transversal.

Let $A \in \mathscr{F}^{<\lambda}$. Put $X_0 = A, X_{n+1} = \bigcup \{B \in \mathscr{F}^{<\lambda} : B \cap X_n \neq \emptyset\}$ $(n < \omega)$, $X = \bigcup_{n<\omega} X_n$. Then, by induction on n, we have $|X_n| < \lambda$ $(n < \omega)$ and hence $|X| < \lambda$. Hence the λ-component of $\mathscr{F}^{<\lambda}$ containing A, $\mathscr{G}(A) = \langle B \in \mathscr{F}^{<\lambda} : B \cap X \neq \emptyset \rangle$, has cardinality $< \lambda$. Since λ is weakly inaccessible, it follows that $\mu(\mathscr{G}(A)) < \lambda$ and hence $\mathscr{G}(A)$ has a transversal by (5.1). Since $\mathscr{F}^{<\lambda}$ is the union of all its λ-components which are pairwise strongly disjoint, it follows that Trans $(\mathscr{F}^{<\lambda}) \neq \emptyset$.

Case 3. cf $(\lambda) = \kappa < \lambda$: Let $\langle \lambda_\alpha : \alpha \leqq \kappa \rangle$ be a continuous increasing sequence of cardinals,

$$\kappa < \lambda_0 < \lambda_1 < \ldots < \lambda_\kappa = \lambda = \lim_{\alpha<\kappa} \lambda_\alpha .$$

Denote by $\mathscr{C}_\alpha$ the set of all the large λ_α-components of $\mathscr{F}$, and let $\mathscr{C} = \bigcup_{\alpha \leqq \kappa} \mathscr{C}_\alpha$. If $\mathscr{G} \in \mathscr{C}_\alpha$, then $|\mathscr{G}| \leqq \lambda_\alpha$ and we may write

$$\mathscr{G}^{\lambda_\alpha} = \langle G_\nu : \nu < \xi(\mathscr{G}) \rangle$$

where $\xi(\mathscr{G})$ is some initial ordinal $\leqq \lambda_\alpha$. For any ordinal β put

$$\mathscr{G}(\beta) = \begin{cases} \langle G_\nu : \nu < \beta \rangle, & \text{if } \beta \leqq \xi(\mathscr{G}), \\ \mathscr{G}^{\lambda_\alpha}, & \text{if } \beta > \xi(\mathscr{G}). \end{cases}$$

If $\mathscr{G}, \mathscr{G}' \in \mathscr{C}$, $\mathscr{G} \neq \mathscr{G}'$ and β, β' are ordinals, then

(5.2) $\quad \mathscr{G}(\beta) \cap \mathscr{G}'(\beta') = \emptyset.$

For, there are $\alpha, \alpha' \leqq \kappa$ such that $\mathscr{G} \in \mathscr{C}_\alpha$, $\mathscr{G}' \in \mathscr{C}_{\alpha'}$. If $\alpha = \alpha'$ then $\mathscr{G}$ and $\mathscr{G}'$ are disjoint since a set $F \in \mathscr{F}^{\lambda_\alpha}$ is a member of exactly one large λ_α-component; if $\alpha \neq \alpha'$ then $\mathscr{G}^{\lambda_\alpha}$ and $\mathscr{G}'^{\lambda_{\alpha'}}$ are disjoint since members of these families have cardinalities λ_α and $\lambda_{\alpha'}$ respectively.

For $\alpha \leqq \kappa$ put

$$\mathscr{F}_\alpha^* = \bigcup_{\mathscr{G} \in \mathscr{C}} \bigcup_{\gamma < \alpha} \mathscr{G}(\lambda_\gamma), \quad \mathscr{F}_\alpha^{**} = \mathscr{F}^{<\lambda_\alpha} \cup \mathscr{F}_\alpha^* .$$

It is easy to see that

(5.3) $\quad \mathscr{F}_{\alpha_0}^{**} = \bigcup_{\alpha<\alpha_0} \mathscr{F}_\alpha^{**}$

if α_0 is a limit ordinal. For, if $A \in \mathscr{F}^{\lambda_{\alpha_0}}$, then there is a large λ_{α_0}-component $\mathscr{G} \in \mathscr{C}$ and $\gamma < \alpha_0$ so that $A \in \mathscr{G}(\lambda_\gamma)$ and hence $A \in \mathscr{F}_{\gamma+1}^{**}$. We also remark that (put $\mathscr{F}_{\kappa+1}^* = \mathscr{F}_\kappa^*$)

(5.4) $\quad |\langle B \in \mathscr{F}_{\alpha+1}^* : A \cap B \neq \emptyset \rangle| \leqq \lambda_\alpha \quad (\alpha \leqq \kappa, A \in \mathscr{F}).$

For, to each $\rho \leqq \kappa$ there is at most one large λ_ρ-component containing A, and if $|B| = \lambda_\rho$ and A, B are members of different large λ_ρ-components then $A \cap B = \emptyset$. Thus $|\langle B \in \mathscr{F}_{\alpha+1}^* : A \cap B \neq \emptyset\rangle| \leqq \kappa \cdot \lambda_\alpha = \lambda_\alpha$.

We are going to define functions φ_α for $\alpha \leqq \kappa$ by transfinite induction so that

(i) *φ_α is a transversal of $\mathscr{F}_\alpha^{**}$, and*

(ii) *φ_α is an extension of φ_γ for $\gamma < \alpha$.*

Then φ_κ will be a transversal of $\mathscr{F} = \mathscr{F}_\kappa^{**}$ as required.

Let $\alpha_0 \leqq \kappa$ and assume that φ_α has already been defined for $\alpha < \alpha_0$ so that (i) and (ii) hold. If $\alpha_0 = 0$, then $\mathscr{F}_{\alpha_0}^{**} = \mathscr{F}^{\leqq\lambda_0}$ has a transversal φ_0 by (5.1). If α_0 is a limit ordinal, then $\varphi_{\alpha_0} = \bigcup\varphi_\alpha$ is a transversal of $\mathscr{F}_{\alpha_0}^{**}$ of the required kind by (5.3) and (ii). It only remains to define φ_{α_0} in the case when α_0 is a successor ordinal, say $\alpha_0 = \alpha + 1$.

First we show that

(5.5) $\quad |A \cap \text{range } (\varphi_\alpha)| \leqq \lambda_\alpha \quad (A \in \mathscr{F}).$

We may assume $A \in \mathscr{F}^{>\lambda_\alpha}$. Then $|A \cap S^{\leqq\lambda_\alpha}| < \lambda_\alpha$ and each element $x \in A \cap S^{\leqq\lambda_\alpha}$ is a member of at most λ_α different sets $B \in \mathscr{F}$. Therefore,

$$|\langle B \in \mathscr{F}^{\leqq\lambda_\alpha} : A \cap B \neq \emptyset\rangle| \leqq \lambda_\alpha.$$

This and (5.4) proves (5.5).

Put

$$\mathscr{F}_1 = \mathscr{F}^{\leqq\lambda_{\alpha+1}} - \mathscr{F}_\alpha^{**}, \quad \mathscr{F}_2 = \mathscr{F}_{\alpha+1}^* - (\mathscr{F}_\alpha^{**} \cup \mathscr{F}_1).$$

Then $\mathscr{F}_{\alpha+1}^{**}$ is the disjoint union of $\mathscr{F}_\alpha^{**}$, $\mathscr{F}_1$ and $\mathscr{F}_2$. The members of $\mathscr{F}_1$ all have cardinality $\lambda_{\alpha+1}$ and so, by (5.5) and Lemma 3 (ii), there is a transversal ψ_1 of $\mathscr{F}_1$ which is disjoint from φ_α. We shall extend $\varphi_\alpha' = \varphi_\alpha \cup \psi_1$ to a transversal of $\mathscr{F}_{\alpha+1}^{**}$ by selecting suitable elements from each set $F \in \mathscr{F}_2$. We do this component by component.

Let $\mathscr{C} = \{\mathscr{G}_\sigma : \sigma < \tau\}$. Let $\sigma < \tau$ and suppose we have already defined a transversal χ, say, of $\mathscr{G}_\sigma^* = \bigcup_{\rho<\sigma}\mathscr{G}_\rho(\lambda_\alpha) - \mathscr{F}_\alpha^{**} \cup \mathscr{F}_1$ which is disjoint from φ_α'. If

$$A \in \mathscr{G}' = \mathscr{G}_\sigma(\lambda_\alpha) - (\mathscr{G}_\sigma^* \cup \mathscr{F}_\alpha^{**} \cup \mathscr{F}_1),$$

then $|A| > \lambda_{\alpha+1}$. Therefore, $|A \cap S^{\leqq\lambda_{\alpha+1}}| < \lambda_{\alpha+1}$ and so

$$|A \cap \text{range } \psi_1| \leqq \lambda_\alpha.$$

Also, by (5.4) we have

$$|A \cap \text{range } (\chi)| \leqq \lambda_\alpha.$$

These two inequalities together with (5.5) show that

(5.6) $\quad |A \cap \text{range } (\varphi_\alpha \cup \psi_1 \cup \chi)| < \lambda_{\alpha+1} \quad (A \in \mathscr{G}').$

Since $|\mathscr{G}'| \leqq \lambda_\alpha < |A|$ $(A \in \mathscr{G}')$, it follows from (5.6) and Corollary 1 that $\mathscr{G}'$ has a transversal χ' disjoint from $\varphi_\alpha \cup \psi_1 \cup \chi$. It follows, by transfinite induction on $\sigma < \tau$, that $\mathscr{F}_2$ has a transversal ψ_2 disjoint from $\varphi_\alpha \cup \psi_1$. Then

$\varphi_{\alpha+1} = \varphi_\alpha \cup \psi_1 \cup \psi_2$ is a transversal of $\mathscr{F}_{\alpha+1}^{**}$ which extends φ_α. This completes the proof of Theorem 2.

6. Proof of Theorem 3. We assume the special case of this theorem (proved in [**13**; **2**]):

(6.1) *if* $\mathscr{F}'$ *is a countable family of countable sets, then*

$$\mathscr{F}' \in \mathscr{L} \Rightarrow \text{Trans}\ (\mathscr{F}') \neq \emptyset.$$

Now let $\mathscr{F}$ be an arbitrary family satisfying condition $\mathscr{L}$. By Lemma 5 there is $\mathscr{F}_1 \subset\subset \mathscr{F}$ such that $\mathscr{F}_1^{\leqq\omega}$ and $\mathscr{F}_1^{>\omega}$ are strongly disjoint, $\mathscr{F}_1^{\leqq\omega} \in \mathscr{L}$ and $\mathscr{F}_1^{>\omega} \in \mathscr{K}^+$.

The ω-components of $\mathscr{F}_1^{\leqq\omega}$ are countable and strongly disjoint and every such component has a transversal by (6.1). $\mathscr{F}_1^{>\omega}$ has a transversal by Theorem 2. Therefore $\mathscr{F}_1$, and hence $\mathscr{F}$, has a transversal.

7. A generalization. We shall now prove a generalization of Theorem 3 using a different idea. A family $\mathscr{F}$ has property $\mathscr{P}$ if and only if the following three conditions are satisfied:

$\mathscr{P}_1$. $\mathscr{F}^{<\omega} \in \mathscr{L}$;

$\mathscr{P}_2$. $|\mathscr{F}^\lambda(x)| \leqq \lambda$ *for* $x \in S(\mathscr{F})$ *and* $\lambda \geqq \omega$;

$\mathscr{P}_3$. *If* λ *is inaccessible and* $x \in S(\mathscr{F})$, *then* $\{\mu < \lambda : \mathscr{F}^\mu(x) \neq \emptyset\}$ *is a non-stationary subset of* λ.

It is clear that if $\mathscr{F} \in \mathscr{L}$, then $\mathscr{F} \in \mathscr{P}$ (if λ inaccessible, $x \in S(\mathscr{F})$ and $\mathscr{F}^\kappa(x) \neq \emptyset$, then $|\{\mu < \lambda : \mathscr{F}^\mu(x) \neq \emptyset\}| \leqq \kappa$). It is also easy to verify that

(7.1) *if* $\mathscr{F} \in \mathscr{P}$ *and* $g(F) \subset F$, $|g(F)| = |F| (F \in \mathscr{F})$, *then*

$$\mathscr{F}_1 = \langle g(F) : F \in \mathscr{F} \rangle \in \mathscr{P}.$$

THEOREM 4. $\mathscr{F} \in \mathscr{P} \Rightarrow \text{Trans}\ (\mathscr{F}) \neq \emptyset$.

Proof of Theorem 4. For each infinite cardinal μ, the μ-components of $\mathscr{F}$ are pairwise strongly disjoint. Every such component has cardinality $\leqq \mu$ and so, by Lemma 1, the μ-sets of a μ-component can be replaced by subsets of power μ which are pairwise disjoint. By (7.1) the family thus obtained still enjoys property $\mathscr{P}$. So we may assume without loss of generality that

(7.2) if $\lambda \geqq \omega$ and $A, B \in \mathscr{F}^\lambda$, $A \neq B$, then $A \cap B = \emptyset$.

As in the proof of Theorem 2, we shall prove the theorem by transfinite induction on $\mu(\mathscr{F})$. If $\mu(\mathscr{F}) = \omega$, then $\mathscr{F} \in \mathscr{L}$ and Trans $(\mathscr{F}) \neq \emptyset$ by Theorem 3. Suppose $\mu(\mathscr{F}) = \lambda > \omega$.

Case 1. $\lambda = \kappa^+$: By (7.2) the members of $\mathscr{F}$ having power κ^+ are pairwise disjoint. Therefore, if we replace every such set by a subset of power ω, the resulting family $\mathscr{F}_1$, say, has property $\mathscr{P}$ and $\mu(\mathscr{F}_1) = \kappa$. Thus Trans $(\mathscr{F}_1) \neq \emptyset$ by the induction hypothesis and hence Trans $(\mathscr{F}) \neq \emptyset$.

Case 2. $\lambda = \mu(\mathscr{F})$ *is singular*: Let cf $(\lambda) = \kappa < \lambda$, and let $\langle \lambda_\rho : \rho \leqq \kappa \rangle$ be a closed increasing sequence of ordinals

$$\kappa < \lambda_0 < \ldots < \lambda_\kappa = \lambda = \lim_{\rho < \kappa} \lambda_\rho.$$

Form a new family $\mathscr{F}_1$ from $\mathscr{F}$ by replacing each set $A \in \bigcup_{\rho \leqq \kappa} \mathscr{F}^{\lambda_\rho}$ by a subset $g(A) \subset A$ of power κ. Any element $x \in S(\mathscr{F})$ belongs to at most κ new sets of power κ and so $\mathscr{F}_1 \in \mathscr{P}$. We may as well assume that $\mathscr{F} = \mathscr{F}_1$, i.e.

(7.3) $\bigcup_{\rho \leqslant \kappa} \mathscr{F}^{\lambda_\rho} = \emptyset.$

If $A \in \mathscr{F}^{\leqq \lambda_\rho}$, let $\mathscr{G}_\rho(A)$ be the unique λ_ρ-component of $\mathscr{F}$ which contains A; if $A \in \mathscr{F}^{> \lambda_\rho}$, let $\mathscr{G}_\rho(A) = \emptyset$, the empty family. Then

$$\mathscr{G}_\rho(A) \subset \mathscr{G}_\sigma(A) \quad \text{for } \rho < \sigma \leqq \kappa.$$

Also, by (7.3),

$$\mathscr{G}_\sigma(A) = \bigcup_{\rho < \sigma} \mathscr{G}_\rho(A) \quad \textit{if } \sigma \textit{ is a limit ordinal} \leqslant \kappa.$$

Put $S_\rho(A) = S(\mathscr{G}_\rho(A))(\rho \leqq \kappa)$. Since $|\mathscr{G}_\rho(A)| \leqq \lambda_\rho$ and $|B| < \lambda_\rho$ $(B \in \mathscr{G}_\rho(A))$, it follows that $|S_\rho(A)| \leqq \lambda_\rho$ $(\rho < \kappa)$. For $B \in \mathscr{F} - \mathscr{G}_\rho(A)$ we have that either (i) $|B| \leqq \lambda_\rho$ and $B \cap S_\rho(A) = \emptyset$ or (ii) $|B| > \lambda_\rho$. Therefore, by (7.1),

$$\mathscr{G}_\rho^*(A) = \langle B - S_\rho(A) : B \in \mathscr{G}_{\rho+1}(A) - \mathscr{G}_\rho(A) \rangle \in \mathscr{P}$$

for $\rho < \kappa$. Now $\mathscr{G}_0(A)$ has a transversal and so does $\mathscr{G}_\rho^*(A)$ $(\rho < \kappa)$ since $\mu(\mathscr{G}_\rho^*(A)) \leqq \lambda_{\rho+1} < \lambda$. Therefore, since the families $\mathscr{G}_0(A)$, $\mathscr{G}_\rho^*(A)$ $(\rho < \kappa)$ are pairwise strongly disjoint, the family

$$\mathscr{G}'(A) = \mathscr{G}_0(A) \cup \bigcup_{\rho < \kappa} \mathscr{G}_\rho^*(A)$$

has a transversal. This clearly implies that the λ-component, $\mathscr{G}_\kappa(A)$, containing A also has a transversal. This holds for any $A \in \mathscr{F}$ and so $\mathscr{F}$ has a transversal since the λ-components of $\mathscr{F}$ are strongly disjoint.

Case 3. λ *is weakly inaccessible*: Since the λ-components of $\mathscr{F}$ are strongly disjoint, we may assume that $\mathscr{F}$ has but a single λ-component. Then $|\mathscr{F}| \leqq \lambda$ and $|S(\mathscr{F})| \leqq \lambda$ and so we can assume further that $\mathscr{F}$ is a family of subsets of λ. (As usual, an ordinal is the set of all smaller ordinals.) Now by (7.2) the members of $\mathscr{F}$ which have power λ are pairwise disjoint and, if we replace these by subsets of power ω, the resulting family still has property $\mathscr{P}$. Thus we can assume that

(7.4) $|A| < \lambda \quad (A \in \mathscr{F}).$

By $\mathscr{P}_3$, for each $x \in \lambda$ there is a function $f_x : \lambda \to \lambda$ such that

$$f_x(\alpha) \leqq \alpha \quad (\alpha < \lambda),$$
$$x \leqq f_x(|A|) < |A| \quad (A \in \mathscr{F}(x),\ |A| > x),$$

(7.5) $|\{\alpha < \lambda : f_x(\alpha) = \gamma\}| < \lambda \quad (\gamma < \lambda).$

We now define a function $g : \lambda \to \lambda$ by putting

$$g(\alpha) = \sup (\alpha \cup \{y \in \lambda : (\exists x < \alpha)(\exists A \in \mathscr{F}(x))(y \in A \text{ and } f_x(|A|) < \alpha)\}).$$

(If C is a set of ordinals then $\sup C$ is the smallest ordinal β such that $\beta > \gamma$ for all $\gamma \in C$.) We immediately have from the definition of g, (7.5) and $\mathscr{P}_2$, that

(7.6) $\alpha \leqq g(\alpha) \leqq g(\beta) < \lambda$ for $\alpha < \beta < \lambda$.

If α is a limit ordinal such that $g(\gamma) < \alpha$ for all $\gamma < \alpha$, then $g(\alpha) = \alpha$. Put

$$C = \{0\} \cup \{\alpha < \lambda : \alpha \text{ a limit ordinal}, g(\alpha) = \alpha\}.$$

Now C is a cofinal subset of λ. For if $\gamma < \lambda$, put $\alpha_0 = \gamma$, $\alpha_{n+1} = g(\alpha_n + 1)$ $(n < \omega)$. Then $\gamma < \alpha = \lim_{n<\omega}\alpha_n$ and $\alpha \in C$. Therefore, we may write

$$C = \{\beta_\nu : \nu < \lambda\},$$

where $0 = \beta_0 < \beta_1 < \ldots < \lambda = \lim_{\nu<\lambda}\beta_\nu$ and β_ν is a limit ordinal satisfying $g(\beta_\nu) = \beta_\nu$ $(\nu < \lambda)$.

We will prove that, for $A \in \mathscr{F}$ there is $\nu = \nu(A) < \lambda$ such that

(7.7) $|A \cap [\beta_\nu, \beta_{\nu+1})| = |A|$.

Let x be the first element of A. If $|A| \leqq x$, then $f_x(|A|) \leqq x$ and hence $A \subset [x, g(x + 1))$. Now there is $\nu < \lambda$ such that $x \in [\beta_\nu, \beta_{\nu+1})$. Then $g(x + 1) \leqq \beta_{\nu+1} = g(\beta_{\nu+1})$ and (7.7) holds. Now suppose that $|A| > x$. There is $\nu < \lambda$ such that $f_x(|A|) \in [\beta_\nu, \beta_{\nu+1})$. Hence, there is γ such that

$$x \leqq f_x(|A|) < \gamma < \beta_{\nu+1}.$$

Then $A \subset [x, g(\gamma))$. Since $g(\gamma) < \beta_{\nu+1}$ and $\beta_\nu \leqq f_x(|A|) < |A|$, we again obtain (7.7).

By (7.7) and (7.1) we can replace each set $A \in \mathscr{F}$ by the subset $g(A) = A \cap [\beta_\nu, \beta_{\nu+1})$ to obtain a family $\mathscr{F}_1$ also with property $\mathscr{P}$. For $A \in \mathscr{F}$, if $\mathscr{G}(A)$ is the λ-component of $\mathscr{F}_1$ containing $g(A)$, then $\mathscr{G}(A)$ is a family of subsets of $[\beta_{\nu(A)}, \beta_{\nu(A)+1})$. Thus $\mu(\mathscr{G}(A)) < \lambda$ and so $\mathscr{G}(A)$ has a transversal. Since different λ-components of $\mathscr{F}_1$ are strongly disjoint, it follows that $\mathscr{F}_1$ (and hence $\mathscr{F}$) has a transversal.

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