

The last forcing standing with diamonds

by

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Abstract. This article continues our previous work [Int. J. Math. Math. Sci. 28 (2001), Arch. Math. Logic 52 (2013)]. We introduce a new property of $(<\lambda)$ -strategically complete forcing notions which implies that their λ -support iterations do not collapse λ^+ .

1. Introduction. While there are still a lot of open problems left in the theory of forcing iterated with finite and/or countable supports and we still need to expand our preservation theorems, there is a feeling that we *understand these iterations pretty well*. Therefore it is natural to look at iterations with uncountable supports and ask for parallel theorems. The first attempt could be to do nothing special and just repeat what has been done for CS iterations. We could start in the way suggested already in Shelah [21] (but not used there).

DEFINITION 1.1. Let $\lambda = \lambda^{<\lambda}$. A notion of forcing \mathbb{P} is said to be λ -proper in the standard sense (or just λ -proper) if for all sufficiently large regular cardinals χ , there is some $x \in \mathcal{H}(\chi)$ such that whenever M is an elementary submodel of $\mathcal{H}(\chi)$ satisfying

$$|M| = \lambda, \quad \mathbb{P}, x \in M, \quad M^{<\lambda} \subseteq M,$$

and p is an element of $M \cap \mathbb{P}$, then there is a condition $q \geq p$ such that

$$q \Vdash "M[G_{\mathbb{P}}] \cap \text{Ord} = M \cap \text{Ord}".$$

The λ -properness may seem to be a straightforward generalization of “proper”, with the right consequences in place. For instance:

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THEOREM 1.2 (Folklore; cf. Hyttinen and Rautila [8, Section 3]). *Assume $\lambda^{<\lambda} = \lambda$ is an uncountable cardinal.*

- (1) *If a forcing notion \mathbb{P} is either strategically $(\leq\lambda)$ -complete or it satisfies the λ^+ -chain condition, then \mathbb{P} is λ -proper.*
- (2) *If \mathbb{P} is λ -proper, $p \in \mathbb{P}$, \underline{A} is a \mathbb{P} -name for a set of ordinals and $p \Vdash |\underline{A}| \leq \lambda$, then there are a condition $q \in \mathbb{P}$ stronger than p and a set B of size λ such that $q \Vdash \underline{A} \subseteq B$.*
- (3) *If \mathbb{P} is λ -proper, then*

$$\Vdash_{\mathbb{P}} \text{“}(\lambda^+)^{\mathbf{V}} \text{ is a regular cardinal”}.$$

Moreover, if \mathbb{P} is also strategically $(<\lambda)$ -complete, then the forcing with \mathbb{P} preserves stationary subsets of λ^+ .

Also chain condition results look similarly:

THEOREM 1.3 (Folklore; cf. Eisworth [3, Proposition 3.1] and Abraham [1, Theorems 2.10, 2.12]). *Assume $\lambda^{<\lambda} = \lambda$, $2^\lambda = \lambda^+$, and let $\bar{\mathbb{P}} = \langle \mathbb{P}_i, \underline{Q}_i : i < \lambda^{++} \rangle$ be a λ -support iteration such that*

- (a) *\mathbb{P}_i is λ -proper for $i \leq \lambda^{++}$, and*
- (b) *$\Vdash_{\mathbb{P}_i} \text{“}|\underline{Q}_i| \leq \lambda^+ \text{”}.$*

Then

- (1) *$\mathbb{P}_{\lambda^{++}}$ satisfies the λ^{++} -chain condition, and*
- (2) *for each $i < \lambda^{++}$ we have $\Vdash_{\mathbb{P}_i} \text{“}2^\lambda = \lambda^+ \text{”}.$*

Proof. Abraham [1, Theorem 2.10, Theorem 2.12] gives the full proof of the theorem for CS iterations. Eisworth [3, Proposition 3.1] presents a careful justification of 1.3(1). Let us sketch arguments justifying 1.3(2).

To argue that $\Vdash_{\mathbb{P}_i} \text{“}2^\lambda = \lambda^+ \text{”}$ suppose towards a contradiction that $p \in \mathbb{P}_i$, $i < \lambda^{++}$, and $p \Vdash_{\mathbb{P}_i} \text{“}x : \lambda^{++} \rightarrow {}^\lambda 2 \text{ is one-to-one”}.$ For each $\xi < \lambda^{++}$ fix a model $M_\xi \prec \mathcal{H}(\chi)$ such that

$$\xi, \bar{\mathbb{P}}, i, p, x \in M_\xi, \quad |M_\xi| = \lambda \quad \text{and} \quad <^\lambda M_\xi \subseteq M_\xi.$$

Applying a “cleaning procedure” as in [3, proof of Proposition 3.1] we may choose a set $I \subseteq \lambda^{++}$ of size λ^{++} , a set $H \subseteq \lambda^{++}$ and mappings $\pi_{\xi,\zeta}$ (for $\xi, \zeta \in I$ satisfying $\xi < \zeta$) such that:

- (a) $\pi_{\xi,\zeta} : M_\xi \xrightarrow{\text{onto}} M_\zeta$ is an \in -isomorphism, $\pi_{\xi,\zeta}(\xi) = \zeta$, $\pi_{\xi,\zeta}(x) = x$, $\pi_{\xi,\zeta}(p) = p$,
- (b) $M_\xi \cap M_\zeta \cap \lambda^{++} = H$, $M_\xi \cap \lambda^+ = M_\zeta \cap \lambda^+ = H \cap \lambda^+$,
- (c) $H \subseteq \min(M_\xi \cap \lambda^{++} \setminus H)$ and $M_\xi \cap \lambda^{++} \subseteq \min(M_\zeta \cap \lambda^{++} \setminus H)$, and
- (d) $i \in H$, $\pi_{\xi,\zeta} \upharpoonright H = \text{id}_H$, $\lambda \subseteq H$.

The following claim will immediately complete the proof.

CLAIM 1.3.1. *If $r \in \mathbb{P}_i$ is (M_ξ, \mathbb{P}_i) -generic and $r \geq p$ and $\xi < \zeta$ are from I , then $r \Vdash_{\mathbb{P}_i} \underline{x}(\xi) = \underline{x}(\zeta)$.*

Proof of the Claim. Suppose towards a contradiction that $r^* \geq r$, $r^* \Vdash \underline{x}(\xi) \neq \underline{x}(\zeta)$. Take $r^+ \geq r^*$ and $\ell < 2$, $\alpha < \lambda$ such that

$$r^+ \Vdash \underline{x}(\xi)(\alpha) = \ell \text{ and } \underline{x}(\zeta)(\alpha) = 1 - \ell.$$

The condition r^+ is (M_ξ, \mathbb{P}_i) -generic, so there is $s \in M_\xi \cap \mathbb{P}_i$ such that $s \Vdash_{\mathbb{P}_i} \underline{x}(\xi)(\alpha) = \ell$ and r^+, s are compatible, say $q \geq r^+, s$. Then q is (M_ξ, \mathbb{P}_i) -generic, so by [3, Claim 3.4] we also have $q \geq \pi_{\xi, \zeta}(s)$. But $\pi_{\xi, \zeta}(s) \Vdash \pi_{\xi, \zeta}(\underline{x}(\xi)(\alpha))(\pi_{\xi, \zeta}(\alpha)) = \ell$, so $\pi_{\xi, \zeta}(s) \Vdash \underline{x}(\zeta)(\alpha) = \ell$. Hence $q \Vdash \underline{x}(\xi)(\alpha) = \underline{x}(\zeta)(\alpha)$, contradicting the choice of r^+ and $q \geq r^+$. ■

This completes the proof of Theorem 1.3. ■

What is missing? The main point of properness is the preservation theorem for CS iterations. If one tries to repeat the proof of the preservation theorem for λ -support iterations of λ -proper forcing notions, then one faces difficulties at limit stages of cofinality less than λ caused by the fact that it is inconvenient to diagonalize λ objects in fewer than λ steps. This is a more serious obstacle than just a technicality. Let us consider the following forcing notion.

EXAMPLE 1.4 (Shelah [22, Appendix]). Assume that $\lambda = \lambda^{<\lambda}$ is an uncountable cardinal and let $\mathcal{S}_\lambda^{\lambda^+} := \{\delta < \lambda^+ : \text{cf}(\delta) = \lambda\}$. Suppose that a sequence $\langle A_\delta, h_\delta : \delta \in \mathcal{S}_\lambda^{\lambda^+} \rangle$ is such that for each $\delta \in \mathcal{S}_\lambda^{\lambda^+}$:

- (a) $A_\delta \subseteq \delta$, $\text{otp}(A_\delta) = \lambda$ and A_δ is a club of δ , and
- (b) $h_\delta : A_\delta \rightarrow 2$.

The forcing notion $\mathbb{Q}^* = \mathbb{Q}^*(\langle A_\delta, h_\delta : \delta \in \mathcal{S}_\lambda^{\lambda^+} \rangle)$ is defined as follows: a condition in \mathbb{Q}^* is a tuple $p = (u^p, v^p, \bar{e}^p, h^p)$ such that

- (a) $u^p \in [\lambda^+]^{<\lambda}$, $v^p \subseteq \mathcal{S}_\lambda^{\lambda^+} \cap u^p$,
- (b) $\bar{e}^p = \langle e_\delta^p : \delta \in v^p \rangle$, where each e_δ^p is a closed bounded non-empty subset of A_δ , and $e_\delta^p \subseteq u^p$,
- (c) if $\delta \in v^p$, then $\max(e_\delta^p) = \sup(u^p \cap \delta) > \sup(v^p \cap \delta)$,
- (d) $h^p : u^p \rightarrow 2$ is such that for each $\delta \in v^p$ we have $h^p \upharpoonright e_\delta^p \subseteq h_\delta$.

The order \leq of \mathbb{Q}^* is such that $p \leq q$ if and only if $u^p \subseteq u^q$, $h^p \subseteq h^q$, $v^p \subseteq v^q$, and for each $\delta \in v^p$ the set e_δ^q is an end-extension of e_δ^p .

Plainly, under the assumptions of 1.4, the following holds true.

OBSERVATION 1.5. *The forcing notion \mathbb{Q}^* is $(<\lambda)$ -complete and $|\mathbb{Q}^*| = \lambda^+$. It satisfies the λ^+ -chain condition, so it is also λ -proper.*

If $\lambda = \lambda^{<\lambda}$ is not inaccessible, $2^{\lambda^+} = \lambda^{++}$ and $2^\lambda = \lambda^+$, then some λ -support iterations of forcing notions like \mathbb{Q}^* are not λ -proper, as a matter

of fact this bad effect happens quite often. The problem comes from the fact that if λ -support iterations of forcings of type \mathbb{Q}^* were λ -proper, we could use Theorem 1.3 and a suitable bookkeeping device to build a forcing notion which forces “ $\lambda = \lambda^{<\lambda}$ is not inaccessible and uniformization for continuous ladder systems holds true”. However, this is not possible:

THEOREM 1.6 (Shelah [22], [23, Appendix, Theorem 3.6(2)]). *Assume $\theta < \lambda = \text{cf}(\lambda)$, $2^\theta = 2^{<\lambda} = \lambda$. Furthermore suppose that for each $\delta \in \mathcal{S}_\lambda^{\lambda^+}$ we have a club A_δ of δ . Then we can find a sequence $\langle d_\delta : \delta \in \mathcal{S}_\lambda^{\lambda^+} \rangle$ of colorings such that*

- $d_\delta : A_\delta \rightarrow 2$, and
- for any $h : \lambda^+ \rightarrow \{0, 1\}$ for stationarily many $\delta \in \mathcal{S}_\lambda^{\lambda^+}$, the set $\{i \in A_\delta : d_\delta(i) \neq h(i)\}$ is stationary in A_δ .

An explicit example of iteration of λ -proper forcings with limit collapsing λ^+ is given in [14].

Many positive results concerning non-collapsing cardinals in iterations with uncountable supports have already been presented in the literature. For instance, Kanamori [10] considered iterations of λ -Sacks forcing notion (similar to the forcing $\mathbb{Q}^{2, \bar{E}}$; see Definition 2.18 and Remark 2.19) and he proved that under some circumstances these iterations preserve λ^+ . Several conditions ensuring that λ^+ is not collapsed in λ -support iterations were introduced in Shelah [24, 25]. Fusion properties of iterations of tree-like forcing notions were used in Friedman and Zdomskyy [5] and Friedman, Honzik and Zdomskyy [4].

Numerous strong versions of λ -properness were studied by the present authors in a series of articles [15–20]. Each of those conditions was meant to be applicable to some natural forcing notions adding a new member of ${}^\lambda\lambda$ without adding new elements of ${}^{<\lambda}\lambda$. In some sense, they explained why the relevant forcings can be iterated (without collapsing cardinals).

Also Eisworth [3] gave another (simpler) relative of the preservation theorem of [15].

The properties introduced in [15] and subsequent works were typically applicable to either very bounding forcing notions or forcings with trees with splittings on a stationary co-stationary set of levels. While those properties were quite general, we had problems in including the forcing notion \mathbb{Q}_λ^2 (see 2.18 below) in that framework. In [20] we managed to formulate a suitable property and show a relevant iteration theorem covering the forcing notions we were interested in (including \mathbb{Q}_λ^2) in the case when λ is strongly inaccessible. Here we present a framework covering those forcings, including our “last forcing standing”, under a very mild demand on λ : it should admit dia-

monds ⁽¹⁾. In a sense, we generalize (and correct, see Remark 3.7) the property considered in [15]. Our property, *the purely sequential properness over semi-diamonds*, is also close to the fuzzy properness over quasi-diamonds of [17, Definition A.3.6].

Notation and terminology. Our notation is rather standard and compatible with that of classical textbooks (like Jech [9]). However, in forcing we keep the convention that *a stronger condition is the larger one*.

- (1) Ordinal numbers will be denoted by the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ , and also by ξ, i, j (with possible sub- and superscripts).

Cardinal numbers will be called κ, λ ; λ will always be assumed to be a regular uncountable cardinal such that $\lambda^{<\lambda} = \lambda$.

Also, χ will denote a sufficiently large regular cardinal; $\mathcal{H}(\chi)$ is the family of all sets hereditarily of size less than χ . Moreover, we fix a well ordering $<_{\chi}^*$ of $\mathcal{H}(\chi)$.

- (2) We will consider several games of two players. One player will be called *Generic* or *Complete*, and we will refer to this player as “she”. Her opponent will be called *Antigeneric* or *Incomplete* and will be referred to as “he”.
- (3) For a forcing notion \mathbb{P} , all \mathbb{P} -names for objects in the extension via \mathbb{P} will be denoted with a tilde below (e.g., $\tilde{\tau}, \tilde{X}$), and $\dot{G}_{\mathbb{P}}$ will stand for the canonical \mathbb{P} -name for the generic filter in \mathbb{P} . The weakest element of \mathbb{P} will be denoted by $\emptyset_{\mathbb{P}}$ (and we will always assume that there is one, and that there is no other condition equivalent to it).

By λ -support iterations we mean iterations in which domains of conditions are of size $\leq \lambda$. However, we will pretend that conditions in a λ -support iteration $\bar{\mathbb{Q}} = \langle \mathbb{P}_{\zeta}, \mathbb{Q}_{\zeta} : \zeta < \zeta^* \rangle$ are total functions on ζ^* and for $p \in \text{lim}(\bar{\mathbb{Q}})$ and $\alpha \in \zeta^* \setminus \text{dom}(p)$ we will stipulate $p(\alpha) = \emptyset_{\mathbb{Q}_{\alpha}}$.

- (4) A *filter* on λ is a non-empty family of subsets of λ closed under supersets and intersections and not containing \emptyset . A filter is $(<\kappa)$ -complete if it is closed under intersections of $<\kappa$ members. (Note: we do allow principal filters or even $\{\lambda\}$.)

For a filter D on λ , the family of all D -positive subsets of λ is called D^+ . (So $A \in D^+$ if and only if $A \subseteq \lambda$ and $A \cap B \neq \emptyset$ for all $B \in D$.) By a *normal filter* on λ we mean a proper uniform filter closed under diagonal intersections.

- (5) By a *sequence* we mean a function whose domain is a set of ordinals. For two sequences η, ν we write $\nu \triangleleft \eta$ whenever ν is a proper initial segment of η , and $\nu \trianglelefteq \eta$ when either $\nu \triangleleft \eta$ or $\nu = \eta$. The length of a sequence η is the order type of its domain and it is denoted by $\text{lh}(\eta)$.

⁽¹⁾ This should explain the title of this paper.

The set of all sequences with domain α and with values in A is denoted by ${}^\alpha A$ and we put ${}^{<\alpha} A = \bigcup_{\beta < \alpha} {}^\beta A$.

- (6) A *tree* is a \triangleleft -downward closed set of sequences. A *complete* λ -*tree* is a tree $T \subseteq {}^{<\lambda} \lambda$ such that every \triangleleft -chain of size less than λ has a \triangleleft -bound in T and for each $\eta \in T$ there is $\nu \in T$ such that $\eta \triangleleft \nu$.

Let $T \subseteq {}^{<\lambda} \lambda$ be a tree. For $\eta \in T$ we define

$$\text{succ}_T(\eta) = \{\alpha < \lambda : \eta \frown \langle \alpha \rangle \in T\}, \quad (T)_\eta = \{\nu \in T : \nu \triangleleft \eta \text{ or } \eta \trianglelefteq \nu\}.$$

We also let $\text{root}(T)$ be the shortest $\eta \in T$ such that $|\text{succ}_T(\eta)| > 1$ and $\text{lim}_\lambda(T) = \{\eta \in {}^\lambda \lambda : (\forall \alpha < \lambda)(\eta \upharpoonright \alpha \in T)\}$.

2. Preliminaries

2.1. Iterations of strategically complete forcing notions

DEFINITION 2.1. Let \mathbb{P} be a forcing notion.

- (1) For an ordinal α , let $\mathfrak{D}_0^\alpha(\mathbb{P})$ be the following game of two players, *Complete* and *Incomplete*: the game lasts at most α moves and during a play the players attempt to construct a sequence $\langle (p_i, q_i) : i < \alpha \rangle$ of pairs of conditions from \mathbb{P} in such a way that

$$(\forall j < i < \alpha)(p_j \leq q_j \leq p_i)$$

and at stage $i < \alpha$ of the game, first *Incomplete* chooses p_i and then *Complete* chooses q_i . *Complete* wins if and only if for every $i < \alpha$ there are legal moves for both players.

- (2) A winning strategy \mathbf{st}^0 of *Complete* in $\mathfrak{D}_0^\alpha(\mathbb{P})$ is *regular* if it instructs *Complete* to play $\emptyset_{\mathbb{P}}$ as long as *Incomplete* plays $\emptyset_{\mathbb{P}}$.
- (3) The forcing notion \mathbb{P} is *strategically* ($<\lambda$)-*complete* (*strategically* ($\leq\lambda$)-*complete*, respectively) if *Complete* has a winning strategy in the game $\mathfrak{D}_0^\lambda(\mathbb{P})$ ($\mathfrak{D}_0^{\lambda+1}(\mathbb{P})$, respectively). Note that then *Complete* also has a regular winning strategy.
- (4) Let a model $N \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ be such that ${}^{<\lambda} N \subseteq N$, $|N| = \lambda$ and $\mathbb{P} \in N$. We say that a condition $p \in \mathbb{P}$ is (N, \mathbb{P}) -*generic in the standard sense* (or just (N, \mathbb{P}) -*generic*) if for every \mathbb{P} -name $\tau \in N$ for an ordinal we have $p \Vdash \tau \in N$.

For the rest of this subsection we assume that:

- $\bar{\mathbb{Q}} = \langle \mathbb{P}_\zeta, \mathbb{Q}_\zeta : \zeta < \zeta^* \rangle$ is a λ -support iteration of strategically ($<\lambda$)-complete forcing notions,
- for $\xi < \zeta^*$, \mathbf{st}_ξ^0 is the $<_\chi^*$ -first \mathbb{P}_ξ -name for a regular winning strategy of *Complete* in $\mathfrak{D}_0^\lambda(\mathbb{Q}_\xi)$, and
- a model $N \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ is such that $|N| = \lambda$, ${}^{<\lambda} N \subseteq N$ and $\bar{\mathbb{Q}}, \dots \in N$.

OBSERVATION 2.2. \mathbb{P}_{ζ^*} is *strategically* ($<\lambda$)-*complete*.

OBSERVATION 2.3. Suppose that $\zeta \in (\zeta^* + 1) \cap N$, $\langle q_\alpha : \alpha < \delta \rangle \subseteq N \cap \mathbb{P}_\zeta$, $\delta < \lambda$, $r \in \mathbb{P}_\zeta$ is (N, \mathbb{P}_ζ) -generic and $q_\beta \leq r$ for all $\beta < \delta$. Then there are conditions $q \in N \cap \mathbb{P}_\zeta$ and $r^+ \in \mathbb{P}_\zeta$ such that $q \leq r^+$, $r \leq r^+$ and $(\forall \beta < \delta)(q_\beta \leq q)$.

LEMMA 2.4. Suppose that $\zeta \in (\zeta^* + 1) \cap N$ is a limit ordinal of cofinality $\text{cf}(\zeta) < \lambda$ and $r \in \mathbb{P}_\zeta$ is such that

$$(\forall \varepsilon \in \zeta \cap N)(r \upharpoonright \varepsilon \text{ is } (N, \mathbb{P}_\varepsilon)\text{-generic}).$$

Assume that conditions $q_\alpha \in N \cap \mathbb{P}_\zeta$ (for $\alpha < \delta < \lambda$) satisfy

- (a) if $\alpha < \beta < \delta$, then $q_\alpha \leq q_\beta \leq r$, and
- (b) if $\varepsilon \in N \cap \zeta$, $s \in \mathbb{P}_\varepsilon$ and $(\forall \alpha < \delta)(q_\alpha \upharpoonright \varepsilon \leq s)$, then

$$s \Vdash_{\mathbb{P}_\varepsilon} \text{“the sequence } \langle q_\alpha(\varepsilon) : \alpha < \delta \rangle \text{ has an upper bound in } \mathbb{Q}_\varepsilon \text{”}.$$

Then there are conditions $q \in N \cap \mathbb{P}_\zeta$ and $r^+ \in \mathbb{P}_\zeta$ such that $q \leq r^+$, $r \leq r^+$ and $(\forall \beta < \delta)(q_\beta \leq q)$.

Proof. The arguments here are similar to those for [15, Proposition 3.3], except that our iterands are *strategically* ($< \lambda$)-complete only (and not necessarily ($< \lambda$)-complete). This introduces some additional complexity.

Let $\langle i_\gamma : \gamma < \text{cf}(\zeta) \rangle \subseteq N \cap \zeta$ be a strictly increasing continuous sequence cofinal in ζ with $i_0 = 0$. By induction on $\gamma < \text{cf}(\zeta)$ we choose r_γ^- , r_γ and r_γ^* such that

- (i) $r_\gamma^- \in \mathbb{P}_{i_\gamma} \cap N$ is above (in \mathbb{P}_{i_γ}) of all $q_\beta \upharpoonright i_\gamma$ for $\beta < \delta$,
- (ii) $r_\gamma, r_\gamma^* \in \mathbb{P}_{i_\gamma}$, $r_\gamma^- \leq r_\gamma \leq r_\gamma^*$ and $r \upharpoonright i_\gamma \leq r_\gamma$,
- (iii) if $\gamma < \varepsilon < \text{cf}(\zeta)$, then $r_\gamma^- \leq r_\varepsilon^-$, $r_\gamma^* \leq r_\varepsilon^*$, and
- (iv) for each $\varepsilon < \zeta$ and $\gamma < \text{cf}(\zeta)$ we have

$$r_\gamma^* \upharpoonright \varepsilon \Vdash \text{“} \langle r_\alpha(\varepsilon), r_\alpha^*(\varepsilon) : \alpha \leq \gamma \rangle \text{ is a legal partial play of } \mathfrak{D}_0^\lambda(\mathbb{Q}_\varepsilon) \text{”}$$

in which Complete uses her regular winning strategy $\mathbf{st}_\varepsilon^0$ ”.

So, at stage $\gamma+1$ of the construction we apply Observation 2.3 to the sequence $\langle r_\gamma^- \widehat{\sim} q_\beta \upharpoonright [i_\gamma, i_{\gamma+1}) : \beta < \delta \rangle \subseteq N \cap \mathbb{P}_{i_{\gamma+1}}$ and the $(N, \mathbb{P}_{i_{\gamma+1}})$ -generic condition $r_\gamma^* \widehat{\sim} r \upharpoonright [i_\gamma, i_{\gamma+1})$. This will give us conditions $r_{\gamma+1}^- \in N \cap \mathbb{P}_{i_{\gamma+1}}$ and $r_{\gamma+1} \in \mathbb{P}_{i_{\gamma+1}}$ so that the demands (i)–(iii) are satisfied. Then $r_{\gamma+1}^* \in \mathbb{P}_{i_{\gamma+1}}$ is chosen according to demand (iv), remembering that the $\mathbf{st}_\varepsilon^0$ are \mathbb{P}_ε -names for *regular* winning strategies. Next, if γ is limit then we first pick $r' \in \mathbb{P}_{i_\gamma}$ that is stronger than all r_α^* for $\alpha < \gamma$ (exists by (iv)). This r' is $(N, \mathbb{P}_{i_\gamma})$ -generic, stronger than all r_α^- for $\alpha < \gamma$ and stronger than all $q_\beta \upharpoonright i_\gamma$ for $\beta < \delta$. So we may use Observation 2.3 to find r_γ^- and $r_\gamma \geq r'$ as needed for (i)–(iii) and then pick r_γ^* to fulfill (iv).

Let $r^+ \in \mathbb{P}_\zeta$ be an upper bound of $\langle r_\gamma : \gamma < \text{cf}(\zeta) \rangle$ (remember clause (iv) above); then also $r \leq r^+$. Now we are going to define a condition $q \in \mathbb{P}_\zeta \cap N$.

We let

$$\text{dom}(q) = \bigcup \{ \text{dom}(r_{\gamma+1}^-) \cap [i_\gamma, i_{\gamma+1}) : \gamma < \text{cf}(\zeta) \},$$

and for $\varepsilon \in \text{dom}(q)$, $i_\gamma \leq \varepsilon < i_{\gamma+1}$, we let $q(\varepsilon)$ be the $<^*_\chi$ -first \mathbb{P}_ε -name for the following object in $\mathbf{V}[G_{\mathbb{P}_\varepsilon}]$.

- (A) If $r_{\gamma+1}^-(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$ is an upper bound on $\{q_\beta(\varepsilon)[G_{\mathbb{P}_\varepsilon}] : \beta < \delta\}$ in $\mathbb{Q}_\varepsilon[G_{\mathbb{P}_\varepsilon}]$, then $q(\varepsilon)[G_{\mathbb{P}_\varepsilon}] = r_{\gamma+1}^-(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$.
- (B) If not (A) but $\{q_\beta(\varepsilon)[G_{\mathbb{P}_\varepsilon}] : \beta < \delta\}$ has an upper bound in $\mathbb{Q}_\varepsilon[G_{\mathbb{P}_\varepsilon}]$, then $q(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$ is such a bound.
- (C) If neither (A) nor (B), then $q(\varepsilon)[G_{\mathbb{P}_\varepsilon}] = q_0(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$.

It should be clear that $q \in \mathbb{P}_\zeta \cap N$. Now,

- $q \leq r^+$.

Why? By induction on $\varepsilon \in \zeta \cap N$ we show that $q \upharpoonright \varepsilon \leq r^+ \upharpoonright \varepsilon$. The steps “ $\varepsilon = 0$ ” and “ ε is limit” are clear, so suppose that we have proved $q \upharpoonright \varepsilon \leq r^+ \upharpoonright \varepsilon$, $i_\gamma \leq \varepsilon < i_{\gamma+1}$ (and we are interested in restrictions to $\varepsilon + 1$). Assume that $G_{\mathbb{P}_\varepsilon} \subseteq \mathbb{P}_\varepsilon$ is a generic filter over \mathbf{V} such that $r^+ \upharpoonright \varepsilon \in G_{\mathbb{P}_\varepsilon}$. Since $q_\beta \upharpoonright i_{\gamma+1} \leq r_{\gamma+1}^- \leq r_{\gamma+1} \leq r^+$, we also have $q_\beta \upharpoonright \varepsilon \in G_{\mathbb{P}_\varepsilon}$ (for $\beta < \delta$) and $r_{\gamma+1}^- \upharpoonright \varepsilon \in G_{\mathbb{P}_\varepsilon}$. Hence $r_{\gamma+1}^-(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$ is an upper bound of $\{q_\beta(\varepsilon)[G_{\mathbb{P}_\varepsilon}] : \beta < \delta\}$. Therefore

$$q(\varepsilon)[G_{\mathbb{P}_\varepsilon}] = r_{\gamma+1}^-(\varepsilon)[G_{\mathbb{P}_\varepsilon}] \leq r_{\gamma+1}(\varepsilon)[G_{\mathbb{P}_\varepsilon}] \leq r^+(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$$

(see (A) above), and we are done.

The proof of the lemma will be finished once we show that

- $(\forall \beta < \delta)(q_\beta \leq q)$.

Why does this hold? By induction on $\varepsilon \in \zeta \cap N$ we show that $q_\beta \upharpoonright \varepsilon \leq q \upharpoonright \varepsilon$ for all $\beta < \delta$. The steps “ $\varepsilon = 0$ ” and “ ε is limit” are as usual clear, so suppose that we have proved $q_\beta \upharpoonright \varepsilon \leq q \upharpoonright \varepsilon$ for $\beta < \delta$, $i_\gamma \leq \varepsilon < i_{\gamma+1}$ (and we are interested in restrictions to $\varepsilon + 1$). Assume that $G_{\mathbb{P}_\varepsilon} \subseteq \mathbb{P}_\varepsilon$ is a generic filter over \mathbf{V} such that $q \upharpoonright \varepsilon \in G_{\mathbb{P}_\varepsilon}$. Then, by the inductive hypothesis and the assumption (b) of the lemma, we know that the sequence $\langle q_\beta(\varepsilon)[G_{\mathbb{P}_\varepsilon}] : \beta < \delta \rangle$ has an upper bound in $\mathbb{Q}_\varepsilon[G_{\mathbb{P}_\varepsilon}]$. Therefore, by (A)+(B), $q(\varepsilon)[G_{\mathbb{P}_\varepsilon}] \geq q_\beta(\varepsilon)[G_{\mathbb{P}_\varepsilon}]$ for all $\beta < \delta$, and we are done. ■

LEMMA 2.5. *Suppose that $\zeta \in (\zeta^* + 1) \cap N$ and conditions $r \in \mathbb{P}_\zeta$ and $p_\alpha, q_\alpha \in \mathbb{P}_\zeta \cap N$ (for $\alpha < \delta < \lambda$) satisfy*

- (i) $p_\alpha \leq q_\alpha \leq p_\beta \leq q_\beta \leq r$ for all $\alpha < \beta < \delta$,
- (ii) for each $\xi \in \zeta \cap N$ and $\beta < \delta$ we have

$q_\beta \upharpoonright \xi \Vdash_{\mathbb{P}_\xi}$ “ $\langle p_\alpha(\xi), q_\alpha(\xi) : \alpha \leq \beta \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\xi)$ in which Complete uses her regular winning strategy st_ξ^0 ”;

(iii) either

(α) r is (N, \mathbb{P}_ζ) -generic, or

(β) ζ is a limit ordinal of cofinality $\text{cf}(\zeta) < \lambda$ and for each $\xi \in \zeta \cap N$ the condition $r \upharpoonright \xi$ is (N, \mathbb{P}_ξ) -generic.

Let $\mathcal{I} \in N$ be an open dense subset of \mathbb{P}_ζ and $p^- \in N \cap \mathbb{P}_\zeta$, $p^- \leq r$. Then there are conditions $p', q' \in \mathbb{P}_\zeta \cap N$ and $r' \in \mathbb{P}_\zeta$ such that

(a) $r \leq r'$ and $q_\alpha \leq p' \leq q' \leq r'$ for all $\alpha < \delta$, and

(b) for each $\xi \in \zeta \cap N$ we have

$q' \upharpoonright \xi \Vdash_{\mathbb{P}_\xi}$ “ $\langle p_\alpha(\xi), q_\alpha(\xi) : \alpha < \delta \rangle \frown \langle p'(\xi), q'(\xi) \rangle$ is a partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\xi)$ in which Complete uses her regular winning strategy \mathbf{st}_ξ^0 ”;

(c) under the assumption of (iii)(α), also $p' \in \mathcal{I}$,

(d) under the assumption of (iii)(β), if $q_\alpha \leq p^-$ for all $\alpha < \delta$, then we may still require that $p^- \leq p'$.

Proof. Under the assumption of (iii)(α), the conclusion of the lemma should be clear (including clause (c)).

So suppose we are in the case described in (iii)(β) and let $\langle i_\gamma : \gamma < \text{cf}(\zeta) \rangle \subseteq N \cap \zeta$ be an increasing continuous sequence cofinal in ζ , with $i_0 = 0$. First, assume that the condition $p^- \in N \cap \mathbb{P}_\zeta$ satisfies $q_\alpha \leq p^- \leq r$ for all $\alpha < \delta$. By induction on $\gamma < \text{cf}(\zeta)$ we will pick conditions $p^\gamma, q^\gamma, r^\gamma, s^\gamma$ so that

(*)₁ $p^\gamma, q^\gamma \in N \cap \mathbb{P}_{i_\gamma}$, $r^\gamma, s^\gamma \in \mathbb{P}_{i_\gamma}$,

(*)₂ $p^- \upharpoonright i_\gamma \leq p^\gamma \leq q^\gamma$, $r \upharpoonright i_\gamma \leq r^\gamma \leq s^\gamma$ and $q^\gamma \leq r^\gamma$,

(*)₃ $q^{\gamma'} \leq p^\gamma \upharpoonright i_{\gamma'}$ and $s^{\gamma'} \leq r^\gamma \upharpoonright i_{\gamma'}$ for $\gamma' < \gamma$, and

(*)₄ if $i_\gamma \leq \xi < i_{\gamma+1}$ and $\gamma < \gamma^* < \text{cf}(\zeta)$, then

$q^{\gamma^*} \upharpoonright \xi \Vdash_{\mathbb{P}_\xi}$ “the sequence $\langle p_\alpha(\xi), q_\alpha(\xi) : \alpha < \delta \rangle \frown \langle p^{\gamma'}(\xi), q^{\gamma'}(\xi) : \gamma < \gamma' \leq \gamma^* \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\xi)$ in which

Complete uses her regular winning strategy \mathbf{st}_ξ^0 ”;

and

$s^{\gamma^*} \upharpoonright \xi \Vdash_{\mathbb{P}_\xi}$ “ $\langle r^{\gamma'}(\xi), s^{\gamma'}(\xi) : \gamma' \leq \gamma^* \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\xi)$ in which Complete uses her regular winning strategy \mathbf{st}_ξ^0 ”.

When we arrive at stage $\gamma < \text{cf}(\zeta)$ of the construction, after having determined $p^{\gamma'}, q^{\gamma'}, r^{\gamma'}, s^{\gamma'}$ for $\gamma' < \gamma$, we first pick a condition $r^* \in \mathbb{P}_{i_\gamma}$ stronger than all $s^{\gamma'}$ for $\gamma' < \gamma$ and such that $r \upharpoonright i_\gamma \leq r^*$. [There is such an r^* by the second part of the demand (*)₄ at previous stages.] Then the condition r^* is $(N, \mathbb{P}_{i_\gamma})$ -generic so we may use arguments as for the first part of the lemma (case (iii)(α)) to pick p^γ, q^γ and $s^\gamma \geq r^\gamma \geq r^*$ so that the demands in (*)₁–(*)₄ are satisfied.

After the construction is carried out we define $p' \in \mathbb{P}_\zeta$ so that $\text{dom}(p') = \bigcup_{\gamma < \text{cf}(\zeta)} \text{dom}(q^\gamma)$ and if $\xi \in \text{dom}(p')$, $i_\gamma \leq \xi < i_{\gamma+1}$, $\gamma < \text{cf}(\zeta)$, then $p'(\xi)$ is the $<_{\chi^*}$ -first \mathbb{P}_ξ -name such that the condition $p' \upharpoonright \xi$ forces (in \mathbb{P}_ξ) that

“if $p^-(\xi) \leq p^{\gamma+1}(\xi)$ then $p'(\xi) = p^{\gamma+1}(\xi)$, otherwise $p'(\xi) = p^-(\xi)$ ”.

Then $p' \in N$ is a condition stronger than p^- (and so also stronger than all q_α for $\alpha < \delta$). Let $q' \in \mathbb{P}_\zeta \cap N$ be such that $\text{dom}(q') = \text{dom}(p')$ and for each $\xi \in \text{dom}(q')$ we have

$q' \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \langle p_\alpha(\xi), q_\alpha(\xi) : \alpha < \delta \rangle \frown \langle p'(\xi), q'(\xi) \rangle$ is a partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\xi)$
in which Complete uses her regular winning strategy \mathbf{st}_ξ^0 ”.

Plainly, $p' \leq q'$. Let $r' \in \mathbb{P}_\zeta$ be an upper bound on $\langle s^\gamma : \gamma < \text{cf}(\zeta) \rangle$ (exists by the second part of the demands in $(*)_4$). Then $r' \geq r$. Now we show inductively that $r' \upharpoonright \xi \geq q' \upharpoonright \xi$ for $\xi < \zeta$. So suppose $\xi \in \text{dom}(q')$, $i_\gamma \leq \xi < i_{\gamma+1}$, and $q' \upharpoonright \xi \leq r' \upharpoonright \xi$. Then $r' \upharpoonright \xi \Vdash \langle p'(\xi) = p^{\gamma+1}(\xi) \rangle$ (by $(*)_2$) and so also $r' \upharpoonright \xi \Vdash \langle q'(\xi) = q^{\gamma+1}(\xi) \leq s^{\gamma+1}(\xi) \leq r'(\xi) \rangle$. Hence $p^- \leq p' \leq q' \leq r'$.

If the condition p^- does not satisfy the assumptions of clause (d), then we may use Lemma 2.4 to find $q \in N \cap \mathbb{P}_\zeta$ and $r^+ \in \mathbb{P}_\zeta$ such that $r \leq r^+$, $q \leq r^+$ and $q_\alpha \leq q$ for all $\alpha < \delta$. Now carry out the above arguments for q, r^+ in place of p^-, r . ■

DEFINITION 2.6.

- (1) An *RS-condition* ⁽²⁾ in \mathbb{P}_{ζ^*} is a pair (p, w) such that $w \in [(\zeta^* + 1)]^{<\lambda}$ is a closed set, $0, \zeta^* \in w$, p is a function with domain $\text{dom}(p) \subseteq \zeta^*$, and
 - (\otimes) for any two successive members $\varepsilon' < \varepsilon''$ of the set w , $p \upharpoonright [\varepsilon', \varepsilon'')$ is a $\mathbb{P}_{\varepsilon''}$ -name of an element of $\mathbb{P}_{\varepsilon''}$ whose support is included in the interval $[\varepsilon', \varepsilon'')$.

The family of all RS-conditions in \mathbb{P}_{ζ^*} is denoted by $\mathbb{P}_{\zeta^*}^{\text{RS}}$.

- (2) If $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ and $G \subseteq \mathbb{P}_{\zeta^*}$ is a generic filter over \mathbf{V} , then we define

$$(p, w)^G = \bigcup \{ (p \upharpoonright [\varepsilon', \varepsilon'']) \upharpoonright [G \cap \mathbb{P}_{\varepsilon'}] : \varepsilon' < \varepsilon'' \text{ are successive members of } w \}$$

Note that $(p, w)^G \in \mathbb{P}_{\zeta^*}$. Also, we write $(p, w) \in' G$ whenever $(p, w)^G \in G$.

- (3) If $(p_1, w_1), (p_2, w_2) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$, then we write $(p_1, w_1) \leq' (p_2, w_2)$ whenever

- (\oplus) for every generic $G \subseteq \mathbb{P}_{\zeta^*}$ over \mathbf{V} ,
if $(p_2, w_2) \in' G$ then $(p_1, w_1)^G \leq_{\mathbb{P}_{\zeta^*}} (p_2, w_2)^G$.

REMARK 2.7. In 2.6(1)(\otimes), $p \upharpoonright [\varepsilon', \varepsilon'')$ is a sequence of $\mathbb{P}_{\varepsilon''}$ -names, so it is a $\mathbb{P}_{\varepsilon''}$ -name for some sequence. This (resulting) sequence is required to belong to $\mathbb{P}_{\varepsilon''}$, and just also to \mathbf{V} . We may therefore think about an RS-condition (p, w) as follows. First, for every $\varepsilon \in w \cap \zeta^*$ we have

⁽²⁾ The RS stands for “revised support”.

- a maximal antichain $\{p_i^\varepsilon : i \in I_\varepsilon\}$ of \mathbb{P}_ε , and
- a function $f^\varepsilon : I_\varepsilon \rightarrow \mathbb{P}_{\zeta^*}$ such that $f^\varepsilon(i)$ is a condition with support included in the interval $[\varepsilon, \min(w \setminus (\varepsilon + 1))]$.

(They describe a \mathbb{P}_ε -name for a condition in \mathbb{P}_{ζ^*} with support in the right interval.) Next, if $\varepsilon \leq \xi < \varepsilon'$ where $\varepsilon, \varepsilon'$ are successive members of w , then we may fix a \mathbb{P}_ξ -name $p(\xi)$ for a member of \mathbb{Q}_ξ such that $p_i^\varepsilon \Vdash_{\mathbb{P}_\xi} p(\xi) = f^\varepsilon(i)(\xi)$. Note however that if we apply this approach to each ξ , we may not end up with a condition in \mathbb{P}_{ζ^*} because of the support! Nevertheless we will think of p as a function on ζ^* where each $p(\xi)$ is a \mathbb{P}_ξ -name for a member of \mathbb{Q}_ξ .

LEMMA 2.8.

- (1) If $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ and $q \in \mathbb{P}_{\zeta^*}$, then there is $q^* \in \mathbb{P}_{\zeta^*}$ stronger than q and such that for any successive members $\varepsilon' < \varepsilon''$ of w the condition $q^* \upharpoonright \varepsilon'$ decides $p \upharpoonright [\varepsilon', \varepsilon'']$ (i.e., $q^* \upharpoonright \varepsilon' \Vdash "p \upharpoonright [\varepsilon', \varepsilon''] = p_{\varepsilon', \varepsilon''}"$ for some $p_{\varepsilon', \varepsilon''} \in \mathbb{P}_{\zeta^*}$).
- (2) For each $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ there is $q \in \mathbb{P}_{\zeta^*}$ such that $(p, w) \leq' (q, \{0, \zeta^*\})$.
- (3) Let $(p_i, w_i) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$ (for $i < \delta < \lambda$), and $q \in \mathbb{P}_{\zeta^*} \cap N$, $r \in \mathbb{P}_{\zeta^*}$ be such that

$$q \leq r \quad \text{and} \quad (\forall i < j < \delta)((p_i, w_i) \leq' (p_j, w_j) \leq' (r, \{0, \zeta^*\})).$$

Assume that either r is $(N, \mathbb{P}_{\zeta^*})$ -generic, or ζ^* is a limit ordinal of cofinality $\text{cf}(\zeta^*) < \lambda$ and for every $\zeta \in \zeta^* \cap N$ the condition $r \upharpoonright \zeta$ is (N, \mathbb{P}_ζ) -generic. Then there are conditions $q' \in N \cap \mathbb{P}_{\zeta^*}$ and $r' \in \mathbb{P}_{\zeta^*}$ such that $q \leq q' \leq r'$, $r \leq r'$ and $(\forall i < \delta)((p_i, w_i) \leq' (q', \{0, \zeta^*\}))$.

Proof. It is a slight generalization of [15, Proposition 3.6].

(1, 2) Straightforward (as \mathbb{P}_{ζ^*} is strategically $(< \lambda)$ -complete).

(3) If r is $(N, \mathbb{P}_{\zeta^*})$ -generic, then the conclusion is immediate. So let us consider the case when $\aleph_0 \leq \text{cf}(\zeta^*) < \lambda$ and $r \upharpoonright \zeta$ is (N, \mathbb{P}_ζ) -generic for each $\zeta \in \zeta^* \cap N$.

Let $\langle i_\gamma : \gamma < \text{cf}(\zeta^*) \rangle \subseteq N \cap \zeta^*$ be a strictly increasing continuous sequence cofinal in ζ^* , with $i_0 = 0$. For $\gamma < \text{cf}(\zeta^*)$ and $i < \delta$ we put $p_i^\gamma = p_i \upharpoonright i_\gamma$ and $w_i^\gamma = (w_i \cap i_\gamma) \cup \{i_\gamma\}$ (clearly $(p_i^\gamma, w_i^\gamma) \in \mathbb{P}_{i_\gamma}^{\text{RS}}$).

Now, we inductively pick conditions $q_\gamma, q_\gamma^+, r_\gamma, r_\gamma^+$ (for $\gamma < \text{cf}(\zeta^*)$) so that

- $q_\gamma, q_\gamma^+ \in \mathbb{P}_{i_\gamma} \cap N$, $r_\gamma, r_\gamma^+ \in \mathbb{P}_{i_\gamma}$,
- $q \upharpoonright i_\gamma \leq q_\gamma \leq q_\gamma^+ \leq r_\gamma$, $r \upharpoonright i_\gamma \leq r_\gamma \leq r_\gamma^+$, and for each $\beta < \gamma$ we have

$$q_\beta^+ \leq q_\gamma \quad \text{and} \quad r_\beta^+ \leq r_\gamma,$$

(iii) for every $\xi < i_\gamma$ we have

$$q_\gamma^+ \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} " \langle q_\beta(\xi), q_\beta^+(\xi) : \beta \leq \gamma \rangle \text{ is a legal partial play of } \mathfrak{D}_0^\lambda(\mathbb{Q}_\xi) \text{ in which Complete uses her regular winning strategy } \mathfrak{st}_\xi^0",$$

and

$r_\gamma^+ \upharpoonright \xi \Vdash_{\mathbb{P}_\xi}$ “ $\langle r_\beta(\xi), r_\beta^+(\xi) : \beta \leq \gamma \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\xi)$ in which Complete uses the strategy \mathbf{st}_ξ^0 ”,

(iv) if $i < \delta$ and $\varepsilon' < \varepsilon''$ are two successive members of w_i with $\varepsilon' < i_\gamma$, then the condition $q_\gamma \upharpoonright \varepsilon'$ decides $p_i \upharpoonright [\varepsilon', \varepsilon'']$ and $(p_i^\gamma, w_i^\gamma) \leq' (q_\gamma, \{0, i_\gamma\})$.

At a limit stage γ of the construction, we first pick an upper bound $r^* \in \mathbb{P}_{i_\gamma}$ on $\langle r_\beta^+ : \beta < \gamma \rangle$ (exists by clause (iii) at previous stages). By (ii) we know that $r \upharpoonright i_\gamma \leq r^*$, so r^* is $(N, \mathbb{P}_{i_\gamma})$ -generic. Consequently, we may use Lemma 2.5 to choose $q_\gamma, q_\gamma^+ \in \mathbb{P}_{i_\gamma} \cap N$ and $r_\gamma \in \mathbb{P}_{i_\gamma}$ satisfying the relevant demands in (ii)+(iii), and then we pick $r_\gamma^+ \geq r_\gamma$ so that the second part of (iii) holds. Note that the demand in (iv) will then follow by the inductive hypothesis.

Suppose now we are at a successor stage $\gamma+1$, so we assume $q_\gamma, q_\gamma^+, r_\gamma, r_\gamma^+$ have already been chosen. We note that a condition $r^* = r_\gamma^+ \frown (r \upharpoonright [i_\gamma, i_{\gamma+1})) \in \mathbb{P}_{i_{\gamma+1}}$ is stronger than r , so it is $(N, \mathbb{P}_{i_{\gamma+1}})$ -generic. Therefore we may apply Lemma 2.5 to r^* , the sequence $\langle q_\beta, q_\beta^+ : \beta \leq \gamma \rangle$ considered as a sequence of conditions in $\mathbb{P}_{i_{\gamma+1}}$ and a suitable open dense subset \mathcal{I} of $\mathbb{P}_{i_{\gamma+1}}$ from N to choose $q_{\gamma+1}, q_{\gamma+1}^+$ and $r_{\gamma+1}, r_{\gamma+1}^+$ so that the demands in (i)–(iv) are satisfied. (For clause (iv) remember that by our assumptions we have $(p_i^{\gamma+1}, w_i^{\gamma+1}) \leq' (r^*, \{0, i_{\gamma+1}\})$.)

Let $r^* \in \mathbb{P}_{\zeta^*}$ be stronger than all r_γ 's. Apply Lemma 2.4 to the sequence $\langle q_\gamma^+ : \gamma < \text{cf}(\zeta^*) \rangle$ and the condition r^* to choose $q' \in N \cap \mathbb{P}_{\zeta^*}$ and $r' \in \mathbb{P}_{\zeta^*}$ such that

$$q' \leq r', \quad r^* \leq r', \quad \text{and} \quad (\forall \gamma < \text{cf}(\zeta^*)) (q_\gamma^+ \leq q').$$

Then r', q' are as required. ■

PROPOSITION 2.9. *Assume that*

- (1) $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$, $\zeta \in w$ and $\text{sup}(w \cap \zeta) < \xi < \zeta$,
- (2) $w' \subseteq \zeta^* + 1$ is a closed set such that $|w'| < \lambda$ and $w \cup \{\xi\} \subseteq w'$, and
- (3) \mathcal{T} is a \mathbb{P}_ζ -name for an ordinal.

Then there are p' and \mathcal{T}' such that

- (a) $(p', w') \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ and $(p, w) \leq' (p', w')$,
- (b) $p' \upharpoonright \xi = p \upharpoonright \xi$ and $p' \upharpoonright [\zeta, \zeta^*) = p \upharpoonright [\zeta, \zeta^*)$, and
- (c) \mathcal{T}' is a \mathbb{P}_ξ -name, and
- (d) if $G \subseteq \mathbb{P}_{\zeta^*}$ is generic over \mathbf{V} and $(p', w') \in' G$, then $\mathcal{T}'[G] = \mathcal{T}[G]$.

Proof. First we declare that $p' \upharpoonright \xi = p \upharpoonright \xi$ and $p' \upharpoonright [\zeta, \zeta^*) = p \upharpoonright [\zeta, \zeta^*)$. Now, $p \upharpoonright [\xi, \zeta)$ is a \mathbb{P}_ξ -name, so we may choose a maximal antichain $\mathcal{B} \subseteq \mathbb{P}_\xi$ and a function $g : \mathcal{B} \rightarrow \mathbb{P}_\zeta$ such that for each $r \in \mathcal{B}$ we have

- (α) $\text{dom}(g(r)) \subseteq [\xi, \zeta]$,
- (β) r decides the value of $p \upharpoonright [\xi, \zeta]$ and $r \Vdash_{\mathbb{P}_\xi} p \upharpoonright [\xi, \zeta] \leq g(r)$, and
- (γ) $r \widehat{\ } g(r)$ forces a value to \mathcal{T} , say $r \widehat{\ } g(r) \Vdash \mathcal{T} = \tau(r)$.

Let \mathcal{T}' be a \mathbb{P}_ξ -name such that $r \Vdash_{\mathbb{P}_\xi} \mathcal{T}' = \tau(r)$ for each $r \in \mathcal{B}$, and let $p' \upharpoonright [\xi, \zeta]$ be a \mathbb{P}_ξ -name such that $r \Vdash p' \upharpoonright [\xi, \zeta] = g(r)$ for $r \in \mathcal{B}$. Then (p', w') and \mathcal{T}' are as required in (a)–(d). ■

It will be convenient for the proof of our main result 4.1 to look at RS-conditions also in a slightly different way.

DEFINITION 2.10. Let $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$. A standard representation of (p, w) is a triple (\mathcal{A}, f, w) such that

- (i) $\mathcal{A} \subseteq \mathbb{P}_{\zeta^*}$ is a maximal antichain,
- (ii) $f : \mathcal{A} \rightarrow \mathbb{P}_{\zeta^*}$,
- (iii) for each $r \in \mathcal{A}$, either $f(r) \leq r$ or for some $\varepsilon \in \text{dom}(r)$ we have $f(r) \upharpoonright \varepsilon \leq r \upharpoonright \varepsilon$ and $r \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $f(r)(\varepsilon), r(\varepsilon)$ are incompatible in \mathbb{Q}_ε ”, and
- (iv) if $r \in \mathcal{A}$, $\varepsilon \in w \cap \zeta^*$ and $\varepsilon' = \min(w \setminus (\varepsilon + 1))$, then $r \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $p \upharpoonright \varepsilon' = f(r) \upharpoonright \varepsilon'$ ”.

OBSERVATION 2.11. For each $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ and an open dense set $\mathcal{I} \subseteq \mathbb{P}_{\zeta^*}$ there is a standard representation (\mathcal{A}, f, w) of (p, w) such that $\mathcal{A} \subseteq \mathcal{I}$.

OBSERVATION 2.12. Assume that $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ and (\mathcal{A}, f, w) is a standard representation of (p, w) .

- (v) If $s, r \in \mathcal{A}$, $\varepsilon \in w \cap \zeta^*$, $\varepsilon' = \min(w \setminus (\varepsilon + 1))$ and the conditions $s \upharpoonright \varepsilon$ and $r \upharpoonright \varepsilon$ are compatible in \mathbb{P}_ε , then $f(s) \upharpoonright \varepsilon' = f(r) \upharpoonright \varepsilon'$.
- (vi) If $r \in \mathcal{A}$, $\varepsilon < \zeta^*$ and $f(r) \upharpoonright \varepsilon \leq r \upharpoonright \varepsilon$, then there is $s \in \mathcal{A}$ such that $r \upharpoonright \varepsilon$ and $s \upharpoonright \varepsilon$ are compatible and $f(s) \leq s$.

PROPOSITION 2.13. Let $w \in [\zeta^* + 1]^{<\lambda}$ be a closed set containing $0, \zeta^*$. Assume that (\mathcal{A}, f, w) satisfies (i)–(iii) of Definition 2.10 and (v) of Observation 2.12. Then there is $(p, w) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ such that (\mathcal{A}, f, w) is a standard representation of (p, w) .

Proof. Define (p, w) so that

$$r \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} “p \upharpoonright [\varepsilon, \varepsilon'] = f(r) \upharpoonright [\varepsilon, \varepsilon']”$$

for $\varepsilon \in w \cap \zeta^*$, $\varepsilon' = \min(w \setminus (\varepsilon + 1))$ and $r \in \mathcal{A}$. Now check. ■

DEFINITION 2.14. Let $(\mathcal{A}_1, f_1, w_1), (\mathcal{A}_2, f_2, w_2)$ be standard representations. We write $(\mathcal{A}_1, f_1, w_1) \preceq (\mathcal{A}_2, f_2, w_2)$ whenever

- (\otimes)₁ $(\forall r \in \mathcal{A}_2)(\exists s \in \mathcal{A}_1)(s \leq r)$ and
- (\otimes)₂ if $r \in \mathcal{A}_2$, $s \in \mathcal{A}_1$, $s \leq r$ and $f_2(r) \leq r$, then $f_1(s) \leq f_2(r)$.

PROPOSITION 2.15. The relation \preceq is transitive on standard representations.

Proof. Assume $(\mathcal{A}_1, f_1, w_1) \preceq (\mathcal{A}_2, f_2, w_2) \preceq (\mathcal{A}_3, f_3, w_3)$. Suppose that $r_i \in \mathcal{A}_i$ (for $i = 1, 2, 3$) are such that $r_1 \leq r_2 \leq r_3$ and $f_3(r_3) \leq r_3$. Then also $f_2(r_2) \leq f_3(r_3)$ and consequently $f_2(r_2)$ and r_2 are compatible. By 2.10(iii) we have $f_2(r_2) \leq r_2$ and therefore also $f_1(r_1) \leq f_2(r_2)$. Together $f_1(r_1) \leq f_2(r_2) \leq f_3(r_3)$, and the transitivity of \preceq follows. ■

OBSERVATION 2.16. *Suppose that $(p_1, w_1), (p_2, w_2) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ and let $(\mathcal{A}_1, f_1, w_1)$ and $(\mathcal{A}_2, f_2, w_2)$ be their standard representations. Assume $(\mathcal{A}_1, f_1, w_1) \preceq (\mathcal{A}_2, f_2, w_2)$. Then $(p_1, w_1) \leq' (p_2, w_2)$.*

PROPOSITION 2.17. *Let $(p_1, w_1), (p_2, w_2) \in \mathbb{P}_{\zeta^*}^{\text{RS}}$ be such that $(p_1, w_1) \leq' (p_2, w_2)$. Suppose that $(\mathcal{A}_1, f_1, w_1)$ is a standard representation of (p_1, w_1) . Then there is a standard representation $(\mathcal{A}_2, f_2, w_2)$ for (p_2, w_2) such that $(\mathcal{A}_1, f_1, w_1) \preceq (\mathcal{A}_2, f_2, w_2)$.*

Proof. Choose a standard representation $(\mathcal{A}_2, f_2, w_2)$ of (p_2, w_2) such that $(\forall s \in \mathcal{A}_2)(\exists r \in \mathcal{A}_1)(r \leq s)$ (possible by Observation 2.11). Then plainly 2.14(⊗)₂ is also satisfied, so $(\mathcal{A}_1, f_1, w_1) \preceq (\mathcal{A}_2, f_2, w_2)$. ■

2.2. The forcings. Let us recall the definitions of the main forcing notions we are interested in. These forcing notions generalize the well known classical forcings with trees used in the set theory of the reals. Thus both \mathbb{Q}_λ^1 and \mathbb{Q}_λ^2 (defined below in 2.18) represent possible generalizations of Miller's rational perfect set forcing (see Miller [13]). The Laver forcing notion (see Laver [11]) is generalized by forcings \mathbb{Q}_λ^4 . Due to flexibility of our parameters, the schemes presented here generalize the Cohen and the Sacks forcing notions as well (see Remark 2.19). The many "bounded" variations of the classical forcings with trees have suitable nice generalizations if λ is inaccessible (see Definition 2.20). In Definition 2.22(1) we will introduce generalizations of the Silver forcing notion (or perhaps rather the Grigorieff forcing, see [6, §3]). Finally, relatives of the Hechler forcing (Hechler [7]) and Eventually Different Real forcing (Miller [12, §5]) are introduced in Definition 2.22(2, 3).

DEFINITION 2.18. Suppose that $\bar{E} = \langle E_t : t \in {}^{<\lambda}\lambda \rangle$ is a system of $(<\lambda)$ -complete filters on a regular cardinal λ and E is a normal filter on λ . We define forcing notions $\mathbb{Q}^{\ell, \bar{E}}$ (for $\ell = 1, 2, 3, 4$) and $\mathbb{Q}_E^{1, \bar{E}}$ and $\mathbb{Q}_E^{3, \bar{E}}$ as follows.

- (1) A condition in $\mathbb{Q}^{2, \bar{E}}$ is a complete λ -tree $T \subseteq {}^{<\lambda}\lambda$ such that
 - (a) if $t \in T$, then either $|\text{succ}_T(t)| = 1$ or $\text{succ}_T(t) \in E_t$,
 - (b) $(\forall t \in T)(\exists s \in T)(t \triangleleft s \ \& \ |\text{succ}_T(s)| > 1)$, and
 - (c)² if $j < \lambda$ and a sequence $\langle t_i : i < j \rangle \subseteq T$ is \triangleleft -increasing, $|\text{succ}_T(t_i)| > 1$ for all $i < j$ and $t = \bigcup_{i < j} t_i$, then $(t \in T \text{ and } |\text{succ}_T(t)| > 1)$,

the order \leq of $\mathbb{Q}^{2,\bar{E}}$ is inverse inclusion, i.e., $T_1 \leq T_2$ if and only if $T_1, T_2 \in \mathbb{Q}^{2,\bar{E}}$ and $T_2 \subseteq T_1$.

- (2) Forcing notions $\mathbb{Q}^{1,\bar{E}}, \mathbb{Q}^{3,\bar{E}}, \mathbb{Q}^{4,\bar{E}}$ are defined analogously, but the demand (c)² is replaced by the respective (c)^ℓ:

(c)¹ for every λ -branch $\eta \in \lim_\lambda(T)$ the set

$$\{\alpha < \lambda : |\text{succ}_T(\eta \upharpoonright \alpha)| > 1\}$$

includes a closed unbounded set,

- (c)³ for some closed unbounded set $C \subseteq \lambda$ consisting of limit ordinals we have

$$(\forall t \in T)(\text{lh}(t) \in C \Leftrightarrow |\text{succ}_T(t)| > 1),$$

(c)⁴ $(\forall t \in T)(\text{root}(T) \triangleleft t \Rightarrow |\text{succ}_T(t)| > 1)$.

- (3) The forcing notions $\mathbb{Q}_E^{1,\bar{E}}$ and $\mathbb{Q}_E^{3,\bar{E}}$ are defined like $\mathbb{Q}^{1,\bar{E}}$ and $\mathbb{Q}^{3,\bar{E}}$, but in demands (c)¹ and (c)³ we replace “a closed unbounded set” by “a set of limit ordinals belonging to the filter E ”.
- (4) If each E_t is the club filter of λ (for all $t \in {}^{<\lambda}\lambda$), then we omit \bar{E} and we write \mathbb{Q}_λ^ℓ instead of $\mathbb{Q}^{\ell,\bar{E}}$.

REMARK 2.19. (1) Note that our definition of \mathbb{Q}_λ^3 slightly differs from the one in [20], but the forcing defined here is a dense subset of the one defined there.

(2) The forcing notion $\mathbb{Q}^{2,\bar{E}}$ was studied by Brown and Groszek [2], who described when this forcing adds a generic of minimal degree.

(3) Remember that in Definition 2.18 we allow the filters E_t to be principal. Thus if $E_t = \{\lambda\}$ for each $t \in {}^{<\lambda}\lambda$, then $\mathbb{Q}^{4,\bar{E}}$ is the λ -Cohen forcing \mathbb{C}_λ and $\mathbb{Q}^{2,\bar{E}}$ is the forcing \mathbb{D}_λ from [15, Proposition 4.10]. If for each $t \in {}^{<\lambda}\lambda$ we let E_t be the filter of all subsets of λ including $\{0, 1\}$, then the forcing notion $\mathbb{Q}^{2,\bar{E}}$ will be equivalent to Kanamori’s λ -Sacks forcing of [10, Definition 1.1].

(4) A relative of \mathbb{Q}_λ^2 was used in iterations in Friedman and Zdomsky [5] and Friedman, Honzik and Zdomsky [4]. It was called Miller(λ) there and the main difference between the two forcings is in condition [5, Definition 2.1(vi)].

(5) The property introduced in this paper does not “capture” \mathbb{Q}_λ^1 . In a subsequent work we will modify it to include more forcings of the form $\mathbb{Q}_E^{1,\bar{E}}$. However, one should note that the forcing notions \mathbb{Q}_λ^1 and \mathbb{Q}_λ^2 are very similar. If in the demand 2.18(2)(c)¹ we replace “includes a closed unbounded set” with “is closed unbounded” then we clearly get an equivalent definition of \mathbb{Q}_λ^2 . In [20, Section 4] we showed that consistently the forcing notions \mathbb{Q}_λ^1 and \mathbb{Q}_λ^2 are equivalent, but also consistently they are not equivalent.

DEFINITION 2.20. Assume that

- λ is weakly inaccessible, and $\varphi : \lambda \rightarrow \lambda$ is a strictly increasing function such that each $\varphi(\alpha)$ is a regular uncountable cardinal and $\alpha < \varphi(\alpha)$ (for $\alpha < \lambda$),
- $\bar{F} = \langle F_t : t \in \bigcup_{\alpha < \lambda} \prod_{\beta < \alpha} \varphi(\beta) \rangle$ where F_t is a $< \varphi(\alpha)$ -complete filter on $\varphi(\alpha)$ whenever $t \in \prod_{\beta < \alpha} \varphi(\beta)$, $\alpha < \lambda$, and
- E is a normal filter on λ .

(1) We define a forcing notion $\mathbb{Q}_{\varphi, \bar{F}}^2$ as follows.

A condition in $\mathbb{Q}_{\varphi, \bar{F}}^2$ is a complete λ -tree $T \subseteq \bigcup_{\alpha < \lambda} \prod_{\beta < \alpha} \varphi(\beta)$ such that

- (a) for every $t \in T$, either $|\text{succ}_T(t)| = 1$ or $\text{succ}_T(t) \in F_t$,
- (b) $(\forall t \in T)(\exists s \in T)(t \triangleleft s \ \& \ |\text{succ}_T(s)| > 1)$, and
- (c)² if $j < \lambda$ and a sequence $\langle t_i : i < j \rangle \subseteq T$ is \triangleleft -increasing, $|\text{succ}_T(t_i)| > 1$ for all $i < j$ and $t = \bigcup_{i < j} t_i$, then $(t \in T \text{ and } |\text{succ}_T(t)| > 1)$.

The order of $\mathbb{Q}_{\varphi, \bar{F}}^2$ is reverse inclusion.

(2) Forcing notions $\mathbb{Q}_{\varphi, \bar{F}}^\ell$ for $\ell = 1, 3, 4$ are defined similarly, but the demand (c)² is replaced by the respective (c) ^{ℓ} :

- (c)¹ for every $\eta \in \lim_\lambda(T)$ the set $\{\alpha < \lambda : |\text{succ}_T(\eta \upharpoonright \alpha)| > 1\}$ contains a closed unbounded subset of λ ,
- (c)³ for some closed unbounded set $C \subseteq \lambda$ consisting of limit ordinals we have

$$(\forall t \in T)(\text{lh}(t) \in C \Leftrightarrow |\text{succ}_T(t)| > 1),$$

- (c)⁴ $(\forall t \in T)(\text{root}(T) \triangleleft t \Rightarrow |\text{succ}_T(t)| > 1)$.

(3) Replacing “a closed unbounded set” in (c)¹ and (c)³ by “a set of limit ordinals belonging to the filter E ” will define forcing notions $\mathbb{Q}_{\varphi, \bar{F}, E}^1$ and $\mathbb{Q}_{\varphi, \bar{F}, E}^3$, respectively.

REMARK 2.21. If λ is strongly inaccessible and φ, \bar{F} are as in 2.20, then $\mathbb{Q}_{\varphi, \bar{F}}^3$ is a dense subset of $\mathbb{Q}_{\varphi, \bar{F}}^2$.

DEFINITION 2.22.

(1) Assume that $\psi : \lambda \rightarrow (\lambda + 1) \setminus \{0, 1\}$ and E is a normal filter on λ . We define a forcing notion \mathbb{S}_E^ψ as follows.

A condition in \mathbb{S}_E^ψ is a function p such that $\text{dom}(p) \subseteq \lambda$, $\lambda \setminus \text{dom}(p) \in E$ and $p(i) < \psi(i)$ for each $i \in \text{dom}(p)$.

The order \leq of \mathbb{S}_E^ψ is inclusion, i.e., $p \leq q$ if and only if $p, q \in \mathbb{S}_E^\psi$ and $p \subseteq q$.

For $p \in \mathbb{S}_E^\psi$ we also set $\text{rt}(p) = \min(\lambda \setminus \text{dom}(p))$ (the notation rt points to analogy with “the length of the root”).

- (2) A condition in a forcing notion \mathbb{H}_λ is a pair (s, g) such that $s \in {}^{\lambda>}\lambda$ and $g \in {}^\lambda\lambda$.

The order \leq of \mathbb{H}_λ is defined by: $(s, g) \leq (s', g')$ if and only if $((s, g), (s', g')) \in \mathbb{H}_\lambda$ and $s \sqsubseteq s'$, $g(\alpha) \leq s'(\alpha)$ for each $\alpha \in [\text{lh}(s), \text{lh}(s'))$ and $g(\alpha) \leq g'(\alpha)$ for all $\alpha \in [\text{lh}(s'), \lambda)$.

- (3) A condition in a forcing notion \mathbb{E}_λ is a pair (s, \bar{G}) such that $s \in {}^{\lambda>}\lambda$, $\bar{G} = \langle G_\beta : \beta < \lambda \rangle$ and for some $\mu < \lambda$, for every $\beta < \lambda$ we have $G_\beta \in [\lambda]^{\leq \mu}$.

The order \leq of \mathbb{E}_λ is defined by: $(s, \bar{G}) \leq (s', \bar{G}')$ if and only if $((s, \bar{G}), (s', \bar{G}')) \in \mathbb{E}_\lambda$ and $s \sqsubseteq s'$, $s'(\alpha) \notin G_\alpha$ for each $\alpha \in [\text{lh}(s), \text{lh}(s'))$ and $G_\alpha \subseteq G'_\alpha$ for all $\alpha \in [\text{lh}(s'), \lambda)$.

OBSERVATION 2.23. Assume $\lambda^{<\lambda} = \lambda$.

- (1) For \bar{E} as in 2.18 and $\ell \in \{1, 2, 3, 4\}$, the forcing notion $\mathbb{Q}^{\ell, \bar{E}}$ is $(<\lambda)$ -lub-complete (i.e., increasing sequences of length $<\lambda$ have least upper bounds). Likewise for $\mathbb{Q}_E^{1, \bar{E}}$ and $\mathbb{Q}_E^{3, \bar{E}}$.
- (2) For φ, \bar{F} as in 2.20 and $\ell \in \{1, 2, 3, 4\}$, the forcing notion $\mathbb{Q}_{\varphi, \bar{F}}^\ell$ is strategically $(<\lambda)$ -complete. Moreover, if $\bar{T} = \langle T_\alpha : \alpha < \delta \rangle \subseteq \mathbb{Q}_{\varphi, \bar{F}}^\ell$ is $\leq_{\mathbb{Q}_{\varphi, \bar{F}}^\ell}$ -increasing and $\text{root}(T_\alpha) \triangleleft \text{root}(T_\beta)$ for $\alpha < \beta < \delta < \lambda$, then $\bigcap_{\alpha < \delta} T_\alpha \in \mathbb{Q}_{\varphi, \bar{F}}^\ell$ is the least upper bound on \bar{T} . Similarly for $\mathbb{Q}_{\varphi, \bar{F}, E}^1$ and $\mathbb{Q}_{\varphi, \bar{F}, E}^3$.
- (3) For E and ψ as in 2.22, the forcing notion \mathbb{S}_E^ψ is $(<\lambda)$ -lub-complete.
- (4) The forcing notions \mathbb{H}_λ and \mathbb{E}_λ are $(<\lambda)$ -lub-complete.

Arguably, one of the most important properties of forcing notions with trees in the set theory of the reals is the possibility to have fusion for suitable ω -sequences of trees. Similar properties hold for our forcing notions (with respect to λ -sequences of conditions). The next two lemmas are actually fusion lemmas exemplifying this similarity.

LEMMA 2.24. Assume that either

- \bar{E} and E are as in 2.18, $1 \leq \ell \leq 4$, and \mathbb{P} is $\mathbb{Q}^{\ell, \bar{E}}$ or $\mathbb{Q}_E^{1, \bar{E}}$ or $\mathbb{Q}_E^{3, \bar{E}}$, or
- $\lambda, \varphi, \bar{F}$ and E are as in 2.20, $1 \leq \ell \leq 4$ and \mathbb{P} is $\mathbb{Q}_{\varphi, \bar{F}}^\ell$ or $\mathbb{Q}_{\varphi, \bar{F}, E}^1$ or $\mathbb{Q}_{\varphi, \bar{F}, E}^3$.

Suppose that $\gamma \leq \lambda$ and $T^\delta \in \mathbb{P}$ for $\delta < \gamma$ are such that

- (i) $T^{\delta+1} \subseteq T^\delta$ and $T^\delta \cap {}^\delta\lambda = T^{\delta+1} \cap {}^\delta\lambda$,
- (ii) if δ is limit, then $T^\delta = \bigcap_{i < \delta} T^i$, and
- (iii) if $t \in T^\delta \cap {}^\delta\lambda$ and $|\text{succ}_{T^\delta}(t)| > 1$, then $|\text{succ}_{T^{\delta+1}}(t)| > 1$.

Then $T^\gamma := \bigcap_{\delta < \gamma} T^\delta \in \mathbb{P}$.

Proof. For \mathbb{Q}_λ^ℓ this was proved in [20, Lemma 3.5]; for other cases the arguments are essentially the same. ■

LEMMA 2.25. *Assume that ψ, E are as in 2.22. Suppose that $p_\delta \in \mathbb{S}_E^\psi$ for $\delta < \lambda$ are such that*

- (i) $p_\delta \subseteq p_{\delta+1}$ and $p_\delta \upharpoonright (\delta + 1) = p_{\delta+1} \upharpoonright (\delta + 1)$, and
- (ii) if δ is limit, then $p_\delta = \bigcup_{i < \delta} p_i$.

Then $p_\lambda := \bigcup_{\delta < \lambda} p_\delta \in \mathbb{S}_E^\psi$.

Proof. Clearly p_λ is a function with $\text{dom}(p) = \bigcup_{\delta < \lambda} \text{dom}(p_\delta) \subseteq \lambda$ and it satisfies $p_\lambda(i) < \psi(i)$ for each $i \in \text{dom}(p_\lambda)$. We should argue that $\lambda \setminus \text{dom}(p_\lambda) \in E$. For this we just note that if $\delta \in \Delta_{\alpha < \lambda}(\lambda \setminus \text{dom}(p_\alpha))$ is a limit ordinal, then $\delta \notin \bigcup_{\alpha < \delta} \text{dom}(p_\alpha) = \text{dom}(p_\delta)$ (by (ii)) and hence $\delta \notin \text{dom}(p_{\delta+1})$ (by (i)). The assumption (i) also implies that $\delta \notin \text{dom}(p_\lambda)$. ■

3. Sequential purity with diamonds. For the rest of the paper we assume the following context.

CONTEXT 3.1.

- (1) λ is a regular uncountable cardinal with $\lambda^{<\lambda} = \lambda$.
- (2) D is a normal filter on λ .
- (3) A set $\mathcal{S} \in D^+$ contains all successor ordinals below λ , $0 \notin \mathcal{S}$ and $\lambda \setminus \mathcal{S}$ is unbounded in λ . For an ordinal $\gamma < \lambda$ we define $\mathcal{S}[\gamma] = \mathcal{S} \setminus \{\delta \leq \gamma : \delta \text{ is limit}\}$.
- (4) \mathcal{R} is the closure of $\lambda \setminus \mathcal{S}$ and $\bar{\gamma} = \langle \gamma_\alpha : \alpha < \lambda \rangle$ is the increasing enumeration of \mathcal{R} (so the sequence $\bar{\gamma}$ is increasing continuous, $\gamma_0 = 0$ and all other terms of $\bar{\gamma}$ are limit ordinals).

DEFINITION 3.2. A sequence $\bar{f} = \langle f_\delta : \delta \in \mathcal{S} \rangle$ is a (D, \mathcal{S}) -diamond if $f_\delta \in {}^\delta \delta$ for $\delta \in \mathcal{S}$ and $(\forall \eta \in {}^\lambda \lambda)(\{\delta \in \mathcal{S} : f_\delta \triangleleft \eta\} \in D^+)$.

OBSERVATION 3.3. *Let \mathbb{P} be a strategically $(<\lambda)$ -complete forcing notion, and D be a normal filter on λ .*

- (1) $\Vdash_{\mathbb{P}}$ “The family of all supersets of diagonal intersections of members of $D^{\mathbf{V}}$ is a (proper) normal filter on λ ”. (Abusing our notation, the (name for the) normal filter generated by D in $\mathbf{V}^{\mathbb{P}}$ will also be denoted by D or sometimes by $D^{\mathbf{V}[\mathbb{G}_{\mathbb{P}}]}$.)
- (2) If \bar{f} is a (D, \mathcal{S}) -diamond, then $\Vdash_{\mathbb{P}}$ “ \bar{f} is a $(D^{\mathbf{V}[\mathbb{G}_{\mathbb{P}}]}, \mathcal{S})$ -diamond”.

A pure extension of a tree is a subtree with the same root. This concept appears in several places in the classical forcing with trees, e.g., in the Laver forcing every sentence can be decided by passing to a pure extension of a condition. The relations \leq_{pr} of pure extensions were important ingredients of the properties studied in [18, Section 2] and [20]. The property introduced in

the present paper also involves pure extensions, but for technical reasons they are considered for bounds of increasing sequences of conditions. However, for our “test tree forcing notions” the intuitions behind the sequential purity R^{pr} introduced in Definition 3.4 below should be that it is essentially the same as \leq_{pr} . For instance, in \mathbb{Q}_λ^2 we say that $T \in \mathbb{Q}_\lambda^2$ is a *pure bound* on $\bar{T} = \langle T_\alpha : \alpha < \delta \rangle$ if \bar{T} is an increasing sequence of conditions from \mathbb{Q}_λ^2 , $\delta < \lambda$ is a limit ordinal, $\alpha \leq \text{lh}(\text{root}(T_\alpha)) < \delta$ and T is a pure extension of $\bigcap_{\alpha < \delta} T_\alpha$. (We will then write $\bar{T} R^{\text{pr}} T$.) Plainly, an increasing sequence of conditions in \mathbb{Q}_λ^2 of length $< \lambda$, all with the same root t , has an upper bound with root t . Therefore, the relation R^{pr} introduced above is actually a λ -sequential⁺ purity on \mathbb{Q}_λ^2 .

DEFINITION 3.4. Let \mathbb{Q} be a forcing notion. A binary relation R^{pr} is called a λ -sequential purity on \mathbb{Q} whenever $\bar{r} R^{\text{pr}} r$ implies

- (a) $\bar{r} = \langle r_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbb{Q}}$ -increasing sequence of conditions from \mathbb{Q} of limit length $\delta < \lambda$, and
- (b) $r \in \mathbb{Q}$ is an upper bound of \bar{r} (i.e., $r_\alpha \leq_{\mathbb{Q}} r$ for all $\alpha < \delta$).

If, additionally, the relation R^{pr} satisfies

- (c) if $\bar{r} = \langle r_\alpha : \alpha < \delta \rangle$, $\bar{r} R^{\text{pr}} s_\beta$ for $\beta < \xi$, $\xi < |\delta|^+$ and $s_\beta \leq s_\gamma$ for $\beta < \gamma < \xi$, then there is a condition $s \in \mathbb{Q}$ stronger than all s_β (for $\beta < \xi$) and such that $\bar{r} R^{\text{pr}} s$,

then we say that R^{pr} is a λ -sequential⁺ purity on \mathbb{Q} .

The next definition introduces the main technical concept of the paper: the game $\mathfrak{D}_\gamma^S(r, N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{q})$ between two players, Generic and Antigeneric. Suppose that $\mathbb{Q} = \mathbb{Q}_\lambda^2$. In a play of \mathfrak{D}_γ^S the two players construct two descending (with respect to inclusion) sequences of trees $\langle T_i^- : i < \lambda \rangle \subseteq \mathbb{Q}_\lambda^2 \cap N$ and $\langle T_i : i < \lambda \rangle \subseteq \mathbb{Q}_\lambda^2$ with the property that oftentimes $T_i \subseteq T_i^-$. If $i \in S[\gamma]$ then the trees T_i^-, T_i are chosen by Generic, otherwise they are picked by Antigeneric. It is not unreasonable to think that for $i < j$ we have

$$\text{root}(T_i^-) \trianglelefteq \text{root}(T_i) \triangleleft \text{root}(T_j^-) \trianglelefteq \text{root}(T_j).$$

Then for many limit $\delta < \lambda$, a diamond sequence \bar{f} guesses (via some coding function h) the sequence $\langle T_i^- : i < \delta \rangle$, and $\bigcap_{i < \delta} T_i^- \in \mathbb{Q}_\lambda^2 \cap N$ is a tree with

$$\text{root}\left(\bigcap_{i < \delta} T_i^-\right) = \bigcup_{i < \delta} \text{root}(T_i^-) = \bigcup_{i < \delta} \text{root}(T_i).$$

The task of Generic will be to make sure that oftentimes for these limit δ ,

$$T_\delta \subseteq q_\delta \quad \text{and} \quad \text{root}(T_\delta) = \text{root}(q_\delta) = \text{root}\left(\bigcap_{i < \delta} T_i^-\right).$$

DEFINITION 3.5 (Compare [15, Definition 2.1]). Let (\mathbb{Q}, \leq) be a strategically $(< \lambda)$ -complete forcing notion and R^{pr} be a λ -sequential purity on \mathbb{Q} . Suppose that a model $N \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ is such that $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ and $\lambda, \mathbb{Q}, D, \mathcal{S}, \dots \in N$ (but note we do not demand $R^{\text{pr}} \in N$) and a function $h : \lambda \rightarrow N$ is such that its range $\text{rng}(h)$ includes $\mathbb{Q} \cap N$. Also, let $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle$ list all dense open subsets of \mathbb{Q} belonging to N and let $\gamma < \lambda$.

- (1) We say that a sequence $\bar{f} = \langle f_\delta : \delta \in \mathcal{S} \rangle$ is a (D, \mathcal{S}, h) -semi-diamond for \mathbb{Q} over N if $f_\delta \in {}^\delta \delta$ for $\delta \in \mathcal{S}$ and

(*) for every $\leq_{\mathbb{Q}}$ -increasing sequence $\bar{p} = \langle p_\alpha : \alpha < \lambda \rangle \subseteq \mathbb{Q} \cap N$ we have

$$\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h(f_\delta(\alpha)) = p_\alpha)\} \in D^+.$$

Below, let \bar{f} be a (D, \mathcal{S}, h) -semi-diamond for \mathbb{Q} over N .

- (2) An $(N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{\mathcal{I}})$ -candidate is a sequence $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ limit} \rangle$ of conditions from $N \cap \mathbb{Q}$ satisfying, for each limit $\delta \in \mathcal{S}$,

(a) if $h \circ f_\delta = \langle h(f_\delta(\alpha)) : \alpha < \delta \rangle \subseteq \mathbb{Q} \cap N$ and it has an upper bound in \mathbb{Q} , then $h(f_\delta(\alpha)) \leq q_\delta$ for all $\alpha < \delta$,

(b) if moreover $h \circ f_\delta \in \text{dom}(R^{\text{pr}})$, then also $h \circ f_\delta R^{\text{pr}} q_\delta$, and

(c) if there is $q \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$ such that $h \circ f_\delta R^{\text{pr}} q$, then also $q_\delta \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$.

If $N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{\mathcal{I}}$ are obvious from the context we may just say a candidate.

- (3) Let $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \ \& \ \delta \text{ is limit} \rangle$ be a candidate and $r \in \mathbb{Q}$. We define a game $\mathfrak{D}_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{q})$ of two players, *Generic* and *Antigeneric*, as follows. A play lasts $\leq \lambda$ moves and in the i th move the players try to choose conditions $r_i^-, r_i \in \mathbb{Q}$ and a set $C_i \in D$ so that ⁽³⁾:

(a) $r \leq r_i, r_i^- \in N$, and if $i \notin \mathcal{S}[\gamma] \cap \mathcal{R}$ then $r_i^- \leq r_i$,

(b) $(\forall i < j < \lambda)(r_i \leq r_j \ \& \ r_i^- \leq r_j^-)$, and

(c) *Generic* chooses r_i^-, r_i, C_i if $i \in \mathcal{S}[\gamma]$, and *Antigeneric* chooses r_i^-, r_i, C_i if $i \notin \mathcal{S}[\gamma]$.

At the end *Generic* wins the play whenever both players always had legal moves (so the game lasted λ steps) and

(*) if $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i < \delta} C_i$ is a limit ordinal and $h \circ f_\delta$ is an increasing sequence of conditions in \mathbb{Q} such that for all $\alpha < \delta$ we have $h(f_\delta(\alpha + 1)) = r_{\alpha+1}^-$, then $q_\delta \leq r_\delta$ and $h \circ f_\delta R^{\text{pr}} r_\delta$.

- (4) Let \bar{q} be a candidate (for $N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}$ and $\bar{\mathcal{I}}$). A condition $r \in \mathbb{Q}$ is *generic for the candidate \bar{q} over $N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$* if *Generic* has a winning strategy in the game $\mathfrak{D}_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{q})$.

⁽³⁾ Remember Context 3.1(3,4).

DEFINITION 3.6 (Compare [15, Definition 2.3]). Let \mathbb{Q} be a strategically $(< \lambda)$ -complete forcing notion.

(1) We say that \mathbb{Q} is *purely sequentially proper over (D, \mathcal{S}) -semi-diamonds* whenever the following condition (\odot) is satisfied:

(\odot) Assume that χ is a large enough regular cardinal and $N \prec (\mathcal{H}(\chi), \in, <_\chi^*)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ and $\lambda, \mathbb{Q}, D, \mathcal{S}, \dots \in N$. Then there exists a λ -sequential purity R^{PF} on \mathbb{Q} such that for every ordinal $\gamma < \lambda$, a condition $p \in \mathbb{Q} \cap N$ and every $\bar{\mathcal{I}}, h, \bar{f}, \bar{q}$ satisfying

- $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle$ lists all open dense subsets of \mathbb{Q} from N ,
- a function $h : \lambda \rightarrow N$ is such that $\mathbb{Q} \cap N \subseteq \text{rng}(h)$,
- a sequence \bar{f} is a (D, \mathcal{S}, h) -semi-diamond for \mathbb{Q} , and
- \bar{q} is an $(N, h, \mathbb{Q}, R^{\text{PF}}, \bar{f}, \bar{\mathcal{I}})$ -candidate,

there is a condition $r \geq p$ generic for \bar{q} over $N, h, \mathbb{Q}, R^{\text{PF}}, \bar{f}, \mathcal{S}, \gamma$.

(2)(a) If in (\odot) above the relation R^{PF} can be required to be a λ -sequential⁺ purity, then we say that \mathbb{Q} is *purely sequentially⁺ proper over (D, \mathcal{S}) -semi-diamonds*,

(b) if the relation R^{PF} does not depend on N , then we add the adjective *uniformly* (so we then say *uniformly purely sequentially proper* etc.).

REMARK 3.7. The game $\mathfrak{D}_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\text{PF}}, \bar{f}, \bar{q})$ is very similar to the game $\mathfrak{D}(r, N, h, \mathbb{Q}, \bar{f}, \bar{q})$ of [15, Definition 2.1(3)]. Some of the differences in the definition of candidates are caused by the larger generality and the use of R^{PF} . But one technical point is actually correcting an error in [15]. When comparing [15, Definition 2.1(3)(\otimes)] and Definition 3.5 here, one notices that in the former paper the condition (\otimes) is “active” if the semi-diamond guessed $\langle r_i^- : i < \delta \rangle$, while here we “activate” this demand if *the successor terms of the sequence* were guessed correctly. In other words, the current (\otimes) is “active” more often. Now, in the proof of [15, Main Claim 3.10] we actually use this more often “active” version of (\otimes) : on page 73, in the paragraph between (3.14) and (3.15), we set $r_{j_0}^\ominus(\varepsilon_\gamma) = s^-(\varepsilon_\gamma)$ and not $r_{j_0}^\ominus(\varepsilon_\gamma) = r_{j_0}^-(\varepsilon_\gamma)$. This makes “guessing on coordinate ε_γ ” different from “guessing in \mathbb{P}_{ζ^*} ”. To avoid the problem we make limit terms of semi-diamond guessing irrelevant. This change should be introduced to [15, Definition 2.1(3)(\otimes)] and this condition should read:

$(\otimes)^{\text{corr}}$ if $\delta \in \mathcal{S} \cap \bigcap_{i < \delta} C_i$ is a limit ordinal and $h \circ F_\delta$ is an increasing sequence of conditions satisfying $h(F_\delta(\alpha + 1)) = r_{\alpha+1}^-$ for all $\alpha < \delta$, then $q_\delta \leq r_\delta$.

Then *the properness over (D, \mathcal{S}) -semi-diamonds* of [15, Definition 2.3] becomes a potentially stronger property, but the examples presented there still satisfy it.

Now,

(\otimes) if \mathbb{P} is proper over $(D, \mathcal{S}[\gamma])$ -semi-diamonds (see [15, Definition 2.3] plus the correction stated above) for each $\gamma < \lambda$, then \mathbb{P} is uniformly purely sequentially⁺ proper over (D, \mathcal{S}) -semi-diamonds.

Why? First note that \mathbb{P} is $(<\lambda)$ -complete and define a binary relation R^{pr} by $\bar{p} R^{\text{pr}} p$ if and only if

- $\bar{p} = \langle p_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbb{P}}$ -increasing sequence of conditions in \mathbb{P} , $\delta < \lambda$ is a limit ordinal and p is an upper bound on \bar{p} .

Then R^{pr} is a λ -sequential⁺ purity. Suppose $N, h, \bar{f}, \bar{\mathcal{I}}, \gamma, \bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ is limit} \rangle$ and p are as in 3.6(1)(\odot). For $\delta \in \mathcal{S}[\gamma]$ let $q_\delta^* \in \mathbb{P}$ be such that

- if δ is limit, $h \circ f_\delta \subseteq \mathbb{P} \cap N$ is a $\leq_{\mathbb{P}}$ -increasing sequence and q_δ is its upper bound, then $q_\delta^* = q_\delta$; otherwise q_δ^* is the $<_{\chi}^*$ -first member of $\bigcap_{i < \delta} \mathcal{I}_i$.

Plainly, $\bar{q}^* = \langle q_\delta^* : \delta \in \mathcal{S}[\gamma] \rangle$ is an (N, h, \mathbb{P}) -candidate over \bar{f} in the sense of [15, Definition 2.1(2)] (and if $\delta \in \mathcal{S}[\gamma]$ is limit and $h \circ f_\delta$ is an increasing sequence of conditions from \mathbb{P} , then $q_\delta^* = q_\delta$). Let st be a winning strategy of Generic in the game $\mathfrak{D}(r, N, h, \mathbb{P}, \bar{f}, \bar{q}^*)$ of [15, Definition 2.1]. The same strategy can be used by Generic in $\mathfrak{D}_\gamma^{\mathcal{S}}(r, N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{q})$ and clearly it is a winning strategy in this game too.

PROPOSITION 3.8. *Let $\mathbb{Q}, N, h, \bar{\mathcal{I}}, R^{\text{pr}}, \gamma$ be as in Definition 3.5.*

- (1) *Every (D, \mathcal{S}) -diamond is a (D, \mathcal{S}, h) -semi-diamond for \mathbb{Q} over N . Below, let \bar{f} be a (D, \mathcal{S}, h) -semi-diamond for \mathbb{Q} over N .*
- (2) *There exists an $(N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{\mathcal{I}})$ -candidate. Let \bar{q} be one.*
- (3) *If \mathbb{Q} is $(<\lambda)$ -lub-complete⁽⁴⁾, then for any condition $r \in \mathbb{Q}$ both players have always legal moves in the game $\mathfrak{D}_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{q})$ satisfying $r_j^- \leq r_j$ provided that for each $i \in \mathcal{S}[\gamma] \cap \mathcal{R} \cap j$ the conditions r_i^-, r_i are compatible.*
- (4) *If \mathbb{Q} is $(<\lambda)$ -complete and $r \in \mathbb{Q}$ is (N, \mathbb{Q}) -generic in the standard sense, then also both players have always legal moves in the game $\mathfrak{D}_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{q})$ satisfying $r_j^- \leq r_j$ provided that for each $i \in \mathcal{S}[\gamma] \cap \mathcal{R} \cap j$ the conditions r_i^-, r_i are compatible.*
- (5) *If $r \in \mathbb{Q}$ is generic for \bar{q} over $N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$, then r is (N, \mathbb{Q}) -generic in the standard sense.*
- (6) *Consequently, if a forcing notion \mathbb{Q} is purely sequentially proper over (D, \mathcal{S}) -semi-diamonds and there exists a (D, \mathcal{S}) -diamond, then \mathbb{Q} is λ -proper in the standard sense.*

Proof. (5) Let \bar{q} be an $(N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{\mathcal{I}})$ -candidate and $r \in \mathbb{Q}$ be generic for \bar{q} over $N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$. Suppose that $\mathcal{I} \in N$ is an open dense subset

⁽⁴⁾ That is, increasing sequences of length $<\lambda$ have least upper bounds.

of \mathbb{Q} , say $\mathcal{I} = \mathcal{I}_{j_0}$ (remember that $\bar{\mathcal{I}} = \langle \mathcal{I}_i : i < \lambda \rangle$ lists all open dense subsets of \mathbb{Q} belonging to N). We want to argue that $\mathcal{I} \cap N$ is predense above r , so suppose $r_0 \geq r$.

Consider a play of $\partial_\gamma^{\mathcal{S}}(r, N, h, \mathbb{Q}, R^{\text{pr}}, \bar{f}, \bar{q})$ in which Generic follows her winning strategy and Antigeneric plays as follows.

- At stage $i = 0$, Antigeneric sets $C_0 = \lambda$, $r_0^- = \emptyset_{\mathbb{Q}}$ and r_0 is as fixed above.
- At stage $i \notin \mathcal{S}[\gamma]$, $i > 0$, Antigeneric first picks any legal move $(\bar{5})$ C_i, r_i^-, r'_i and then “corrects” it by choosing a condition $r_i \geq r'_i$ such that $r_i \in \bigcap_{j < i} \mathcal{I}_j$ (remember that \mathbb{Q} is strategically $(< \lambda)$ -complete).

After the play is completed and a sequence $\langle C_i, r_i^-, r_i : i < \lambda \rangle$ is constructed, we know that the condition $(\bar{*})$ of Definition 3.5(3) is satisfied. Also, since \bar{f} is a (D, \mathcal{S}, h) -semi-diamond for \mathbb{Q} over N , we know that $\{\delta \in \mathcal{S} : (\forall \alpha < \delta)(h(f_\delta(\alpha)) = r_\alpha^-)\} \in D^+$. Pick a limit ordinal $\delta \in \mathcal{S}[\gamma] \cap \Delta_{i < \lambda} C_i$ such that $\delta > j_0$, δ is a limit of elements of $\lambda \setminus \mathcal{S}$ and $h \circ f_\delta = \langle r_\alpha^- : \alpha < \delta \rangle$. Then by $(\bar{*})$ we have $q_\delta \leq r_\delta$ and $h \circ f_\delta R^{\text{pr}} r_\delta$. Moreover, since $r_\alpha \leq r_\delta$ for all $\alpha < \delta$ and since δ is a limit of points from $\lambda \setminus \mathcal{S}$, we get $r_\delta \in \bigcap_{j < \delta} \mathcal{I}_j$. Therefore (by 3.5(2)(c)) $q_\delta \in \bigcap_{j < \delta} \mathcal{I}_j$, so in particular $q_\delta \in \mathcal{I}_{j_0} \cap N$. But the condition r_δ is stronger than q_δ and it is also stronger than r_0 , so r_0 is compatible with q_δ . ■

PROPOSITION 3.9. *Assume that a forcing notion \mathbb{P} is one of the following types:*

- (a) $\mathbb{P} = \mathbb{S}_E^\psi$ where ψ, E are as in Definition 2.22(1) and

$$A \in E \Rightarrow (\lambda \setminus \mathcal{S}) \cup A \in D,$$

- (b) $\mathbb{P} = \mathbb{Q}^{\ell, \bar{E}}$ where \bar{E} is as in Definition 2.18 and $1 < \ell \leq 4$,

- (c) $\mathbb{P} = \mathbb{Q}_E^{3, \bar{E}}$ where \bar{E}, E are as in Definition 2.18, and

$$A \in E \Rightarrow (\lambda \setminus \mathcal{S}) \cup A \in D,$$

- (d) $\mathbb{P} = \mathbb{Q}_{\varphi, \bar{F}}^\ell$ where $\lambda, \varphi, \bar{F}$ are as in Definition 2.20, $1 < \ell \leq 4$,

- (e) $\mathbb{P} = \mathbb{Q}_{\varphi, \bar{F}, E}^3$ where \bar{E}, E are as in Definition 2.20 and

$$A \in E \Rightarrow (\lambda \setminus \mathcal{S}) \cup A \in D,$$

- (f) $\mathbb{P} = \mathbb{S}_E^\psi$ where ψ, E are as in Definition 2.22, and $\lambda \setminus \mathcal{S} \in E$,

- (g) $\mathbb{P} = \mathbb{Q}_E^{\ell, \bar{E}}$ or $\mathbb{P} = \mathbb{Q}_{\varphi, \bar{F}, E}^\ell$, where $\ell = 1$ or $\ell = 3$, \bar{E}, E are as in Definition 2.18 or $\lambda, \varphi, \bar{F}, E$ are as in Definition 2.20, respectively, and $\lambda \setminus \mathcal{S} \in E$,

- (h) $\mathbb{P} = \mathbb{H}_\lambda$ or $\mathbb{P} = \mathbb{E}_\lambda$ (see Definition 2.22(2, 3)),

- (i) \mathbb{P} is strategically $(\leq \lambda)$ -complete and $(< \lambda)$ -complete.

⁽⁵⁾ Note that since Generic uses her winning strategy, there are always legal moves.

Then the forcing notion \mathbb{P} is uniformly purely sequentially⁺ proper over (D, \mathcal{S}) -semi-diamonds.

Proof. (a) The arguments here are very similar to those for $\mathbb{Q}_{\varphi, \bar{F}, E}^3$ below, but much simpler. We will use the fact that the forcing notion \mathbb{S}_E^ψ is $(<\lambda)$ -lub-complete without explicitly saying this. Define a binary relation R^{pr} by $\bar{p} R^{\text{pr}} p$ if and only if

- $\bar{p} = \langle p_\alpha : \alpha < \delta \rangle$ is an increasing sequence of conditions from \mathbb{S}_E^ψ such that $\delta < \lambda$ is a limit ordinal, and
- $p \in \mathbb{S}_E^\psi$ is an upper bound of the sequence \bar{p} with $\text{rt}(p) = \text{rt}(\bigcup_{\alpha < \delta} p_\alpha)$.

Plainly, R^{pr} is a λ -sequential⁺ purity on \mathbb{S}_E^ψ . Now, let $N, h, \bar{f}, \bar{\mathcal{I}}, \gamma$ be as in the assumptions of 3.6(1)(\odot) and let $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ is limit} \rangle$ be an appropriate candidate. Suppose $p \in \mathbb{S}_E^\psi \cap N$. We will define a condition $p_\lambda \geq p$ generic for \bar{q} . For this, by induction on $\alpha < \lambda$ we choose conditions $p_\alpha \in \mathbb{S}_E^\psi \cap N$ as follows.

- (\blacktriangle)₁ $p_0 = p$ and if δ is a limit ordinal, then $p_\delta = \bigcup_{\xi < \delta} p_\xi$.
- (\blacktriangle)₂ Suppose that $\alpha = \delta + 1$, $\delta \in \mathcal{S}[\gamma]$ is a limit ordinal, and $h \circ f_\delta$ is an increasing sequence of conditions from $\mathbb{S}_E^\psi \cap N$ such that $p_\delta \leq \bigcup_{\beta < \delta} h(f_\delta(\beta))$. Then we let $p_\alpha = (p_\delta \upharpoonright \alpha) \cup (q_\delta \upharpoonright [\alpha, \lambda))$.
- (\blacktriangle)₃ If $\alpha = \delta + 1$ and the clause (\blacktriangle)₂ does not apply, then $p_\alpha = p_\delta$.

Under the assumptions of (\blacktriangle)₂ we have $p_\delta \leq \bigcup_{\beta < \delta} h(f_\delta(\beta)) \leq q_\delta$. Therefore, it should be clear that the construction can be carried out and that the resulting sequence $\langle p_\alpha : \alpha < \lambda \rangle$ satisfies the assumptions of Lemma 2.25. Consequently, $p_\lambda := \bigcup_{\alpha < \lambda} p_\alpha \in \mathbb{S}_E^\psi$.

Let us show that p_λ is generic for \bar{q} over $N, h, \mathbb{S}_E^\psi, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$. Consider the following strategy **st** of Generic in the game $\mathcal{D}_\gamma^{\mathcal{S}}(p_\lambda, N, h, \mathbb{S}_E^\psi, R^{\text{pr}}, \bar{f}, \bar{q})$. At stage $i \in \mathcal{S}[\gamma]$, when a sequence $\langle r_j^-, r_j, C_j : j < i \rangle$ has already been constructed, Generic picks r_i^-, r_i, C_i so that

- (\blacktriangle)₄ if i is limit, then $r_i^- = \bigcup_{j < i} r_j^-$, $r_i = \bigcup_{j < i} r_j$ and $C_i = \bigcap_{j < i} C_j$,
- (\blacktriangle)₅ if i is a successor, say $i = j + 1$, then $r_i^-, r_i \in \mathbb{S}_E^\psi$ are chosen so that $r_j \leq r_i$, $\text{rt}(r_j) < \text{rt}(r_i^-) = \text{rt}(r_i)$, $r_j^- \leq r_i^-$, $p_i \leq r_i^-$ and $r_i^- \leq r_i$. Also, $C_i = \lambda \setminus (\text{dom}(r_i) \cap \mathcal{S})$.

Note that at stage $i = j + 1$, since $r_j^- \leq r_j$ and $p_i \leq p_\lambda \leq r_j$, we know that $r_j^- \cup p_i \in \mathbb{S}_E^\psi \cap N$ and $r_j^- \cup p_i \leq r_j$. Therefore Generic may easily pick r_i^-, r_i as required. Also, by our assumption on E , we are sure that $C_i \in D$.

Suppose now that $\langle r_i^-, r_i, C_i : i < \lambda \rangle$ is a play of $\mathcal{D}_\gamma^{\mathcal{S}}(T^*, N, h, \mathbb{S}_E^\psi, R^{\text{pr}}, \bar{f}, \bar{q})$ in which Generic follows the strategy **st** described above. Let $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i < \delta} C_i$ be a limit ordinal such that $h \circ f_\delta$ is an increasing sequence of

conditions from \mathbb{S}_E^ψ satisfying $h(f_\delta(i+1)) = r_{i+1}^-$ for all $i < \delta$. By $(\blacktriangle)_5 + (\blacktriangle)_4$ we have $i \leq \text{rt}(h(f_\delta(i))) < \delta$ and $p_i \leq_{\mathbb{S}_E^\psi} h(f_\delta(i))$ (for all successor $i < \delta$) and $\delta \notin \text{dom}(\bigcup_{j < \delta} r_j^-) = \text{dom}(\bigcup_{j < \delta} h(f_\delta(j)))$. Therefore $q_\delta \upharpoonright [\delta + 1, \lambda) \subseteq p_{\delta+1} \subseteq p_\lambda \subseteq r_\delta$, $q_\delta \upharpoonright \delta = (\bigcup_{j < \delta} h(f_\delta(j))) \upharpoonright \delta = r_\delta \upharpoonright \delta$ and $\delta = \text{rt}(r_\delta) = \text{rt}(q_\delta) = \text{rt}(\bigcup_{j < \delta} h(f_\delta(j)))$. Consequently, $q_\delta \leq r_\delta$ and $h \circ f_\delta R^{\text{pr}} r_\delta$.

(b, c, d, e) We know that the forcing notion \mathbb{P} is strategically $(< \lambda)$ -complete (see Observation 2.23). Define a binary relation R^{pr} by $\bar{T} R^{\text{pr}} T$ if and only if

- $\bar{T} = \langle T_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbb{P}}$ -increasing sequence of conditions from \mathbb{P} such that $\delta < \lambda$ is a limit ordinal and for $\alpha < \delta$ we have $\alpha \leq \text{lh}(\text{root}(T_\alpha)) < \delta$, and
- $T \in \mathbb{P}$ is a $\leq_{\mathbb{P}}$ -upper bound of the sequence \bar{T} with $\text{root}(T) = \text{root}(\bigcap_{\alpha < \delta} T_\alpha)$. (Note that the previous bullet implies $\bigcap_{\alpha < \delta} T_\alpha \in \mathbb{P}$. In cases (b) and (d) we also know that $\bigcup_{\alpha < \delta} \text{root}(T_\alpha) = \text{root}(\bigcap_{\alpha < \delta} T_\alpha)$.)

It is easy to see that R^{pr} is a λ -sequential⁺ purity on \mathbb{P} . Suppose that $N \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ and $h, \bar{f}, \bar{\mathcal{I}}, \gamma$ are as in the assumptions of 3.6(1)(\odot). Let $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ is limit} \rangle$ be an $(N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{\mathcal{I}})$ -candidate and let $T \in \mathbb{P} \cap N$. We want to find a condition $T^* \geq T$ generic for \bar{q} over $N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$. To this end we choose by induction on $\alpha < \lambda$ conditions $T_\alpha \in \mathbb{P} \cap N$ as follows.

(Δ)₁ $T_0 = T$ and if δ is a limit ordinal, then $T_\delta = \bigcap_{\xi < \delta} T_\xi$.

(Δ)₂ Suppose that $\alpha = \delta + 1$, $\delta \in \mathcal{S}[\gamma]$ is a limit ordinal, and $h \circ f_\delta$ is a $\leq_{\mathbb{P}}$ -increasing sequence of conditions from $\mathbb{P} \cap N$ such that for all successors $\beta < \delta$ we have

$$\beta \leq \text{lh}(\text{root}(h(f_\delta(\beta)))) < \delta \quad \text{and} \quad T_\beta \leq h(f_\delta(\beta)).$$

Let $T_\alpha^* = \{t \in T_\delta : \text{root}(\bigcap_{\beta < \delta} h(f_\delta(\beta))) \leq t \Rightarrow t \in q_\delta\}$. If $\ell \neq 3$ then $T_\alpha^* \in \mathbb{P}$ and we put $T_\alpha = T_\alpha^*$. Otherwise, we choose $T_\alpha \in \mathbb{P}$ such that $T_\alpha \subseteq T_\alpha^*$ and $T_\alpha \cap \alpha \lambda = T_\alpha^* \cap \alpha \lambda$ and $\text{root}(\bigcap_{\beta < \delta} h(f_\delta(\beta))) \in T_\alpha$.

(Δ)₃ If $\alpha = \delta + 1$ and the clause (Δ)₂ does not apply, then $T_\alpha = T_\delta$.

Suppose that $\alpha = \delta + 1$ satisfies the assumptions of (Δ)₂. Then $\bigcap_{\beta < \delta} h(f_\delta(\beta)) \in \mathbb{P}$ (see Observation 2.23) and $T_\delta \leq \bigcap_{\beta < \delta} h(f_\delta(\beta)) \leq q_\delta$ and $\text{root}(q_\delta) = \text{root}(\bigcap_{\beta < \delta} h(f_\delta(\beta))) \in T_\delta$. Let $t_\delta = \text{root}(\bigcap_{\beta < \delta} h(f_\delta(\beta)))$. If $\text{lh}(t_\delta) = \delta$, then also $t_\delta = \bigcup_{\beta < \delta} \text{root}(h(f_\delta(\beta)))$ and $\text{succ}_{T_{\delta+1}}(t_\delta) = \text{succ}_{q_\delta}(t_\delta)$. (Note that in cases (b) and (d) this must happen.) If $\text{lh}(t_\delta) > \delta$, then $\bigcup_{\beta < \delta} \text{root}(h(f_\delta(\beta))) = t_\delta \upharpoonright \delta \triangleleft t_\delta$ and $\text{succ}_{T_{\delta+1}}(t_\delta \upharpoonright \delta) = \text{succ}_{T_\delta}(t_\delta \upharpoonright \delta)$. (This may happen only in cases (c) and (e).)

Therefore we may conclude that the sequence $\langle T_\alpha : \alpha < \lambda \rangle$ satisfies the assumptions of Lemma 2.24. At limit stages $\delta < \lambda$ this implies that

$T_\delta = \bigcap_{\beta < \delta} T_\beta \in \mathbb{P}$ (so the construction can be carried out) and for λ we get $T^* := \bigcap_{\alpha < \lambda} T_\alpha \in \mathbb{P}$.

CLAIM 3.9.1. T^* is (N, \mathbb{P}) -generic in the standard sense.

Proof of the Claim. Suppose towards a contradiction that there are $\xi_0 < \lambda$ and a condition $T^+ \geq T^*$ such that T^+ is incompatible with every member of $\mathcal{I}_{\xi_0} \cap N$. Construct inductively a sequence $\langle S_\alpha : \alpha < \lambda \rangle$ of conditions in \mathbb{P} such that

- $T^+ \leq S_\beta \leq S_\alpha$, $\text{lh}(\text{root}(S_\beta)) < \text{lh}(\text{root}(S_\alpha))$ for $\beta < \alpha < \lambda$, and
- $S_{\alpha+1} \in \bigcap_{\xi \leq \alpha} \mathcal{I}_\xi$ and if α is limit, then $S_\alpha = \bigcap_{\beta < \alpha} S_\beta$ (for all $\alpha < \lambda$).

Let $t_\alpha = \text{root}(S_\alpha)$. Since $S_\alpha \subseteq T^+ \subseteq T^* \subseteq T_\alpha$ we see that $t_\alpha \in T_\alpha$ and $(T_\alpha)_{t_\alpha} \leq S_\alpha$. Now we claim that

(\heartsuit) there is a limit ordinal $\delta \in \mathcal{S}[\gamma]$ such that $\xi_0 < \delta$ and

$$h \circ f_\delta = \langle (T_\alpha)_{t_\alpha} : \alpha < \delta \rangle, \quad t_\delta = \bigcup_{\alpha < \delta} t_\alpha, \quad \text{lh}(t_\delta) = \delta \quad \text{and} \quad S_\delta \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha.$$

This should be clear in cases (b) and (d) (as closed unbounded subsets of λ belong to D). In cases (c) and (e) we note that

$$A := \{\delta \in \mathcal{S}[\gamma] : \delta \text{ limit and } h \circ f_\delta = \langle (T_\alpha)_{t_\alpha} : \alpha < \delta \rangle\} \in D^+.$$

Letting $\eta = \bigcup_{\alpha < \lambda} t_\alpha$ we have $\eta \in \lim_\lambda(S_\alpha)$, so

$$A_\alpha := \{\delta < \lambda : |\text{succ}_{S_\alpha}(\eta \upharpoonright \delta)| > 1\} \in E$$

(for each $\alpha < \lambda$). By our additional assumptions on E we conclude that

$$A \cap \bigtriangle_{\alpha < \lambda} A_\alpha \cap \{\delta \in \mathcal{S}[\gamma] \setminus (\xi_0 + 1) : (\forall \alpha < \delta)(\text{lh}(t_\alpha) < \delta)\} \neq \emptyset.$$

Then any δ from the set above will witness (\heartsuit).

So let δ be as given by (\heartsuit). Then $h \circ f_\delta R^{\text{pr}} S_\delta$, so by 3.5(2) we conclude $q_\delta \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$ (in particular $q_\delta \in \mathcal{I}_{\xi_0} \cap N$). But $q_\delta \leq (T_{\delta+1})_{t_\delta} \leq (T^*)_{t_\delta} \leq (T^+)_{t_\delta}$, contradicting the choice of T^+ . ■

Now we are ready to argue that T^* is generic for \bar{q} over $N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$ and for this let us consider the following strategy **st** of Generic in the game $\mathcal{D}_\gamma^{\mathcal{S}}(T^*, N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{q})$. At stage $i \in \mathcal{S}[\gamma]$, when a sequence $\langle r_j^-, r_j, C_j : j < i \rangle$ has already been constructed, Generic picks r_i^-, r_i, C_i so that

(Δ)₄ if i is limit, then $r_i^- = \bigcap_{j < i} r_j^-$, $r_i = \bigcap_{j < i} r_j$ and $C_i = \bigcap_{j < i} C_j$,

(Δ)₅ if i is a successor, say $i = j + 1$, then r_i^-, r_i are chosen so that $r_j \leq r_i$, $\text{root}(r_j) \triangleleft \text{root}(r_i^-) = \text{root}(r_i)$, $r_j^- \leq r_i^-$, $T_i \leq r_i^-$ and $r_i^- \leq r_i$. Also, $C_i = \lceil \text{lh}(\text{root}(r_i)) + 1001, \lambda \rceil$ in cases (b), (d), and $C_i = (\lambda \setminus \mathcal{S}) \cup \{\alpha < \lambda : (\exists t \in r_i)(\text{lh}(t) = \alpha \wedge |\text{succ}_{r_i}(t)| > 1)\}$ in cases (c), (e).

Why are the choices possible? By (Δ)₅ at previous stages, for each limit $i < \lambda$ the intersection of the conditions played so far is in \mathbb{P} (remember

Observation 2.23). Note also that then $r_i^- \leq r_i$. At stage $i = j + 1$, since $r_j^- \leq r_j$, $T_i \leq T^* \leq r_j$ and T^* is (N, \mathbb{P}) -generic, we may find $r^i \in \mathbb{P} \cap N$ stronger than both r_j^- and T_i and compatible with r_j . Then we may choose r_i^* stronger than both r^i and r_j . Pick $t \in r_i^*$ such that $|\text{succ}_{r_i^*}(t)| > 1$ and $\text{lh}(t) > \text{lh}(\text{root}(r^i))$ and set $r_i^- = (r^i)_t$ and $r_i = (r_i^*)_t$.

Note also that there are always legal moves for Antigeneric (use Observation 2.23 and the choices in $(\Delta)_5$).

Now suppose that $\langle r_i^-, r_i, C_i : i < \lambda \rangle$ is a play of $\mathfrak{D}_\gamma^{\mathcal{S}}(T^*, N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{q})$ in which Generic follows the strategy **st** described above. Let $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i < \delta} C_i$ be a limit ordinal such that $h \circ f_\delta$ is an increasing sequence of conditions in \mathbb{P} satisfying

$$h(f_\delta(i)) = r_i^- \quad \text{for all successor } i < \delta.$$

Then our choices in $(\Delta)_5 + (\Delta)_4$ imply that $i \leq \text{lh}(\text{root}(h(f_\delta(i)))) < \delta$ and $h(f_\delta(i)) \geq T_i$ (for all successor $i < \delta$). Let $t = \bigcup_{i < \delta} \text{root}(r_i^-)$. Since $\text{lh}(t) = \delta \in C_i$ for all $i < \delta$, we know that $|\text{succ}_{r_i}(t)| > 1$ for each $i < \delta$ and therefore $t = \text{root}(\bigcap_{i < \delta} r_i) = \text{root}(\bigcap_{i < \delta} r_i^-)$. Now using $(\Delta)_2 + (\Delta)_4$ we conclude $q_\delta = (T_{\delta+1})_t \leq (T^*)_t \leq (r_\delta)_t = r_\delta$ and $h \circ f_\delta R^{\text{pr}} r_\delta$.

(f) As in the proof of (a) above we note that \mathbb{S}_E^ψ is $(< \lambda)$ -lub-complete and we will use this fact several times (without saying this). We define a binary relation R^{pr} by $\bar{p} R^{\text{pr}} p$ if and only if

- $\bar{p} = \langle p_\alpha : \alpha < \delta \rangle \subseteq \mathbb{S}_E^\psi$ is an increasing sequence of conditions such that $\delta < \lambda$ is a limit ordinal, and
- $p \in \mathbb{S}_E^\psi$ is an upper bound of the sequence \bar{p} .

Then R^{pr} is a λ -sequential⁺ purity on \mathbb{S}_E^ψ . Let $N, h, \bar{f}, \bar{\mathcal{I}}, \gamma$ be as in the assumptions of 3.6(1)(\odot) and \bar{q} be a candidate. Suppose $p \in \mathbb{S}_E^\psi \cap N$. Choose inductively conditions $p_\alpha \in \mathbb{S}_E^\psi \cap N$ (for $\alpha < \lambda$) so that

- (\blacktriangledown)₁ $p_0 \geq p$ satisfies $\mathcal{S} \subseteq \text{dom}(p_0)$ and if δ is limit, then $p_\delta = \bigcup_{\xi < \delta} p_\xi$,
- (\blacktriangledown)₂ if $\alpha = \delta + 1$, $\delta \in \mathcal{S}[\gamma]$ is a limit ordinal, and $h \circ f_\delta$ is an increasing sequence of conditions from $\mathbb{S}_E^\psi \cap N$ such that $p_\delta \leq \bigcup_{\beta < \delta} h(f_\delta(\beta))$, then $p_\alpha = (p_\delta \upharpoonright \alpha) \cup (q_\delta \upharpoonright [\alpha, \lambda))$, and
- (\blacktriangledown)₃ if $\alpha = \delta + 1$ and the clause (\blacktriangledown)₂ does not apply, then $p_\alpha = p_\delta$.

It should be clear that the construction can be completed. We note that if $\alpha = \delta + 1$ satisfies the assumptions of (\blacktriangledown)₂, then (by the choice of p_0) $\delta \in \text{dom}(p_\delta)$. Clearly the sequence $\langle p_\alpha : \alpha < \lambda \rangle$ satisfies the assumptions of Lemma 2.25 and hence $p_\lambda := \bigcup_{\alpha < \lambda} p_\alpha \in \mathbb{S}_E^\psi$.

We claim that p_λ is generic for \bar{q} . To see this, consider the following strategy **st** for Generic in the game $\mathfrak{D}_\gamma^{\mathcal{S}}(T^*, N, h, \mathbb{S}_E^\psi, R^{\text{pr}}, \bar{f}, \bar{q})$. At stage $i \in \mathcal{S}[\gamma]$,

when a sequence $\langle r_j^-, r_j, C_j : j < i \rangle$ has already been constructed, Generic picks r_i^-, r_i, C_i so that

- (\blacktriangledown)₄ if i is limit, then $r_i^- = \bigcup_{j < i} r_j^-$, $r_i = \bigcup_{j < i} r_j$ and $C_i = \bigcap_{j < i} C_j$, and
- (\blacktriangledown)₅ if i is a successor, say $i = j + 1$, then r_i^-, r_i are chosen so that $r_j \leq r_i$, $\text{rt}(r_j) < \text{rt}(r_i^-)$, $r_j^- \leq r_i^-$, and $p_i \leq r_i^- \leq r_i$. Also, $C_i = [\text{rt}(r_i) + 1001, \lambda)$.

Suppose now that $\langle r_i^-, r_i, C_i : i < \lambda \rangle$ is a play of $\mathfrak{D}_\gamma^S(T^*, N, h, \mathbb{S}_E^\psi, R^{\text{pr}}, \bar{f}, \bar{q})$ in which Generic follows the strategy **st**. Let $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i < \delta} C_i$ be a limit ordinal such that $h \circ f_\delta$ is an increasing sequence of conditions in \mathbb{S}_E^ψ satisfying $h(f_\delta(i + 1)) = r_{i+1}^-$ for all $i < \delta$. Then, by the description of **st**, we have $i \leq \text{rt}(h(f_\delta(i))) < \delta$ and $p_i \leq h(f_\delta(i))$ (for all successor $i < \delta$). Therefore $q_\delta \upharpoonright [\delta + 1, \lambda) \subseteq p_{\delta+1} \subseteq p_\lambda \subseteq r_\delta$ and $q_\delta \upharpoonright (\delta + 1) = r_\delta \upharpoonright (\delta + 1)$. Consequently, $q_\delta \leq r_\delta$ and clearly $h \circ f_\delta R^{\text{pr}} r_\delta$.

(g) The forcing notion \mathbb{P} is strategically ($< \lambda$)-complete by Observation 2.23. Define a binary relation R^{pr} by $\bar{T} R^{\text{pr}} T$ if and only if

- $\bar{T} = \langle T_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbb{P}}$ -increasing sequence of conditions from \mathbb{P} such that $\delta < \lambda$ is a limit ordinal and for $\alpha < \delta$ we have $\alpha \leq \text{lh}(\text{root}(T_\alpha)) < \delta$, and
- $T \in \mathbb{P}$ is a $\leq_{\mathbb{P}}$ -upper bound of the sequence \bar{T} . (Note that the previous bullet implies $\bigcap_{\alpha < \delta} T_\alpha \in \mathbb{P}$ is the least upper bound of \bar{T} .)

Clearly R^{pr} is a λ -sequential⁺ purity on \mathbb{P} . Let $N, h, \bar{f}, \bar{T}, \gamma$ be as in the assumptions of 3.6(1)(\odot) and $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ is limit} \rangle$ be a candidate. Suppose $T \in \mathbb{P} \cap N$. To construct a condition $T^* \geq T$ generic for \bar{q} we choose by induction on $\alpha < \lambda$ conditions $T_\alpha \in \mathbb{P} \cap N$ as follows.

(∇)₁ $T_0 \geq T$ satisfies

$$(\forall t \in T_0)(|\text{succ}_{T_0}(t)| > 1 \Rightarrow \text{lh}(t) \notin \mathcal{S}),$$

and if δ is a limit ordinal, then $T_\delta = \bigcap_{\xi < \delta} T_\xi$.

(∇)₂ Suppose that $\alpha = \delta + 1$, $\delta \in \mathcal{S}[\gamma]$ is a limit ordinal, and $h \circ f_\delta$ is a $\leq_{\mathbb{P}}$ -increasing sequence of conditions from $\mathbb{P} \cap N$ such that for all successor $\beta < \delta$ we have

$$\beta \leq \text{lh}(\text{root}(h(f_\delta(\beta)))) < \delta \quad \text{and} \quad T_\beta \leq h(f_\delta(\beta)).$$

Let $t_\delta = \bigcup_{\beta < \delta} \text{root}(h(f_\delta(\beta)))$ and $T_\alpha^* = \{t \in T_\delta : t_\delta \leq t \Rightarrow t \in q_\delta\}$.

Pick $T_\alpha \in \mathbb{P}$ so that $T_\alpha \subseteq T_\alpha^*$ and $T_\alpha \cap \alpha \lambda = T_\alpha^* \cap \alpha \lambda$.

(∇)₃ If $\alpha = \delta + 1$ and the clause (∇)₂ does not apply, then $T_\alpha = T_\delta$.

Suppose that $\alpha = \delta + 1$ satisfies the assumptions of (∇)₂. Then $\bigcap_{\beta < \delta} h(f_\delta(\beta)) \in \mathbb{P}$ and $T_\delta \subseteq \bigcap_{\beta < \delta} h(f_\delta(\beta)) \leq q_\delta$ and $t_\delta \leq \text{root}(\bigcap_{\beta < \delta} h(f_\delta(\beta))) \leq \text{root}(q_\delta)$. Also $\text{lh}(t_\delta) = \delta \in \mathcal{S}$ so, by (∇)₁, $|\text{succ}_{T_\delta}(t_\delta)| = 1$. Hence we may conclude that the sequence $\langle T_\alpha : \alpha < \lambda \rangle$ satisfies the assumptions of Lemma 2.24. At

limit stages $\delta < \lambda$ this implies that $T_\delta = \bigcap_{\beta < \delta} T_\beta \in \mathbb{P}$ (so the construction can be carried out) and for λ we get $T^* := \bigcap_{\alpha < \lambda} T_\alpha \in \mathbb{P}$.

CLAIM 3.9.2. T^* is (N, \mathbb{P}) -generic in the standard sense.

Proof of the Claim. Assume that $\xi_0 < \lambda$ and $T^+ \geq_{\mathbb{P}} T^*$. Build inductively a sequence $\langle S_\alpha : \alpha < \lambda \rangle$ of conditions in \mathbb{P} such that

- $T^+ \leq S_\beta \leq S_\alpha$, $\text{lh}(\text{root}(S_\beta)) < \text{lh}(\text{root}(S_\alpha))$ for $\beta < \alpha < \lambda$, and
- $S_{\alpha+1} \in \bigcap_{\xi \leq \alpha} \mathcal{I}_\xi$, and if α is limit then $S_\alpha = \bigcap_{\beta < \alpha} S_\beta$ (for all $\alpha < \lambda$).

Let $s_\alpha = \text{root}(S_\alpha)$. Since $T_\alpha \supseteq T^+ \supseteq S_\alpha$ we see that $s_\alpha \in T_\alpha$ and $(T_\alpha)_{s_\alpha} \leq S_\alpha$. For some limit ordinal $\delta \in \mathcal{S}[\gamma]$ we have $\xi_0 < \delta$ and $\text{lh}(s_\alpha) < \delta$ for $\alpha < \delta$ and $h \circ f_\delta = \langle (T_\alpha)_{s_\alpha} : \alpha < \delta \rangle$. Then $h \circ f_\delta R^{\text{pr}} S_\delta$ and $S_\delta \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$. Hence also $q_\delta \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$ (in particular $q_\delta \in \mathcal{I}_{\xi_0} \cap N$). Since for $t_\delta = \bigcup_{\alpha < \delta} s_\alpha$ we have $q_\delta \leq (T_{\delta+1})_{t_\delta} \leq (T^*)_{t_\delta} \leq (T^+)_{t_\delta}$ we may conclude $\mathcal{I}_{\xi_0} \cap N$ is predense above T^* . ■

Let us show that T^* is generic for \bar{q} over $N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$. Consider the following strategy **st** for Generic in the game $\mathfrak{D}_\gamma^{\mathcal{S}}(T^*, N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{q})$. At stage $i \in \mathcal{S}[\gamma]$, when a sequence $\langle r_j^-, r_j, C_j : j < i \rangle$ has already been constructed, Generic picks r_i^-, r_i, C_i so that

- (∇)₄ if i is limit, then $r_i^- = \bigcap_{j < i} r_j^-$, $r_i = \bigcap_{j < i} r_j$ and $C_i = \bigcap_{j < i} C_j$, and
- (∇)₅ if i is a successor, say $i = j + 1$, then r_i^-, r_i are chosen so that $r_j \leq r_i$, $\text{root}(r_j) \triangleleft \text{root}(r_i^-)$, $r_j^- \leq r_i^-$, $T_i \leq r_i^-$ and $r_i^- \leq r_i$. Also, $C_i = [\text{lh}(\text{root}(r_i)) + 1001, \lambda)$.

As in the previous part, one uses Claim 3.9.2 to show that the choices are indeed possible and that there are always legal moves for Antigeneric.

Suppose now that $\langle r_i^-, r_i, C_i : i < \lambda \rangle$ is a play of $\mathfrak{D}_\gamma^{\mathcal{S}}(T^*, N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{q})$ in which Generic follows the strategy **st** described above. Let $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i < \delta} C_i$ be a limit ordinal such that $h \circ f_\delta$ is an increasing sequence of conditions in \mathbb{P} satisfying

$$h(f_\delta(i)) = r_i^- \quad \text{for all successor } i < \delta.$$

Then, by (∇)₅ + (∇)₄, we have $i \leq \text{lh}(\text{root}(h(f_\delta(i)))) < \delta$ and $h(f_\delta(i)) \geq T_i$ (for all successor $i < \delta$). Therefore, letting $t_\delta = \bigcup_{i < \delta} \text{root}(r_i^-)$ and using (∇)₂ we conclude $q_\delta \leq (T_{\delta+1})_{t_\delta} \leq (T^*)_{t_\delta} \leq (r_\delta)_{t_\delta} = r_\delta$ and $h \circ f_\delta R^{\text{pr}} r_\delta$.

(h) In both cases the arguments are almost identical, so we will consider $\mathbb{P} = \mathbb{H}_\lambda$ only. Define a binary relation R^{pr} by $\bar{p} R^{\text{pr}} p$ if and only if

- $\bar{p} = \langle (s_\alpha, g_\alpha) : \alpha < \delta \rangle$ is a $\leq_{\mathbb{H}_\lambda}$ -increasing sequence of conditions from \mathbb{H}_λ of limit length $\delta < \lambda$, and
- $p = (s, g) \in \mathbb{H}_\lambda$ is an upper bound of \bar{p} with $s = \bigcup_{\alpha < \delta} s_\alpha$.

Clearly, R^{pr} is a λ^+ -sequential purity on \mathbb{H}_λ . Suppose $N, h, \bar{f}, \bar{\mathcal{I}}, \gamma$ are as in 3.6(⊙) and let \bar{q} be a candidate for these parameters. We claim that every

condition from $\mathbb{H}_\lambda \cap N$ is generic for \bar{q} . So let $(s, g) \in \mathbb{H}_\lambda \cap N$. Consider the following strategy **st** for Generic in $\mathfrak{D}_\gamma^{\mathcal{S}}((s, g), N, h, \mathbb{H}_\lambda, R^{\text{pr}}, \bar{f}, \bar{q})$. At stage $i \in \mathcal{S}[\gamma]$ of the play, after a sequence $\langle (s_j^-, g_j^-), (s_j, g_j), C_j : j < i \rangle$ was already constructed, Generic is instructed to choose (s_i^-, g_i^-) , (s_i, g_i) and C_i as follows.

- (\bullet)₁ If $i = i_0 + 1$ is a successor, then $g_i^- = g_{i_0}^-$, $g_i = g_{i_0}$, $s_i^- = s_i = s_{i_0} \frown \langle g_{i_0}(\text{lh}(s_{i_0})) \rangle$ and $C_i = [\text{lh}(s_{i_0}) + 1001, \lambda)$.
- (\bullet)₂ If $i \in \mathcal{S}[\gamma]$ is limit and $h(f(j)) = (s_j^-, g_j^-)$ for each successor $j < i$, and $g_i = (s^*, g^*)$ (note that necessarily $s^* = \bigcup_{j < i} s_j^-$), then $s_i^- = s_i = s^*$, $C_i = \lambda$ and for $\alpha < \lambda$,

$$g_i(\alpha) = \sup\{g_j(\alpha), g^*(\alpha) : j < i\},$$

$$g_i^-(\alpha) = \sup\{g_j^-(\alpha), g^*(\alpha) : j < i\}.$$

- (\bullet)₃ If $i \in \mathcal{S}[\gamma]$ is limit but we are not in the previous case, then $s_i^- = s_i = \bigcup_{j < i} s_j^-$, $g_i(\alpha) = \sup\{g_j(\alpha) : j < i\}$, $g_i^-(\alpha) = \sup\{g_j^-(\alpha) : j < i\}$ for each $\alpha < \lambda$.

Now check.

(i) Define a binary relation R^{pr} by $\bar{p} R^{\text{pr}} p$ if and only if $\bar{p} = \langle p_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbb{P}}$ -increasing sequence of conditions from \mathbb{P} which has an upper bound in \mathbb{P} , $\delta < \lambda$ is a limit ordinal and $p \in \mathbb{P}$ is an upper bound on \bar{p} .

Plainly, R^{pr} is a λ -sequential⁺ purity on \mathbb{P} . (Remember \mathbb{P} is $(< \lambda)$ -complete.) Assume $N, h, \bar{f}, \bar{\mathcal{L}}, \gamma, \bar{q}$ are as in 3.6(1)(\odot) and let $p \in N$. Let **st**^{*} $\in N$ be a winning strategy of Complete in the game $\mathfrak{D}_0^{\lambda+1}(\mathbb{P})$. By induction on $\alpha \leq \lambda$ choose conditions $p_\alpha^*, q_\alpha^* \in \mathbb{P}$ so that

- (\square)₁ $\langle p_\alpha^*, q_\alpha^* : \alpha \leq \lambda \rangle$ is a legal play of $\mathfrak{D}_0^{\lambda+1}(\mathbb{P})$ in which Complete follows her winning strategy **st**^{*},
- (\square)₂ $p_0^* = p$, $p_\alpha^*, q_\alpha^* \in N$ for all $\alpha < \lambda$,
- (\square)₃ $p_{\alpha+1}^* \in \bigcap_{\beta < \alpha} \mathcal{I}_\beta$ for each $\alpha < \lambda$, and
- (\square)₄ if $\delta \in \mathcal{S}$ is limit and $q_\delta \geq p_\alpha^*$ for all $\alpha < \delta$, then $p_\delta^* = q_\delta$.

(Clearly the construction is possible.) We will argue that the condition p_λ^* is generic for the candidate \bar{q} over $N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \mathcal{S}, \gamma$. Consider a strategy **st**⁺ of Generic in $\mathfrak{D}_\gamma^{\mathcal{S}}(p_\lambda^*, N, h, \mathbb{P}, R^{\text{pr}}, \bar{f}, \bar{q})$ which instructs her to choose r_i^-, r_i, C_i (for $i \in \mathcal{S}[\gamma]$) and side ordinals $\alpha_i < \lambda$ (for $i < \lambda$) so that the following demands (\square)₅–(\square)₉ are satisfied:

- (\square)₅ $r_i^- \in \mathbb{P} \cap N$ and $r_i \in \mathbb{P}$.
- (\square)₆ $r_i^- \leq r_i$ and if $j < i$ then $r_j \leq r_i$ and $\gamma < \alpha_j < \alpha_i$.
- (\square)₇ $\alpha_i < \lambda$ is the smallest ordinal above $\{\gamma + 1\} \cup \{\alpha_j + 1 : j < i\}$ such that $r_i^- \leq p_{\alpha_i}^*$ and then $r_{i+1}^- = p_{\alpha_i}^*$, $r_{i+1} = r_i$ and $C_{i+1} = [\alpha_i + 1001, \lambda)$.

- (□)₈ If $\delta \in \mathcal{S}[\gamma]$ is limit and $q_\delta \geq r_i^-$ for all $i < \delta$, and the family $\{q_\delta\} \cup \{r_i : i < \delta\}$ has an upper bound in \mathbb{P} , then $r_\delta^- = q_\delta$ and $r_\delta \in \mathbb{P}$ is the $\langle \chi^* \rangle$ -first condition stronger than q_δ and stronger than all r_i for $i < \delta$, and $C_i = \lambda$.
- (□)₉ If $\delta \in \mathcal{S}[\gamma]$ is limit but we are not in the case of (□)₈, then $(r_\delta^-, r_\delta, C_\delta)$ is the $\langle \chi^* \rangle$ -first legal move of Generic after $\langle r_i^-, r_i, C_i : i < \delta \rangle$ which satisfies also $r_\delta^- \leq r_\delta$.

The above conditions fully define a strategy \mathbf{st}^+ for Generic. Concerning clause (□)₇ and the choice of α_i , note that by (□)₃ there is an $\alpha < \lambda$ such that either the conditions p_α^*, r_i^- are incompatible, or $p_\alpha^* \geq r_i^-$. Since $r_i \geq p_\lambda^* \geq p_\alpha^*$ and $r_i \geq r_i^-$, the conditions p_α^*, r_i^- are always compatible, and hence there is $\alpha_i < \lambda$ such that $p_{\alpha_i}^* \geq r_i^-$. Note that the condition p_λ^* is (N, \mathbb{P}) -generic in the standard sense (by (□)₃) and also for each $j < \lambda$ there is an upper bound on $\{r_i : i < j\}$ (and this upper bound is (N, \mathbb{P}) -generic). Hence, for both players, there are always legal moves during a play in which Generic followed \mathbf{st}^+ so far (satisfying $r_i^- \leq r_i$). So, in particular, clause (□)₉ completes the description of \mathbf{st}^+ .

Let us argue that \mathbf{st}^+ is a winning strategy for Generic. To this end, assume $\langle r_i^-, r_i, C_i : i < \lambda \rangle$ is a play in which Generic follows \mathbf{st}^+ and let $\langle \alpha_i : i < \lambda \rangle$ be the side ordinals chosen by her. Suppose $\delta \in \mathcal{S}[\gamma] \cap \bigcap_{i < \delta} C_i$ is a limit ordinal such that $h \circ f_\delta$ is an increasing sequence of conditions from $\mathbb{P} \cap N$ satisfying

$$h(f_\delta(i+1)) = r_{i+1}^- \quad \text{for all } i < \delta.$$

Then also $i \leq \alpha_i < \delta$ for each $i < \delta$, so by (□)₇ we have

$$(\forall i < \delta)(\exists \alpha < \delta)(r_i^- \leq p_\alpha^*) \quad \text{and} \quad (\forall \alpha < \delta)(\exists i < \delta)(p_\alpha^* \leq r_i^-).$$

Thus the sequences $\langle p_\alpha^* : \alpha < \delta \rangle$ and $\langle r_i^- : i < \delta \rangle$ and also $h \circ f_\delta$ have the same upper bounds. Now, by 3.5(2)(a, b), we have $q_\delta \geq h \circ f_\delta(i+1) = r_{i+1}^-$ for all $i < \delta$, and hence q_δ is stronger than all p_α^* for $\alpha < \delta$. By (□)₄ we get $q_\delta = p_\delta^* \leq p_\lambda^* \leq r_i$ (for $i < \delta$). Thus clause (□)₈ applies and $r_\delta^- = q_\delta \leq r_\delta$, so also $h \circ f_\delta R^{\text{pr}} r_\delta$. ■

4. The iteration theorem. Here we will prove the main result: pure sequential⁺ properness is almost preserved in λ -support iterations. The proof of the theorem is somewhat similar to that of [15, Theorem 3.7], but there are major differences in the concepts involved.

THEOREM 4.1. *Assume Context 3.1. Let $\bar{\mathbb{Q}} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \zeta^* \rangle$ be a λ -support iteration such that for each $\alpha < \zeta^*$,*

$$\Vdash_{\mathbb{P}_\alpha} \text{“} \mathbb{Q}_\alpha \text{ is purely sequentially}^+ \text{ proper over } (D, \mathcal{S})\text{-semi-diamonds”}.$$

Then

- (1) $\mathbb{P}_{\zeta^*} = \lim(\bar{\mathbb{Q}})$ is purely sequentially proper over (D, \mathcal{S}) -semi-diamonds.
 (2) If additionally, for each $\alpha < \zeta^*$,

$$\Vdash_{\mathbb{P}_\alpha} \text{“}\mathbb{Q}_\alpha \text{ is } (<\lambda)\text{-complete”}$$

then \mathbb{P}_{ζ^*} is purely sequentially⁺ proper over (D, \mathcal{S}) -semi-diamonds.

Proof. By adding trivial iterands to the tail of $\bar{\mathbb{Q}}$ we may assume that $\zeta^* \geq \lambda^+$ (this demand is just to avoid slight complications in enumerations in $(\boxtimes)_1, (\boxtimes)_2$ below).

Assume that χ is a large enough regular cardinal and $N \prec (\mathcal{H}(\chi), \in, <_\chi^*)$, $|N| = \lambda$, ${}^{<\lambda}N \subseteq N$ and $\lambda, \bar{\mathbb{Q}}, D, \mathcal{S}, \dots \in N$. For $\xi < \zeta^*$ let \mathbf{st}_ξ^0 be the $<_\chi^*$ -first \mathbb{P}_ξ -name for a regular winning strategy of Complete in $\partial_0^\lambda(\mathbb{Q}_\xi)$ (so if $\xi \in \zeta^* \cap N$ then also $\mathbf{st}_\xi^0 \in N$). To define a λ -sequential purity R^{pr} on \mathbb{P}_{ζ^*} we first fix

- $(\boxtimes)_1$ a list $\langle (\tau_i, \zeta_i) : i < \lambda \rangle$ of all pairs $(\tau, \zeta) \in N$ such that $\zeta \leq \zeta^*$, $\text{cf}(\zeta) \geq \lambda$ and τ is a \mathbb{P}_ζ -name for an ordinal, and
 $(\boxtimes)_2$ an increasing sequence $\bar{w} = \langle w_i : i < \lambda \rangle$ of closed subsets of $\zeta^* + 1$ such that

$$(\boxtimes)_2^a \quad w_0 = \{0, \zeta^*\}, \bigcup_{i < \lambda} w_i = (\zeta^* + 1) \cap N \text{ and for each } i < \lambda \text{ we have } |w_i| \leq 2 + i,$$

$$(\boxtimes)_2^b \quad \text{if } i \text{ is limit then } w_i \text{ is the closure of } \bigcup_{j < i} w_j, \text{ and}$$

$$(\boxtimes)_2^c \quad \text{if } i \text{ is a successor, say } i = j + 1, \text{ then for some } \xi \in w_i \cap \zeta_j \text{ we have } \sup(w_j \cap \zeta_j) < \xi.$$

We also define a function $\Upsilon : (\zeta^* + 1) \cap N \rightarrow \lambda$ by

$$(\boxtimes)_3 \quad \Upsilon(\varepsilon) = \min\{i < \lambda : \varepsilon \in w_i\} \text{ for } \varepsilon \in (\zeta^* + 1) \cap N.$$

Then for every $\zeta \in \zeta^* \cap N$ we fix a \mathbb{P}_ζ -name R_ζ^{pr} for a binary relation such that

$\Vdash_{\mathbb{P}_\zeta}$ “ R_ζ^{pr} is a λ -sequential⁺ purity on \mathbb{Q}_ζ such that if $N[G_{\mathbb{P}_\zeta}]$ is a model of the right form, then R_ζ^{pr} witnesses the pure sequential⁺ properness of \mathbb{Q}_ζ for $N[G_{\mathbb{P}_\zeta}]$ ”.

Now, for $\zeta \leq \zeta^*$, we define a binary relation $R^{\text{pr}} \upharpoonright \zeta$ by

$$(\boxtimes)_4 \quad \bar{p} R^{\text{pr}} \upharpoonright \zeta p \text{ if and only if}$$

$$(\boxtimes)_4^a \quad \bar{p} = \langle p_\alpha : \alpha < \delta \rangle \text{ is an increasing sequence of conditions from } \mathbb{P}_\zeta \text{ of a limit length } \delta < \lambda, p \in \mathbb{P}_\zeta \text{ is an upper bound on } \bar{p} \text{ and}$$

$$(\boxtimes)_4^b \quad \text{for every } \xi \in \zeta \cap N \text{ such that } {}^{(6)} \Upsilon(\xi) \cdot \omega \leq \delta \text{ we have}$$

$$p \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \text{“}\langle p_{\Upsilon(\xi) + \alpha}(\xi) : \alpha < \delta \rangle R_\xi^{\text{pr}} p(\xi)\text{”}.$$

The relation R^{pr} is $R^{\text{pr}} \upharpoonright \zeta^*$.

One may wonder why we use $\Upsilon \cdot \omega$ in $(\boxtimes)_4^b$. When playing the game ∂_1^S , we will play its variant on each coordinate from $N \cap \zeta^*$. However, at stage δ

⁽⁶⁾ The “ \cdot ” stands for ordinal multiplication.

of the main game, we will look at coordinates from w_δ only, so we will start our coordinate games with some delay. At stage $j = \mathcal{Y}(\xi) + i$, the length of the play constructed so far at coordinate ξ is i . It may happen that $i \in \mathcal{S}$ but $\mathcal{Y}(\xi) + i \notin \mathcal{S}$. Since we do want to control who is making a move at every coordinate, it makes sense to demand that $\delta \setminus \mathcal{Y}(\xi)$ has order type δ whenever δ is of any potential importance. This translates to the demand that at coordinate $\xi \in N \cap \zeta^*$ we play the game $\partial_{\mathcal{Y}(\xi) \cdot \omega}^{\mathcal{S}}$ and the relevant δ s are not smaller than $\mathcal{Y}(\xi) \cdot \omega$.

Clearly, for each $\zeta \leq \zeta^*$,

- $R^{\text{Pr}} \upharpoonright \zeta$ is a λ -sequential purity on \mathbb{P}_ζ ,
- if each \mathbb{Q}_α is (forced to be) $(< \lambda)$ -complete ⁽⁷⁾, then $R^{\text{Pr}} \upharpoonright \zeta$ is a λ -sequential⁺ purity, and
- if $\langle p_\alpha : \alpha < \delta \rangle (R^{\text{Pr}} \upharpoonright \zeta) p$ and $\zeta' < \zeta$ then $\langle p_\alpha \upharpoonright \zeta' : \alpha < \delta \rangle (R^{\text{Pr}} \upharpoonright \zeta') p \upharpoonright \zeta'$.

We will show that the relation R^{Pr} witnesses the pure sequential properness of \mathbb{P}_{ζ^*} for N . So let $h, \bar{\mathcal{I}}, \bar{f}, \bar{q}$ and p be as in 3.6(1)(\odot), that is, they satisfy the following conditions $(\boxtimes)_5$ – $(\boxtimes)_8$:

- $(\boxtimes)_5$ A function $h : \lambda \rightarrow N$ is such that $\mathbb{P}_{\zeta^*} \cap N \subseteq \text{rng}(h)$.
- $(\boxtimes)_6$ A sequence $\bar{f} = \langle f_\delta : \delta \in \mathcal{S} \rangle$ is a (D, \mathcal{S}, h) -semi-diamond for \mathbb{P}_{ζ^*} over N .
- $(\boxtimes)_7$ $\bar{\mathcal{I}} = \langle \mathcal{I}_\alpha : \alpha < \lambda \rangle$ lists all open dense subsets of \mathbb{P}_{ζ^*} from N .
- $(\boxtimes)_8$ $\bar{q} = \langle q_\delta : \delta \in \mathcal{S} \text{ is limit} \rangle$ is an $(N, h, \mathbb{P}_{\zeta^*}, R^{\text{Pr}}, \bar{f}, \bar{\mathcal{I}})$ -candidate, and $p \in \mathbb{P}_{\zeta^*} \cap N$.

We also have an ordinal $\gamma < \lambda$, but replacing \mathcal{S} with $\mathcal{S}[\gamma]$ in the following arguments makes not much difference, so we may pretend that $\gamma = 1$ and $\mathcal{S}[\gamma] = \mathcal{S}$.

We will define a condition $r \geq p$ such that Generic will have a winning strategy in the game $\partial_1^{\mathcal{S}}(r, N, h, \mathbb{P}_{\zeta^*}, R^{\text{Pr}}, \bar{f}, \bar{q})$, that is, r will be generic for the candidate \bar{q} . This condition will be defined essentially by considering the game $\partial_{\mathcal{Y}(\zeta) \cdot \omega}^{\mathcal{S}}$ on each coordinate $\zeta \in \zeta^* \cap N$. The proof that it has the required properties will be by induction on ζ . Therefore, we have to introduce both the restrictions of our parameters to ζ (denoted with superscript $[\zeta]$) and the parameters at coordinate ζ (denoted with superscript $\langle \zeta \rangle$). We will also need other auxiliary notions, definitions and claims. First note that if we modify f_δ (and q_δ) for all limit $\delta \in \mathcal{S}$ such that $h \circ f_\delta$ is not an increasing sequence of conditions from \mathbb{P}_{ζ^*} , then we can make the winning for Generic only more difficult. So we may assume that for each limit $\delta \in \mathcal{S}$ we have

- $(\boxtimes)_9$ $h \circ f_\delta$ is a $\leq_{\mathbb{P}_{\zeta^*}}$ -increasing sequence of members of $\mathbb{P}_{\zeta^*} \cap N$.

For $\zeta \in N \cap (\zeta^* + 1)$ we define

⁽⁷⁾ We need this assumption to get bounds on coordinates $\xi < \zeta$ for which $(\boxtimes)_4^b$ is not “active”.

- (\boxtimes)₁₀ $h^{[\zeta]} : \lambda \rightarrow N$ is a function such that for each $\alpha < \lambda$, $h^{[\zeta]}(\alpha) = h(\alpha) \upharpoonright \zeta$ provided $h(\alpha)$ is a function, and $h^{[\zeta]}(\alpha) = \emptyset$ otherwise,
 (\boxtimes)₁₁ $\bar{\mathcal{I}}^{[\zeta]} = \langle \mathcal{I}_\alpha^{[\zeta]} : \alpha < \lambda \rangle$, where $\mathcal{I}_\alpha^{[\zeta]} = \{p \upharpoonright \zeta : p \in \mathcal{I}_\alpha\}$.

It may happen that for some limit $\delta \in \mathcal{S}$ there is no condition $q \in \mathbb{P}_{\zeta^*}$ satisfying $q \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$ and $h \circ f_\delta R^{\text{pr}} q$ (see 3.5(2)(c)), but there are such conditions for $\mathbb{P}_\zeta, \bar{\mathcal{I}}^{[\zeta]}$ and $h^{[\zeta]} \circ f_\delta$. Consequently, the sequence $\langle q_\delta \upharpoonright \zeta : \delta \in \mathcal{S} \text{ limit} \rangle$ does not have to be an $(N, h^{[\zeta]}, \mathbb{P}_\zeta, R^{\text{pr}} \upharpoonright \zeta, \bar{f}, \bar{\mathcal{I}}^{[\zeta]})$ -candidate. To define suitable $q^{[\zeta]}$ we need the concept of saturated conditions. Suppose that $\delta \in \mathcal{S}$ is a limit ordinal, $\zeta \in w_\delta$ and $q^* \in \mathbb{P}_\zeta$. We say that the condition q^* is *saturated over* $w_\delta, h, \bar{\mathcal{I}}$ at ζ if one of the following two possibilities (\boxtimes)₁₂^a or (\boxtimes)₁₂^b holds:

- (\boxtimes)₁₂^a (i) $(h^{[\zeta]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \zeta) q^*$ and $q^* \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\zeta]}$, but
 (ii) if $\zeta' \in w_\delta \setminus (\zeta + 1)$ then there is no $q' \in \mathbb{P}_{\zeta'}$ such that $q^* \leq q'$ and

$$(h^{[\zeta']} \circ f_\delta) (R^{\text{pr}} \upharpoonright \zeta') q' \quad \text{and} \quad q' \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\zeta']}.$$

- (\boxtimes)₁₂^b (i) $0 < \zeta = \sup(w_\delta \cap \zeta)$ and for each $\xi \in w_\delta \cap \zeta$ we have

$$(h^{[\xi]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \xi) q^* \upharpoonright \xi \quad \text{and} \quad q^* \upharpoonright \xi \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\xi]}, \quad \text{but}$$

- (ii) there is no condition $q' \in \mathbb{P}_\zeta$ such that $q^* \leq q'$ and

$$(h^{[\zeta]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \zeta) q' \quad \text{and} \quad q' \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\zeta]}.$$

CLAIM 4.1.1. *Suppose $\delta \in \mathcal{S}$ is a limit ordinal. Then there exist $\zeta \in w_\delta$ and $q^* \in \mathbb{P}_\zeta \cap N$ such that q^* is saturated over $w_\delta, h, \bar{\mathcal{I}}$ at ζ .*

Proof of the Claim. Let $\langle \alpha_\xi : \xi \leq \xi^* \rangle$ be the increasing enumeration of w_δ (so it is a continuous sequence and $\alpha_{\xi^*} = \zeta^*$ and $\xi^* < |\delta|^+$). We construct the condition q^* as an upper bound on an increasing sequence $\langle q_\xi^+ : \xi < \xi_0 \rangle$, $\xi_0 \leq \xi^*$. Conditions q_ξ^+ are chosen so that $q_\xi^+ \in \mathbb{P}_{\alpha_\xi} \cap N$ and they satisfy (\boxtimes)₁₂^a(i). To make sure that we can pass the limit steps, at coordinates $\varepsilon \in N \cap \zeta^*$ which are not relevant for R^{pr} (i.e., with $\mathcal{Y}(\varepsilon) \cdot \omega > \delta$) we employ strategic completeness. (This forces us to introduce auxiliary conditions q_ξ^* .) We do not have to worry about coordinates relevant for R^{pr} (i.e., with $\mathcal{Y}(\varepsilon) \cdot \omega \leq \delta$) as there the appropriate bounds exist by 3.4(c). The construction may stop at some stage $\xi_0 < \xi^*$ because we have arrived at a saturated condition. If the procedure does not stop before ξ^* , then we get a condition saturated at ζ^* .

Let us present this argument with all technical details. So we attempt to choose inductively a sequence $\langle q_\xi^+, q_\xi^* : \xi \leq \xi^* \rangle$ so that $q_\xi^+, q_\xi^* \in \mathbb{P}_{\alpha_\xi} \cap N$ and

- (i) if $\xi' < \xi'' \leq \xi^*$ then $q_{\xi'}^+ \leq q_{\xi'}^* \leq q_{\xi''}^+ \leq q_{\xi''}^*$,
- (ii) if $\varepsilon \in N \cap \zeta^*$ and $\Upsilon(\varepsilon) \cdot \omega > \delta$, then for every $\xi \leq \xi^*$,
- $$q_\xi^* \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“} \langle q_{\xi'}^+(\varepsilon), q_{\xi'}^*(\varepsilon) : \xi' \leq \xi \rangle \text{ is a legal partial play of } \mathfrak{D}_0^\lambda(\mathbb{Q}_\varepsilon)$$
- in which Complete uses her regular winning strategy $\mathfrak{st}_\varepsilon^0$ ”,
- (iii) if $\varepsilon \in N \cap \zeta^*$ and $\Upsilon(\varepsilon) \cdot \omega \leq \delta$, then $q_\xi^+(\varepsilon) = q_\xi^*(\varepsilon)$,
- (iv) $(h^{[\alpha_\xi]} \circ f_\delta)(R^{\text{pr}} \upharpoonright \alpha_\xi) q_\xi^+$ and $q_\xi^+ \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\alpha_\xi]}$.

If we have arrived at a limit stage ξ_0 of the construction and we have successfully defined $\langle q_\xi^+, q_\xi^* : \xi < \xi_0 \rangle$ (so that conditions (i)–(iv) are satisfied for all $\xi < \xi_0$), then, remembering ${}^{<\lambda}N \subseteq N$ and $\delta < \lambda$, we may find a condition $q_{\xi_0}^\diamond \in \mathbb{P}_{\alpha_{\xi_0}} \cap N$ which is above all q_ξ^+ (for $\xi < \xi_0$) and satisfies ⁽⁸⁾ $(h^{[\alpha_{\xi_0}]} \circ f_\delta)(R^{\text{pr}} \upharpoonright \alpha_{\xi_0}) q_{\xi_0}^\diamond$. Then also $q_{\xi_0}^\diamond \upharpoonright \alpha_\xi \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\alpha_\xi]}$ for all $\xi < \xi_0$, so either $q_{\xi_0}^\diamond$ is saturated by $(\boxtimes)_{12}^b$ (and then we stop), or else we may pick $q_{\xi_0}^+ \in \mathbb{P}_{\alpha_{\xi_0}} \cap N$ stronger than $q_{\xi_0}^\diamond$ and satisfying the demand in (iv), and then we pick $q_{\xi_0}^* \in \mathbb{P}_{\alpha_{\xi_0}} \cap N$ by (ii)+(iii).

Now, if we have arrived at a successor stage $\xi = \xi_0 + 1 \leq \xi^*$ and we have defined $q_{\xi_0}^+, q_{\xi_0}^*$, then either $q_{\xi_0}^*$ is saturated by $(\boxtimes)_{12}^a$ (and then we stop), or else we may pick $q_{\xi_0+1}^+ \in \mathbb{P}_{\alpha_{\xi_0+1}} \cap N$ stronger than $q_{\xi_0}^*$ and such that (iv) holds, and then we choose $q_{\xi_0+1}^*$ by (ii)+(iii). (We also stipulate $q_0^+ = q_0^* = \emptyset_{\mathbb{P}_0} \in \mathbb{P}_0$.)

If we did manage to carry out the construction up to ξ^* and we successfully defined $q_{\xi^*}^+$, then it is vacuously saturated by $(\boxtimes)_{12}^a$. ■

Now, for each limit $\delta \in \mathcal{S}$ we fix a pair $(q_\delta^*, \zeta_\delta^*) \in N$ so that

- $(\boxtimes)_{13}^a$ q_δ^* is saturated over $w_\delta, h, \bar{\mathcal{I}}$ at ζ_δ^* , and
- $(\boxtimes)_{13}^b$ if the condition q_δ given by $(\boxtimes)_8$ is saturated over $w_\delta, h, \bar{\mathcal{I}}$ at ζ^* , then $q_\delta^* = q_\delta, \zeta_\delta^* = \zeta^*$.

Next for $\zeta \in (\zeta^* + 1) \cap N$ we define

$(\boxtimes)_{14}$ $q^{[\zeta]} = \langle q_\delta^{[\zeta]} : \delta \in \mathcal{S} \text{ is limit} \rangle$, where for a limit $\delta \in \mathcal{S}$ we set

- if $\zeta < \zeta_\delta^*$ then $q_\delta^{[\zeta]} = q_\delta^* \upharpoonright \zeta$,
- if $\zeta = \zeta_\delta^*$, $q_\delta^* \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\zeta]}$ and $(h^{[\zeta]} \circ f_\delta)(R^{\text{pr}} \upharpoonright \zeta) q_\delta^*$, then $q_\delta^{[\zeta]} = q_\delta^*$,
- otherwise, $q_\delta^{[\zeta]}$ is the $<^*_\chi$ -first condition in \mathbb{P}_ζ satisfying the demands in (a)–(c) of 3.5(2) with $h^{[\zeta]}, R^{\text{pr}} \upharpoonright \zeta, \bar{\mathcal{I}}^{[\zeta]}$ here in place of $h, R^{\text{pr}}, \bar{\mathcal{I}}$ there.

⁽⁸⁾ Here we use the assumption that $R_\varepsilon^{\text{pr}}$ are sequential⁺ purities rather than just sequential purities.

CLAIM 4.1.2. *Let $\zeta \in N \cap (\zeta^* + 1)$. Then*

- (1) $h^{[\zeta]} : \lambda \rightarrow N$ is such that $\mathbb{P}_\zeta \cap N \subseteq \text{rng}(h^{[\zeta]})$,
- (2) \bar{f} is a $(D, \mathcal{S}, h^{[\zeta]})$ -semi-diamond sequence for \mathbb{P}_ζ ,
- (3) $\bar{\mathcal{I}}^{[\zeta]}$ lists all open dense subsets of \mathbb{P}_ζ belonging to N , and
- (4) $\bar{q}^{[\zeta]}$ is an $(N, h^{[\zeta]}, \mathbb{P}_\zeta, R^{\text{pr}} \upharpoonright \zeta, \bar{f}, \bar{\mathcal{I}}^{[\zeta]})$ -candidate.

Proof of the Claim. Straightforward from the definitions. ■

For $\zeta \in \zeta^* \cap N$ we set

- (⊠)₁₅ $\bar{f}^{(\zeta)} = \langle f_\delta^{(\zeta)} : \delta \in \mathcal{S} \rangle$, where for $\delta \in \mathcal{S}$ and $\alpha < \delta$ we let

$$f_\delta^{(\zeta)}(\alpha) = \begin{cases} f_\delta(\alpha) & \text{if } \delta < \Upsilon(\zeta) \cdot \omega, \\ f_\delta(\Upsilon(\zeta) + \alpha) & \text{if } \delta \geq \Upsilon(\zeta) \cdot \omega, \end{cases}$$

and then we define \mathbb{P}_ζ -names $h^{(\zeta)}, \bar{\mathcal{I}}^{(\zeta)}, \bar{q}^{(\zeta)}$ so that

- (⊠)₁₆ $\bar{\mathcal{I}}^{(\zeta)} = \langle \bar{\mathcal{I}}_\alpha^{(\zeta)} : \alpha < \lambda \rangle$ where $\bar{\mathcal{I}}_\alpha^{(\zeta)} = \{p(\zeta) : p \in \mathcal{I}_\alpha \ \& \ p \upharpoonright \zeta \in G_{\mathbb{P}_\zeta}\}$,
- (⊠)₁₇ $h^{(\zeta)}$ is a name for a function with domain λ and such that for each $\gamma < \lambda$, if $h(\gamma)$ is a function, $\zeta \in \text{dom}(h(\gamma))$ and $(h(\gamma))(\zeta)$ is a \mathbb{P}_ζ -name, then $h^{(\zeta)}(\gamma) = (h(\gamma))(\zeta)$, otherwise it is $\emptyset_{\mathbb{Q}_\zeta}$, and
- (⊠)₁₈ $\bar{q}^{(\zeta)} = \langle q_\delta^{(\zeta)} : \delta \in \mathcal{S} \text{ is limit} \rangle$, where for a limit $\delta \in \mathcal{S}$, $q_\delta^{(\zeta)}$ is the $<_\chi^*$ -first \mathbb{P}_ζ -name for a condition in \mathbb{Q}_ζ such that if $\zeta < \zeta_\delta^*$ and the condition $q_\delta^*(\zeta) = q_\delta^{[\zeta+1]}(\zeta)$ satisfies the demands in 3.5(2)(a)–(c) with $h^{(\zeta)}, f_\delta^{(\zeta)}, R_\zeta^{\text{pr}}, \bar{\mathcal{I}}^{(\zeta)}$ here in place of $h, f_\delta, R^{\text{pr}}, \bar{\mathcal{I}}$ there, then $q_\delta^{(\zeta)} = q_\delta^*(\zeta)$, otherwise $q_\delta^{(\zeta)}$ is any condition in \mathbb{Q}_ζ satisfying those demands.

Note that

- (⊠)₁₉ if $\delta \in \mathcal{S}$ is limit and $\zeta \in \zeta_\delta^* \cap N$ and $\Upsilon(\zeta) \cdot \omega \leq \delta$, then $q_\delta^* \upharpoonright \zeta \Vdash_{\mathbb{P}_\zeta} "q_\delta^{(\zeta)} = q_\delta^*(\zeta)"$.

CLAIM 4.1.3. *Assume that $\zeta \in N \cap \zeta^*$ and $r \in \mathbb{P}_\zeta$ is (N, \mathbb{P}_ζ) -generic in the standard sense. Then the condition r forces (in \mathbb{P}_ζ) that:*

- (1) $N[G_{\mathbb{P}_\zeta}]$ is a model of the right form, $\bar{\mathcal{I}}^{(\zeta)}$ lists all open dense subsets of \mathbb{Q}_ζ which belong to $N[G_{\mathbb{P}_\zeta}]$ and the function $h^{(\zeta)}$ is such that $N[G_{\mathbb{P}_\zeta}] \cap \mathbb{Q}_\zeta \subseteq \text{rng}(h^{(\zeta)})$,
- (2) $\bar{f}^{(\zeta)}$ is a $(D, \mathcal{S}, h^{(\zeta)})$ -semi-diamond for \mathbb{Q}_ζ over $N[G_{\mathbb{P}_\zeta}]$, and
- (3) $\bar{q}^{(\zeta)}$ is an $(N[G_{\mathbb{P}_\zeta}], h^{(\zeta)}, \mathbb{Q}_\zeta, \bar{f}^{(\zeta)}, \bar{\mathcal{I}}^{(\zeta)})$ -candidate.

Proof of the Claim. (2) Assume that $r^* \in \mathbb{P}_\zeta$, $r^* \geq r$, and $\bar{q}' = \langle q'_\alpha : \alpha < \lambda \rangle$ is a \mathbb{P}_ζ -name for an increasing sequence of conditions from $\mathbb{Q}_\zeta \cap N[G_{\mathbb{P}_\zeta}]$ (so q'_α is a name for an object in $N[G_{\mathbb{P}_\zeta}]$, but it need not belong to N). Suppose also that A_ξ (for $\xi < \lambda$) are \mathbb{P}_ζ -names for members of $D \cap \mathbf{V}$.

Construct inductively a sequence $\langle r_i^-, q_i'', r_i, r_i^+, A_i : i < \lambda \rangle$ so that

- (i) $r_i^- \in \mathbb{P}_\zeta \cap N$, $r_i, r_i^+ \in \mathbb{P}_\zeta$, $r^* \leq r_0$, $r_i^- \leq r_i \leq r_i^+$, $r_i^- \leq r_j^-$ for $i < j < \lambda$,
- (ii) for each $\varepsilon < \zeta$ and $j < \lambda$:

$r_j^+ \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $\langle r_i(\varepsilon), r_i^+(\varepsilon) : i \leq j \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\varepsilon)$ in which Complete uses her regular winning strategy $\mathfrak{st}_\varepsilon^0$ ”,

- (iii) $q_i'' \in N$ is a \mathbb{P}_ζ -name for a condition in \mathbb{Q}_ζ and $r_i \Vdash_{\mathbb{P}_\zeta}$ “ $q_i' = q_i''$ ”,
- (iv) $A_i \in D \cap \mathbf{V}$ and $r_i \Vdash_{\mathbb{P}_\zeta}$ “ $A_i = A_i$ ”,
- (v) $r_j^- \Vdash_{\mathbb{P}_\zeta}$ “the sequence $\langle q_i'' : i \leq j \rangle$ is $\leq_{\mathbb{Q}_\zeta}$ -increasing”.

[Why is the construction possible? First, after arriving at step $j < \lambda$ we note that the sequence $\langle r_i, r_i^+ : i < j \rangle$ has an upper bound by (ii). So we may choose a condition r_j' stronger than all r_i, r_i^+ (for $i < j$) and deciding the values of q_j', A_j , say $r_j' \Vdash_{\mathbb{P}_\zeta}$ “ $q_j' = q_j'' \ \& \ A_j = A_j$ ”, where $q_j'' \in N$ and $A_j \in D \cap \mathbf{V}$. Since $r_j' \geq r_i^-$ (for all $i < j$) and $r_j' \Vdash_{\mathbb{P}_\zeta}$ “the sequence $\langle q_i'' : i \leq j \rangle$ is increasing”, and since r_j' is (N, \mathbb{P}_ζ) -generic, we may choose conditions $r_j^- \in N \cap \mathbb{P}_\zeta$ and $r_j \in \mathbb{P}_\zeta$ so that $r_i^- \leq r_j^- \leq r_j$ for $i \leq j$ and $r_j' \leq r_j$ and $r_j^- \Vdash_{\mathbb{P}_\zeta}$ “the sequence $\langle q_i'' : i \leq j \rangle$ is increasing”. Finally, $r_j^+ \in \mathbb{P}_\zeta$ is defined essentially by (ii).]

Now define

$$q_i^* = \begin{cases} \emptyset_{\mathbb{Q}_\zeta} & \text{if } i < \mathcal{Y}(\zeta), \\ q_i'' & \text{if } i = \mathcal{Y}(\zeta) + \alpha. \end{cases}$$

Clearly, $\langle r_i^- \widehat{\ } \langle q_i^* : i < \lambda \rangle$ is an increasing sequence of conditions from $\mathbb{P}_{\zeta^*} \cap N$. Therefore, as D is normal and $A_i \in D$ and \bar{f} is a (D, \mathcal{S}, h) -semi-diamond, we may find a limit ordinal $\delta \in \mathcal{S} \cap \Delta_{i < \lambda} A_i$ such that $\delta > \mathcal{Y}(\zeta) \cdot \omega$ and $\langle h \circ f_\delta(i) : i < \delta \rangle = \langle r_i^- \widehat{\ } \langle q_i^* : i < \delta \rangle$. Then for each $\alpha < \delta$ we have

$$h^{(\zeta)} \circ f_\delta^{(\zeta)}(\alpha) = h^{(\zeta)}(f_\delta(\mathcal{Y}(\zeta) + \alpha)) = h(f_\delta(\mathcal{Y}(\zeta) + \alpha))(\zeta) = q_{\mathcal{Y}(\zeta) + \alpha}^* = q_\alpha''.$$

Also, r_δ is stronger than all r_i (for $i < \delta$) so it forces that $(\forall i < \delta)(A_i = A_i \ \& \ q_i' = q_i'')$. Hence

$$r_\delta \Vdash \text{“} \delta \in \mathcal{S} \cap \Delta_{\xi < \lambda} A_\xi \ \& \ h^{(\zeta)} \circ f_\delta^{(\zeta)} = \bar{q} \upharpoonright \delta \text{”}.$$

The rest should be clear.

(1, 3) Straightforward. ■

Recall that $\langle (\tau_i, \zeta_i) : i < \lambda \rangle$ was fixed in $(\boxtimes)_1$, $\langle w_i : i < \lambda \rangle$ was chosen in $(\boxtimes)_2$, the function \mathcal{Y} was defined in $(\boxtimes)_3$, and \bar{q}, p are from $(\boxtimes)_8$. Also, q_i^* are the saturated conditions picked in $(\boxtimes)_{13}$.

CLAIM 4.1.4. *There is a sequence $\langle p_i, p_i^+ : i < \lambda \rangle \subseteq N$ such that for all $i < \lambda$ we have:*

- (\boxtimes) $_{20}^a$ $(p_i, w_i), (p_i^+, w_i) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$,
 (\boxtimes) $_{20}^b$ $(p, \{0, \zeta^*\}) \leq' (p_j, w_j) \leq' (p_j^+, w_j) \leq' (p_i, w_i)$ for all $j < i$, and if $\varepsilon \in \zeta^* \cap N$ and $\Upsilon(\varepsilon) > i$, or $\varepsilon \in \zeta^* \setminus N$, and if $G \subseteq \mathbb{P}_{\zeta^*}$ is generic over \mathbf{V} such that $(p_i^+, w_i) \in' G$, then

$(p_i^+, w_i)^G \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “the sequence $\langle (p_j, w_j)^G(\varepsilon), (p_j^+, w_j)^G(\varepsilon) : j \leq i \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\varepsilon)$ in which Complete uses her regular strategy $\mathbf{st}_\varepsilon^0$ ”.

- (\boxtimes) $_{20}^c$ if $\varepsilon \in N \cap \zeta^*$, then $p_{\Upsilon(\varepsilon)}(\varepsilon) = p_j(\varepsilon) = p_j^+(\varepsilon)$ for all $j \geq \Upsilon(\varepsilon)$,
 (\boxtimes) $_{20}^d$ if $i \in \mathcal{S}$ is limit and $\varepsilon \in \text{dom}(q_i^*)$, $\Upsilon(\varepsilon) \geq i$, then $p_i(\varepsilon)$ is such that for every generic $G \subseteq \mathbb{P}_{\zeta^*}$ over \mathbf{V} with $(p_i, w_i) \in' G$, and two successive members $\varepsilon', \varepsilon''$ of the set w_i such that $\varepsilon' \leq \varepsilon < \varepsilon''$ we have:
- if $\{p_j(\varepsilon)[G \cap \mathbb{P}_{\varepsilon'}][G \cap \mathbb{P}_\varepsilon] : j < i\} \cup \{q_i^*(\varepsilon)[G \cap \mathbb{P}_\varepsilon]\}$ has an upper bound in $\mathbb{Q}_\varepsilon[G \cap \mathbb{P}_\varepsilon]$, then $p_i(\varepsilon)[G \cap \mathbb{P}_{\varepsilon'}][G \cap \mathbb{P}_\varepsilon]$ is such a bound,

- (\boxtimes) $_{20}^e$ for some $\xi \in w_{i+1} \cap \zeta_i \setminus (\sup(w_i \cap \zeta_i) + 1)$ we have $p_{i+1} \upharpoonright \xi = p_i^+ \upharpoonright \xi$ and for some a \mathbb{P}_ξ -name $\tau \in N$, for every generic $G \subseteq \mathbb{P}_{\zeta^*}$ over \mathbf{V} with $(p_{i+1}, w_{i+1}) \in' G$, we have $\tau_i[G \cap \mathbb{P}_{\zeta_i}] = \tau[G \cap \mathbb{P}_\xi]$.

Proof of the Claim. By induction on $i < \lambda$ we choose $p_i, p_i^+, \mathcal{A}_i, \mathcal{A}_i^+, f_i, f_i^+$ such that the following demands $(*)_1$ – $(*)_6$ are satisfied:

- (*) $_1$ $(p_i, w_i), (p_i^+, w_i) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$ and $(\mathcal{A}_i, f_i, w_i), (\mathcal{A}_i^+, f_i^+, w_i) \in N$ are their standard representations, respectively, and $(p_0, w_0) = (p, \{0, \zeta^*\})$.
 (*) $_2$ If $i < j < \lambda$ then $(\mathcal{A}_i, f_i, w_i) \preceq (\mathcal{A}_i^+, f_i^+, w_i) \preceq (\mathcal{A}_j, f_j, w_j)$.
 (*) $_3$ If $j < \lambda$, $\varepsilon \in \zeta^* \setminus w_j$, $s_j \in \mathcal{A}_j^+$, $f_j^+(s_j) \leq s_j$ and for $i \leq j$ the conditions $r_i \in \mathcal{A}_i$, $s_i \in \mathcal{A}_i^+$ are such that $r_i \leq s_i \leq s_j$, then

$f_j^+(s_j) \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $\langle f_i(r_i)(\varepsilon), f_i^+(s_i)(\varepsilon) : i \leq j \rangle$ is a legal partial play of $\mathcal{D}_0^\lambda(\mathbb{Q}_\varepsilon)$ in which Complete uses her regular winning strategy $\mathbf{st}_\varepsilon^0$ ”.

- (*) $_4$ If $\varepsilon \in w_j$, $j < \lambda$ and $r \in \mathcal{A}_j$, $r' \in \mathcal{A}_j^+$ and $r'' \in \mathcal{A}_{\Upsilon(\varepsilon)}$ are such that $r'' \leq r \leq r'$, then

$$p_j(\varepsilon) = p_j^+(\varepsilon) = p_{\Upsilon(\varepsilon)}(\varepsilon) \quad \text{so} \quad f_j(r)(\varepsilon) = f_j^+(r')(\varepsilon) = f_{\Upsilon(\varepsilon)}(r'')(\varepsilon).$$

- (*) $_5$ If $j \in \mathcal{S}$ is limit and $\varepsilon \in \text{dom}(q_j^*) \setminus \bigcup_{i < j} w_i$, $r \in \mathcal{A}_j$ and $r_i \in \mathcal{A}_i$ are such that $r_i \leq r$ for $i < j$ and $f_j(r) \leq r$ then

$\Vdash_{\mathbb{P}_\varepsilon}$ “if $\{f_i(r_i)(\varepsilon) : i < j\} \cup \{q_j^*(\varepsilon)\}$ has an upper bound in \mathbb{Q}_ε then $f_j(r)(\varepsilon)$ is such a bound”.

- (*) $_6$ Let $\xi = \sup(w_{i+1} \cap \zeta_i)$. Then $p_{i+1} \upharpoonright \xi = p_i^+ \upharpoonright \xi$ and for some \mathbb{P}_ξ -name $\tau \in N$ we have: if $r \in \mathcal{A}_{i+1}$ and $f_{i+1}(r) \leq r$, then $r \Vdash$ “ $\tau = \tau_i$ ”.

As declared in $(*)_1$ we set $p_0 = p$ and we choose $p_0^+ \in \mathbb{P}_{\zeta^*} \cap N$ essentially by $(*)_3$. Thus we let $\text{dom}(p_0^+) = \text{dom}(p_0)$, $p_0(0) = p_0^+(0)$ and for

$\varepsilon \in \text{dom}(p_0^+) \setminus \{0\}$ we let $p_0^+(\varepsilon)$ be the $<^*_\chi$ -first \mathbb{P}_ε -name such that

$$p_0^+ \upharpoonright \varepsilon \Vdash \text{“} p_0^+(\varepsilon) \text{ is the answer to } p_0(\varepsilon) \text{ given by the strategy } \mathbf{st}_\varepsilon^0 \text{”}.$$

Next we use 2.11 and 2.17 to choose standard representations $(\mathcal{A}_0, f_0, w_0) \in N$ and $(\mathcal{A}_0^+, f_0^+, w_0) \in N$ for $(p_0, w_0), (p_0^+, w_0)$ such that $(\mathcal{A}_0, f_0, w_0) \preceq (\mathcal{A}_0^+, f_0^+, w_0)$.

Suppose we have chosen $p_i, p_i^+, \mathcal{A}_i, \mathcal{A}_i^+, f_i, f_i^+ \in N$ for $i < j$ so that the relevant demands in $(*)_{1-6}$ are satisfied.

CASE: j is a successor ordinal, say $j = i_0 + 1$. Let $\xi = \sup(w_j \cap \zeta_{i_0})$ and $\zeta = \min(w_{i_0} \setminus \zeta_{i_0})$. By $(\boxtimes)_2^\zeta$ we know $\sup(w_{i_0} \cap \zeta_{i_0}) < \xi$. Working in N apply Proposition 2.9 to $\xi, \zeta, w_{i_0}, w_j, \tau_{i_0}$ and $(p_{i_0}^+, w_{i_0})$ to get p_j and τ such that

- $(\otimes)_1$ $(p_j, w_j) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$ and $(p_{i_0}^+, w_{i_0}) \leq' (p_j, w_j)$,
- $(\otimes)_2$ $p_j \upharpoonright (\xi \cup [\zeta, \zeta^*)) = p_{i_0}^+ \upharpoonright (\xi \cup [\zeta, \zeta^*))$,
- $(\otimes)_3$ $\tau \in N$ is a \mathbb{P}_{ξ} -name, and
- $(\otimes)_4$ if $G \subseteq \mathbb{P}_{\zeta^*}$ is generic over \mathbf{V} and $(p_j, w_j) \in' G$, then $\tau[G] = \tau_{i_0}[G]$.

By Proposition 2.17 (used in N) we may choose a representation $(\mathcal{A}_j, f_j, w_j) \in N$ for (p_j, w_j) such that $(\mathcal{A}_{i_0}^+, f_{i_0}^+, w_{i_0}) \preceq (\mathcal{A}_j, f_j, w_j)$. Necessarily,

- $(\otimes)_5$ if $r \in \mathcal{A}_j$ and $f_j(r) \leq r$, then $r \Vdash \text{“} \tau = \tau_{i_0} \text{”}$.

Since

- $(\otimes)_6$ if $\varepsilon \in w_{i_0}$, then either $\varepsilon < \xi$ or $\varepsilon \geq \zeta$,

we have $p_j(\varepsilon) = p_{i_0}^+(\varepsilon)$ whenever $\varepsilon \in w_{i_0}$. Therefore if $\varepsilon \in \zeta^* \cap N$, $\Upsilon(\varepsilon) < j$, $r \in \mathcal{A}_j$, $r^+ \in \mathcal{A}_{i_0}^+$, $r' \in \mathcal{A}_{\Upsilon(\varepsilon)}$ and $r'' \in \mathcal{A}_{\Upsilon(\varepsilon)}^+$ are such that $r' \leq r'' \leq r^+ \leq r$, then $(p_j(\varepsilon) = p_{i_0}^+(\varepsilon) = p_{\Upsilon(\varepsilon)}(r))$ and

$$f_j(r)(\varepsilon) = f_{i_0}^+(r^+)(\varepsilon) = f_{\Upsilon(\varepsilon)}^+(r'')(\varepsilon) = f_{\Upsilon(\varepsilon)}(r')(\varepsilon).$$

To define \mathcal{A}_j^+, f_j^+ and p_j^+ we first note that

- $(\otimes)_7$ if $r \in \mathcal{A}_j$, $f_j(r) \leq r$ and $r_i \in \mathcal{A}_i$, $s_i \in \mathcal{A}_i^+$ (for $i < j$), are such that $r_i \leq s_i \leq r$, and $\varepsilon \in \zeta^* \setminus w_j$, then

$$f_j(r) \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{“} \langle f_i(r_i)(\varepsilon), f_i^+(s_i)(\varepsilon) : i < j \rangle \frown \langle f_j(r)(\varepsilon) \text{ is a partial play of } \mathcal{Q}_\varepsilon^\lambda \text{ in which Complete uses her strategy } \mathbf{st}_\varepsilon^0 \text{”}.$$

Let $r \in \mathcal{A}_j$ and let $r_i \in \mathcal{A}_i$, $s_i \in \mathcal{A}_i^+$ (for $i < j$) be such that $r_i \leq s_i \leq r$. We define $g(r) \in \mathbb{P}_{\zeta^*}$ so that $\text{dom}(g(r)) = \text{dom}(f_j(r))$ and

- $(\otimes)_8^a$ if $\varepsilon \in \text{dom}(g(r)) \setminus w_j$, then $g(r)(\varepsilon)$ is the $<^*_\chi$ -first \mathbb{P}_ε -name for a condition in \mathcal{Q}_ε such that it is forced that

- (a) if $\langle f_i(r_i)(\varepsilon), f_i^+(s_i)(\varepsilon) : i < j \rangle \frown \langle f_j(r)(\varepsilon) \rangle$ is a legal partial play of $\mathcal{Q}_\varepsilon^\lambda$ in which Complete follows her regular winning strategy $\mathbf{st}_\varepsilon^0$,

then $g(r)(\varepsilon)$ is the answer to this partial play given to Complete by $\mathbf{st}_\varepsilon^0$,

(b) if the assumptions of (a) are not satisfied then $g(r)(\varepsilon) = f_j(r)(\varepsilon)$,
 $(\otimes)_8^b$ for $\varepsilon \in \text{dom}(g(r)) \cap w_j$ we set $g(r)(\varepsilon) = f_j(r)(\varepsilon)$.

Clearly, g is a well defined function from \mathcal{A}_j to \mathbb{P}_{ζ^*} and $g \in N$. We note that $g(r) \geq f_j(r)$ for $r \in \mathcal{A}_j$ and

$(\otimes)_9$ if $\alpha < \beta$ are successive elements of w_j , $r^0, r^1 \in \mathcal{A}_j$ are such that $r^0 \upharpoonright \alpha$ and $r^1 \upharpoonright \alpha$ are compatible, then $g(r^0) \upharpoonright \beta = g(r^1) \upharpoonright \beta$.

[Why? Let $r_i^{\ell,+} \in \mathcal{A}_i^+$, $r_i^\ell \in \mathcal{A}_i$ for $i < j$ be such that $r_i^\ell \leq r_i^{\ell,+} \leq r^\ell$. Then $r_i^0 \upharpoonright \alpha$ and $r_i^1 \upharpoonright \alpha$ are compatible and so are $r_i^{0,+} \upharpoonright \alpha$ and $r_i^{1,+} \upharpoonright \alpha$. Therefore, by $(*)_1$ and 2.12(v), $f_i(r_i^0) \upharpoonright \beta = f_i(r_i^1) \upharpoonright \beta$ and $f_i^+(r_i^{0,+}) \upharpoonright \beta = f_i^+(r_i^{1,+}) \upharpoonright \beta$. Since the definition of $g(r^\ell)(\varepsilon)$ involves only $\langle f_i(r_i^\ell)(\varepsilon), f_i^+(r_i^{\ell,+})(\varepsilon) : i < j \rangle$ and $f_j(r^\ell)(\varepsilon), \mathbf{st}_\varepsilon^0, w_j$, we conclude that $g(r^0)(\varepsilon) = g(r^1)(\varepsilon)$ for all $\varepsilon < \beta$.]

Pick a maximal antichain $\mathcal{A}_j^+ \in N$ of \mathbb{P}_{ζ^*} such that

$(\otimes)_{10}^a$ $(\forall s \in \mathcal{A}_j^+)(\exists r \in \mathcal{A}_j)(r \leq s)$, and

$(\otimes)_{10}^b$ if $s \in \mathcal{A}_j^+$, $r \in \mathcal{A}_j$ and $r \leq s$, then either $g(r) \leq s$, or for some $\varepsilon \in \text{dom}(s)$ we have $g(r) \upharpoonright \varepsilon \leq s \upharpoonright \varepsilon$ and

$s \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “the conditions $g(r)(\varepsilon), s(\varepsilon)$ are incompatible in \mathbb{Q}_ε ”.

Define $f_j^+ : \mathcal{A}_j^+ \rightarrow \mathbb{P}_{\zeta^*}$ by $f_j^+(s) = g(r)$ where $r \in \mathcal{A}_j$ is the unique member satisfying $r \leq s$. Plainly, the function f_j^+ belongs to N and it satisfies condition (v) of Observation 2.12 (by $(\otimes)_9$). Hence $(\mathcal{A}_j^+, f_j^+, w_j)$ is a standard representation of some $(p_j^+, w_j) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$ such that $p_j^+(\varepsilon) = p_j(\varepsilon)$ whenever $\varepsilon \in w_j$. Note also that if $s \in \mathcal{A}_j^+$ and $f_j^+(s) \leq s$, then letting $r \in \mathcal{A}_j$ be such that $s \geq r$ we see that $s \geq g(r) \geq f_j(r)$. Thus $r, f_j(r)$ are compatible and hence $r \geq f_j(r)$. This in turn implies (by $(\otimes)_7$) that for each $\varepsilon \in \text{dom}(g(r)) \setminus w_j$ the condition $f_j(r) \upharpoonright \varepsilon$ forces the assumptions of case (a) of $(\otimes)_8$. Consequently, the demand $(*)_3$ (for $f_j^+(s)$) holds. One easily verifies that also the other relevant demands from $(*)_1$ – $(*)_6$ are satisfied by $\mathcal{A}_j, f_j, p_j, \mathcal{A}_j^+, f_j^+, p_j^+$.

CASE: j is a limit ordinal. If $j \in \mathcal{S}$ then let $q^* = q_j^*$, otherwise $q^* = \emptyset_{\mathbb{P}_{\zeta^*}}$. Let $\mathcal{A}_j^0 \in N$ be a maximal antichain of \mathbb{P}_{ζ^*} such that

$$(\forall s \in \mathcal{A}_j^0)(\forall i < j)(\exists r \in \mathcal{A}_i)(r \leq s).$$

We are going to define a function $g : \mathcal{A}_j^0 \rightarrow \mathbb{P}_{\zeta^*}$. Let $r \in \mathcal{A}_j^0$ and for $i < j$ let $r_i \in \mathcal{A}_i$, $s_i \in \mathcal{A}_i^+$ be the unique elements such that $r_i \leq s_i \leq r$. We define $g(r)$ so that $\text{dom}(g(r)) = \bigcup_{i < j} \text{dom}(f_i(r_i)) \cup \text{dom}(q^*)$ and

(\otimes)₁₁^a if $\varepsilon \in \text{dom}(g(r)) \setminus \bigcup_{i < j} w_i$, then $g(r)(\varepsilon)$ is the $<^*_\chi$ -first \mathbb{P}_ε -name for a condition in \mathbb{Q}_ε for which it is forced that

- (a) if $\{f_i(r_i)(\varepsilon) : i < j\} \cup \{q^*(\varepsilon)\}$ has an upper bound in \mathbb{Q}_ε , then $g(r)(\varepsilon)$ is such a bound,
- (b) if the assumption of (a) is not satisfied but $\{f_i(r_i)(\varepsilon) : i < j\}$ has an upper bound in \mathbb{Q}_ε , then $g(r)(\varepsilon)$ is such a bound,
- (c) otherwise, $g(r)(\varepsilon) = \emptyset_{\mathbb{Q}_\varepsilon}$,

(\otimes)₁₁^b for $\varepsilon \in \text{dom}(g(r)) \cap \bigcup_{i < j} w_i$ we set $g(r)(\varepsilon) = f_{r(\varepsilon)}(r_{r(\varepsilon)})(\varepsilon)$.

Clearly, $g : \mathcal{A}_j^0 \rightarrow \mathbb{P}_{\zeta^*}$ is a well defined function, $g \in N$.

(\otimes)₁₂ Assume $r \in \mathcal{A}_j^0$, $r_i \in \mathcal{A}_i$, $r_i \leq r$ for $i < j$.

- (a) If $f_i(r_i) \leq r_i$ for all $i < j$, then $f_i(r_i) \leq g(r)$ for all $i < j$.
- (b) If $f_i(r_i), r_i$ are incompatible for some $i < j$, then r and $g(r)$ are incompatible.

[Why? Clause (a) follows immediately from the inductive hypothesis ($*$)₃ and the definition of g . To show clause (b) assume that $\varepsilon < \zeta^*$ and $i_0 < j$ are such that

- $f_i(r_i) \upharpoonright \varepsilon \leq r_i \upharpoonright \varepsilon$ for all $i < j$, and
- $r_{i_0} \upharpoonright \varepsilon \Vdash$ “ $f_{i_0}(r_{i_0})(\varepsilon), r_{i_0}(\varepsilon)$ are incompatible”

(see 2.10(iii)). By Observation 2.12(vi) for each $i < j$ there is $s_i \in \mathcal{A}_i$ such that $r_i \upharpoonright \varepsilon$ and $s_i \upharpoonright \varepsilon$ are compatible and $f_i(s_i) \leq s_i$. Then also $f_i(r_i) \upharpoonright (\varepsilon + 1) = f_i(s_i) \upharpoonright (\varepsilon + 1) \leq s_i \upharpoonright (\varepsilon + 1)$ and for each $\alpha \in (\varepsilon + 1) \setminus \bigcup_{i < j} w_i$ we have

$f_i(r_i) \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha}$ “ $\langle f_{i'}(r_{i'}) \upharpoonright \alpha : i' < i \rangle$ are innings of Incomplete in a play of \mathcal{Q}_α^0 in which Complete follows \mathbf{st}_α^0 ”.

(Note that if $i' < i$ and $t \in \mathcal{A}_{i'}$ is such that $t \leq s_i$, then $t \upharpoonright \varepsilon$ and $r_{i'} \upharpoonright \varepsilon$ are compatible, so $f_{i'}(t) \upharpoonright (\varepsilon + 1) = f_{i'}(r_{i'}) \upharpoonright (\varepsilon + 1)$.) Consequently, by induction on $\alpha \leq \varepsilon + 1$ we may show that $f_i(r_i) \upharpoonright \alpha \leq g(r) \upharpoonright \alpha$ for all $i < j$ and $\alpha \leq \varepsilon$. In particular, $f_{i_0}(r_{i_0}) \upharpoonright (\varepsilon + 1) \leq g(r) \upharpoonright (\varepsilon + 1)$ and hence, by the choice of i_0 and ε , the conditions $g(r)$ and r_{i_0} are incompatible. Hence also $g(r)$ and r are incompatible.]

In the same way as for (\otimes)₉ we may argue that

(\otimes)₁₃ if $\alpha < \beta$ are successive elements of w_j , and $r^0, r^1 \in \mathcal{A}_j^0$ are such that $r^0 \upharpoonright \alpha$ and $r^1 \upharpoonright \alpha$ are compatible, then $g(r^0) \upharpoonright \beta = g(r^1) \upharpoonright \beta$.

Now pick a maximal antichain $\mathcal{A}_j \in N$ of \mathbb{P}_{ζ^*} such that

(\otimes)₁₄^a $(\forall s \in \mathcal{A}_j)(\exists r \in \mathcal{A}_j^0)(r \leq s)$, and

(\otimes)₁₄^b if $s \in \mathcal{A}_j$, $r \in \mathcal{A}_j^0$ and $r \leq s$ then either $g(r) \leq s$, or for some $\varepsilon \in \text{dom}(s)$ we have $g(r) \upharpoonright \varepsilon \leq s \upharpoonright \varepsilon$ and $s \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $g(r)(\varepsilon), s(\varepsilon)$ are incompatible in \mathbb{Q}_ε ”.

Define $f_j : \mathcal{A}_j \rightarrow \mathbb{P}_{\zeta^*}$ by $f_j(s) = g(r)$ where $r \in \mathcal{A}_j^0$ is unique such that $r \leq s$ for $s \in \mathcal{A}_j^0$. Using $(\otimes)_{12+}(\otimes)_{13}$ and Proposition 2.13 we may easily argue that $(\mathcal{A}_j, f_j, w_j) \in N$ is a standard representation of some $(p_j, w_j) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$ such that $p_j(\varepsilon) = p_{\mathcal{Y}(\varepsilon)}(\varepsilon)$ for $\varepsilon \in w_j$ and $(\mathcal{A}_i, f_i, w_i) \preceq (\mathcal{A}_i^+, f_i^+, w_i) \preceq (\mathcal{A}_j, f_j, w_j)$ for all $i < j$. Finally, exactly as in the successor case ($(\otimes)_{7-}(\otimes)_{10}$ and later) we define \mathcal{A}_j, f_j^+ and p_j^+ . We easily verify that the demands $(*)_{1-}(*)_6$ are satisfied.

This completes the construction of $\langle p_i, p_i^+, \mathcal{A}_i, \mathcal{A}_i^+, f_i, f_i^+ : i < \lambda \rangle$. It should be self-evident that the conditions $(*)_{1-}(*)_6$ imply $(\boxtimes)_{20-}(\boxtimes)_{20}^e$. ■

Let $\langle p_i^*, p_i^+ : i < \lambda \rangle$ be given by Claim 4.1.4. Recalling Remark 2.7, we define a condition $r \in \mathbb{P}_{\zeta^*}$ by declaring that its domain (support) is $\text{dom}(r) = \zeta^* \cap N$ and for each $\zeta \in \zeta^* \cap N$,

$$r \restriction \zeta \Vdash \text{“} r(\zeta) \geq p_{\mathcal{Y}(\zeta)}(\zeta) \text{ is generic for } \bar{q}^{(\zeta)} \text{ over } N[G_{\mathbb{P}_\zeta}], h^{(\zeta)}, \mathbb{Q}_\zeta, R_\zeta^{\text{PR}}, \bar{f}^{(\zeta)}, \mathcal{S}, \mathcal{Y}(\zeta) \cdot \omega \text{”}.$$

MAIN CLAIM 4.1.5. *For every $\zeta \in (\zeta^* + 1) \cap N$, Generic has a winning strategy in the game $\mathfrak{D}_1^S(r \restriction \zeta, N, h^{[\zeta]}, \mathbb{P}_\zeta, R^{\text{PR}} \restriction \zeta, \bar{f}, \bar{q}^{[\zeta]})$.*

Proof of the Main Claim. We prove the claim by induction on $\zeta \in (\zeta^* + 1) \cap N$, so suppose that $\zeta \in (\zeta^* + 1) \cap N$ and we know that for all $\xi \in N \cap \zeta$ the condition $r \restriction \xi$ is generic for $\bar{q}^{[\xi]}$ over $N, h^{[\xi]}, \mathbb{P}_\xi, R^{\text{PR}} \restriction \xi, \bar{f}, \mathcal{S}, 1$. Note that the inductive hypothesis implies that

- (\boxplus)₀ $r(\xi)$ is well-defined for $\xi \in N \cap \zeta$ (by Claim 4.1.3), and so
- (\boxplus)₁ for each $\xi \in N \cap \zeta$ we may fix a \mathbb{P}_ξ -name \mathbf{st}_ξ such that the condition $r \restriction \xi$ forces that it is a winning strategy of Generic in the game $\mathfrak{D}_{\mathcal{Y}(\xi), \omega}^S(r(\xi), N[G_{\mathbb{P}_\xi}], h^{(\xi)}, \mathbb{Q}_\xi, R_\xi^{\text{PR}}, \bar{f}^{(\xi)}, \bar{q}^{(\xi)})$, and
- (\boxplus)₂ $(p_i^\zeta, w_i^\zeta) \leq' (r \restriction \zeta, \{0, \zeta\})$ for all $i < \lambda$, where $p_i^\zeta = p_i \restriction \zeta$ and $w_i^\zeta = (w_i \cap \zeta) \cup \{\zeta\}$.

SUBCLAIM 4.1.5.1. *Assume that $(p_i^*, v_i) \in \mathbb{P}_{\zeta^*}^{\text{RS}} \cap N$ (for $i < \delta < \lambda$), and $p^* \in \mathbb{P}_\zeta \cap N$, $r' \in \mathbb{P}_\zeta$ are such that*

$$r \restriction \zeta \leq r', \quad p^* \leq r' \quad \text{and} \quad (\forall j < i < \delta)((p_j^*, v_j) \leq' (p_i^*, v_i) \leq' (r', \{0, \zeta\})).$$

Then there are conditions $p^+ \in N \cap \mathbb{P}_\zeta$ and $r^+ \in \mathbb{P}_\zeta$ such that

$$p^* \leq p^+ \leq r^+, \quad r' \leq r^+ \quad \text{and} \quad (\forall i < \delta)((p_i^*, v_i) \leq' (p^+, \{0, \zeta\})).$$

Proof of the Subclaim. By the inductive hypothesis (of Main Claim 4.1.5) we know that $r \restriction \xi$ is (N, \mathbb{P}_ξ) -generic for all $\xi \in \zeta \cap N$. Therefore, if ζ is a successor or a limit ordinal of cofinality $\geq \lambda$, then we immediately find that $r \restriction \zeta$ is (N, \mathbb{P}_ζ) -generic (see 3.8(5) and use $(\boxtimes)_1$ and $(\boxtimes)_{20}^e$ of 4.1.4). Hence either r' is (N, \mathbb{P}_ζ) -generic, or ζ is a limit ordinal of cofinality $\text{cf}(\zeta) < \lambda$

and for all $\xi < \zeta$ the condition $r' \upharpoonright \xi$ is (N, \mathbb{P}_ξ) -generic. In either case Lemma 2.8(3) applies. ■

Recalling that $\bar{\gamma} = \langle \gamma_\alpha : \alpha < \lambda \rangle$ was defined in 3.1(4) we set

$$E_0^\zeta := \{\delta < \lambda : \delta = \gamma_\delta > 0 \text{ and } \delta > \mathcal{Y}(\zeta) \text{ and } (\forall \alpha < \delta)(\alpha \cdot \omega < \delta)\}$$

(plainly, $E_0^\zeta \in D$).

We will give a strategy **st** for Generic in $\mathcal{D}_1^S(r \upharpoonright \zeta, N, h^{[\zeta]}, \mathbb{P}_\zeta, R^{\text{pr}} \upharpoonright \zeta, \bar{f}, \bar{q}^{[\zeta]})$. Essentially, for each $\varepsilon \in \zeta \cap N$, Generic is going to play a suitable game at coordinate ε , but at each time she is playing the game on less than λ coordinates. Her actions will be described on the intervals $[\gamma_\alpha, \gamma_{\alpha+1})$ separately. We may assume that, for $\xi \in N \cap \zeta$, the sets $C_i \in D^{\mathbf{V}[\mathbb{G}_{\mathbb{P}_\xi}^{\mathbf{V}}]}$ given to Generic by her winning strategy **st** $_\xi$ are of the form $\Delta_{\alpha < \lambda} \underline{A}_\alpha$, where $\underline{A}_\alpha \in D^{\mathbf{V}}$. To keep the “past and future plays” under control, in addition to the innings (r_i^-, r_i, C_i) of the game, Generic will construct aside

$$(\boxplus)_3^\alpha \quad s_\alpha, z_\alpha, z_\alpha^\diamond, r_j^*, r_j^+, r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon), \underline{A}_i^\xi(\varepsilon), \mathcal{I}_{i,\varepsilon}, A_{i,\varepsilon}^\xi \text{ for } j < \gamma_{\alpha+1}, \xi < \lambda \text{ and } \varepsilon \in N \cap \zeta \text{ with } \mathcal{Y}(\varepsilon) < \gamma_{\alpha+1} \text{ and } i \text{ such that } \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}.$$

The primary roles played by these objects are as follows:

- $r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon), \Delta_{\xi < \lambda} \underline{A}_i^\xi(\varepsilon)$ are the innings at stage i of the game played at coordinate ε ,
- $s_\alpha \in \mathbb{P}_\zeta$ are conditions deciding $r_i^\ominus(\varepsilon), \underline{A}_i^\xi(\varepsilon)$ and $\mathcal{I}_{i,\varepsilon}, A_{i,\varepsilon}^\xi$ are the values forced; the conditions s_α will be used as r_j 's too,
- $r_j^*, r_j^+ \in \mathbb{P}_\zeta$ form a strategic completeness play and are used to guarantee that the sequences $\langle r_j : j < \gamma_\alpha \rangle$ have upper bounds,
- $z_\alpha, z_\alpha^\diamond \in N \cap \mathbb{P}_\zeta$ are conditions forming a strategic completeness play and are used to guarantee that the sequences $\langle r_j^-(\varepsilon) : j < \gamma_\alpha \rangle$ (are forced to) have upper bounds.

Thus we require that the following demands $(\boxplus)_4 - (\boxplus)_{10}$ are satisfied in addition to the rules of the game:

$$(\boxplus)_4 \quad s_\alpha, r_j^*, r_j^+ \in \mathbb{P}_\zeta, z_\alpha, z_\alpha^\diamond \in N \cap \mathbb{P}_\zeta, \text{ and } r_j^+ \leq s_\alpha, r_i^* \leq r_i^+ \leq r_j^* \text{ for all } i < j < \gamma_{\alpha+1}.$$

$$(\boxplus)_5 \quad \text{If } \gamma_\alpha \notin \mathcal{S} \text{ then } r_{\gamma_\alpha} \leq r_{\gamma_\alpha}^* \text{ and if } \gamma_\alpha \in \mathcal{S} \text{ then } r_{\gamma_\alpha} = r_{\gamma_\alpha}^*. \text{ Also, } s_\alpha \in \bigcap_{i < \gamma_{\alpha+1}} \mathcal{I}_i^{[\zeta]} \text{ and if } \gamma_\alpha < j < \gamma_{\alpha+1}, \text{ then } r_j = s_\alpha \text{ and}$$

$$C_j = E_0^\zeta \cap \bigcap \{A_{i,\varepsilon}^\xi : \xi < \gamma_{\alpha+1} \ \& \ \varepsilon \in N \cap \zeta \ \& \ \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}\}.$$

$$(\boxplus)_6 \quad \text{If } \varepsilon \in \zeta \setminus \{\varepsilon' \in N \cap \zeta : \mathcal{Y}(\varepsilon') < \gamma_{\alpha+1}\}, j < \gamma_{\alpha+1}, \text{ then}$$

$$r_j^+ \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{ “} \langle r_i^*(\varepsilon), r_i^+(\varepsilon) : i \leq j \rangle \text{ is a legal partial play of } \mathcal{D}_0^\lambda(\mathbb{Q}_\varepsilon) \text{ in which Complete uses her regular winning strategy } \underline{\mathbf{st}}_\varepsilon^0 \text{”}.$$

(\boxplus)₇ If $\varepsilon \in N \cap \zeta$ and $\mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}$, then $r_i^\ominus(\varepsilon)$ is a \mathbb{P}_ε -name for a condition in $N[G_{\mathbb{P}_\varepsilon}] \cap \mathbb{Q}_\varepsilon$, $r_i^\oplus(\varepsilon)$ is a \mathbb{P}_ε -name for a condition in \mathbb{Q}_ε and $A_i^\xi(\varepsilon)$ are \mathbb{P}_ε -names for elements of $D \cap \mathbf{V}$ (for $\xi < \lambda$) such that

$r \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $\langle r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon), \Delta_{\xi < \lambda} A_i^\xi(\varepsilon) : \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1} \rangle$ is a partial play of the game $\mathfrak{D}_{\mathcal{Y}(\varepsilon)\text{-}\omega}^S(r(\varepsilon), N[G_{\mathbb{P}_\varepsilon}], h^{(\varepsilon)}, \mathbb{Q}_\varepsilon, R_\varepsilon^{\text{Pr}}, \bar{f}^{(\varepsilon)}, \bar{q}^{(\varepsilon)})$ in which Generic uses her winning strategy \mathbf{st}_ε ”.

(\boxplus)₈ If $\varepsilon \in N \cap \zeta$ and $\gamma_\alpha \leq j = \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}$, then $\mathcal{I}_{i,\varepsilon} \in N$ is a \mathbb{P}_ε -name for a condition in \mathbb{Q}_ε , $A_{i,\varepsilon}^\xi \in D$ for $\xi < \gamma_{\alpha+1}$, and

$s_\alpha \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $A_{i,\varepsilon}^\xi(\varepsilon) = A_{i,\varepsilon}^\xi$ and if $j \in \mathcal{S}$ then $r_j^-(\varepsilon) = r_i^\ominus(\varepsilon) = \mathcal{I}_{i,\varepsilon}$ ”.

Also, $r_j^* \upharpoonright \varepsilon \Vdash r_j^*(\varepsilon) = r_j^+(\varepsilon) = r_i^\oplus(\varepsilon)$.

(\boxplus)₉ If $\gamma_\alpha \notin \mathcal{S}$, then $(p_j^\zeta, w_j^\zeta) \leq' (r_{\gamma_\alpha+1}^-, \{0, \zeta\})$ for all $j < \gamma_{\alpha+1}$ (where $p_j^\zeta = p_j \upharpoonright \zeta$ and $w_j^\zeta = (w_j \cap \zeta) \cup \{\zeta\}$, see (\boxplus)₂).

(\boxplus)₁₀ If $\gamma_\alpha \notin \mathcal{S}$, then $r_{\gamma_\alpha}^- \leq z_\alpha \leq z_\alpha^\diamond \leq r_{\gamma_\alpha+1}^-$ and $z_\beta^\diamond \leq z_\alpha$ for $\beta < \alpha$, $\gamma_\beta \notin \mathcal{S}$, and for each $\varepsilon \in N \cap \zeta$ we have

$z_\alpha^\diamond \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $\langle z_\beta(\varepsilon), z_\beta^\diamond(\varepsilon) : \beta \leq \alpha, \gamma_\beta \notin \mathcal{S} \rangle$ is a legal partial play of $\mathfrak{D}_0^\lambda(\mathbb{Q}_\varepsilon)$ in which Complete uses her winning strategy $\mathbf{st}_\varepsilon^0$ ”.

Suppose that the players arrived at a stage γ_α of the play and the sequence $\langle r_i^-, r_i, C_i : i < \gamma_\alpha \rangle$ has been constructed and the objects listed in (\boxplus)₃ ^{β} have been chosen for all $\beta < \alpha$. The procedure applied now by Generic depends on whether the inning at γ_α is given by Generic or Antigeneric, so we will have two cases below.

CASE 1: $\gamma_\alpha \notin \mathcal{S}$. The inning $(r_{\gamma_\alpha}^-, r_{\gamma_\alpha}, C_{\gamma_\alpha})$ is chosen by Antigeneric, but we have to argue first that

(\boxplus)₁₁ there exists a legal move for Antigeneric.

For this we note that there is a condition $r' \in \mathbb{P}_\zeta$ stronger than all r_j for $j < \gamma_\alpha$. [Why? If $\alpha = \beta + 1$ then s_β works: see (\boxplus)₅. If α is limit, then look at the conditions r_j^*, r_j^+ for $j < \gamma_\alpha$: by (\boxplus)₆–(\boxplus)₈ the sequence $\langle r_j^* : j < \gamma_\alpha \rangle$ has an upper bound and by (\boxplus)₅ this bound will be above all r_j for $j < \gamma_\alpha$.] Then the condition r' is stronger than $r \upharpoonright \zeta$ and stronger than all r_j^- (for $j < \gamma_\alpha$) and we may use Subclaim 4.1.5.1 to pick conditions $r^- \in \mathbb{P}_\zeta \cap N$ and $r^* \in \mathbb{P}_\zeta$ so that

$$r^- \leq r^*, \quad r' \leq r^* \quad \text{and} \quad (\forall j < \gamma_\alpha)(r_j^- \leq r^-).$$

Then (r^-, r^*, λ) is a legal inning of Antigeneric at this stage.

So, let $(r_{\gamma_\alpha}^-, r_{\gamma_\alpha}, C_{\gamma_\alpha})$ be the inning played by Antigeneric at stage γ_α . By Subclaim 4.1.5.1, Generic may pick conditions $r^- \in \mathbb{P}_\zeta \cap N$ and $r^+ \in \mathbb{P}_\zeta$

so that

$$(\boxplus)_{12} \quad r_{\gamma_\alpha}^- \leq r^- \leq r^+, r_{\gamma_\alpha} \leq r^+ \text{ and } (\forall j < \gamma_{\alpha+1})((p_j^\zeta, w_j^\zeta) \leq' (r^-, \{0, \zeta\})).$$

Then $z_\beta^\diamond \leq r_{\gamma_{\beta+1}}^- \leq r_{\gamma_\alpha}^- \leq r^- \leq r^+$ (for $\beta < \alpha$, $\gamma_\beta \notin \mathcal{S}$), so by the argument as in 4.1.5.1 we may apply Lemma 2.5 for $\langle z_\beta, z_\beta^\diamond : \beta < \alpha \ \& \ \gamma_\beta \notin \mathcal{S} \rangle$ and r^-, r^+ . This will give us conditions $z_\alpha, z_\alpha^\diamond \in N \cap \mathbb{P}_\zeta$ and $r_{\gamma_\alpha}^* \in \mathbb{P}_\zeta$ such that $r^- \leq z_\alpha \leq z_\alpha^\diamond \leq r_{\gamma_\alpha}^*$, $r^+ \leq r_{\gamma_\alpha}^*$ and for each $\varepsilon \in \zeta \cap N$ the condition $z_\alpha^\diamond \upharpoonright \varepsilon$ forces that “the sequence $\langle z_\beta(\varepsilon), z_\beta^\diamond(\varepsilon) : \beta \leq \alpha, \gamma_\beta \notin \mathcal{S} \rangle$ is a legal partial play of $\partial_0^\lambda(\mathbb{Q}_\varepsilon)$ in which Complete uses her regular winning strategy $\mathbf{st}_\varepsilon^{0}$ ”. Now, for each $\varepsilon \in N \cap \zeta$ and i with $\Upsilon(\varepsilon) + i < \gamma_{\alpha+1}$, Generic picks $r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon), A_i^\xi(\varepsilon)$ so that (for $\Upsilon(\varepsilon) + i < \gamma_\alpha$ they are the ones chosen at previous stages and)

$$r \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{ “ } \langle r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon), \Delta_{\xi < \lambda} A_i^\xi(\varepsilon) : \Upsilon(\varepsilon) + i < \gamma_{\alpha+1} \rangle \text{ is a partial play of } \\ \partial_{\Upsilon(\varepsilon) \cdot \omega}^S(r(\varepsilon), N[G_{\mathbb{P}_\varepsilon}], h^{(\varepsilon)}, \mathbb{Q}_\varepsilon, R_\varepsilon^{\text{pr}}, f^{(\varepsilon)}, \bar{q}^{(\varepsilon)}) \\ \text{ in which Generic uses her winning strategy } \mathbf{st}_\varepsilon \text{ ”}$$

and

- if $\Upsilon(\varepsilon) + i = \gamma_\alpha$, then ⁽⁹⁾

$$r_{\gamma_\alpha}^* \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{ “ } r_i^\ominus(\varepsilon) = z_\alpha^\diamond(\varepsilon) \ \& \ r_i^\oplus(\varepsilon) = r_{\gamma_\alpha}^*(\varepsilon) \ \& \ A_i^\xi(\varepsilon) = C_{\gamma_\alpha} \text{ ”},$$

- if $\gamma_\alpha < \Upsilon(\varepsilon) < \gamma_{\alpha+1}$, then

$$r_{\gamma_\alpha}^* \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{ “ } r_0^\ominus(\varepsilon) = z_\alpha^\diamond(\varepsilon) \ \& \ r_0^\oplus(\varepsilon) = r_{\gamma_\alpha}^*(\varepsilon) \ \& \ A_0^\xi(\varepsilon) = C_{\gamma_\alpha} \text{ ”}.$$

Next, Generic picks $r_{\gamma_\alpha}^+ \in \mathbb{P}_\zeta$ so that $\text{dom}(r_{\gamma_\alpha}^+) = \text{dom}(r_{\gamma_\alpha}^*)$ and the demands of $(\boxplus)_6$ (for $j = \gamma_\alpha$) are satisfied and $r_{\gamma_\alpha}^+(\varepsilon) = r_{\gamma_\alpha}^*(\varepsilon)$ whenever $\varepsilon \in N \cap \zeta$, $\Upsilon(\varepsilon) < \gamma_{\alpha+1}$. For $\gamma_\alpha < j < \gamma_{\alpha+1}$ she chooses r_j^*, r_j^+ so that $\text{dom}(r_j^*) = \text{dom}(r_j^+) = \text{dom}(r_{\gamma_\alpha}^*)$ and $(\boxplus)_6$ holds (for $\varepsilon \in \zeta \setminus \{\varepsilon' \in N \cap \zeta : \Upsilon(\varepsilon') < \gamma_{\alpha+1}\}$) and for $\varepsilon \in N \cap \zeta$ with $\Upsilon(\varepsilon) < \gamma_{\alpha+1}$ we have

- if $j < \Upsilon(\varepsilon)$, then $r_j^*(\varepsilon) = r_j^+(\varepsilon) = r_{\gamma_\alpha}^*(\varepsilon)$, and
- if $j = \Upsilon(\varepsilon) + i$, then $r_j^* \upharpoonright \varepsilon \Vdash \text{ “ } r_j^*(\varepsilon) = r_j^+(\varepsilon) = r_i^\oplus(\varepsilon) \text{ ”}.$

By $(\boxplus)_6$ and $(\boxplus)_7$ the sequence $\langle r_j^+ : j < \gamma_{\alpha+1} \rangle$ has an upper bound in \mathbb{P}_ζ , so Generic may choose $s_\alpha \in \mathbb{P}_\zeta$ and $\tau_{i,\varepsilon} \in N$, $A_{i,\varepsilon}^\xi \in D$ (for $\varepsilon \in N \cap \zeta$, $\Upsilon(\varepsilon) + i < \gamma_{\alpha+1}$ and $\xi < \gamma_{\alpha+1}$) such that

- s_α is stronger than all r_j^+ (for $j < \gamma_{\alpha+1}$) and $s_\alpha \in \bigcap_{j < \gamma_{\alpha+1}} \mathcal{I}_j^{[\zeta]}$, and
- $s_\alpha \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon} \text{ “ } r_i^\ominus(\varepsilon) = \tau_{i,\varepsilon} \ \& \ A_i^\xi(\varepsilon) = A_{i,\varepsilon}^\xi \text{ ”}.$

⁽⁹⁾ Note that necessarily i is *not* a successor in this case. Hence if $\gamma_\alpha < \Upsilon(\varepsilon) \cdot \omega$, then $i \notin \mathcal{S}[\Upsilon(\varepsilon) \cdot \omega]$ and otherwise $i = \gamma_\alpha \notin \mathcal{S}$. Consequently, in the game $\partial_{\Upsilon(\varepsilon) \cdot \omega}^S$ the inning at i belongs to Antigeneric.

She also defines conditions $r_j^- \in \mathbb{P}_\zeta \cap N$ (for $\gamma_\alpha < j < \gamma_{\alpha+1}$) by declaring that $\text{dom}(r_j^-) = \text{dom}(z_\alpha^\diamond) \cup \{\varepsilon' \in N \cap \zeta : \mathcal{Y}(\varepsilon') < \gamma_{\alpha+1}\}$ and

- if $\varepsilon \in \text{dom}(r_j^-) \setminus \{\varepsilon' \in N \cap \zeta : \mathcal{Y}(\varepsilon') < \gamma_{\alpha+1}\}$ or $\varepsilon \in N \cap \zeta$, $j < \mathcal{Y}(\varepsilon) < \gamma_{\alpha+1}$, then $r_j^-(\varepsilon) = z_\alpha^\diamond(\varepsilon)$, and
- if $\varepsilon \in N \cap \zeta$ and $j = \mathcal{Y}(\varepsilon) + i^* < \gamma_{\alpha+1}$, then

$r_j^- \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “if $\langle \mathcal{I}_{i,\varepsilon} : \gamma_\alpha \leq \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1} \rangle$ is an increasing sequence of conditions above $z_\alpha^\diamond(\varepsilon)$, then $r_j^-(\varepsilon) = \mathcal{I}_{i^*,\varepsilon}$, otherwise $r_j^-(\varepsilon) = z_\alpha^\diamond(\varepsilon)$ ”.

Then for $\gamma_\alpha < j < \gamma_{\alpha+1}$ Generic plays r_j^- chosen above, $r_j = s_\alpha$ and $C_j = E_0^\zeta \cap \bigcap \{A_{i,\varepsilon}^\xi : \xi < \gamma_{\alpha+1} \ \& \ \varepsilon \in N \cap \zeta \ \& \ \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}\}$. Observe that the rules of the game and the demands $(\boxplus)_4 - (\boxplus)_{10}$ are obeyed by the choices above.

CASE 2: $\gamma_\alpha \in \mathcal{S}$ (this may happen only if α is limit). Generic's choices are similar to those from the previous case, except that $r_{\gamma_\alpha}, r_{\gamma_\alpha}^-$ need to be treated differently, and this influences the choice of s_α too.

By $(\boxplus)_7$ at previous stages, for each $\varepsilon \in N \cap \zeta$ with $\mathcal{Y}(\varepsilon) < \gamma_\alpha$, Generic may pick $r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon)$ and $A_i^\xi(\varepsilon)$ (for $\gamma_\alpha \leq \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}$) so that $(\boxplus)_7$ still holds. By $(\boxplus)_6 + (\boxplus)_8$ she may choose a condition $r' \in \mathbb{P}_\zeta$ stronger than all r_j for $j < \gamma_\alpha$ and such that $r'(\varepsilon) = r_i^\oplus(\varepsilon)$ when $\varepsilon \in N \cap \zeta$, $\mathcal{Y}(\varepsilon) + i = \gamma_\alpha$, $0 < i$. Now, for $\varepsilon \in \zeta \cap N$ with $\gamma_\alpha \leq \mathcal{Y}(\varepsilon) < \gamma_{\alpha+1}$ we have

$r' \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “the condition $r'(\varepsilon)$ is $(N[G_{\mathbb{P}_\varepsilon}], \mathbb{Q}_\varepsilon)$ -generic and $r'(\varepsilon) \geq r_j^-(\varepsilon)$ for all $j < \gamma_\alpha$ ”.

Therefore, by Observation 2.3, Generic may pick \mathbb{P}_ε -names $r_0^\ominus(\varepsilon), r_0^\oplus(\varepsilon)$ for conditions in \mathbb{Q}_ε such that

$r \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $r_0^\ominus(\varepsilon) \in N[G_{\mathbb{P}_\varepsilon}]$ & $r_0^\ominus(\varepsilon) \leq r_0^\oplus(\varepsilon)$ & $r(\varepsilon) \leq r_0^\oplus(\varepsilon)$ ”,

and

$r' \upharpoonright \varepsilon \Vdash_{\mathbb{P}_\varepsilon}$ “ $r'(\varepsilon) \leq r_0^\oplus(\varepsilon)$ & $(\forall j < \gamma_\alpha)(r_j^-(\varepsilon) \leq r_0^\ominus(\varepsilon))$ ”.

Then for $\varepsilon \in \zeta \cap N$, $\xi < \lambda$ and i such that $\gamma_\alpha \leq \mathcal{Y}(\varepsilon) < \mathcal{Y}(\varepsilon) + i < \gamma_{\alpha+1}$, Generic picks \mathbb{P}_ε -names $r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon)$ and $A_i^\xi(\varepsilon)$ so that $(\boxplus)_7$ holds.

Now she declares that $\text{dom}(r_{\gamma_\alpha}) = \text{dom}(r_{\gamma_\alpha}^*) = \text{dom}(r')$ and for $\varepsilon \in \text{dom}(r_{\gamma_\alpha})$ she sets

$$r_{\gamma_\alpha}(\varepsilon) = r_{\gamma_\alpha}^*(\varepsilon) = \begin{cases} r_0^\oplus(\varepsilon) & \text{if } \varepsilon \in N \cap \zeta, \gamma_\alpha \leq \mathcal{Y}(\varepsilon) < \gamma_{\alpha+1}, \\ r'(\varepsilon) & \text{otherwise.} \end{cases}$$

Then $r_{\gamma_\alpha}^+, r_j^*, r_j^+$ (for $\gamma_\alpha < j < \gamma_{\alpha+1}$) are defined exactly as in the previous case. As before, the sequence $\langle r_j^+ : j < \gamma_{\alpha+1} \rangle$ has an upper bound in \mathbb{P}_ζ , so

Generic may find a condition $s' \in \mathbb{P}_\zeta$, names $\mathcal{I}_{i,\varepsilon} \in N$ for conditions in \mathbb{Q}_ε and sets $A_{i,\varepsilon}^\xi \in D$ (for $\varepsilon \in N \cap \zeta$, $\gamma_\alpha \leq \Upsilon(\varepsilon) + i < \gamma_{\alpha+1}$ and $\xi < \gamma_{\alpha+1}$) such that

- $r_j^+ \leq s'$ for all $j < \gamma_{\alpha+1}$ and $s' \in \bigcap_{j < \gamma_{\alpha+1}} \mathcal{I}_j^{[\zeta]}$, and
- $s' \Vdash_{\mathbb{P}_\varepsilon} \text{“} \mathcal{I}_i^\ominus(\varepsilon) = \mathcal{I}_{i,\varepsilon} \ \& \ A_{i,\varepsilon}^\xi(\varepsilon) = A_{i,\varepsilon}^\xi \text{”}$.

Clearly $r_j^- \leq r_j \leq r' \leq r_{\gamma_\alpha} \leq s'$ for $j < \gamma_\alpha$, so Generic may use Subclaim 4.1.5.1 to pick conditions $s_\alpha \in \mathbb{P}_\zeta$ and $r^- \in N \cap \mathbb{P}_\zeta$ such that $r^- \leq s_\alpha$, $s' \leq s_\alpha$ and $(\forall j < \gamma_\alpha)(r_j^- \leq r^-)$. Moreover, Generic may modify r^- so that, additionally, for each $\varepsilon \in \text{dom}(r^-)$ we have

$\Vdash_{\mathbb{P}_\varepsilon}$ “if the sequence $\langle r_j^-(\varepsilon) : j < \gamma_\alpha \rangle$ has an upper bound in \mathbb{Q}_ε then $r^-(\varepsilon)$ is such a bound”.

Then, for $\gamma_\alpha \leq j < \gamma_{\alpha+1}$, Generic sets $\text{dom}(r_j^-) = \text{dom}(r^-)$ and

- if $\varepsilon \in \text{dom}(r_j^-) \setminus \{\varepsilon' \in N \cap \zeta : \Upsilon(\varepsilon') < \gamma_{\alpha+1}\}$, then $r_j^-(\varepsilon) = r^-(\varepsilon)$, and
- if $\varepsilon \in N \cap \zeta$, $\Upsilon(\varepsilon) < \gamma_{\alpha+1}$ and i^* is 0 if $j \leq \Upsilon(\varepsilon)$ and i^* is such that $j = \Upsilon(\varepsilon) + i^*$ if $\Upsilon(\varepsilon) < j < \gamma_{\alpha+1}$, then $r_j^-(\varepsilon)$ is the $<_\chi^*$ -first name for a condition in \mathbb{Q}_ε such that

$r_j^- \Vdash_{\mathbb{P}_\varepsilon}$ “if $\langle \mathcal{I}_{i,\varepsilon} : \gamma_\alpha \leq \Upsilon(\varepsilon) + i < \gamma_{\alpha+1} \rangle$ is an increasing sequence of conditions above all $r_{j'}^-(\varepsilon)$ for $j' < \gamma_\alpha$, then $r_j^-(\varepsilon) = \mathcal{I}_{i^*,\varepsilon}$, and otherwise $r_j^-(\varepsilon) = r^-(\varepsilon)$ ”.

Clearly the conditions r_j^- (for $\gamma_\alpha \leq j < \gamma_{\alpha+1}$) are well defined and they belong to N . Since in the current case α is a limit ordinal, $(\boxplus)_{10}$ implies that for each $\varepsilon \in N \cap \zeta$ and a condition $z \in \mathbb{P}_\varepsilon$ stronger than all $r_{j'}^- \Vdash_{\mathbb{P}_\varepsilon}$ (for $j' < \gamma_\alpha$) we have

$z \Vdash_{\mathbb{P}_\varepsilon}$ “the sequence $\langle r_{j'}^-(\varepsilon) : j' < \gamma_\alpha \rangle$ has an upper bound in \mathbb{Q}_ε ”,

and thus $z \Vdash_{\mathbb{P}_\varepsilon} \text{“} r_{j'}^-(\varepsilon) \leq r^-(\varepsilon) \text{ for all } j' < \gamma_\alpha \text{”}$ (remember the additional demand on r^-). Consequently ⁽¹⁰⁾, if $j' < \gamma_\alpha$ and $\gamma_\alpha \leq j_1 < j_2 < \gamma_{\alpha+1}$ then $r_{j'}^- \leq r_{j_1}^- \leq r_{j_2}^- \leq s_\alpha$.

Now, for $\gamma_\alpha \leq j < \gamma_{\alpha+1}$ Generic plays r_j^- chosen above, r_{γ_α} and $r_j = s_\alpha$ if $j > \gamma_\alpha$, and $C_j = E_0^\zeta \cap \bigcap \{A_{i,\varepsilon}^\xi : \xi < \gamma_{\alpha+1} \ \& \ \varepsilon \in N \cap \zeta \ \& \ \Upsilon(\varepsilon) + i < \gamma_{\alpha+1}\}$. Clearly the rules of the game ⁽¹¹⁾ and the demands $(\boxplus)_4$ – $(\boxplus)_{10}$ are obeyed by the choices above.

SUBCLAIM 4.1.5.2. *The strategy \mathbf{st} described above is a winning strategy for Generic in $\mathfrak{D}_1^S(r \upharpoonright \zeta, N[G_{\mathbb{P}_\zeta}], h^{[\zeta]}, \mathbb{P}_\zeta, R^{\text{Pr}} \upharpoonright \zeta, \bar{f}, \bar{q}^{[\zeta]})$.*

⁽¹⁰⁾ Without $(\boxplus)_{10}$ it is not clear that $r_j^- \Vdash_{\mathbb{P}_\varepsilon} \text{“} r_{j'}^-(\varepsilon) \leq r^-(\varepsilon) \text{ for all } j' < \gamma_\alpha \text{”}$.

⁽¹¹⁾ In this case it is not required that $r_{\gamma_\alpha}^- \leq r_{\gamma_\alpha}$ as $\gamma_\alpha \in \mathcal{S} \cap \mathcal{R}$.

Proof of the Subclaim. Suppose that $\langle r_j^-, r_j, C_j : j < \lambda \rangle$ is a result of a play of $\mathcal{D}_1^S(r \upharpoonright \zeta, N[G_{\mathbb{P}_\zeta}], h^{[\zeta]}, \mathbb{P}_\zeta, R^{\text{pr}} \upharpoonright \zeta, \bar{f}, \bar{q}^{[\zeta]})$ in which Generic follows the strategy **st**. By what was said in the description of the strategy (specifically in $(\boxplus)_{11}$) both players always had legal moves, so the play really lasted λ steps. Let

$$s_\alpha, z_\alpha, z_\alpha^\diamond, r_j^*, r_j^+, r_i^\ominus(\varepsilon), r_i^\oplus(\varepsilon), A_i^\xi(\varepsilon), \mathcal{I}_{i,\varepsilon}, A_{i,\varepsilon}^\xi$$

be the objects written aside by Generic (so they satisfy $(\boxplus)_{4-}(\boxplus)_{10}$).

We will argue that 3.5(3)(*) holds (i.e., Generic wins).

Assume that a limit ordinal $\delta \in \mathcal{S} \cap \bigcap_{j < \delta} C_j$ is such that $(^{12}) h^{[\zeta]}(f_\delta(j)) = r_j^-$ for each successor $j < \delta$. Then also $\delta \in E_0^\zeta$, so $(\forall \alpha < \delta)(\alpha \cdot \omega < \delta)$ and $\mathcal{Y}(\zeta) < \delta = \gamma_\delta = \sup\{\gamma_\alpha : \alpha < \delta \ \& \ \gamma_\alpha \notin \mathcal{S}\}$. Therefore, by $(\boxplus)_9$,

$$(\forall i < \delta)(\exists j < \delta)((p_i^\zeta, w_i^\zeta) \leq' (r_j^-, \{0, \zeta\})),$$

and consequently if $\zeta' = \min(\zeta_\delta^*, \zeta)$, then $(p_i^{\zeta'}, w_i^{\zeta'}) \leq' (q_\delta^* \upharpoonright \zeta', \{0, \zeta'\})$ for all $i < \delta$. (Remember that q_δ^* is an upper bound on $h^{[\zeta_\delta^*]} \circ f_\delta$, see $(\boxtimes)_{12} + (\boxtimes)_{13}$.) Hence, by $(\boxtimes)_{20}^d + (\boxplus)_2$,

$(\boxplus)_{13}$ if $\varepsilon \in \zeta \cap \zeta_\delta^* \cap N$, $\mathcal{Y}(\varepsilon) \geq \delta$ and $z \in \mathbb{P}_\varepsilon$ is stronger than both $q_\delta^* \upharpoonright \varepsilon$ and $r \upharpoonright \varepsilon$, then $z \Vdash "q_\delta^*(\varepsilon) \leq r(\varepsilon)"$.

Let $\bar{\beta} = \langle \beta(\varepsilon) : \varepsilon \leq \varepsilon^* \rangle$ be the increasing enumeration of $w_\delta \cap (\zeta + 1)$. Then $\beta(0) = 0 \in w_0$ and $\beta(\varepsilon^*) = \zeta \in w_{i^*}$, where $i^* = \mathcal{Y}(\zeta) < \delta$. Since r_δ is stronger than all r_j^-, r_j (for $j < \delta$) we also see that for each $\varepsilon \leq \varepsilon^*$,

$(\boxplus)_{14}^a$ $r_\delta \upharpoonright \beta(\varepsilon)$ is an upper bound on $h^{[\beta(\varepsilon)]} \circ f_\delta$,

$(\boxplus)_{14}^b$ $r_\delta \upharpoonright \beta(\varepsilon) \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\beta(\varepsilon)]}$ (by $(\boxplus)_5$), and

$(\boxplus)_{14}^c$ if $\beta(\varepsilon) \in w_i \cap \zeta$ for some $i < \delta$, then $\mathcal{Y}(\beta(\varepsilon)) \cdot \omega < \delta$ and $\delta \in \bigcap \{A_{j,\beta(\varepsilon)}^\xi : \xi, j < \delta\}$, and hence (by $(\boxplus)_8$) the condition $r_\delta \upharpoonright \beta(\varepsilon)$ forces (in $\mathbb{P}_{\beta(\varepsilon)}$) that

- $\delta \in \Delta_{j < \lambda} \Delta_{\xi < \lambda} A_j^\xi(\beta(\varepsilon)) \cap \mathcal{S}[\mathcal{Y}(\beta(\varepsilon)) \cdot \omega]$,
- $\underline{h}^{(\beta(\varepsilon))} \circ f_\delta^{(\beta(\varepsilon))}$ is an increasing sequence of conditions in $\mathbb{Q}_{\beta(\varepsilon)}$, and
- $\underline{h}^{(\beta(\varepsilon))}(f_\delta^{(\beta(\varepsilon))}(j)) = h(f_\delta(\mathcal{Y}(\beta(\varepsilon)) + j))(\beta(\varepsilon)) = r_{\mathcal{Y}(\beta(\varepsilon)) + j}^-(\beta(\varepsilon)) = r_j^\ominus(\beta(\varepsilon))$ for all successor $j < \delta$.

By induction on $\varepsilon \leq \varepsilon^*$ we are going to show that

$(\boxplus)_{15}^a$ $(h^{[\beta(\varepsilon)]} \circ f_\delta)(R^{\text{pr}} \upharpoonright \beta(\varepsilon))(r_\delta \upharpoonright \beta(\varepsilon))$, and

$(\boxplus)_{15}^b$ $\beta(\varepsilon) \leq \zeta_\delta^*$ and $q_\delta^{[\beta(\varepsilon)]} = q_\delta^* \upharpoonright \beta(\varepsilon) \leq r_\delta \upharpoonright \beta(\varepsilon)$.

To start, we note that $\langle r_j^\ominus(0), r_j^\oplus(0), \Delta_{\xi < \lambda} A_j^\xi(0) : j < \delta \rangle$ is a legal partial play of $\mathcal{D}_0^S(r(0), N, \underline{h}^{(0)}, \mathbb{Q}_0, R_0^{\text{pr}}, \bar{f}^{(0)}, \bar{q}^{(0)})$ in which Generic uses her win-

⁽¹²⁾ $h^{[\zeta]} \circ f_\delta$ is an increasing sequence of conditions by $(\boxtimes)_9$.

ning strategy \mathbf{st}_0 (and all names involved are actually objects from \mathbf{V}). Also $\delta \in \Delta_{j < \lambda} \Delta_{\xi < \lambda} A_j^\xi(0)$, $r_j^\ominus(0) = r_j^-(0)$ for all $j \in \delta \cap \mathcal{S}$ (by $(\boxplus)_8$) and $r_j^-(0) = h^{(0)}(f_\delta(j))$ for all successor $j < \delta$. Therefore, as the play on the coordinate $\varepsilon = 0$ is won by Generic, $r_\delta(0) = r_\delta^*(0) = r_\delta^\oplus(0) \geq q_\delta^{(0)}$ and $(h^{(0)} \circ f_\delta) R_0^{\text{pr}} r_\delta(0)$. Also, by $(\boxplus)_{14}^a$, $r_\delta \upharpoonright \beta(1)$ is an upper bound on $h^{[\beta(1)]} \circ f_\delta$ and hence $(h^{[\beta(1)]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \beta(1)) (r_\delta \upharpoonright \beta(1))$. And since $r_\delta \upharpoonright \beta(1) \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\beta(1)]}$ (see $(\boxplus)_{14}^b$) we may conclude that $\zeta_\delta^* \geq \beta(1)$ and therefore $q_\delta^{[\beta(1)]} = q_\delta^* \upharpoonright \beta(1)$ (as $\beta(1)$ is not a limit of members of w_δ). Now by induction on $\xi \leq \beta(1)$ we argue that $q_\delta^* \upharpoonright \xi \leq r_\delta \upharpoonright \xi$: it follows from $(\boxtimes)_{19}$ and what has been said above that $q_\delta^*(0) = q_\delta^{(0)} \leq r_\delta(0)$. Suppose $0 < \xi < \beta(1)$, $\xi \in N$, and assume that we have shown $q_\delta^* \upharpoonright \xi \leq r_\delta \upharpoonright \xi$. Then, by $(\boxplus)_{13}$, $r_\delta \upharpoonright \xi \Vdash_{\mathbb{P}_\xi}$ “ $q_\delta^*(\xi) \leq r(\xi) \leq r_\delta(\xi)$ ”. This concludes the arguments for $(\boxplus)_{15}$ when $\varepsilon = 1$.

Suppose we have justified $(\boxplus)_{15}$ for $\varepsilon_0 < \varepsilon^*$ and $\mathcal{Y}(\beta(\varepsilon_0)) < \delta$. By $(\boxplus)_7$, the condition $r_\delta \upharpoonright \beta(\varepsilon_0)$ forces that

$$\langle r_i^\ominus(\beta(\varepsilon_0)), r_i^\oplus(\beta(\varepsilon_0)), \Delta_{\xi < \lambda} A_i^\xi(\beta(\varepsilon_0)) : \mathcal{Y}(\beta(\varepsilon_0)) + i < \delta \rangle$$

is a legal partial play of the game

$$\partial_{\mathcal{Y}(\beta(\varepsilon_0))}^{\mathcal{S}}(r(\beta(\varepsilon_0)), N[G_{\mathbb{P}_{\beta(\varepsilon_0)}}], h^{(\beta(\varepsilon_0))}, \mathbb{Q}_{\beta(\varepsilon_0)}, R_{\beta(\varepsilon_0)}^{\text{pr}}, \bar{f}^{(\beta(\varepsilon_0))}, \bar{q}^{(\beta(\varepsilon_0))})$$

in which Generic uses the winning strategy $\mathbf{st}_{\beta(\varepsilon_0)}$. Therefore, using $(\boxplus)_{14}^c$ and $(\boxplus)_5 + (\boxplus)_8$, we find that $r_\delta \upharpoonright \beta(\varepsilon_0)$ forces (in $\mathbb{P}_{\beta(\varepsilon_0)}$) that

$$r_\delta(\beta(\varepsilon_0)) = r_\delta^\oplus(\beta(\varepsilon_0)) \geq q_\delta^{(\beta(\varepsilon_0))} \quad \text{and} \quad (h^{(\beta(\varepsilon_0))} \circ f_\delta^{(\beta(\varepsilon_0))}) R_{\beta(\varepsilon_0)}^{\text{pr}} r_\delta(\beta(\varepsilon_0)).$$

Since $h^{(\beta(\varepsilon_0))} \circ f_\delta^{(\beta(\varepsilon_0))}(j) = h(f_\delta(\mathcal{Y}(\beta(\varepsilon_0)) + j)(\beta(\varepsilon_0)))$ and $r_\delta \upharpoonright \beta(\varepsilon_0 + 1)$ is an upper bound on $h^{[\beta(\varepsilon_0 + 1)]} \circ f_\delta$ (by $(\boxplus)_{14}^a$), we may use our inductive hypothesis of $(\boxplus)_{15}^a$ for ε_0 to conclude that $(h^{[\beta(\varepsilon_0 + 1)]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \beta(\varepsilon_0 + 1)) (r_\delta \upharpoonright \beta(\varepsilon_0 + 1))$ (i.e., $(\boxplus)_{15}^a$ for $\varepsilon = \varepsilon_0 + 1$). By $(\boxplus)_{14}^b$ and the inductive hypothesis of $(\boxplus)_{15}^b$ for ε_0 we see now that $\zeta_\delta^* \geq \beta(\varepsilon_0 + 1)$ and also, as $\beta(\varepsilon_0 + 1)$ is not the limit of elements of w_δ , $q_\delta^{[\beta(\varepsilon_0 + 1)]} = q_\delta^* \upharpoonright \beta(\varepsilon_0 + 1)$. As before we may use $(\boxplus)_{13} + (\boxtimes)_{19}$ to show inductively that for all $\xi \leq \beta(\varepsilon_0 + 1)$ we have $q_\delta^* \upharpoonright \xi \leq r_\delta \upharpoonright \xi$, getting $(\boxplus)_{15}^b$ for $\varepsilon = \varepsilon_0 + 1$.

Suppose that we have shown $(\boxplus)_{15}$ for $\varepsilon_0 < \varepsilon^*$ and $\mathcal{Y}(\beta(\varepsilon_0)) = \delta$ (so $\beta(\varepsilon_0)$ is the limit of points from $\bigcup_{i < \delta} w_i$). The assumption of $(\boxplus)_{15}^a$ for ε_0 implies immediately that $(h^{[\beta(\varepsilon_0 + 1)]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \beta(\varepsilon_0 + 1)) (r_\delta \upharpoonright \beta(\varepsilon_0 + 1))$ as there are no new “active” coordinates here (see $(\boxtimes)_4^b$ and $(\boxplus)_{14}^a$). Just as before we argue that $(\boxplus)_{15}^b$ holds for $\varepsilon = \varepsilon_0 + 1$ as well.

Now suppose that $\varepsilon \leq \varepsilon^*$ is a limit ordinal and that we have shown $(\boxplus)_{15}$ for all $\varepsilon' < \varepsilon$. Since $\beta(\varepsilon) = \sup\{\beta(\varepsilon') : \varepsilon' < \varepsilon\}$, the assumption of $(\boxplus)_{15}^a$ for $\varepsilon' < \varepsilon$ implies that $(h^{[\beta(\varepsilon)]} \circ f_\delta) (R^{\text{pr}} \upharpoonright \beta(\varepsilon)) (r_\delta \upharpoonright \beta(\varepsilon))$. The assumption of $(\boxplus)_{15}^b$

for $\varepsilon' < \varepsilon$ implies that $\zeta_\delta^* \geq \beta(\varepsilon)$ and $q_\delta^* \upharpoonright \beta(\varepsilon) \leq r_\delta \upharpoonright \beta(\varepsilon)$. Hence, remembering $(\boxplus)_{14}^b$, we also conclude that $q_\delta^* \upharpoonright \beta(\varepsilon) \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha^{[\beta(\varepsilon)]}$ and therefore $q_\delta^{[\beta(\varepsilon)]} = q_\delta^* \upharpoonright \beta(\varepsilon)$. Consequently, the proof of $(\boxplus)_{15}$ is complete.

Finally, $(\boxplus)_{15}$ for ε^* implies that Generic has won the play under consideration, finishing the proof of Subclaim 4.1.5.2. ■

This completes the proof of Main Claim 4.1.5. ■

To deduce the theorem we will argue that the strategy **st** for Generic in $\mathcal{D}_1^S(r, N, h, \mathbb{P}_{\zeta^*}, R^{\text{pr}}, \bar{f}, \bar{q}^{[\zeta^*]})$ described in the proof of Claim 4.1.5 for $\zeta = \zeta^*$ is also a winning strategy for Generic in $\mathcal{D}_1^S(r, N, h, \mathbb{P}_{\zeta^*}, R^{\text{pr}}, \bar{f}, \bar{q})$. Note that if $\zeta = \zeta^*$ and in the proof of Subclaim 4.1.5.2 we look at $\varepsilon = \varepsilon^*$ (i.e., $\beta(\varepsilon) = \zeta^*$), then we deduce that for relevant δ ,

- (α) r_δ is an upper bound on $h \circ f_\delta$ (by $(\boxplus)_{14}^a$),
- (β) $r_\delta \in \bigcap_{\alpha < \delta} \mathcal{I}_\alpha$ (by $(\boxplus)_{14}^b$), and
- (γ) $(h \circ f_\delta) R^{\text{pr}} r_\delta$ (by $(\boxplus)_{15}^a$).

Therefore, q_δ must have the same properties (α)–(γ), and by $(\boxtimes)_{13}^b$ we get $\zeta_\delta^* = \zeta^*$ and $q_\delta^* = q_\delta$. Now $(\boxplus)_{15}^b$ implies $q_\delta^{[\zeta^*]} = q_\delta$ and we see that Generic wins indeed.

Thus the condition r is generic for \bar{q} over $N, h, \mathbb{P}_{\zeta^*}, R^{\text{pr}}, \bar{f}$ and this completes the proof of Theorem 4.1. ■

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