Existence of rigid-like families of abelian p - groups

S. Shelah

Dedicated to the memory of A. Robinson

ABSTRACT: We prove that for arbitrarily large λ, there are large families of abelian groups, with only the necessary Homomorphisms between them.

INTRODUCTION: Here a group means an abelian group. Improving results of Fuchs (see [Fu2], [Fu4]), Shelah [Sh] proved the existence of 2^λ rigid groups of cardinality λ; i.e. for every cardinal λ, there are groups $G_i (i < 2^\lambda)$ each of cardinality λ, such that if h is a non-zero homomorphism from G_i into G_j, then $i = j$, and $h(x) = nx$ for some integer n.

We try to generalize this theorem to separable p - groups.

We cannot have rigid systems of separable p-groups because any basic subgroup of a separable p-group G, is the image of an endomorphism of G. Also the multiplication by a p-adic integer is an endomorphism. Weakening accordingly the notion of rigid systems, we prove existence theorems in §1, §2 (for possible extensions, see a remark at the end of section 2).

Pierce [P] asked, and this is repeated in [Fu.2], p.55, problem 55, whether there are essentially indecomposable p-groups of arbitrarily large cardinalities (G is essentially indecomposable if $G = G_1 \oplus G_2$ implies that G_1, or G_2 is bounded). Our result implies a positive answer (because each member of a rigid-like family is essentially indecomposable).

Fuchs [Fu.2], p.55, problem 53 asked to construct large systems of p-groups such that all homomorphisms between different members are small. As a zero-like homomorphism is the same thing as a small homomorphism (as defined in [Fu.1] 46.3 p.195) theorem 5.1 answers this question. The construction in theorem 1.2 gives for $\mu = \lambda^{\aleph_0} = 2^{\lambda} > 2^{\aleph_0}$, a family of 2^{μ} separable p-groups of power μ so that any homomorphism between different members has range of power $\leq\lambda$.

We assume knowledge of naive set theory, and of separable p-groups as in [Fu. 1],VI;[Fu.2], XI.

<u>Notation:</u> Let λ, μ, κ denote infinite cardinals, α, β, γ, δ, i, j ordinals, δ a limit ordinal, k, ℓ, m,n, M, N natural numbers or integers, ω the first infinite ordinal. We let η, τ, ν be sequences of ordinals. Let $\ell(\eta)$ be the length of η, $\eta(i)$ its ith element. Let $cf[\alpha]$ be the cofinality of α.

G, H and sometimes K, I, R are groups, h, g are homomorphisms, p, q are prime natural numbers, r a rational or sometimes a p-adic integer. Here a group means a reduced separable p-group.

When notations become complex, $a_i(j)$ is written as $a[j,i]$, a_i as $a[i]$.

1 Rigid-like systems of p-groups

DEFINITION 1.1:

(1) A homomorphism $h:G \to H$ is called zero-like if there are no $m < \omega$ and $a_n \in G$ for $n < \omega$ such that a_n has exponent $n+m$ and $h(a_n)$ has exponent $\geq n+1$. We call h semi-zero-like if there are no $m < \omega$, $a_n^i \in G$ for $n < \omega$, $i < (2^{\aleph_0})^+$, such that a_n^i has exponent $n+m$, $h(a_n^i)$ has order $\geq n+1$, and $p^n h(a_n^i) \neq p^n h(a_n^j)$ for $i \neq j$.

(2) Let G be a subgroup of H, $h:G \to H$ a homomorphism. Then h is called simple if $h = h_1 + h_2$, h_1 is zero-like, h_2 is a multiplication by a p-adic integer. Similarly h is semi-simple when h_1 is semi-zero-like.

(3) A family $\{G_i : i < i_0\}$ (of separable p-groups) is called rigid-like if whenever $h: G_i \to G_j$ is a non-zero-like homomorphism then $i = j$ and h is simple. Similarly a semi-rigid-like family is defined.

DEFINITION 1.2: G is essentially indecomposable if $G = G_1 \oplus G_2$ implies G_1 or G_2 is bounded.

CLAIM 1.1:

(1) Suppose $\bigoplus_{\substack{i<\beta(n) \\ n<\omega}} \langle x_i^n \rangle$ is a basic subgroup of G, x_i^n of exponent $n+1$, $h: G \to H$ a homomorphism. Then h is zero-like iff $f(n) = \min_i\{n -$ "the exponent of $h(x_i^n)$"$\}$ goes to infinity.

(2) Suppose I is a basic subgroup of G, and $|p^m I| \geq \lambda$ for $m < \omega$. Then for any group H of cardinality $\leq \lambda$ there is a zero-like homomorphism from G onto H.

(3) If G belongs to a rigid-like family *then* G is essentially indecomposable.

(4) If G belongs to a semi-rigid-like family and $G = G_1 \oplus G_2$ then for some $m < \omega$, $\ell = 1$ or 2, $|p^m G_\ell| \leq 2^{\aleph_0}$.

PROOF: Immediate (Part (2) is similar to a theorem in [Fu. 1]).

THEOREM 1.2: Assume $\mu = \lambda^{\aleph_0} = 2^\lambda > 2^{\aleph_0}$.

(A) There is a semi-rigid-like family of 2^μ groups of cardinality μ, with basic subgroups of cardinality $\leq \lambda$.

(B) Moreover if G, H are members of the family, I a pure subgroup of G, closed in it, and $p^m I$ has power $\geq \lambda$ for each m, and $h: I \to H$ is a non-semi-zero-like homomorphism then $G = H$, and h is semi-simple.

REMARK: In (A) we can demand the basic subgroups have cardinality λ.

PROOF

NOTATION: W.l.o.g. $\chi < \lambda$ implies $\chi^{\aleph_0} < \mu$, hence λ has cofinality ω and $\chi < \lambda$ implies $\chi^{\aleph_0} < \lambda$. So let $\lambda = \sum_{n<\omega} \lambda_n$, $2^{\aleph_0} < \lambda_0 < \lambda_1 < \ldots$, each λ_n regular, $\lambda_n^{\aleph_0} = \lambda_n$. Let G be the group generated by x_i^n, $i < \lambda_n^+$ $0 \leq n < \omega$, $p^{n+1} x_i^n = 0$. Let H be the torsion-completion of G, so each $a \in H$ is of the form $\Sigma k_i^n x_i^n$, where $\{i: k_i^n \neq 0\}$ is finite for each n, and for some m $p^m k_i^n x_i^n = 0$, for every n, i. Let $d(a) = \{x_i^n: k_i^n x_i^n \neq 0,\ n < \omega,\ i < \lambda_n^+\}$ and $d_m(a) = \{x_i^n : p^m k_i^n x_i^n \neq 0,\ p^{m+1} k_i^n x_i^n = 0,\ n < \omega,\ i < \lambda_n^+\}$.

If $d(a)$ is infinite $d_*(a)$ is $d_m(a)$ for the maximal m for which $d_m(a)$ is infinite. We attribute properties to a instead of $d(a)$, sometimes. Let d_1, d_2 be almost disjoint if $d_1 \cap d_2$ is finite; let d_1 be almost included in d_2 if $d_1 - d_2$ is finite.

Let $X_m = \{x_i^m: i < \lambda_m^+\}$. If $A \subseteq H$, let $PC(A)$ be the smallest subgroup I of H such that $A \subseteq I$, and $\sum_{i,n} pk_i^n x_i^n \in I$, where

$k_i^n x_i^n \neq 0 \Rightarrow pk_i^n x_i^n \neq 0$, implies $\Sigma k_i^n x_i^n \in I$. Clearly I is a pure subgroup of H. Note that if I is any pure subgroup of H then any homomorphism $h: I \to H$ has an extension to a homomorphism $h: H \to H$. If I is dense (in the p-adic topology) the extension is unique. Also each closed pure subgroup I of H is determined by any basic subgroup of it (so its cardinality is either μ or $\leq\lambda$).

Hence, H has $2^\lambda = \mu$ pure closed subgroups, and there are μ homomorphisms from H into H. Let $\{ (h_i, I_i) : i < \mu\}$ be a list of all pairs of homomorphisms $h: H \to H$ and pure closed subgroups I of H, each pair appearing μ times.

We now define by induction on $i < \mu$, $a_i^*, b_i^* \in H$ which will satisfy the following induction assumptions, and then for $C \subseteq \mu$ let $G(C) = PC[G \cup \{a_i^*: i \in C\}]$; our family will be $\subseteq \{G(C): C \subseteq \mu\}$.

The induction assumptions are

(1) $d_*(a_\alpha^*)$ is not almost included in a finite union of $d(a_i^*)$, $d(b_i^*)$, $(i < \alpha)$

(2) a_α^* has exponent $m+1$ when $d_*(a_\alpha^*) = d_m(a_\alpha^*)$.

(3) If for every m $|p^m I_\alpha| \geq \lambda$ and $h_\alpha | I_\alpha$ is not semi-simple <u>then</u> $h(a_\alpha^*) = b_\alpha^*$ and $b_\alpha^* \notin PC[G \cup \{a_i^*: i \leq \alpha\}]$

(4) If for every m $|p^m I_\alpha| \geq \lambda$ and $h_\alpha | I_\alpha$ is semi-simple but not zero-like <u>then</u> $h_\alpha(a_\alpha^*) = b_\alpha^*$, $b_\alpha^* \notin PC[G \cup \{a_i^*: i < \alpha\}]$

Notice that (1), (2) implies $\{a_\alpha^* + G : \alpha < \mu\}$ is an independent family. We first prove:

<u>CLAIM 1.3</u>: Suppose μ_n are regular cardinals, $\mu_n \leq \mu_{n+1}$, and $\kappa < \mu_n \to \kappa^{\aleph_0} < \mu_n$.

(1) If K is a subgroup of H, and $|p^m K| \geq \sum_n \mu_n$ for any m, <u>then</u> we can find $y_i^n \in K$ $(n < \omega,\ i < \mu_n)$ which are pairwise

disjoint, and y_i^n has exponent $\geq n+1$. Hence we can assume the $d(y_i^n)$'s are disjoint to some prescribed set of cardinality $< \mu_n$.

(2) Suppose $Y^n \subseteq K$, and $z_1 \neq z_2 \in Y_n$ implies $p^n z_1 \neq p^n z_2$, and $|Y^n| = \mu_n$. Then we can find y_i^n which is $z_1 - z_2$ for some $z_1, z_2 \in Y^n$; for $n < \omega$, $i < \mu_n$, such that these y_i^n's satisfy the conclusion of (1).

PROOF:

(1) As $|p^m K| \geq \sum_n \mu_n$, we can find $Y^m \subseteq K$ such that $z_1, z_2 \in Y^m \to p^m z_1 \neq p^m z_2$, hence it suffices to prove (2).

(2) By Erdös and Rado [ER], (as $|Y^n| = \mu_n$, μ_n regular, $\kappa < \mu_n \to \kappa^{\aleph_0} < \kappa_n$ and as $d(z)$ is countable for $z \in Y^n$) there are $Y_n \subseteq Y^n$, $|Y_n| = \mu_n$ and a set d^n such that for $z_1 \neq z_2 \in Y_n$, $d(z_1) \cap d(z_2) = d^n$. As necessarily $\mu_n > 2^{\aleph_0}$, we can assume that if $x_j^m \in d^{n(0)}$, $z_1, z_2 \in Y_{n(0)}$, $z_\ell = \sum_{n,i} k_i^{n,\ell} x_i^n$ then $k_j^{m,1} = k_j^{m,2}$. Now notice that if z_ℓ $\ell = 1, \dots, 4$, are distinct members of Y_n, then $p^n(z_1 - z_2) \neq p^n(z_3 - z_4)$.

Define by induction on $\alpha < \sum_n \mu_n$ and on $m < \omega$, the y_α^m $(\alpha < \mu_m)$. If we have defined y_β^n for $\beta < \alpha$, $\beta < \mu_n$ then for $\beta = \alpha < \mu_n$, $n < m$ we define y_α^m as follows:
Clearly the number of $z \in Y_m$ such that $d(z) \cap d(y_\beta^n) \nsubseteq d^m$ for some n, β as above, is $\leq |\alpha| + \aleph_0 < \mu_m$, hence choose $z_1, z_2 \in Y_m$ which do not satisfy it, and let $y_\alpha^m = z_1 - z_2$.

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Suppose we have defined a_j^*, b_j^* for $j < \alpha$, and we shall define them for α.

CASE I: $h_\alpha | I_\alpha$ is semi-zero-like or $p^n I_\alpha$ has power $< \lambda$ for some n.

If for every m, $p^m I_\alpha$ has power $\geq \lambda$ let $K = I_\alpha$, otherwise let $K = H$. Then by Claim 1.3 we can find in

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K elements y_i^n, $i < \lambda_n^+$, $n < \omega$, y_i^n of exponent $n+1$, the $d(y_i^n)$'s pairwise disjoint. For any $\eta \in \prod_{n<\omega} \lambda_n^+$ let $y_\eta = \prod_{n<\omega} p^n y^n_{\eta(n)} \in K$, so we can find a_i $i < \mu$, such that $pa_i = 0$ and $d_*(a_i) = d(a_i)$, and no $d(a_i)$ is almost included in a finite union of $d(a_j)$ $j \neq i$, $j < \mu$. Hence for any set $A \subseteq H$, of power $<\mu$, some $d(a_i)$ is not almost included in a finite union of $d(a)$, $a \in A$. (Otherwise for each a_i there is a corresponding finite set $A_i \subseteq A$; so for some A^* $\{i < \mu : A_i = A^*\}$ has power $> 2^{\aleph_0}$, so a countable set has $> 2^{\aleph_0}$ distinct subsets, contradiction).

So we can define $a^* \in \{a_i : i < \mu\}$ to satisfy (1), (2), and b^* so that $d_*(b^*)$ is not almost included in a finite union of $d(a_i^*), d(b_j^*)$ $(i \leq \alpha, j < \alpha)$. Clearly (3), (4) are satisfied.

CASE II: Not Case 1, but $p^m K_\alpha$ has power $\geq \lambda$ for every m, where K_α is the kernel of $h_\alpha | I_\alpha$.

Let the image of $h_\alpha | I_\alpha$ be R_α and as not case 1, $h_\alpha | I_\alpha$ is not semi-zero-like; hence there are $m < \omega$, $z_i^n \in K$ of exponent $n+m$ for $i < (2^{\aleph_0})^+$, such that $p^n y_i^n \neq p^n y_j^n$ for $i \neq j$ where $y_i^n = h_\alpha(z_i^n)$. Letting $Y^n = \{y_i^n : i < (2^{\aleph_0})^+\}$ we can, by Claim 1.3 and renaming assume the y_i^n are pairwise disjoint. Let $y_n = y_0^n$; as R is reduced, $h_\alpha(\Sigma k_n z_0^n) = \Sigma k_n y_n$.

So $\Sigma k_n y_n$ exists and belongs to R_α whenever for some ℓ $p^\ell k_n y_n = 0$ for every n. Hence, $\Sigma k_n p^n y_n \in R$ for every k_n, and $p^n y_n \neq 0$. Now for any sequence $\bar{k} = \langle \dots, k_n, \dots \rangle$, $-p < k_n < p$, let $b_{\bar{k}} = \Sigma k_n p^n y_n$ only when $d(\bar{k}) = \{n : k_n \neq 0\}$ is infinite. Notice that $d(b_{\bar{k}}) = \bigcup_{n \in d(\bar{k})} d(y_n)$. We shall find a $\bar{k}$ such that $b_{\bar{k}} \notin PC[G \cup \{a_i^* : i < \alpha\}]$. Suppose there is no such $\bar{k}$; then there are $m = m(\bar{k})$, and $\alpha > i(0,\bar{k}) > i(1,\bar{k}) > \dots > i(m,\bar{k})$ and integers $M_\ell = M(\ell,\bar{k})$ such that:

$$(*) \quad b_{\bar{k}} + G = M_0 a^*_{i(0,\bar{k})} + \dots + M_m a^*_{i(m,\bar{k})} + G \quad (\text{where } M_\ell a^*_{i(\ell,\bar{k})} \neq 0).$$

Notice that by condition (1), this expression is determined uniquely.

Suppose $d(\bar{k}) \subseteq d(\bar{\ell})$ but $i(0,\bar{k}) > i(0,\bar{\ell})$ Then:

(A) $d_*(a^*_{i(0,\bar{k})}) \subseteq^* d(b_{\bar{k}}) \cup \bigcup_{0<n\leq m(\bar{k})} d(a^*_{i(n,\bar{k})})$ [as $pb_{\bar{k}} = 0$, $d(M_o a^*_{i(0,\bar{k})}) = d_*(a^*_{i(0,\bar{k})})$ by conditions (1), (2)]. ($\subseteq^*$- almost included).

(B) $d(b_{\bar{k}}) \subseteq d(b_{\bar{\ell}})$ (by the expression for $d(b_{\bar{k}})$).

(C) $d(b_{\bar{\ell}}) \subseteq \bigcup_{0\leq n\leq m(\bar{\ell})} d(a^*_{i(n,\bar{\ell})})$

Combining we get $d_*(a^*_{i(0,\bar{k})})$ is almost included in a finite union of $d(a^*_j)$, $j < i(0,\bar{k})$, contradicting condition (1). Hence $d(\bar{k}) \subseteq d(\bar{\ell})$ implies $i(0,\bar{k}) \leq i(0,\bar{\ell})$.

So choose $\bar{k}$ with minimal $i(0,\bar{k})$; as $d(\bar{k})$ is infinite, there are $\bar{k}^1$, $\bar{k}^2$ such that $d(\bar{k}) = d(\bar{k}^1) \cup d(\bar{k}^2)$, $d(\bar{k}^1) \cap d(\bar{k}^2) = \emptyset$, and $n \in d(\bar{k}^1) \to k^1_n = k_n$, $n \in d(\bar{k}^2) \to k^2_n = k_n$.

By the previous statement and the choice of $\bar{k}$, $i(0,\bar{k}) = i(0,\bar{k}^1) = i(0,\bar{k}^2)$. By condition (1) necessarily $M(0,\bar{k}^1) = M(0,\bar{k}) = M(0,\bar{k}^2)$. Define $\bar{k}^*$ so that $n \in d(\bar{k}^1) \to k^*_n = k_n$, $n \in d(\bar{k}^2) \to n = -k_n$, $n \notin d(\bar{k}) \to k^*_n = 0$. Then $\bar{b}_{\bar{k}^*} = b_{\bar{k}^1} - b_{\bar{k}^2}$, hence the expression for $b_{\bar{k}^*}$ in (*) can be obtained by subtracting those of $\bar{b}_{\bar{k}^1}$, $\bar{b}_{\bar{k}^2}$. Then $a^*_{i(0,k)}$ vanishes, and we get a contradiction to the choice of $\bar{k}$.

Hence for some $\bar{k}$, $b_{\bar{k}} \notin PC[G \cup \{a_i : i < \alpha\}]$, and we define $b^*_\alpha = b_{\bar{k}}$.

As h_α is into a reduced group, K_α is closed in H's topology (but is not necessarily a pure subgroup), hence K_α is closed in its (p-adic) topology hence it is torsion-complete. Remember $p^m K_\alpha$ has power $\geq \lambda$ for every m. Let $a \in I_\alpha$ be such that $h_\alpha(a) = b^*_\alpha$, and let its exponent be m. So by claim 1.3, as in Case I, we can find $a' \in K$ such that $d_*(a') = d_m(a')$ is not almost contained in a finite union of $d(a^*_i)$, $d(b^*_j)$, $d(a)$ $(i < \alpha,\ j \leq \alpha)$ and let $a^*_\alpha = a' + a$.

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Case III: Neither case I, nor case II, but $|p^m K_\alpha| > 2^{\aleph_0}$ for every m

So $p^m I_\alpha$ has cardinality $\geq \lambda$ for every m, and for some m(*) $p^{m(*)}K_\alpha$ has cardinality $<\lambda$, and as mentioned before it is torsion complete; hence $K_\alpha = K^1 \oplus K^2$, K^1 bounded, K^2 torsion-complete, unbounded and of power $< \lambda$, but $> 2^{\aleph_0}$. The situation is dual to that of Case II, the kernel and image interchanging roles. Using claim 1.3 twice we get $a_i \in I_\alpha$ $(i < \mu)$ such that a_i has exponent m+1, $h_\alpha(a_i)$ has exponent 1, and no $d_*(a_i) = d_m(a_i)$ is almost included in a finite union of $d(a_j)$ $j \neq i$; and similarly for the $d(h_\alpha(a_i)) = d_*(h_\alpha(a_i))$ and $d(a_i), d(h_\alpha(a_i))$ are disjoint to $d(b)$, $b \in K^2$. Hence there is $a \in \{a_i : i < \mu\}$ so that $d_*(a)$, $d_*(h_\alpha(a))$ are not almost included in a finite union of $d(a_i^*)$, $d(b_i^*)$ $(i < \alpha)$. We let $b_\alpha^* = h_\alpha(a)$. Suppose we let $a_\alpha^* = a$; the only thing that can go wrong is that $b_\alpha^* \in PC[G \cup \{a_i^* : i \leq \alpha\}]$, hence, as in Case II (**) $b_\alpha^* + G = M_0 a_{i(0)}^* + \ldots + k a_\alpha^* + G$ where $\alpha > i(0) > \ldots$

Multiplying by p, $pb_\alpha^* = 0$, hence $pka_\alpha^* \in G$ hence [as $d_*(a_\alpha^*) = d_m(a_\alpha^*)$ m+1 is the exponent of a_α^*] $pka_\alpha^* = 0$, but $ka_\alpha^* \neq 0$. So $k = p^m k_1$, k_1 not divisible by p.

As in the definition of the y_n's in Case II, we can find $w_n \in K^2$ of order n+1, $d(w_n)$ pairwise disjoint; $k_n, n < \omega$; and $J \subseteq \omega - \{0,\ldots,m\}$ so that $\sum_{n \in J} k_n p^n w_n \notin PC[G \cup \{a_i^* : i < \alpha\}]$ and let $a_\alpha^* = a + \sum_{n \in J} p^{n-m} w_n$. Then $h_\alpha(a_\alpha^*) = b_\alpha^*$, and suppose we get (**) again, with $M'_\ell, i'(0), \ldots, k'$. Then as $d(a)$, $d(h(a))$ are disjoint to $d(\sum_{n \in J} p^{n-m} w_n)$, $k = k'$. Subtracting the equations we get a contradiction to the definition of the w_n's. So in any case we can define a_α^*.

Case IV: Not cases I, II, III, but $h_\alpha | I_\alpha$ is not semi-simple.

Let G^* be the smallest subgroup of H such that $PC(G^*) = G^*$,

$G \subseteq G^*$, $a^*_i, b^*_i \in G^*$ $(i < \alpha)$; $a_i \in G^*$, $d(b) \subseteq \bigcup_{i=1}^{n} d(a_i)$ implies $b \in G^*$ and for some m $p^m \mathrm{Ker}(h_\alpha | I_\alpha) \subseteq G^*$ (as not case III, $p^m \mathrm{Ker}(h_\alpha | I_\alpha) \leq 2^{\aleph_0}$ for some m) and $a \in G^* \Longrightarrow h_\alpha(a) \in G^*$ for $a \in G^* + I_\alpha$. Clearly the power of G is $\lambda + |\alpha|$. Let $I^* = G^* + I_\alpha$; for every $a \in I^* - G^*$ there are no $a_i \in G^*$ such that $d(a) \subseteq \bigcup_{i=1}^{n} d(a_i)$. We can find $a' \in a+G^*$ so that $a + G^*$, a' have the same order. Hence $d_*(a') \not\subseteq \bigcup_{i=1}^{n} d(a_i)$ for any $a_i \in G^*$. Notice that $h_\alpha(a) + G^* = h_\alpha(a') + G^*$. If for some such a we let $a^*_\alpha = a'$, $b^*_\alpha = h_\alpha(a')$, the only thing that can go wrong is that $h_\alpha(a') \in PC[G^* \cup \{a'\}]$, hence for some rational $r = r_a$, $h_\alpha(a') - ra \in G^*$ hence $h_\alpha(a) - ra \in G^*$; and for $a \in G^*$ we let $r_a = 0$; and r is a p-adic integer as the order of $h_\alpha(a') + G^*$ is $\leq$ the order of $a' + G$. For the same reason if r, r' are suitable r_a's then $r - r'$ is divisible by p^m where m is the exponent of $h(a') + G^*$ (divisibility among the p-adic integers). So if we choose the minimal integer r_a, r_a is defined uniquely.

If $b = p^\ell a$, $r_b - r_a$ is divisible by $p^{m-\ell}$. Also if $b = ra$ r, an integer, $(r,p) = 1$ then $r_a = r_b$. If also $h(b)+G^*$ has exponent $m+1$, $b \in I^*$, then $h(a-b) - (r_a a - r_b b) + G^* = 0$ and $h(a-b) - r_{a-b}(a-b) + G^* = 0$ hence $(r_a - r_{a-b})a + G^* = (r_b - r_{a-b})b + G^*$.

If we choose b so that $d(a)$, $d(b)$ are almost disjoint this implies that p^m divides $r_a - r_{a-b}$ and $r_b - r_{a-b}$; hence it divides $r_a - r_b$, hence $r_a = r_b$. As for every such a, b, there is $c \in I^*$ such that $h(c) + G^*$ has exponent m, and $d(c)$ is almost disjoint from $d(a)$ and $d(b)$, then $r_a = r_c = r_b$.

Combining we get a p-adic integer r such that $h_\alpha(a) - ra \in G^*$ for $a \in I^*$ so if $h^*(x) = rx$ then $h = (h_\alpha - h^*) | I_\alpha$ has range of power $< \mu$. By assumption h is not semi-zero-like hence as in case II, for some $b = h(a)$, $a \in I_\alpha$,

$b \notin PC[G \cup \{a^*_i: i < \alpha\}]$, $pb = 0$. Also there is $x \in I_\alpha \cap \ker(h)$, $d_*(x) = d_*(x+a)$, and x is not almost included in a finite union of $d(a^*_i)$, $d(b^*_i)$ $(i < \alpha)$ and $d(x)$ is disjoint to $d(a)$, $d(b)$.

Then let $a^*_\alpha = a+x$, $b^*_\alpha = h_\alpha(a+x) = h_\alpha(a) + h_\alpha(x) = (b+h^*(a)+h^*(x) = b+ra^*_\alpha$

The only thing that can go wrong is $b^*_\alpha \in PC[G \cup \{a^*_i, i < \alpha\} \cup \{a^*_\alpha\}]$ i.e. $p^\ell b^*_\alpha + G = Ma^*_\alpha + .. + M_i a^*_{\alpha(i)} + ... + G$ $(\alpha(j) < \alpha)$ where $d(p^\ell b^*_\alpha)$, $d(b^*_\alpha)$ are equal up to a finite set. Using $d_*(x)$ we see that necessarily $p^\ell rx = Mx$; hence we get a contradiction to the definition of b, and x.

Case V Not any of the previous cases.

So $h_\alpha | I_\alpha$ is semi-simple; let $h_\alpha | I_\alpha = h^1 + h^2$, h^1 semi-zero-like, $h^2(x) = rx$, r a p-adic integer. Let $r = p^m r_1$, r_1 a p-adic unit. As in Case I, by claim 1.3, we can find $\{a_i: i < \mu\} \subseteq I_\alpha$ such that $d_*(a_i) = d_m(a_i)$ is not almost included in a finite union of $d(a_j)$ $(j \neq i)$, $d(a^*_j)$, $d(b^*_j)$, $(j < \alpha)$. The image of h^1 is of cardinality $\leq \lambda$, so w.l.o.g. $h^1(a_i)$ does not depend on i. So $a^*_\alpha = a_3 - a_2$, $b^*_\alpha = h_\alpha(a^*_\alpha)$ will satisfy our demands.

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We have defined, for $J \subseteq \mu$, $G(J) = PC[G \cup \{a^*_i: i \in J\}]$. It is easy to check that $b^*_i \notin G(J)$ when $h_\alpha | I_\alpha$ is not semi-simple; and $a^*_j \in G(J)$ iff $j \in J$. Suppose $J^* \subseteq J_0, J_1$ where $J^* = \{a: \text{ not case V}\}$ and $h: G(J_0) \to G(J_1)$ is a homomorphism, so, for some α $I_\alpha = H$, $h_\alpha | G(J_0) = h$ and so $h(a^*_\alpha) \neq b^*_\alpha$. As in cases II, III, IV equality holds, h_α is zero-like (case I), or simple (case V.) In the second case it is easy to check this implies $J_0 \subseteq J_1$. So if $\{J_i: i < 2^\mu\}$ is a family of subsets J of μ, $J^* \subseteq J$, no one included in the other, then $\{G(J_i) : i < 2^\mu\}$ is the required family in (1.2) (A).

For 1.2(B) it suffices to choose $\{J_i : i < 2^\mu\}$ so that if $I \subseteq H$ is a closed pure subgroup, $p^m I$ has power $\geq \lambda$ for every m, $h: I \to H$ is a simple homomorphism, $i \neq j < 2^\mu$, then for some α $I_\alpha = I$, $h = h_\alpha | I_\alpha$, $\alpha \in J_i$, $\alpha \notin J_j$, [so by $h(a^*_\alpha)$ we get the non-existence of the homomorphism h from $I \cap G(J_i)$ into $G(J_j)$]. This can be easily done as in the list $\{(I_\alpha, h_\alpha): \alpha < \mu\}$ each pair appears μ times.

Conclusion 1.4: For $\mu = \lambda^{\aleph_0} = 2^\lambda > 2^{\aleph_0}$ there is a family of 2^μ groups each of cardinality μ, such that any homomorphism from one member to another has range of cardinality $\leq \lambda$.

A complement to 1.1(3) is:

Claim 1.5: Suppose $|p^m G| = \lambda$ for every m, but $\lambda^{\aleph_0} > \lambda \geq 2^{\aleph_0}$. Then $G = G_1 \oplus G_2$ where for every $m < \omega$, $\ell = 1,2$ $|p^m G_\ell|^{\aleph_0} \geq \lambda$. (So G_1, G_2 are unbounded, hence G is essentially decomposable).

Proof: Let $\bigoplus_{\substack{i<\beta(n)\\ n<\omega}} \langle x^n_i \rangle$ be a basic subgroup of G, where x^n_i has exponent $n + 1$. Let $\lambda_0 = \min_n \sum_{m \geq n} |\beta(n)|$, so by the assumption $\lambda_0^{\aleph_0} = \lambda^{\aleph_0}$. Define for $a \in G$, $d(a)$ as in the proof of 1.2. We can easily find $Y_j \subseteq \{x^n_i : n < \omega,\ i < \beta(n)\}$ for $j < \lambda_0^{\aleph_0}$ such that $\min_n \sum_{m \geq n} |Y_j \cap \{x^m_i : i < \beta(m)\}|$ is λ_0, and $\alpha \neq \beta$ implies $Y_\alpha \cap Y_\beta$ is a subset of $\{x^m_i : i < \beta(m),\ m < n\}$ for some n. As for each $a \in G$, the number of α's such that $Y_\alpha \cap d(a)$ is infinite, is $\leq 2^{\aleph_0}$ (by the "almost-disjointness" of the Y_α's), clearly

$$\sum_\alpha |\{a: d(a) \cap Y_\alpha \text{ is infinite}\}| \leq |G| + 2^{\aleph_0} < \lambda^{\aleph_0}$$

As the number of α's is $\lambda^{\aleph_0}$, for some α for no $a \in G$ is $d(a) \cap Y_\alpha$ infinite. So let G_1 be the closure in G of the subgroup generated by Y_α, and G_2 be the closure in G of the subgroup generated by $\{x^n_i : i < \beta(n),\ n < \omega,\ x^n_i \notin Y_\alpha\}$. Clearly $G = G_1 \oplus G_2$, $|p^m G_\ell| \geq \lambda_0$.

2. Large rigid-like systems

A group means an (abelian) reduced separable p-group.

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Theorem 2.1: Suppose μ is a strong limit cardinal of cofinality $> \aleph_0$. Then there is a rigid-like system of 2^μ groups each of cardinality μ.

Remark: Here we elaborate less than §1.

Proof: Let G be $\oplus\langle x_i^n\rangle$, $i < \mu$, $n < \omega$, where x_i^n has exponent $n + 1$, and H the torsion completion of G. We use the notation of Th. 1.2, and in addition $G_\alpha = \bigoplus_{\substack{i<\alpha\\ n<\omega}} \langle x_i^n\rangle$, H_α the torsion completion of G_α, and $H_\delta^* = \bigcup_{\alpha<\delta} H_\alpha$, $X_n^\alpha = \{x_i^n: i < \alpha\}$, $X^\alpha = \bigcup_{n<\omega} X_n^\alpha$.

We define $a_\alpha^\lambda, b_\alpha^\lambda \in H$ for $\lambda < \mu$ a strong limit cardinal of cofinality $\aleph_0$, $\alpha < 2^\lambda$, so that

(1) $a_\alpha^\lambda, b_\alpha^\lambda \in H_\lambda - H_\lambda^*$

(2) $b_\alpha^\lambda \notin PC[G_\lambda \cup \{a_\beta^\kappa: (\kappa,\beta) < (\lambda,\alpha)\}]$ where $(\kappa,\beta) < (\lambda,\alpha)$ means $\kappa < \lambda$ or $\kappa = \lambda, \beta < \alpha$.

(3) The intersection of $d_*(a_\alpha^\lambda)$ with any $d(a_\beta^\kappa), d(b_\beta^\kappa)$, $(\kappa,\beta) < (\lambda,\alpha)$ is included in some X^γ, $\gamma < \lambda$.

(4) $d_*(a_\alpha^\lambda)$ is not almost included in any X_γ, $\gamma < \lambda$.

We define them by induction on λ, so suppose $a_\alpha^\kappa, b_\alpha^\kappa$ were defined for $\kappa < \lambda$. Let $\{(I_\alpha^\lambda, h_\alpha^\lambda): \alpha < 2^\lambda\}$ be a list of all pairs of closed pure subgroups I of H_λ, and a homomorphism $h : I \to H_\lambda$, each appearing 2^λ times. Now we define by induction on α, as follows. If there are $a_\alpha^\lambda \in I_\alpha^\lambda$, and $b_\alpha^\lambda = h_\alpha^\lambda(a_\alpha^\lambda)$ which satisfy conditions (1), (2), (3), (4) and $b_\alpha^\lambda \notin PC[G \cup \{a_\beta^\kappa: (\kappa,\beta) \leq (\lambda,\alpha)\}]$ we choose them in this way. If not but there are $a_\alpha^\lambda \in I_\alpha$, $b_\alpha^\lambda = h_\alpha^\lambda(a_\alpha)$ which satisfy conditions (1) - (4) we choose them in this way. Otherwise we choose $a_\alpha^\lambda, b_\alpha^\lambda$ so that $a_\alpha^\lambda, b_\alpha^\lambda$ are almost disjoint, and almost disjoint with any $a_\beta^\kappa, b_\beta^\kappa$ $(\kappa,\beta) < (\lambda,\alpha)$ and (1) - (4) are satisfied. So we have three possibilities which we denote respectively by A, B, C. Let $J_0^* = \{(\lambda,\alpha)$: possibility A or B holds$\}$ and $J^* = \{(\lambda,\alpha)$: possibility A holds$\}$.

Let $G(J) = PC[G \cup \{a_\alpha^\lambda: (\lambda,\alpha) \in J\}]$. Suppose $J^* \subseteq J_1, J_2 \subseteq J_0^*$

and , h: $G(J_1) \to G(J_2)$ a non-zero-like homomorphism. Let h^c be the unique extension of h to a homomorphism $h^c: H \to H$. We shall show that h is simple and that for arbitrarily large $\lambda < \mu$ there is α, $h^\lambda_\alpha = h^c | I^\lambda_\alpha$ and for (λ,α) possibility B holds, hence

$$(I^\lambda_\alpha, h^\lambda_\alpha) = (I^\lambda_\beta, h^\lambda_\beta), (\lambda,\beta) \in J_1 \Rightarrow (\lambda,\beta) \in J_2$$

From this it will be easy to draw our conclusion, as in 1.2.

Notice that by conditions (1) - (4), $b^\lambda_\alpha \in G(J_\ell)$, only if (λ,α) satisfies possibility B and $(\lambda,\alpha) \in J$. Also $a^\lambda_\alpha \in G(J_\ell)$ iff $(\lambda,\alpha) \in J_\ell$ (for $\ell = 1,2$).

But first we prove:

<u>Claim 2.2</u>: Assume $I' \subseteq H$ is a closed subgroup, $h': I' \to H$ a non-zero-like homomorphism. Then for some $a \in I', h'(a) \notin PC(G \cup \{a^\kappa_\beta: (\kappa,\beta) < (\mu,\mu)\})$ and $h'(a)$ has exponent 1.

<u>Proof</u>: As h' is not zero-like, there are $m < \omega$ and $z_n \in I'$ such that z_n has order $n + m$, and $y_n = h'(z_n)$ has order $n + 1$. Let $\zeta(n) = \min\{\zeta: d(p^n y_n) \subseteq X^\zeta\}$; as the $\zeta(n)$ are ordinals there are $n_0 < n_1 < \dots$ such that $\zeta(n_0) < \zeta(n_1) < \dots$ or $\zeta(n_0) = \zeta(n_1) = \dots$ As we can replace y_{n_ℓ} by $p^{n_\ell - \ell} y_{n_\ell}$, we can assume $n_\ell = \ell$. Let $\delta = \sup_n \zeta(n)$. As $d(p^n y_n)$ is countable each $\zeta(n)$, hence δ has cofinality ω or is a successor. Define δ_n so that if $\mathrm{cf}\ \delta = \omega$ then $\delta_0 < \delta_1 < \dots$, $\delta = \bigcup_{n<\omega} \delta_n$ and $\delta_\ell < \zeta(\ell)$; and if δ is a successor $\delta_0 = \delta_1 = \dots$; $\delta = \delta_0 + 1$; and clearly then $\zeta(n) = \delta$, hence $\delta_n < \zeta(n)$. We now define inductively $n(\ell)$, $t_\ell \notin d[p^{n(\ell)} y_{n(\ell)}]$ such that $n(\ell) < n(\ell+1)$, and when $\ell < m$, $t_\ell \in d(p^{n(m)} y_{n(m)})$ and $t_\ell \notin X^{\delta_\ell}$. Let $n(0) = 0$, and suitable t_0 exists as $\delta_n < \zeta(n)$. If we have defined for ℓ, let $n(\ell+1)$ be greater than the orders of $t_0,\dots,t_\ell$, and than $n(\ell)$; hence $t_0,\dots,t_\ell \notin d(p^{n(\ell+1)} y_{n(\ell+1)})$, and we can choose $t_{\ell+1} \in d(p^{n(\ell+1)} y_{n(\ell+1)})$, $t_{\ell+1} \notin X^{\delta_{\ell+1}}$ as $\delta_{\ell+1} < \zeta(n(\ell+1))$. By renaming assume $n(\ell) = \ell$. As we can replace y_n

by $y_n + \sum_{\omega>\ell\geq n+1} k_\ell p^{\ell-n} y_\ell$, we can assume $t_m \in d(p^n y_n)$ iff $n = m$. For $J \subseteq \omega$ let $b_J = \sum_{n\in J} p^n y_n$; then $d(b_J) \subseteq \bigcup_n d(p^n y_n)$ hence $d(b_J) \subseteq X^\delta$, and $t_n \in d(b_J)$ iff $n \in J$. Also $b_J \in$ Range h' $[b_J = h'(\Sigma p^n z_n)]$. We shall show that for some J $b_J \notin PC[G \cup \{a^\kappa_\gamma : (\kappa,\gamma) < (\mu,\mu)\}]$. If δ is a successor every infinite J suffices, by the induction assumptions. If δ has cofinality ω, and $J = \omega$ is not sufficient, necessarily δ is a strong limit cardinal of cofinality ω; and for some $w \in \{a^\delta_\gamma : \gamma < 2^\delta\}$, $d = d_*(w) \cap \{t_n : n < \omega\}$ is infinite, and then any J such that $d \cap \{t_n : n \in J\}$, $d \cap \{t_n : n \notin J\}$ are infinite will suffice. As we mentioned $b_J \in$ Range h', so for some $a \in I'$, $h'(a) = b_J$. So we have proved the claim; and we continue with $h: G(J_1) \to G(J_2)$.

<u>Observation 1:</u> If h^* is a zero-like homomorphism, $h^*: H \to H$, then for every m p^mRange$(h^c - h^*)$ has cardinality μ.

For suppose $m = m(*)$ is a counter example, let $h^1 = h^c - h^*$. By the claim 2.2 there is an a such that

$$h^c(a) \notin PC(G \cup \{a^\kappa_\beta : \kappa < \mu,\ \beta < 2^\kappa\})$$

$h^c(a)$ has exponent 1, a has exponent $n(*)$.

We define by induction, strong limit cardinals of cofinality $\aleph_0$, $\lambda_n < \lambda_{n+1} < \mu$, $a \in H_{\lambda_0}$, $\lambda_0 > |p^{m(*)}$ Range $h^1|$; and sets $Y_n \subseteq X^{\lambda_{n+1}}_{n+1}$, $Y_n \cap d(a) = \emptyset$ such that h^1 is constant on $p^{m(*)}$ Y_n, and $|Y_n| = (\lambda_n^{\aleph_0})^+$, and for $y \in Y_n$, $h^1(y) \in H_{\lambda_{n+1}}$. (This is easy to do.) We can also assume, as in 1.3, that for distinct $y_\ell \in Y_n$, $h^*(y_1) - h^*(y_2)$, $h^*(y_3) - h^*(y_4)$ are disjoint, and $\subseteq X^{\lambda_{n+1}}$ and disjoint to X^{λ_n}.

Let $\lambda = \sum_n \lambda_n$, so λ too is a strong limit cardinal of cofinality $\aleph_0$, Let I' be the torsion completion of $PC(\{a\} \cup \bigcup_n Y_n)$ in H and $h' = h^1 | I'$. Clearly $h': I' \to H$, and for some α, $(I', h') = (I^\lambda_\alpha, h^\lambda_\alpha)$. We can find $n(0) < \omega$

$n(0) \geq n(*) + m(*)$ such that $h^*(p^{n-n(*)}y) = 0$ for $n \geq n(0)$, $y \in Y_n$ and for $i < 2^\lambda$, $y^i = \sum_{n(0)<n<\omega} p^{n-n(*)} (y_n^i - z_n^i)$; $y_n^i \neq z_n^i \in Y_n$, and the y^i's are pairwise almost disjoint. Clearly $h^1(y^i) = 0$, and except for $< 2^\lambda$ i's, y^i is almost disjoint from a_β^κ, b_β^κ for $(\kappa,\beta) < (\lambda,\alpha)$ and to a, $h^c(a)$. Notice $h^c(a + y^i) = h^c(a) + h^1(y^i) = h^c(a)$. It is now easy to check that $h^c(a + y^i) \notin PC[G_\lambda \cup \{a_\beta^\kappa: (\kappa,\beta) < (\lambda,\alpha)\} \cup \{a + y^i\}]$ hence for (λ,α) possibility A occurs, hence we get a contradiction to $h: G(J_1) \to G(J_2)$.

Observation II: Suppose h is not simple but h_0 is simple, $h' = h - h_0$; and $y_i^n \in PC(X_n)$ for $i < \mu$ are pairwise disjoint and $\neq 0$; and the exponent of y_i^n is $n + 1$. Then there are $\ell = \ell_0$ and n_0 such that for every $n \geq n_0$, $S \subseteq \mu$ of cardinality μ; the set $\{p^{n-\ell} h'(y_i^n): i \in S\}$ has cardinality μ.

Otherwise we can find $n(\ell)$ $\ell < \omega$ and $S_\ell \subseteq \mu$, $|S_\ell| = \mu$ such that $n(\ell) < n(\ell+1)$ and $|\{p^{n(\ell)-\ell} h'(y_i^n);\ i \in S_\ell\}| < \mu$. Let $h_0 = h_1 + h_2$, h_2 zero-like, $h_1(x) = rx$, r a p-adic integer.

The rest is like the proof of observation I noticing that for every $c \in H_\lambda$ $h^c(c) \in PC[G_\lambda \cup \{a_\beta^\kappa: (\kappa,\beta) < (\lambda,\alpha)\} \cup \{c\}]$ iff $(h^c - h_1)(c) \in PC[G_\lambda \cup \{a_\beta^\kappa: (\kappa,\beta) < (\lambda,\alpha)\} \cup \{c\}]$.

Observation III: We can find an increasing continuous sequence of strong limit cardinals $\lambda_\alpha < \mu$ $\alpha < cf\ \mu$, (say $\lambda_0 = 0$) and $y_i^n \in G$ for $i < \mu$ such that:

(i) y_i^n is $x_{j(i,1)}^n - x_{j(i,2)}^n$

(ii) the y_i^n are pairwise disjoint

(iii) if $\lambda_\alpha \leq i < \lambda_{\alpha+1}$ then $d(h(y_i^n)) \subseteq X^{\lambda_{\alpha+2}}$, but is disjoint to $X^{\lambda_{\alpha+1}}$

(iv) the $h(y_i^n)$ are pairwise disjoint.

The proof is by induction, using a theorem of Erdos-Rado [ER].

* * * * *

For each y_i^n choose a maximal $\ell = \ell(n,i)$ and a rational p-adic integer r_i^n such that $p^{n-\ell}(h(y_i^n) - r_i^n y_i^n) = 0$. (ℓ may be -1, so it is always defined). As the number of possible ℓ's is $\aleph_0$, $\mathrm{cf}\mu > \aleph_0$, we can assume $\ell(n,i) = \ell(n)$; similarly $r_i^n = r_n$.

Observation IV: For every ℓ there is a k_ℓ such that $n, m \geq k_\ell \Rightarrow r_n - r_m$ is divisible by p^ℓ. So for some p-adic integer r, $r_n - r$ is divisible by p^ℓ for $n \geq k_\ell$. Also $\lim_{n\to\infty} \ell(n) = \infty$.

Let ℓ be given. Let $\lambda = \lambda_\omega$ and I be the torsion completion of $PC(\{y_i^n : i < \lambda_\omega, n < \omega\})$ and $\alpha < 2^\lambda$ be such that $(I_\alpha^\lambda, h_\alpha^\lambda) = (I, h^c| \; I)$. Clearly we can find $i(n) < \lambda$ such that $y^\ell = \sum_{\ell \leq n} p^{n-\ell} y_{i(n)}^n$ satisfies the following: if $(\kappa,\beta) < (\lambda,\alpha)$ then for some m, for every $n \geq m$, $p^{n-\ell} y_{i(n)}^n$, $p^{n-\ell} h(y_{i(n)}^n)$ are disjoint to a_β^κ. As (λ,α) is not in possibility A , $h^c(y^\ell) \in PC[G_\lambda \cup \{a_\beta^\kappa : (\kappa,\beta) < (\lambda,\alpha) \cup \{y^\ell\}]$ hence there is an m such that $d(h^c(y^\ell)) = d(p^m h^c(y^\ell))$ and

$$p^m h^c(y^\ell) = My^\ell + M_{i(0)}^{\kappa(0)} a_{i(0)}^{\kappa(0)} + \ldots + M_{i(n)}^{\kappa(n)} a_{i(n)}^{\kappa(n)} + c$$

$c \in G_\lambda$, $(\kappa(0), i(0)), \ldots,$ are $< (\lambda,\alpha)$. Then clearly $M' = M/p^m$ is an integer, and for some k_ℓ for every $k \geq k_\ell$ $h(p^{k-\ell} y_{i(k)}^k) = M' p^{k-\ell} y_{i(k)}^k$ hence $\ell(k) \geq \ell$ and $M' p^{k-\ell} y_{i(k)}^k = p^{k-\ell} r_k y_{i(k)}^k$ hence $(M' - r_k)$ is divisible by p^ℓ. Clearly we have proved the observation.

* * *

So for every ℓ for every n big enough $p^{n-\ell}(h(y_i^n) - r y_i^n) = 0$, so the only way not to contradict II is to assume

Observation V: h is simple.

* * *

So let $h = h_1 + h_2$, $h_1(x) = rx$, h_2 is zero-like. Let $r = p^k r'$, r' a p-adic unit. Hence for some $n_0 < \omega$ for every $n \geq n_0$ $h_2(y_i^n)$ has order $< n-k$. The only thing left to be proved is that if $\lambda^0 < \mu$ there are λ, $\lambda^0 < \lambda < \mu$ and $\alpha < 2^\lambda$ such that $h^\lambda_\alpha = h^c | I^\lambda_\alpha$ and (λ,α) satisfies possibility B. Choose $\lambda = \lambda_\ell > \lambda^0$, δ a limit ordinal and α such that $I^\lambda_\alpha = PC(\{y^n_i : i < \lambda, \ n < \omega\}]$ $h^\lambda_\alpha = h^c | I^\lambda_\alpha$. By the assumption on h, the pair (λ,α) does not satisfy possibility A. Now we can find $y = \sum_{n_0 \leq n} p^{n-k} y^n_{i(n)}$ which is almost disjoint to a^κ_β $(\kappa,\beta) < (\lambda,\alpha)$. Clearly $h_2(y) = 0$, $d(h_1(y) = d(y)$, so it proves (λ,α) satisfies possibility B.

So we have finished the proof of 2.1.

Conclusion 2.3: There are arbitrarily large essentially indecomposable groups. (This answers a question of Pierce [P], which is [Fu. 2], Pr.55, p. 55.)

Remark: We can prove 2.1 also for $\mu = \aleph_{\alpha+n}$ where $\aleph_\alpha$ is a strong limit cardinal of cofinality $> \aleph_0$ and $0 \leq n < \omega$ or $\aleph_\alpha = \lambda^{\aleph_0} = 2^\lambda$, $0 < n < \omega$.

Moreover, if G, H are members of the family, I a closed subgroup of G, $|p^k I| \geq \aleph_{\alpha+n}$ for every k, $h: I \to H$ a homomorphism then $G = H$, and h is simple. (The purity of I is not needed).

Question: Can we prove 2.1 when $\mu = \lambda^{\aleph_0} = 2^\lambda$? (assuming G.C.H this is the only open case). Can we prove 1.2 when $\mu = 2^{\aleph_0}$?

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Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem, Israel