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Zero-one laws for graphs with edge probabilities decaying with distance. Part II

by

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Abstract. Let G_n be the random graph on $[n] = \{1, \ldots, n\}$ with the probability of $\{i, j\}$ being an edge decaying as a power of the distance, specifically the probability being $p_{|i-j|} = 1/|i-j|^\alpha$, where the constant $\alpha \in (0,1)$ is irrational. We analyze this theory using an appropriate weight function on a pair (A, B) of graphs and using an equivalence relation on $B \setminus A$. We then investigate the model theory of this theory, including a "finite compactness". Lastly, as a consequence, we prove that the zero-one law (for first order logic) holds.

Introduction. This continues [Sh 467] which is Part I and will be denoted here by [I], background and a description of the results are given in [I,§0]; as this is the second part, our sections are named §4–§6 and not §1–§3.

Recall that we fix an irrational $\alpha \in (0,1)_{\mathbb{R}}$ and the random graph $\mathscr{M}_n = \mathscr{M}_n^0$ is drawn as follows:

(a) its set of elements is $[n] = \{1, \ldots, n\}$,
(b) for $i < j$ in $[n]$ the probability of $\{i, j\}$ being an edge is $p_{|i-j|}$, where p_l is $1/l^\alpha$ if $l > 1$ and $1/2^\alpha$ if $l = 1$, or just ([1]) $p_l = 1/l^\alpha$ for $l > 1$,
(c) the drawings for the edges are independent,
(d) $\mathscr{K}_n$ is the set of possible values of $\mathscr{M}_n$, $\mathscr{K}$ is the class of graphs.

Our main interest is to prove the 0-1 laws (for first order logic) for this 0-1 context, but also to analyze the limit theory.

We can now explain our intentions.

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([1]) Originally we assume the former but actually also the latter works.

Zero step: We define relations $<^*_x$ on the class of graphs with no apparent relation to the probability side.

First step: We can prove that these $<^*_x$ have the formal properties of $<_x$, like $<^*_i$ is a partial order etc.; this is done in §4, e.g., in 4.17.

Remember from [I, §1] that $A <_a B \Leftrightarrow$ for random enough $\mathscr{M}_n$ and $f : A \hookrightarrow \mathscr{M}_n$, the maximal number of pairwise disjoint $g \supseteq f$ satisfying $g : B \to \mathscr{M}_n$ is $< n^\varepsilon$ (for every fixed ε).

Second step: We shall start dealing with the two versions of $<_a$: the $<_a$ from [I, §1] and $<^*_a$ defined in 4.11(5) below. We intend to prove:

$(*)$ $A <^*_a B \;\Rightarrow\; A <_a B$.

For this it suffices to show that for every $f : A \hookrightarrow \mathscr{M}_n$ and positive real ε, the expected value of the following is $\leq 1/n^\varepsilon$: the number of extensions $g : B \to \mathscr{M}_n$ of f satisfying "the sets $\mathrm{Rang}(g{\restriction}(B \setminus A))$, $f(A)$ are at distance $\geq n^\varepsilon$". Then the expected value of the number of k-tuples of such (pairwise) disjoint g is $\leq 1/n^{k\varepsilon}$. So if $k\varepsilon > |A|$, the expected value of the number of functions f with k pairwise disjoint such extensions g is $< 1/n^{k\varepsilon - |A|}$. Hence for random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow \mathscr{M}_n$ there are no such k-tuples of pairwise disjoint g's. This will help to prove that $<^*_i = <_i$. We do this and more probability arguments in §5. But the full proofs are delayed to [Sh:E48].

Third step: We deduce from §5 that $<^*_x = <_x$ for all relevant x and prove that the context is weakly nice. We then work somewhat more to prove the existential part of nice (the simple goodness (see [I, Definition 2.12(1)]) of appropriate candidate). That is we first prove "weak niceness" by proving that $A <^*_i B$ implies (A, B) satisfies the demand for $<_i$ of [I, §1], and in a strong way the parallel thing for $\leq_s$. Those involve probability estimation, i.e., quoting §5. But we need more: sufficient conditions for appropriate tuples to be simply good, and this is the first part of §6.

Fourth step: This is the universal part from niceness. This does not involve any probability, just weight computations (and previous stages), in other words, purely model-theoretic investigation of the "limit" theory. By the "universal part of nice" we mean (A) of [I, 2.13(1)] which includes:

if $\bar a \in {}^k(\mathscr{M}_n), b \in \mathscr{M}_n$ then there are $m_1 < m$, $B \subseteq \mathrm{cl}^{k,m_1}(\bar a)$ such that $\bar a \subseteq B$ and

$$\mathrm{cl}^k(B) \underset{B}{\overset{\mathscr{M}_n}{\bigcup\!\!\!\!\!\!\;\cdot\;}} (\mathrm{cl}^k(\bar a b, \mathscr{M}_n) \setminus \mathrm{cl}^k(B, \mathscr{M})) \cup B.$$

This is done in the last part of §6.

Because of the request of the referee and editors to shorten the paper, the computational part (in §5) in full was moved to [Sh:E48], and the generalization to the case we have a successor function (which was §7) was moved to [Sh:E49].

4. Applications. We intend to apply the general theorems ([I, Lemmas 2.17, 2.19]) to our problem. That is, we try to answer: does the main context $\mathscr{M}_n^0$ with $p_i = 1/i^\alpha$ for $i > 1$ satisfy the 0-1 law? So here our irrational number $\alpha \in (0,1)_{\mathbb{R}}$ is fixed. We work in Main Context (see 4.1 below, the other one, $\mathscr{M}_n^1$, would work out as well, see §7).

4.1. *Context.* A particular case of [I, 1.1]: $p_i = 1/i^\alpha$ for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_{\mathbb{R}}$ is a fix irrational) and the nth random structure is $\mathscr{M}_n = \mathscr{M}_n^0 = ([n], R)$ (i.e. only the graph with the probability of $\{i,j\}$ being $p_{|i-j|}$).

4.2. Fact. (1) *For any graph* A,

$$1 = \lim_n \operatorname{Prob}(A \text{ is embeddable into } \mathscr{M}_n).$$

(2) *Moreover* (2), *for every* $\varepsilon > 0$,

$$1 = \lim_n \operatorname{Prob}(A \text{ has} \geq n^{1-\varepsilon} \text{ disjoint copies in } \mathscr{M}_n).$$

This is easy, still, before proving it, note that since by our definition of closure $A \subseteq \mathrm{cl}^{m,k}(\emptyset, \mathscr{M}_n)$ implies that A has $< n^\varepsilon$ embeddings into $\mathscr{M}_n$, we get:

4.3. Conclusion. $\langle \mathrm{cl}^{m,k}_{\mathscr{M}_n}(\emptyset) : n < \omega \rangle$ *satisfies the* 0-1 *law* (*being a sequence of empty models*).

Hence (see [I, Def. 1.4, Conclusion 2.19])

4.4. Conclusion. $\mathscr{K}_\infty = \mathscr{K}$ *and for our main theorem it suffices to prove simple almost niceness of* $\mathfrak{K}$ (see [I, Def. 2.13]).

(Now 4.3 explicates one part of what in fact we always meant by "random enough" in previous discussions.)

Proof of 4.2. Let the nodes of A be $\{a_0, \ldots, a_{k-1}\}$. Let the event $\mathscr{E}_r^n$ be

$$a_l \mapsto 2rk + 2l \text{ is an embedding of } A \text{ into } \mathscr{M}_n.$$

The point is that for various values of r these tries are going to speak on pairwise disjoint sets of nodes, so we get independent events.

4.5. Subfact. $\operatorname{Prob}(\mathscr{E}_r^n) = q > 0$ (*i.e.* > 0 *but it does* not *depend on* n, r).

(Note: this is not true in a close context where the probability of $\{i,j\}$ being an edge when $i \neq j$ is $1/n^\alpha + 1/2^{|i-j|}$, as in that case the probability

(2) Actually also "$\geq cn$" works for $c \in \mathbb{R}^{>0}$ depending on A only.

depends on n. But still, we can have $\geq q > 0$ which suffices.) Here

$$q = \prod_{l<m<k,\ \{l,m\}\text{ edge}} \frac{1}{(2(m-l))^\alpha} \cdot \prod_{l<m<k,\ \{l,m\}\text{ not an edge}} \left(1 - \frac{1}{(2(m-l))^\alpha}\right).$$

(What we need is that all the relevant edges have probability > 0, < 1. Note: if we have retained $p = 1/i^\alpha$ this is false for the pairs $(i, i+1)$, so we have changed p_1. Anyway, in our case we multiplied by 2 to avoid this (in the definition of the event).) For the second case (the probability of edge being $1/n^\alpha + 1/2^{i-j}$),

$$q \geq \prod_{l<m<k,\ \{l,m\}\text{ edge}} \frac{1}{2^{|m-l|}} \cdot \prod_{l<m<k,\ \{l,m\}\text{ not an edge}} \left(1 - \frac{1}{(3/2)^{|m-l|}}\right).$$

So $\mathrm{Prob}(\mathscr{E}^n_r)$ has a positive lower bound which does not depend on r.

Also the events $\mathscr{E}^n_0, \ldots, \mathscr{E}^n_{[n/2k]-1}$ are independent. So the probability that they all fail is

$$\prod_{i<\lfloor n/2k\rfloor} (1 - \mathrm{Prob}(\mathscr{E}^n_i)) \leq \prod_i (1-q) \leq (1-q)^{n/2k},$$

which goes to 0 quite fast. The "moreover" part is left to the reader. $\blacksquare_{4.4}$

4.6. Definition. (1) Let

$$\mathscr{T} = \{(A, B, \lambda) : A \subseteq B \text{ graphs (generally: models from } \mathscr{K}) \text{ and } \lambda \text{ an equivalence relation on } B \setminus A\}.$$

We may write (A, B, λ) instead of $(A, B, \lambda\restriction(B \setminus A))$.

(2) We say that $X \subseteq B$ is λ-*closed* if

$$x \in X \text{ and } x \in B \cap \mathrm{Dom}(\lambda) \text{ implies } x/\lambda \subseteq X.$$

(3) $A \leq^* B$ if ([3]) $A \leq B \in \mathscr{K}_\infty$ (clearly $\leq^*$ is a partial order).

Story: We would like to ask, for any given copy of A in $\mathscr{M}_n$, if there is a copy of B above it, and how many; we hope for a dichotomy: usually none, always few *or* always many. The point of λ is to take distance into account, because for our present distribution being near is important; $b_1 \lambda b_2$ will indicate that b_1 and b_2 are near. Note that being near is not transitive, but "luck" helps us, we will succeed by "pretending" it is. We will look at many candidates for a copy of $B \setminus A$ and compute the expected value. We would like to show that saying "variance small" says that the true value is near the expected value.

([3]) Note: this is in our present specific context, so this definition does not apply to §1, §2, §3, §7; in fact, in §7 we give a different definition for a different context.

4.7. Definition. (1) For $(A,B,\lambda)\in\mathscr{T}$ let

$$\mathbf{v}(A,B,\lambda)=\mathbf{v}_\lambda(A,B)=|(B\setminus A)/\lambda|$$

be the number of λ-equivalence classes in $B\setminus A$ ($\mathbf{v}$ stands for vertices). (This measures degrees of freedom in choosing candidates for B over a given copy of A.)

(2) Let

$$\mathbf{e}(A,B,\lambda)=\mathbf{e}_\lambda(A,B)=|e_\lambda(A,B)|,$$

where

$$e_\lambda(A,B)=\{e: e \text{ an edge of } B,\ e\nsubseteq A, \text{ and } e\nsubseteq x/\lambda \text{ for } x\in B\setminus A\}.$$

[This measures the number of "expensive", "long" edges ($\mathbf{e}$ stands for edges).]

Story: $\mathbf{v}$ larger means that there are more candidates for B; $\mathbf{e}$ larger means that the probability per candidate is smaller.

4.8. Definition. (1) For $(A,B,\lambda)\in\mathscr{T}$ and our given irrational $\alpha\in(0,1)_{\mathbb{R}}$ we define ($\mathbf{w}$ stands for weight)

$$\mathbf{w}(A,B,\lambda)=\mathbf{w}_\lambda(A,B)=\mathbf{v}_\lambda(A,B)-\alpha\mathbf{e}_\lambda(A,B).$$

(2) Let

$$\Xi(A,B)=\{\lambda:(A,B,\lambda)\in\mathscr{T}, \text{ and if } C\subseteq B\setminus A \text{ is a nonempty } \lambda\text{-closed set then } \mathbf{w}_\lambda(A,C\cup A)>0\}.$$

(3) If $A\le^* B$ then we let $\xi(A,B)=\max\{\mathbf{w}_\lambda(A,B):\lambda\in\Xi(A,B)\}$.

4.9. Observation. (1) $(A,B,\lambda)\in\mathscr{T}\ \&\ A\ne B\Rightarrow \mathbf{w}_\lambda(A,B)\ne 0$.

(2) If $A\le^* B\le^* C$ (hence $A\le^* C$) and $(A,C,\lambda)\in\mathscr{T}$ and B is λ-closed then

(a) $(A,B,\lambda\restriction(B\setminus A))\in\mathscr{T}$,
(b) $(B,C,\lambda\restriction(C\setminus B))\in\mathscr{T}$,
(c) $\mathbf{w}_\lambda(A,C)=\mathbf{w}_{\lambda\restriction(B\setminus A)}(A,B)+\mathbf{w}_{\lambda\restriction(C\setminus B)}(B,C)$,
(d) similarly for $\mathbf{v}$ and $\mathbf{e}$.

(3) Note that 4.9(2) legitimizes our writing λ instead of $\lambda\restriction(C\setminus A)$ or $\lambda\restriction(B\setminus(C\cup A))$ when $(A,B,\lambda)\in\mathscr{T}$ and C is a λ-closed subset of B. Thus we may write, e.g., $\mathbf{w}_\lambda(A\cup C,B)$ for $\mathbf{w}(A\cup C,B,\lambda\restriction(B\setminus A\setminus C))$.

(4) If $(A,B,\lambda)\in\mathscr{T}$ and $D\subseteq B\setminus A$ and $D^+=\bigcup\{x/\lambda: x\in D\}$ then $\mathbf{w}_{\lambda\restriction D^+}(A,A\cup D^+)\le\mathbf{w}_{\lambda\restriction D}(A,A\cup D)$ and D^+ is λ-closed.

Proof. (1) As α is irrational and $\mathbf{v}_\lambda(A,B)$ is not zero.

(2) Clauses (a), (b) are immediate, and for a proof of (c), (d) see the proof of 4.16 below.

(3) Left to the reader.

(4) Clearly by the choice of D^+ we have $\mathbf{v}_{\lambda\restriction D^+}(A, A\cup D^+) = |D^+/\lambda| = |D/(\lambda\restriction D)| = \mathbf{v}_{\lambda\restriction D}(A, A\cup D)$ and $\mathbf{e}_{\lambda\restriction D^+}(A, A\cup D^+) \geq \mathbf{e}_{\lambda\restriction D}(A, A\cup D^+)$, hence $\mathbf{w}_{\lambda\restriction D^+}(A, A\cup D^+) \leq \mathbf{w}_{\lambda\restriction D}(A, A\cup D^+)$. $\blacksquare_{4.9}$

4.10. Discussion. Note that $\mathbf{w}_\lambda(A, B)$ measures in a sense the expected value of the number of copies of B over a given copy of A with λ saying when one node is "near to" another. Of course, when λ is the identity this degenerates to the definition in [ShSp 304].

We would like to characterize $\leq_i$ and $\leq_s$ (from [I, Definition 1.4(3), (4)] using $\mathbf{w}$ and to prove that they are O.K. (meaning that they form a nice context). Looking at the expected behaviour, we attempt to give an "effective" definition (depending on α only).

All of this, of course, just says what the intention of these relations and functions is (i.e. $<^*_i$, $<^*_s$, $<^*_{pr}$ and $\mathbf{v}$, $\mathbf{e}$, $\mathbf{w}$ below); we still will not prove anything on the connections to $\leq_i, \leq_s, \leq_{pr}$. We may view it differently: We are, for our fixed α, defining $\mathbf{w}_\lambda(A, B)$ and investigating the $\leq^*_i, \leq^*_s, \leq^*_{pr}$ defined below per se ignoring the probability side.

4.11. Definition. (1) $A \leq B$ means A is a submodel of B, and remember that by Definition 4.6(3), $A \leq^* B$ means [4] $A \leq B \in \mathscr{K}_\infty$.

(2) $A <^*_c B$ if $A <^* B$ and for every λ, we have

$$(A, B, \lambda) \in \mathscr{T} \;\Rightarrow\; \mathbf{w}_\lambda(A, B) < 0.$$

(3) $A \leq^*_i B$ if $A \leq^* B$ and for every A' we have

$$A \leq^* A' <^* B \;\Rightarrow\; A' <^*_c B.$$

Of course, $A <^*_i B$ means $A \leq^*_i B \,\&\, A \neq B$.

(4) $A \leq^*_s B$ if $A \leq^* B$ and for no A' do we have

$$A <^*_i A' \leq^* B.$$

Of course, $A <^*_s B$ means $A \leq^*_s B \,\&\, A \neq B$.

(5) $A <^*_a B$ if $A \leq^* B \,\&\, \neg(A <^*_s B)$ (i.e. $A \leq^* B$ and there is $A' \subseteq B \setminus A$ such that $A <^*_i A \cup A' \leq^* B$),

(6) $A <^*_{pr} B$ if $A \leq^* B$ and $A <^*_s B$ but for no C do we have $A <^*_s C <^*_s B$.

4.12. Remark. We *intend* to prove that usually $\leq^*_x = \leq_x$ but it will take time.

4.13. Lemma. *Suppose $A' <^* B$, $(A', B, \lambda) \in \mathscr{T}$ and $\mathbf{w}_\lambda(A', B) > 0$. Then there is A'' satisfying $A' \leq^* A'' <^* B$ such that A'' is λ-closed and*

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[4] Note: this is in our present specific context, so this definition does not apply to §1, §2, §3, §7; in fact in §7 we give a different definition for a different context.

$(*)_1[A'', B, \lambda]$ *we have* $\mathbf{w}_\lambda(A'', B) > 0$ *and if* $C \subseteq B \backslash A''$, $C \notin \{\emptyset, B \backslash A''\}$ *and* C *is* λ*-closed then* $\mathbf{w}_\lambda(A'', A'' \cup C) > 0$ *and* $\mathbf{w}_\lambda(A'' \cup C, B) < 0$.

Proof. Let C' be a maximal λ-closed subset of $B \setminus A'$ with the property that $\mathbf{w}_\lambda(A' \cup C', B) > 0$. Such a C' exists since $C' = \emptyset$ is as required and B is finite. Let $A'' = A' \cup C'$. Since C' is λ-closed, it follows that $B \setminus A''$ is λ-closed and $(A'', B, \lambda{\restriction}(B \setminus A'')) \in \mathscr{T}$ and clearly $\mathbf{w}_\lambda(A'', B) > 0$. Now suppose $D \subseteq B \setminus A''$ is λ-closed and $D \notin \{\emptyset, B \setminus A''\}$. By the maximality of C', $\mathbf{w}_\lambda(A'' \cup D, B) < 0$. Now (by 4.9(2)(c))

$$\mathbf{w}_\lambda(A'', B) = \mathbf{w}_\lambda(A'', A'' \cup D) + \mathbf{w}_\lambda(A'' \cup D, B),$$

and the left term is positive by the choice of C' and A'', but the right term is negative by the previous sentence, so we conclude $\mathbf{w}_\lambda(A'', A'' \cup D) > 0$, contradicting the maximality of C'. $\blacksquare_{4.13}$

4.14. Claim. *Assume $A <^* B$. The following statements are equivalent*:

(i) $A <^*_i B$,

(ii) *for no* A' *and* λ *do we have*:

$(*)_2 = (*)_2[A, A', B, \lambda]$ *we have* $A \leq^* A' <^* B$, $(A', B, \lambda) \in \mathscr{T}$ *and* $\mathbf{w}_\lambda(A', B) > 0$,

(iii) *for no* A', λ *do we have*:

$(*)_3 = (*)_3[A, A', B, \lambda]$ *we have* $A \leq^* A' <^* B$, $(A', B, \lambda) \in \mathscr{T}$, $\mathbf{w}_\lambda(A', B) > 0$ *and* $(*)_1[A', B, \lambda]$ *of* 4.13.

Proof. For the equivalence of the first and the second clauses read Definition 4.11(2),(3) (remembering 4.9(1)). Trivially $(*)_3 \Rightarrow (*)_2$ and hence the second clause implies the third one. Now we will see that (iii)$\Rightarrow$(ii). So suppose $\neg$(ii); let this be exemplified by A', λ, i.e. they satisfy $(*)_2$. Then by 4.13 there is A'' such that $A' \leq^* A'' <^* B$ and $(*)_1[A'', B, \lambda]$ of 4.13 holds. So A'', λ exemplifies that $\neg$(iii) holds. $\blacksquare_{4.14}$

4.15. Observation. (1) If $(*)_3[A, A', B, \lambda]$ from 4.14(iii) holds, then we have: if $C \subseteq B \backslash A'$ is λ-closed nonempty then $\mathbf{w}(A', A' \cup C, \lambda{\restriction}C) > 0$. [Why? If $C \neq B \setminus A'$ this is stated explicitly, otherwise this means $\mathbf{w}(A', B, \lambda) > 0$, which holds.]

(2) In $(*)_3$ of 4.14(iii), i.e., 4.13$(*)_1[A', B, \lambda]$, we can allow any λ-closed $C \subseteq B \setminus A'$ if we make the inequalities nonstrict. [Why? If $C = \emptyset$ then $\mathbf{w}_\lambda(A', A' \cup C) = \mathbf{w}_\lambda(A', A') = 0$ and $\mathbf{w}_\lambda(A' \cup C, B) = \mathbf{w}_\lambda(A', B) > 0$. If $C = B \setminus A'$ then $\mathbf{w}_\lambda(A', A' \cup C) = \mathbf{w}_\lambda(A', B) > 0$ and $\mathbf{w}_\lambda(A' \cup C, B) = \mathbf{w}_\lambda(B, B) = 0$. Lastly, if $C \notin \{\emptyset, B \setminus A'\}$ we use 4.13$(*)_1[A', B, \lambda]$ itself.]

(3) If $(A, B, \lambda) \in \mathscr{T}$, $A' \leq^* A$, $B' \leq^* B$, $A' \leq^* B'$ and $B \setminus A = B' \setminus A'$ then $(A', B', \lambda) \in \mathscr{T}$, $\mathbf{w}(A', B', \lambda) \geq \mathbf{w}(A, B, \lambda)$, $\mathbf{e}(A', B', \lambda) \leq \mathbf{e}(A, B, \lambda)$, and $\mathbf{v}(A', B', \lambda) = \mathbf{v}(A, B, \lambda)$.

(4) In (3) if in addition $A \bigcup_{A'}^{M} B'$, i.e., there is no edge $\{x, y\}$ with $x \in A \setminus A'$ and $y \in B' \setminus A'$ then the equalities hold.

4.16. Claim. *$A \leq^*_s B$ if and only if either $A = B$ or for some λ we have: $(A, B, \lambda) \in \mathscr{T}$ and $\mathbf{w}_\lambda(A, B) > 0$, moreover for every nonempty λ-closed $C \subseteq B \setminus A$, we have $\mathbf{w}(A, A \cup C, \lambda{\restriction}C) > 0$, that is, $\Xi(A, B) \neq \emptyset$.*

Proof. The "only if" direction. Suppose $A \leq^*_s B$. If $A = B$ we are done. So assume $A <^*_s B$. Let C be minimal such that $A \leq^* C \leq^* B$ and for some λ_0 the triple $(C, B, \lambda_0) \in \mathscr{T}$ satisfies: for every nonempty λ_0-closed $C' \subseteq B \setminus C$ we have $\mathbf{w}(C, C \cup C', \lambda_0{\restriction}C') > 0$ (exists because $C = B$ is O.K. as there is no such C'). By 4.9(4), for every nonempty $C' \subseteq B \setminus C$ we have $\mathbf{w}(C, C \cup C', \lambda_0{\restriction}C') > 0$, hence $\neg(C <^*_i C \cup C')$ by (i)$\Leftrightarrow$(ii) of 4.14. If $C = A$ we have finished by the definition of $\leq^*_s$. Otherwise, the hypothesis $A \leq^*_s B$ implies that $\neg(A <^*_i C)$, hence 4.14(iii) fails, which means that (recalling 4.11(1)) for some C', λ_1 we have $A \leq^* C' <^* C$, $(C', C, \lambda_1) \in \mathscr{T}$, $\mathbf{w}_{\lambda_1}(C', C) > 0$ and for every λ_1-closed $D \subseteq C \setminus C'$ satisfying $D \notin \{\emptyset, C \setminus C'\}$ we have

$$\mathbf{w}(C', C' \cup D, \lambda_1{\restriction}D) > 0, \qquad \mathbf{w}(C' \cup D, C, \lambda_1{\restriction}(C \setminus C' \setminus D)) < 0.$$

Define an equivalence relation λ on $B \setminus C'$: an equivalence class of λ is an equivalence class of λ_0 or an equivalence class of λ_1.

We shall show that (C', B, λ) satisfies the requirement above on C, thus contradicting the minimality of C. Clearly $A \leq^* C' \leq^* B$. So let $D \subseteq B \setminus C'$ be λ-closed and define $D_0 = D \cap (B \setminus C)$ and $D_1 = D \cap (C \setminus C')$. Clearly D_0 is λ_0-closed so $\mathbf{w}(C, C \cup D_0, \lambda{\restriction}D_0) \geq 0$ (see 4.15(2)), and D_1 is λ_1-closed so $\mathbf{w}(C', C' \cup D_1, \lambda{\restriction}D_1) \geq 0$ (this follows from: for every λ_1-closed $D \subseteq C \setminus C'$ with $D \notin \{\emptyset, C \setminus C'\}$ we have $\mathbf{w}_\lambda(C', C' \cup D, \lambda_1{\restriction}D) > 0$, and by 4.15(2)). Now (in the last line we change C' to C twice), by 4.15(3) we will get

$$\begin{aligned}
\mathbf{v}(C', C' \cup D, \lambda) &= |D/\lambda| = |D_1/\lambda_1| + |D_0/\lambda_0| \\
&= \mathbf{v}(C', C' \cup D_1, \lambda{\restriction}D_1) + \mathbf{v}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda{\restriction}D_0) \\
&= \mathbf{v}(C', C' \cup D_1, \lambda{\restriction}D_1) + \mathbf{v}(C, C \cup D_0, \lambda{\restriction}D_0),
\end{aligned}$$

and (using 4.15(3))

$$\begin{aligned}
\mathbf{e}(C', C' \cup D, \lambda) = \mathbf{e}&(C', C' \cup D_1, \lambda{\restriction}D_1) \\
&+ \mathbf{e}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda{\restriction}D_0) \\
\leq \mathbf{e}&(C', C' \cup D_1, \lambda{\restriction}D_1) + \mathbf{e}(C, C \cup D_0, \lambda{\restriction}D_0),
\end{aligned}$$

and hence

$$\begin{aligned}
\mathbf{w}(C', C' \cup D, \lambda) &= \mathbf{v}(C', C' \cup D, \lambda) - \alpha \mathbf{e}(C', C' \cup D, \lambda) \\
&= \mathbf{v}(C', C' \cup D_1, \lambda{\restriction}D_1) + \mathbf{v}(C, C \cup D_0, \lambda{\restriction}D_0)
\end{aligned}$$

$$
\begin{aligned}
&\quad - \alpha\mathbf{e}(C', C' \cup D_1, \lambda\restriction D_1) \\
&\quad - \alpha\mathbf{e}(C' \cup D_1, C' \cup D_1 \cup D_0, \lambda\restriction D_0) \\
&\geq \mathbf{v}(C', C' \cup D_1, \lambda\restriction D_1) + \mathbf{v}(C, C \cup D_0, \lambda\restriction D_0) \\
&\quad - \alpha\mathbf{e}(C', C' \cup D_1, \lambda\restriction D_1) - \alpha\mathbf{e}(C, C \cup D_0, \lambda\restriction D_0) \\
&= \mathbf{w}(C', C' \cup D_1, \lambda\restriction D_1) + \mathbf{w}(C, C \cup D_0, \lambda\restriction D_0) \geq 0,
\end{aligned}
$$

and the (strict) inequality holds by the irrationality of α, i.e. by 4.9(1). So actually (C', B, λ) satisfies the requirements on C, λ_0, thus giving a contradiction to the minimality of C.

The "if" direction. As the case $A = B$ is obvious, we can assume that the second half of 4.16 holds. So let λ be as in the second half of 4.16.

Suppose $A <^* C \leq^* B$, and we shall prove that $\neg(A <^*_i C)$, thus finishing by Definition 4.11. We shall show that $(A', \lambda') = (A, \lambda\restriction(C \setminus A))$ satisfies $(*)_2[A, A, C, \lambda']$ from 4.14, thus 4.14(ii) fails, hence 4.14(i) fails, i.e., $\neg(A <^*_i C)$ as required. Let $D = \bigcup\{x/\lambda : x \in C \setminus A\}$, so D is a nonempty λ-closed subset of $B \setminus A$. Hence by the present assumption on A, B, λ we have $\mathbf{w}(A, A \cup D, \lambda\restriction D) > 0$. Now

$$\mathbf{v}(A, C, \lambda\restriction C) = |C/\lambda| = |D/\lambda| = \mathbf{v}(A, D, \lambda\restriction D)$$

and

$$\mathbf{e}(A, C, \lambda\restriction C) \leq \mathbf{e}(A, D, \lambda\restriction D),$$

so $\mathbf{w}(A, C, \lambda\restriction C) \geq \mathbf{w}(A, D, \lambda\restriction D) > 0$ as required. $\blacksquare_{4.16}$

4.17. Claim. (1) *$\leq^*_i$ is transitive.*

(2) *$\leq^*_s$ is transitive.*

(3) *For any $A \leq^* C$, for some B we have $A \leq^*_i B \leq^*_s C$.*

(4) *If $A <^* B$ and $\neg(A \leq^*_s B)$ then $A <^*_c B$ or there is C such that $A <^* C <^* B$ and $\neg(A <^*_s C)$.*

(5) *Smoothness holds (with $<^*_i$ instead of $<_i$, see* [I, 2.5(4)]*), that is,*

(a) *if $A \leq^* C \leq^* M \in \mathscr{K}, A \leq^* B \leq^* M$, $B \cap C = A$ then $A <^*_c B \Rightarrow C <^*_c B \cup C$ and $A \leq^*_i C \Rightarrow B \leq^*_i B \cup C$,*

(b) *if in addition $C \underset{A}{\overset{M}{\bigcup}} B$ then $A <^*_c B \Leftrightarrow C \leq^*_c B \cup C$ and $A \leq^*_i B \Leftrightarrow C \leq^*_i B \cup C$ and $A \leq^*_s B \Leftrightarrow C \leq^*_s B \cup C$.*

(6) *For $A <^* B$ we have $\neg(A \leq^*_s B) \Leftrightarrow (\exists C)(A <^*_c C \leq^* B)$.*

(7) *If $A \leq^* B \leq^* C$ and $A \leq^*_s C$ then $A \leq^*_s B$.*

(8) *If $A_l \leq^*_s B_l$ for $l = 1, 2$, $A_1 \leq^* A_2$, $B_1 \leq^* B_2$ and $B_2 \setminus A_2 = B_1 \setminus A_1$ then*

$$\xi(A_1, B_1) \geq \xi(A_2, B_2).$$

(9) *In* (8), *equality holds iff* A_2, B_1 *are freely amalgamated over* A_1 *inside* B_2.

(10) *If* $A <_s^* B_l$ *for* $l = 1, 2$ *and* $B_1 <_i^* B_2$ *and for some edge* x, y *of* B_2 *we have* $s \in A$, $y \in B_2 \setminus B_1$ *then* $\xi(A, B_1) > \xi(A, B_2)$.

(11) *If* $B_1 <^* B_2$ *and for no* $x \in B_1, y \in B_2 \setminus B_1$ *is* $\{x, y\}$ *an edge of* B_2 *then* $B_1 <_s^* B_2$.

(12) *If* $A \leq^* B \leq^* C$ *and* $A \leq_i^* C$ *then* $B \leq_i^* C$.

(13) *If* $A <_{pr}^* B$ *and* $a \in B \setminus A$ *then* $A \cup \{a\} \leq_i^* B$.

(14) *If* $A_1 <_{pr}^* B_1, A_1 \leq^* A_2 \leq^* B_2$ *and* $B_1 \leq^* B_2$ *and* $B_2 = A_2 \cup B_1$ *then* $A_2 \leq_s^* B_2$ *or* $A_2 <_{pr}^* B_2$.

Proof. (1) Assume $A \leq_i^* B \leq_i^* C$ we shall prove $A \leq_i^* C$. It suffices to prove that 4.14(ii) holds with A, C here standing for A, B there. So assume $A \leq^* A' <^* C$, $(A', C, \lambda) \in \mathscr{T}$ and we shall prove that $\mathbf{w}_\lambda(A', C) \leq 0$; this suffices. Let $A'_1 := A' \cap B$ and $A'_0 := B \cup A' \cup \bigcup\{x/\lambda : x \in B\}$. As $A \leq_i^* B$, by 4.14 + 4.15(2) we have $\mathbf{w}_\lambda(A'_1, B) \leq 0$, by 4.15(3) we have $\mathbf{w}_\lambda(A', B \cup A') \leq \mathbf{w}_\lambda(A'_1, B)$, and by 4.9(4) we have $\mathbf{w}_\lambda(A', A'_0) \leq \mathbf{w}_\lambda(A', A' \cup B)$. Those three inequalities together give $\mathbf{w}_\lambda(A', A'_0) \leq 0$, and as $B \leq_i^* C$, by 4.14 we have $\mathbf{w}_\lambda(A'_0, C) \leq 0$. By 4.9(2)(c) we have $\mathbf{w}_\lambda(A', C) = \mathbf{w}_\lambda(A', A'_0) + \mathbf{w}_\lambda(A'_0, C)$ and by the previous sentence the latter is $\leq 0 + 0 = 0$, so $\mathbf{w}_\lambda(A, A') \leq 0$ as required.

(2) We use the condition from 4.16. So assume $A_0 \leq_s^* A_1 \leq_s^* A_2$ and let λ_l witness $A_l \leq_s^* A_{l+1}$ (i.e. $(A_l, A_{l+1}, \lambda_l)$ is as in 4.16). Let λ be the equivalence relation on $A_2 \setminus A_0$ such that for $x \in A_{l+1} \setminus A_l$ we have $x/\lambda = x/\lambda_l$. It follows easily that $(A_0, A_2, \lambda) \in \mathscr{T}$. Now, by 4.9(2)(c), 4.15(3) and 4.16 the triple (A_0, A_2, λ) satisfies the second condition in 4.16 so $A_0 \leq_s^* A_2$.

(3) Let B be maximal such that $A \leq_i^* B \leq^* C$; such a B exists as C is finite ([5]) and for $B = A$ we get $A \leq_i^* B \leq^* C$. Now if $B \leq_s^* C$ we are done. Otherwise by the definition of $\leq_s^*$ in 4.11(4) there is B' such that $B <_i^* B' \leq^* C$; now by part (1) we have $A \leq_i^* B' \leq^* C$, contradicting the maximality of B, so really $B \leq_s^* C$ and we are done.

(4) We assume $A <^* B$. If $A <_c^* B$ we are done, hence we can assume $\neg(A <_c^* B)$. Clearly there is λ such that $(A, B, \lambda) \in \mathscr{T}$ and $\mathbf{w}_\lambda(A, B) \geq 0$. So by the irrationality of α the inequality is strict and by 4.13 there is C such that $A \leq^* C <^* B$, C is λ-closed, $\mathbf{w}_\lambda(C, B) > 0$ and if $C' \subseteq B \setminus C$ is nonempty λ-closed and $\neq B \setminus C$ then $\mathbf{w}_\lambda(C, C \cup C') > 0$ and $\mathbf{w}_\lambda(C \cup C', B) < 0$. So by 4.16 + inspection, $C <_s^* B$, and thus by 4.17(2), $A \leq_s^* C \Rightarrow A^* \leq_s^* B$; but we know that $\neg(A <_s^* B)$, hence by part (2), $\neg(A \leq_s^* C)$, so the second possibility in the conclusion holds.

([5]) Actually, the finiteness is not needed if for possibly infinite A, B we define $A \leq_i^* B$ iff for every finite $B' \leq^* B$ there is a finite B'' such that $B' \leq^* B'' \leq^* B$, and $B'' \cap A <_i^* B''$.

(5) *Clause* (a): $A \leq_c^* B \Rightarrow C <_c^* B \cup C$ and $A \leq_i^* C \Rightarrow B \leq_i^* B \cup C$. [Why? Note that by our assumption $C <^* B \cup C$ and $B <^* B \cup C$. The first desired conclusion is easier, so we prove the second; hence assume $A \leq_i^* C$. If $B \leq^* D <^* B \cup C$ and $(D, B \cup C, \lambda) \in \mathscr{T}$, then $A \leq^* D \cap C <^* C$, so as $A \leq_i^* C$, by the definition of $\leq_i$ we have $\mathbf{w}_\lambda(D \cap C, C) < 0$. Hence (noting $C \setminus D \cap C = B \cup C \setminus D$) by Observation 4.15(3) we have $\mathbf{w}_\lambda(D, B \cup C) \leq \mathbf{w}_\lambda(D \cap C, C) < 0$ and so (by the definition of $\leq_c^*$) $D \leq_c^* B \cup C$. As this holds for any such D by Definition 4.11(3) we have $B \leq_i^* B \cup C$ as required.]

Clause (b): If in addition $C \bigcup_{A}^{M} B$ then $A <_c^* C \Leftrightarrow B \leq_c^* B \cup C$ and $A \leq_i C \Leftrightarrow B \leq_i B \cup C$ and $A \leq_s^* B \Leftrightarrow B \leq_s^* B \cup C$. [Why? Immediate by 4.15(4), Definition 4.11 and part (a).]

(6) The "only if" direction can be proved by induction on $|B|$, using 4.17(4). For the "if" direction assume that for some C, $A <_c^* C \leq^* B$, and choose a minimal C like that. Now if $A \leq^* A^* <^* C$, and λ_1 is an equivalence relation on $C \setminus A^*$, then let λ_0 be an equivalence relation on $A^* \setminus A$ such that $\mathbf{w}_{\lambda_0}(A, A^*) \geq 0$ (exists by the minimality of C) and let $\lambda = \lambda_0 \cup \lambda_1$. Then $(A, C, \lambda) \in \mathscr{T}$ and by 4.9(2)(i) we have $\mathbf{w}_\lambda(A^*, C) = \mathbf{w}_\lambda(A, C) - \mathbf{w}_\lambda(A, A^*)$; but as $A <_c^* C$ we have $\mathbf{w}_\lambda(A, C) < 0$, and by the choice of λ_0 we have $\mathbf{w}_\lambda(A, A^*) \geq 0$, hence $\mathbf{w}_\lambda(A^*, C) < 0$ so that $\mathbf{w}_{\lambda_1}(A^*, C) = \mathbf{w}_\lambda(A^*, C) < 0$. As λ_1 was any equivalence relation on $C \setminus A^*$, by Definition 4.11(2) we have shown that $A^* <_c^* C$. By the definition of $\leq_i^*$ (4.11(3)), as A^* was arbitrary such that $A \leq^* A^* <^* C$, by Definition 4.11(3) we get $A <_i^* C$, hence by the definition of $\leq_s$ (4.11(4)) we deduce $\neg(A \leq_s^* B)$ as required.

(7) Immediate by Definition 4.11(4).

(8) It is enough to prove that

$\circledast$ if $\lambda \in \Xi(A_2, B_2)$ then $\lambda \in \Xi(A_1, B_1)$ and $\mathbf{w}_\lambda(A_1, B_1) \geq \mathbf{w}_\lambda(A_2, B_2)$.

So assume λ is an equivalence relation over $B_2 \setminus A_2$ which is equal to $B_1 \setminus A_1$. Now for every nonempty λ-closed $C \subseteq B_1 \setminus A_1$ we have

(i) $\mathbf{v}_\lambda(A_1, A_1 \cup C) = |C/\lambda| = \mathbf{v}_\lambda(A_2, A_2 \cup C)$,

(ii) $\mathbf{e}_\lambda(A_1, A_1 \cup C) \leq \mathbf{e}_\lambda(A_2, A_2 \cup C)$ [as any edge in $e_\lambda(A_1, A_1 \cup C)$ belongs to $e_\lambda(A_2, A_2 \cup C)$], hence

(iii) $\mathbf{w}_\lambda(A_1, A_1 \cup C) \geq \mathbf{w}_\lambda(A_2, A_2 \cup C)$.

So by the definition of $\Xi(A_1, B_1)$ we have $\lambda \in \Xi(A_2, B_2) \Rightarrow \lambda \in \Xi(A_1, B_1)$ and, moreover, the desired inequality in $\circledast$ holds.

(9) If $A_2 \bigcup_{A_1}^{B_2} B_1$ in the proof of (8) we get $e_\lambda(A_1, A_1 \cup C) = e_\lambda(A_2, A_2 \cup C)$, hence $\mathbf{w}_\lambda(A_1, A_1 \cup C) = \mathbf{w}_\lambda(A_2, A_2 \cup C)$, in particular $\mathbf{w}_\lambda(A_1, B_1) =$

$\mathbf{w}_\lambda(A_2, B_2)$. Also now the proof of (8) gives $\lambda \in \Xi(A_1, B_1) \Rightarrow \lambda \in \Xi(A_2, B_2)$, so trivially $\xi(A_1, B_1) = \xi(A_2, B_2)$.

If $\neg(A_2 \bigcup_{A_1}^{B_2} B_1)$ then for every equivalence relation λ on $B_1 \setminus A_1 = B_2 \setminus A_2$ we have

(ii)$^+$ $\mathbf{e}_\lambda(A_1, B_1) < \mathbf{e}_\lambda(A_2, B_2)$ [as $e_\lambda(A_1, B_1)$ is a proper subset of $e_\lambda(A_2, B_2)$ by our present assumption], hence

(iii)$^+$ $\mathbf{w}_\lambda(A_1, B_1) > \mathbf{w}_\lambda(A_2, B_2)$.

As the number of such λ is finite and as we have shown $\Xi(A_2, B_2) \subseteq \Xi(A_1, B_1)$ we get $\xi(A_1, B_1) > \xi(A_2, B_2)$.

(10) This follows from $\circledast_1 + \circledast_2$ below and the finiteness of $\Xi(A, B_2)$ upon recalling Definition 4.8(3).

$\circledast_1$ $\lambda \in \Xi(A, B_2) \Rightarrow \lambda \restriction (B_1 \setminus A) \in \Xi(A, B_1)$.

[Why? If λ is an equivalence relation on $B_2 \setminus A$ and $\lambda_1 := \lambda \restriction (B_1 \setminus A)$ then λ_1 is an equivalence relation on $B_1 \setminus A$ and for any nonempty λ_1-closed $C_1 \subseteq B_1 \setminus A$, letting $C_2 = \bigcup\{x/\lambda : x \in C_1\}$ we have $\mathbf{w}_\lambda(A, A \cup C_1) \geq \mathbf{w}_\lambda(A, A \cup C_2)$ by 4.9(4) and the latter is positive because $\lambda \in \Xi(A, B_2)$.]

$\circledast_2$ $\lambda \in \Xi(A, B_2) \Rightarrow \mathbf{w}_\lambda(A, B_2) < \mathbf{w}_\lambda(A, B_1)$.

[Why? Otherwise let λ be from $\Xi(A, B_2)$ and let $C_\lambda = \bigcup\{x/\lambda : x \in B_1 \setminus A\} \cup A$ so $B_1 \leq^* C_\lambda \leq^* B_2$.]

Case 1: $C_\lambda = B_2$. So $\mathbf{v}_\lambda(A, B_1) = \mathbf{v}(A, B_1, \lambda \restriction (B_1 \setminus A)) = |(B_1 \setminus A)/\lambda| = |(B_2 \setminus A)/\lambda| = \mathbf{v}_\lambda(A, B_2, \lambda)$. By an assumption of part (10), for some $x \in A$ and $y \in B_2 \setminus B_1$ the pair $\{x, y\}$ is an edge so $e(A, B_1, \lambda \restriction (B_1 \setminus A))$ is a proper subset of $e(A, B_2, \lambda)$. Hence

$$\mathbf{e}_\lambda(A, B_1) < \mathbf{e}_\lambda(A, B_2)$$

and so

$$\mathbf{w}_\lambda(A, B_1) > \mathbf{w}_\lambda(A, B_2)$$

is as required.

Case 2: $C_\lambda \neq B_2$. As in Case 1, $\mathbf{w}_\lambda(A, B_1) \geq \mathbf{w}_\lambda(A, C_\lambda)$. Now $B_1 \leq^* C_\lambda \leq^* B_2$ (using the case assumption) and $B_1 <^*_i B_2$ by assumption, so by part (12) below we have $C_\lambda \leq^*_i B_2$, hence $C_\lambda <^*_i B_2$, and by 4.14 this implies $\mathbf{w}_\lambda(C_\lambda, B_2) < 0$. So $\mathbf{w}_\lambda(A, B_2) = \mathbf{w}_\lambda(A, C_\lambda) + \mathbf{w}_\lambda(C_\lambda, B_2) \leq \mathbf{w}_\lambda(A, B_1) + \mathbf{w}_\lambda(C_\lambda, B_2) < \mathbf{w}_\lambda(A, B_1)$ as required.

(11) Define λ to be the equivalence relation with exactly one class on $B_2 \setminus B_1$, so $(B_1, B_2, \lambda) \in \mathscr{T}$, $\mathbf{v}_\lambda(B_1, B_2) = 1$, $\mathbf{e}_\lambda(B_1, B_2) = 0$ and thus $\mathbf{w}_\lambda(B_1, B_2) \geq 0$. Hence $\lambda \in \Xi(B_2, B_2)$ so that $B_1 <_s B_2$.

(12) By Definition 4.11(3).

(13) Clearly $A \cup \{a\} \leq^* B$, hence by part (3) for some C we have $A \cup \{a\} \leq^*_i C \leq^*_s B$. If $C = B$ we are done; otherwise $A <^*_s C$ by part (7) so we have $A <^*_s C <^*_s B$, contradiction.

(14) Easy by Definition 4.11(6). $\blacksquare_{4.17}$

5. The probabilistic inequalities. In this section we deal with probabilistic inequalities about the number of extensions for the context $\mathscr{M}^0_n$. Mostly the computations are delayed to [Sh:E48].

Note: the proof of almost simple niceness of $\mathfrak{K}$ is in the next section.

5.1. *Context.* As in §4, so $p_i = 1/i^\alpha$ for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0,1)_\mathbb{R}$ is irrational) and $\mathscr{M}_n = \mathscr{M}^0_n$ (i.e. only the graph).

5.2. Definition. Let $\varepsilon > 0$, $k \in \mathbb{N}$, $\mathscr{M}_n \in \mathscr{K}$ and $A <^* B$ be in $\mathscr{K}_\infty$. Assume $f : A \hookrightarrow \mathscr{M}_n$ is an embedding or just $f : A \hookrightarrow [n]$, which means it is one-to-one. Define

$$\begin{aligned}
&\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n)\\
&\quad := \{\bar g : (1)\ \bar g = \langle g_l : l < k\rangle,\\
&\qquad\qquad (2)\ f \subseteq g_l, g_l \text{ a one-to-one function from } B \text{ into } |\mathscr{M}_n|,\\
&\qquad\qquad (3)\ g_l : B \hookrightarrow_f \mathscr{M}_n \text{ for } l \le k \text{ or just } g_l : B \hookrightarrow_A \mathscr{M}_n,\\
&\qquad\qquad\quad \text{which means: } \{a,b\} \in \mathrm{Edge}(B) \setminus\ \mathrm{Edge}(A)\\
&\qquad\qquad\quad \Rightarrow \{g(b), g(b)\} \in \mathrm{Edge}(\mathscr{M}_n)\\
&\qquad\qquad\quad (\text{and } g \text{ is one-to-one extending } f),\\
&\qquad\qquad (4)\ l_1 \ne l_2 \Rightarrow \mathrm{Rang}(g_{l_1}) \cap \mathrm{Rang}(g_{l_2}) = \mathrm{Rang}(f),\\
&\qquad\qquad (5)\ [l < k\ \&\ x \in B \setminus A\ \&\ y \in A] \Rightarrow |g_l(x) - g_l(y)| \ge n^\varepsilon\}.
\end{aligned}$$

The size of this set has a natural connection with the number of pairwise disjoint extensions $g : B \hookrightarrow \mathscr{M}_n$ of f, hence with $A <_s B$; see 5.3 below.

5.3. Fact. *For every ε and k and $A \leq^* B$ we have:*

(∗) *for every n, k and $M \in \mathscr{K}_n$ and one-to-one $f : A \hookrightarrow_A M_n$ we have: if $\mathscr{G}^{\varepsilon,k}_{A,B}(f, M_n) = \emptyset$ then*

$$\max\{l : \text{there are } g_m : B \hookrightarrow_A M \text{ for } m < l \text{ such that } f \subseteq g_m \text{ and}$$
$$[m_1 < m_2 \Rightarrow \mathrm{Rang}(g_{m_1}) \cap \mathrm{Rang}(g_{m_2}) \subseteq \mathrm{Rang}(f)]\}$$
$$\le 2|A|n^\varepsilon + (k-1).$$

Proof. Assume that there are g_m as above for $m < l^*$, where $l^* > 2|A|n^\varepsilon + k - 1$. By renaming without loss of generality for some $l^{**} \le l^*$ the set $\mathrm{Rang}(g_m) \setminus \mathrm{Rang}(f)$ when $m < l^{**}$ is at distance $\ge n^\varepsilon$ from $\mathrm{Rang}(f)$ but if $l \in [l^{**}, l^*]$ then $\mathrm{Rang}(g_l) \setminus \mathrm{Rang}(f)$ has distance $< n^\varepsilon$ to $\mathrm{Rang}(f)$. Recall that by one of our assumptions $l^{**} \le k - 1$. Now for each $x \in \mathrm{Rang}(f)$, there are $\leq 2n^\varepsilon$ numbers $m \in [l^{**}, l^*)$ such that $\min\{|x - g_m(y)| : y \in B \setminus A\} \leq n^\varepsilon$. So by the demand on l^{**} we have $l^* - l^{**} \leq |A| \cdot (2n^\varepsilon) = 2|A|n^\varepsilon$ and as $l^{**} < k$ we are done. $\blacksquare_{5.3}$

The next theorem is central; it does not yet prove almost niceness but its parallels from [ShSp 304], [BlSh 528] were immediate, and here we see the main additional difficulties: we are looking for copies B over A but we have to take into account the distance, and the closeness of images of points in B under embeddings into $\mathscr{M}_n$. To prove 5.4 we will have to look for different types of g's which satisfy condition (5) from the definition of $\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n)$; restricting ourselves to one kind we will calculate the expected value of a "relevant part" of $\mathscr{G}^{\varepsilon,1}_{A,B}(f, \mathscr{M}_n)$ and we will show that it is small enough.

5.4. Theorem. *Assume $A <^* B$ (so both in $\mathscr{K}_\infty$). Then a sufficient condition for*

$\bigotimes_1$ *for every $\varepsilon > 0$, for some $k \in \mathbb{N}$, for every random enough $\mathscr{M}_n$ we have*:
(∗) *if $f : A \hookrightarrow [n]$ then $\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n) = \emptyset$*

is the following:

$\bigotimes_2$ $A <^*_a B$ (*which by Definition* 4.11(5) *means* $A <^* B \,\&\, \neg(A <_s B)$).

5.5. Remark. From $\bigotimes_1$ we can conclude: for every $\varepsilon \in \mathbb{R}^+$ we have: for every random enough $\mathscr{M}_n$, for every $f : A \hookrightarrow_A \mathscr{M}_n$, there cannot be $\geq n^\varepsilon$ extensions $g : B \hookrightarrow_A \mathscr{M}_n$ of f pairwise disjoint over f.

5.6. *Explanation.* For this, first choose $\varepsilon_1 < \varepsilon$. Note that for any k we have $\mathscr{G}^{\varepsilon,k}_{A,B}(f, \mathscr{M}_n) \subseteq \mathscr{G}^{\varepsilon_1,k}_{A,B}(f, \mathscr{M}_n)$. Choose k_1 for ε_1 by 5.3. Then the number of pairwise disjoint extensions $g : B \hookrightarrow_A \mathscr{M}_n$ of f is $\leq 2|A|n^{\varepsilon_1} + (k_1 - 1)$. For sufficiently large n this is $< n^\varepsilon$.

5.7. Remark. We think of $g : B \hookrightarrow \mathscr{M}_n$ extending f such that, for some constants c_1 and c_2 with $c_2 > 2c_1$,

$$x\lambda y \;\Rightarrow\; |g(x) - g(y)| < c_1$$

and

$$[\{x, y\} \subseteq B \,\&\, \{x, y\} \nsubseteq A \,\&\, \neg x\lambda y] \;\Rightarrow\; |g(x) - g(y)| \geq c_2.$$

5.8. *Explanation.* The number of such g is $\sim n^{|(B \setminus A)/\lambda|} = n^{\mathbf{v}(A,B,\lambda)}$; the probability of each being an embedding, assuming f is one, is $\sim n^{-\alpha \mathbf{e}(A,B,\lambda)}$, hence the expected value is $\sim n^{\mathbf{w}_\lambda(A,B)}$ ($\sim$ means "up to a constant"). So $A <^*_i B$ implies that usually there are few such copies of B over any copy of A, i.e. the expected value is < 1. In [ShSp 304], λ is equality, here things are more complicated.

By 5.4 we have sufficient conditions for: (given $A \leq^* B$) every $f : A \hookrightarrow \mathscr{M}_n$ has few pairwise disjoint extensions to $g : B \hookrightarrow \mathscr{M}_n$. Now we try to get a dual, a sufficient condition for: (given $A \leq^* B$) for every random enough $\mathscr{M}_n$, every $f : A \hookrightarrow \mathscr{M}_n$ has "many" pairwise disjoint extensions to $g : A \hookrightarrow \mathscr{M}_n$.

5.9. Lemma. *Assume*

(A) $(A, B, \lambda) \in \mathscr{T}$ *and* $m \in \mathbb{N}$,
(B) $(\forall B')[A <^* B' \leq^* B \;\&\; B'$ *is* λ*-closed* $\Rightarrow \mathbf{w}_\lambda(A, B') > 0]$ (*recall that "*B' *is* λ*-closed" means* $x\lambda y \;\&\; x \in B' \Rightarrow y \in B'$).

Then there is $\zeta \in \mathbb{R}^+$, *in fact we can let*

$$\zeta := \min\{\mathbf{w}_{\lambda\restriction B'}(A, B') : A \subseteq B' \subseteq B \text{ and } B' \text{ is } \lambda\text{-closed}\},$$

such that:

⊗ *for every small enough* $\varepsilon > 0$, *for every random enough* $\mathscr{M}_n$, *for every* $f : A \hookrightarrow \mathscr{M}_n$ *and* k *with* $0 < k < k + n^{1-\varepsilon} < n$, *there are* $\geq n^{(1-\varepsilon)\cdot\zeta}$ *pairwise disjoint extensions* g *of* f *satisfying*

(i) $g : B \hookrightarrow \mathscr{M}_n$,
(ii) $g(B \setminus A) \subseteq [k, k + n^{1-\varepsilon})$,
(iii) *if* $x\lambda y$ (*so* $x, y \in B \setminus A$) *then* $|x - y| \leq 2|B|$,
(iv) *if* $x \in B \setminus A, y \in B$ *and* $\neg(x\lambda y)$ *then* $|x - y| \geq n^\varepsilon$,
(v) *if* $B <_i B'$, $A <_s B'$ *and* $|B' \setminus B| \leq m$ *then there is no extension* $g' : B' \hookrightarrow \mathcal{M}_n$ *of* g *such that* $(\forall x \in B' \setminus B)(\forall y \in B)(|g'(x) - g'(y)| \geq mn^\varepsilon)$.

Now, 5.4 and 5.9 are enough for proving $<_i^* = <_i$, $<_s^* = <_s$, weakly nice and similar things. But we need more.

5.10. Lemma. *Assume*

(A) $(A, B, \lambda) \in \mathscr{T}$,
(B) $\xi = \xi(A, B) = \mathbf{w}_\lambda(A, B) > 0$ (*see Definition* 4.8(3)),
(C) *if* $A <^* C <^* B$ *and* C *is* λ*-closed then* $\mathbf{w}_\lambda(C, B) < 0$ (*hence necessarily* $\xi \in (0,1)_\mathbb{R}$ *and* $C \neq \emptyset \Rightarrow \mathbf{w}_\lambda(A, C) > 0$ *and even* $\mathbf{w}_\lambda(A, C) > \xi$).

Then for every $\varepsilon \in \mathbb{R}^+$, *every random enough* $\mathscr{M}_n$, *and every* $f : A \hookrightarrow \mathscr{M}_n$, *we have*

(a) *the number of* $g : B \hookrightarrow \mathscr{M}_n$ *extending* f *is at least* $n^{\xi-\varepsilon}$,
(b) *also the maximal number of pairwise disjoint extensions* $g : B \hookrightarrow \mathscr{M}_n$ *of* f *is at least this number.*

5.11. Remark. (1) We can get a reasonably much better bound (see [ShSp 304], [BlSh 528] and [Sh 550]) but this suffices.

(2) In the most interesting cases of 5.10 we have $A <_{pr}^* B$ but it applies to more cases.

5.12. Claim. *Assume*

(A) $(A, B, \lambda) \in \mathscr{T}$,

(B) *if* $C \subseteq B \setminus A$ *is nonempty and* λ*-closed then* $\mathbf{w}_\lambda(A, A \cup C) > 0$.

Then for some $\varepsilon_0 \in \mathbb{R}^+$, *for every* $\varepsilon \in (0, \varepsilon_0)_\mathbb{R}$, *every random enough* $\mathscr{M}_n$, *and every* $f : A \hookrightarrow \mathscr{M}_n$, *we have*

(a) *the number of* $g : B \hookrightarrow \mathscr{M}_n$ *extending* f *is at least* $n^{\mathbf{w}_\lambda(A,B)-\varepsilon}$,

(b) *for every* $X \subseteq [n]$ *with* $|X| \leq n^{\varepsilon_0-\varepsilon}$, *the number of* $g : B \hookrightarrow \mathscr{M}_n$ *extending* f *with* $\mathrm{Rang}(g) \cap X \subseteq \mathrm{Rang}(f)$ *is at least* $n^{\mathbf{w}_\lambda(A,B)-\varepsilon}$.

5.13. Remark. (1) By 4.16 the statement

"for some λ the hypothesis of 5.12 holds"

is equivalent to "$A \leq^*_s B$".

(2) The affinity of this claim to being nice (see [I, §2]) should be clear.

(3) If $|X| \geq n^{1-\varepsilon}$ we can demand $\mathrm{Rang}(g) \subseteq X$ but no need arises.

5.14. Claim. (1) *Assume* $A <^*_{pr} B$, *and let* $\xi = \xi(A, B)$ *that is*

$$\xi = \max\{\mathbf{w}_\lambda(A, B) : (A, B, \lambda) \in \mathscr{T} \text{ and for every } \lambda\text{-closed nonempty } C \subseteq B \setminus A \text{ we have } \mathbf{w}(A, A \cup C, \lambda{\restriction}C) > 0\}.$$

Then for every $\varepsilon \in \mathbb{R}^+$, *every random enough* $\mathscr{M}_n$, *and every* $f : A \hookrightarrow \mathscr{M}_n$, *we have*

$(*)$ *the number of* $g : B \hookrightarrow \mathscr{M}_n$ *extending* f *is at most* $n^{\xi+\varepsilon}$.

(2) *Assume that* $A <^*_s B$, $\lambda \in \Xi(A, B)$ *and* $\xi = \mathbf{w}_\lambda(A, B)$. *Then for any small enough reals* $\zeta, \varepsilon > 0$ *for every random enough* $\mathcal{M}_n$, *for every* $f : A \hookrightarrow \mathcal{M}_n$ *the set* $\mathcal{G}^{\varepsilon,\zeta}_{f,B,\lambda}(\mathcal{M}_n)$ *defined below has* $\leq n^{\xi+\varepsilon}$ *members, where*

$$\mathcal{G}^{\varepsilon,\zeta}_{f,B,\lambda}(\mathcal{M}_n) = \{g : g : B \hookrightarrow \mathcal{M}_n \text{ extends } f \text{ and } x \in B \setminus A \wedge y \in B \wedge \neg(x\lambda y) \Rightarrow n^\zeta \leq |g(x) - g(y)|\}.$$

6. The conclusion

Comment. In this section it is shown that $<^*_i$ and $<^*_s$ (introduced in §4) agree with the $<_i$ and $<_s$ of [I, §1] by using the probabilistic information from §5. Then it is proven that the main context $\mathscr{M}_n$ is simply nice (hence simply almost nice) and it satisfies the 0-1 law.

6.1. *Context.* As in §4 and §5, so $p_i = 1/i^\alpha$ for $i > 1$, $p_1 = p_2$ (where $\alpha \in (0, 1)_\mathbb{R}$ irrational) and $\mathscr{M}_n = \mathscr{M}^0_n$ (only the graph) and $\leq_i$, $\leq_s$, cl are as defined in §1. (So $\mathscr{K}_\infty = \mathscr{K}$ by 4.4.)

Note that actually the section has two parts of distinct flavours: in 6.2–6.5 we use the probabilistic information from §5 to show that the definitions of $<_x$ from [I, §1] and of $<^*_x$ from §4 give the same relation. But to actually prove almost niceness, we need more work on the relations $\le^*_x$ defined in §4; this is done in 6.8, 6.10, 6.11. Lastly, we put everything together.

The argument in 6.2–6.5 parallels that in [BlSh 528], which is more hidden in [ShSp 304]. The most delicate step is to establish clauses (A)(δ) and (ε) of [I, Definition 2.13(1)] (almost simply nice). For this, we consider $f : A \hookrightarrow \mathscr{M}_n$ and try to extend f to $g : B \hookrightarrow \mathscr{M}_n$, where $A \le_s B$, such that Rang(g) and $\mathrm{cl}^k(f(A), \mathscr{M}_n)$ are "freely amalgamated" over Rang(f). The key facts have been established in Section 5. If $\zeta = \mathbf{w}(A, B, \lambda)$ we have shown (Claim 5.12) that for every $\varepsilon > 0$ and every random enough $\mathscr{M}_n$, there are $\ge n^{\zeta-\varepsilon}$ embeddings of B into $\mathscr{M}_n$ extending f. But we also show (using 5.14) that for each obstruction to free amalgamation there is a $\zeta' < \zeta$ such that for every $\varepsilon_1 > 0$ the number of embeddings satisfying this obstruction is $< n^{\zeta'+\varepsilon_1}$, where $\zeta' = \mathbf{w}(A, B', \lambda)$ (for some B' exemplifying the obstruction) with $\zeta' + \alpha \le \zeta$. So if $\alpha > \varepsilon + \varepsilon_1$ we overcome the obstruction. The details of this computation for various kinds of obstructions are carried out in proving Claim 6.5.

6.2. Claim. *Assume $A <^* B$. Then the following are equivalent*:

(i) $A <^*_i B$ (*i.e. from Definition* 4.11(3)),

(ii) *it is not true that*: *for some ε, for every random enough $\mathscr{M}_n$ and every $f : A \hookrightarrow \mathscr{M}_n$, the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is $\ge n^\varepsilon$*,

(iii) *for every $\varepsilon \in \mathbb{R}^+$, every random enough $\mathscr{M}_n$, and every $f : A \hookrightarrow \mathscr{M}_n$, the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is $< n^\varepsilon$ (this is the definition of $A <_i B$ in* [I, §1]).

Proof. We shall use the well known finite Δ-system lemma: if $f_i : B \to [n]$ is one-to-one and $f_i{\restriction}A = f$ for $i < k$ then for some $w \subseteq \{0, \dots, k-1\}$ with $|w| \ge k^{1/2^{|B\setminus A|}}/|B \setminus A|^2$, and $A' \subseteq B$ and f^* we have: $\bigwedge_{i\in w} f_i{\restriction}A' = f^*$ and $\langle \mathrm{Rang}(f_i{\restriction}(B \setminus A') : i \in w\rangle$ are pairwise disjoint (so if the f_i's are pairwise distinct then $B \setminus A' \ne \emptyset$).

We use freely Fact 4.2. First, clearly (iii)⇒(ii). Second, if ¬(i), i.e., $\neg(A <^*_i B)$ then by 4.14 (equivalence of first and last possibilities + 4.13(1)) there are A', λ as there, that is, such that:

$A \le^* A' <^* B$ and $(A', B, \lambda) \in \mathscr{T}$ and if $C \subseteq B \setminus A'$ is nonempty λ-closed then $\mathbf{w}(A', A' \cup C, \lambda{\restriction}C) > 0$ (see 4.14).

So (A', B, λ) satisfies the assumptions of 5.9, which gives ¬(ii), i.e., we have proved (ii)⇒(i).

Lastly, to prove (i)⇒(iii) assume ¬(iii). So for some $\varepsilon \in \mathbb{R}^+$:

$(*)_1$ $0 < \limsup_{n\to\infty} \mathrm{Prob}($for some $f : A \hookrightarrow \mathscr{M}_n$, the number of $g : B \hookrightarrow \mathscr{M}_n$ extending f is $\ge n^\varepsilon)$.

By the first paragraph of this proof it follows that from $(*)_1$ we can deduce that for some $\zeta \in \mathbb{R}^+$,

$(*)_2$ $0 < \limsup_{n\to\infty}$ Prob(for some A' with $A \leq^* A' <^* B$ and $f' : A' \hookrightarrow \mathscr{M}_n$ there are $\geq n^\zeta$ functions $g : B \hookrightarrow \mathscr{M}_n$ which are pairwise disjoint extensions of f').

So for some A' with $A \leq^* A' <^* B$ we have

$(*)_3$ $0 < \limsup_{n\to\infty}$ Prob(for some $f' : A' \hookrightarrow \mathscr{M}_n$ there are $\geq n^\zeta$ functions $g : B \hookrightarrow \mathscr{M}_n$ which are pairwise disjoint extensions of f').

By 5.4 (and 5.3, 5.2) we have $\neg(A' <^*_a B)$, which by Definition 4.11(5) means that $A' <^*_s B$, which (by Definition 4.11(4)) implies $\neg(A <^*_i B)$, so $\neg$(i) holds as required. $\blacksquare_{6.2}$

6.3. Claim. *For $A <^* B \in \mathscr{K}_\infty$, the following conditions are equivalent*:

(i) $A <^*_s B$,

(ii) *it is not true that*: *for every $\varepsilon \in \mathbb{R}^+$, every random enough $\mathscr{M}_n$, and every $f : A \hookrightarrow \mathscr{M}_n$, there are no n^ε pairwise disjoint extensions $g : B \hookrightarrow \mathscr{M}_n$ of f*,

(iii) *for some $\varepsilon \in \mathbb{R}^+$, for every random enough $\mathscr{M}_n$ and every $f : A \hookrightarrow \mathscr{M}_n$, there are $\geq n^\varepsilon$ pairwise disjoint extensions $g : B \hookrightarrow \mathscr{M}_n$ of f.*

Proof. Reflection shows that (iii)⇒(ii).

If $\neg$(i), i.e., $\neg(A <^*_s B)$ then by Definition 4.11(4), $A <^*_i B' \leq^* B$ for some B', hence 6.2 easily yields $\neg$(ii), so (ii)⇒(i).

Lastly, it suffices to prove (i)⇒(iii). Now by (i) and 4.16 for some λ the assumptions of 5.9 hold, and hence its conclusion, which gives clause (iii). $\blacksquare_{6.3}$

6.4. Conclusion. (1) $<^*_s = <_s$ *and* $<^*_i = <_i$, *and* $\mathfrak{K}$ *is weakly nice, where* $<_s, <_i$ *are defined in* [I, 1.4(4),(5)]; *hence* $<^*_{pr} = \leq_{pr}$.

(2) $(\mathscr{K}, \mathrm{cl})$ *is as required in* [I, §2], *and the* $\leq_i$, $\leq_s$ *defined in* [I, §2] *are the same as those defined in* [I,§1] *for our context*, *of course when for* $A \leq B \in \mathscr{K}_\infty$ *we let* $\mathrm{cl}(A, B)$ *be minimal* A' *such that* $A \leq A' \leq_s B$.

(3) *Also* $\mathfrak{K}$ (*that is*, $(\mathscr{K}, \mathrm{cl})$) *is transitive local transparent and smooth* (see [I, 2.2(3), 2.3(2), 2.5(5),(4)]).

Proof. (1) $<^*_s = <_s$ and $<^*_i = <_i$ by 6.2, 6.3 and see definition in [I, §1]. Lastly, $\mathfrak{K}$ being weakly nice follows from 6.3 (see definition in [I, §1]).

(2) By [I, 2.6].

(3) By [I, 2.6] the transitive local and transparent follows (see clauses $(\delta), (\varepsilon), (\zeta)$ there). As for smoothness, use 4.17(5). $\blacksquare_{6.4}$

Note that we are in a "nice" case, in particular no successor function. Toward proving it we characterize "simply good".

6.5. Claim. *If $A \leq^*_s B$ and $k, t \in \mathbb{N}$ satisfies $k + |B| \leq t$, then for every random enough $\mathscr{M}_n$ and every $f : A \hookrightarrow \mathscr{M}_n$, we can find $g : B \hookrightarrow \mathscr{M}_n$ extending f such that*:

(i) $\mathrm{Rang}(g) \cap \mathrm{cl}^t(\mathrm{Rang}(f), \mathscr{M}_n) = \mathrm{Rang}(f)$,

(ii) $\mathrm{Rang}(g) \underset{\mathrm{Rang}(f)}{\overset{\mathscr{M}_n}{\bigcup}} \mathrm{cl}^t(\mathrm{Rang}(f), \mathscr{M}_n)$,

(iii) $\mathrm{cl}^k(\mathrm{Rang}(g), \mathscr{M}_n) \subseteq \mathrm{Rang}(g) \cup \mathrm{cl}^k(\mathrm{Rang}(f), \mathscr{M}_n)$.

6.6. Remark. Note that in clauses (i), (ii) of 6.5 we can replace t by k—this just demands less. We shall use this freely. Have we put t in the second appearance of k in clause (iii) of 6.5 the loss would not be great: just as in [I], we should systematically use [I, 2.12(2)] instead of [I, 2.12(1)].

Proof of Claim 6.5. We prove this by induction on $|B \setminus A|$, but by the character of the desired conclusion, if $A <^*_s B <^*_s C$, to prove it for the pair (A, C) it suffices to prove it for the pairs (A, B) and (B, C). Also, if $B = A$ the statement is trivial (because we can take $f = g$). So, without loss of generality, $A <^*_{pr} B$ (see Definition 4.11(6)).

Let λ be such that $(A, B, \lambda) \in \mathscr{T}$ and for every λ-closed $C \subseteq B \setminus A$ we have $\mathbf{w}_\lambda(A, A \cup C) > 0$ and

$$\begin{aligned} \xi := \mathbf{w}_\lambda(A, B) = \max\{\mathbf{w}_{\lambda_1}(A, B) : (A, B, \lambda_1) \in \mathscr{T} \text{ satisfies :} \\ \text{for every } \lambda_1\text{-closed nonempty } C \subseteq B \setminus A \\ \text{we have } \mathbf{w}_{\lambda_1}(A, A \cup C) > 0\}. \end{aligned}$$

Choose $\varepsilon \in \mathbb{R}^+$ small enough and $k(*)$ large enough. The requirements on ε, $k(*)$ will be clear by the end of the argument.

Let $\mathscr{M}_n$ be random enough, and $f : A \hookrightarrow \mathscr{M}_n$. Now by 6.2 and the definition of cl^t we have $(*)$ and by 5.9 for $\zeta = \varepsilon$ we have $(*)_1$, where

$(*)$ $|\mathrm{cl}^t(f(A), \mathscr{M}_n)| \leq n^{\varepsilon/k(*)}$,
$(*)_1$ $|\mathscr{G}| \geq n^{\xi - \varepsilon/2}$,

where $\mathscr{G}$ is constructed there so in particular

$$\begin{aligned} \mathscr{G} \subseteq \{g : g \text{ extends } f \text{ to an embedding of } B \text{ into } \mathscr{M}_n \\ \text{and satisfies clauses (i)–(v) from 5.9}\}. \end{aligned}$$

Recall that

$\circledast_1$ if $A' \subseteq M \in \mathscr{K}$ and $a \in \mathrm{cl}^k(A', M)$ then for some C we have $C \subseteq \mathrm{cl}^k(A', M)$, $|C| \leq k$, $a \in C$ and $\mathrm{cl}^k(C \cap A', C) = C$

(by the definition of cl^k, see [I, §1]).

We intend to find $g \in \mathscr{G}$ satisfying the requirements in the claim. Now g being an embedding of B into $\mathscr{M}_n$ extending f follows from $g \in \mathscr{G}$. So it is enough to prove that $< n^{\xi-\varepsilon}$ members g of $\mathscr{G}$ fail clause (i) and similarly for clauses (ii) and (iii).

More specifically, let $\mathscr{G}^1 = \{g \in \mathscr{G} : g(B) \cap \mathrm{cl}^t(f(A), \mathscr{M}_n) \neq A\}$, $\mathscr{G}^2 = \{g \in \mathscr{G} : g \notin \mathscr{G}^1$ but clause (ii) fails for $g\}$ and $\mathscr{G}^3 = \{g \in \mathscr{G}$: clause (iii) fails for g but $g \notin \mathscr{G}^1 \cup \mathscr{G}^2\}$. So clearly it is enough to prove $\mathscr{G} \not\subseteq \mathscr{G}^1 \cup \mathscr{G}^2 \cup \mathscr{G}^3$, because: (i) fails for $g \Rightarrow g \in \mathscr{G}^1$, (ii) fails for $g \Rightarrow g \in \mathscr{G}^2 \vee g \in \mathscr{G}^1$, and (iii) fails for $g \Rightarrow g \in \mathscr{G}^3 \vee g \in \mathscr{G}^2 \vee g \in \mathscr{G}^1$.

On the number of $g \in \mathscr{G}^1$: For $a \in B \setminus A$ and $x \in \mathrm{cl}^t(f(A), \mathscr{M}_n)$ let $\mathscr{G}^2_{a,x} = \{g \in \mathscr{G}^2 : g(a) = x\}$, so $\mathscr{G}^2 = \bigcup\{\mathscr{G}^2_{a,x} : a \in B \setminus A$ and $x \in \mathrm{cl}^t(f(A), \mathscr{M}_n)\}$, and by 4.17(13) clearly $A \cup \{a\} \leq_i B$ (as $A <_{pr} B$). The rest is as in the proof for $\mathscr{G}^2$ below, only easier.

On the number of $g \in \mathscr{G}^2$: If $g \in \mathscr{G}^2$ then for some

$$x_g \in \mathrm{cl}^t(\mathrm{Rang}(f), \mathscr{M}_n) \setminus \mathrm{Rang}(f) \text{ and } y \in B \setminus A$$

we have: $\{x_g, g(y)\}$ is an edge of $\mathscr{M}_n$. Note $x_g \notin g(B)$ as $g \in \mathscr{G}_2$.

We now form a new structure $B^2 = B \cup \{x^*\}$ $(x^* \notin B)$ such that $g \cup \{\langle x^*, x_g\rangle\} : B^2 \hookrightarrow \mathscr{M}_n$ and let $A^2 = B^2 {\restriction} (A \cup \{x^*\})$. Now up to isomorphism over B there are a finite number (i.e., with a bound not depending on n) of such B^2, say $\langle B^2_j : j < j^*_2\rangle$.

For $x \in \mathrm{cl}^t(\mathrm{Rang}(f), \mathscr{M}_n)$ and $j < j^*$ let

$$\mathscr{G}^2_{j,x} := \{g : g \text{ is an embedding of } B^2_j \text{ into } \mathscr{M}_n \text{ extending } f \text{ and satisfying } g(x^*) = x\},$$

$$\mathscr{G}^2_j := \bigcup_{x \in \mathrm{cl}^t(f(A), \mathscr{M}_n)} \mathscr{G}^2_{j,x}.$$

So:

$(*)_2$ if $g \in \mathscr{G}^2$ then
$g \in \bigcup\{\{g' {\restriction} B : g' \in \mathscr{G}^2_{j,x}\} : j < j^*_2 \text{ and } x \in \mathrm{cl}^t(f(A), \mathscr{M}_n)\}$.

Now, if $\neg(A^2_j <_s B^2_j)$ then as $A <^*_{pr} B$ it follows easily that $A^2_j <_i B^2_j$, so by 6.2 using $(*)$ (with $\varepsilon/2 - \varepsilon/k(*)$ here standing for ε in (iii) there) we have

$(*)_3$ if $\neg(A^2_j <_s B^2_j)$ then $|\mathscr{G}^2_j| \leq n^{\varepsilon/2}$.

[Why? As $A^2_j <_i B^2_j$, on the one hand for each $x \in \mathrm{cl}^t(\mathrm{Rang}(f), \mathscr{M}_n)$ by 6.2 the number of $g : B^2_j \hookrightarrow \mathscr{M}_n$ extending $f \cup \{\langle x^*, x\rangle\} : A^2_j \hookrightarrow \mathscr{M}_n$ is $< n^{\varepsilon/k(*)}$, and on the other hand the number of candidates for x is $\leq |\mathrm{cl}^t(\mathrm{Rang}(f), \mathscr{M}_n)| \leq n^{\varepsilon/k(*)}$. So $|\mathscr{G}^2_j| \leq n^{\varepsilon/k(*)} \cdot n^{\varepsilon/k(*)} \leq n^{2\varepsilon/k(*)} \leq n^{\varepsilon/2}$.]

If $A_j^2 <_s B_j^2$, then by 4.17(14) still $A_j^2 <_{pr} B_j^2$, and if we let

$$\xi_j^2 := \max\{\mathbf{w}_\lambda(A_j^2, B_j^2) : (A_j^2, B_j^2, \lambda) \in \mathscr{T} \text{ and for every } \lambda\text{-closed nonempty } C \subseteq B_j^2 \setminus A_j^2 \text{ we have } \mathbf{w}(A_j^2, A_j^2 \cup C, \lambda \restriction C) > 0\},$$

then clearly $\xi_j^2 < \xi - 2\varepsilon$ (as we retain the "old" edges, and by at least one we actually enlarge the number of edges but we keep the number of "vertices", i.e., equivalence classes; see 4.17(9)).

So, by 5.14,

$(*)_4$ if $A_j^2 <^*_{pr} B_j^2$ then $|\mathscr{G}_j^2| \le n^{\xi - 2\varepsilon}$.

As $\xi - 2\varepsilon > \varepsilon$ by $(*)_3 + (*)_4$, multiplying by j^*, as n is large enough,

$(*)_5$ $|\mathscr{G}^2|$, the number of $g \in \mathscr{G} \setminus \mathscr{G}^1$ failing clause (ii) of 6.5, is $\le n^{\xi - \varepsilon}$.

On the number of $g \in \mathscr{G}^3$: First if $g \in \mathscr{G}^3$, then there are A^+, B^+, C, g^+ such that

$\otimes_1$ $A \le_i A^+ \le_s B^+$, $B \le B^+$, $B \cap A^+ = A$, $|B^+| \le |B| + k, |A^+| \le |A| + k, C \nsubseteq B \cup A^+, B^+ \setminus B \subseteq C \subseteq B^+$, $C \cap B <_i C$, hence $\mathrm{cl}^k(C \cap B, B^+) \supseteq C$ and $g \subseteq g^+$, $g^+ : B^+ \hookrightarrow \mathscr{M}_n$, $g^+(A^+) \subseteq \mathrm{cl}^t(f(A), \mathscr{M}_n)$.

[Why? As $g \in \mathscr{G}^3$ there is $y_g \in \mathrm{cl}^k(g(B), \mathscr{M}_n)$ such that $y_g \notin g(B)$ and moreover $y_g \notin \mathrm{cl}^k(f(A), \mathscr{M}_n)$. By the first statement (and $\circledast_1$ above) there is $C^* \subseteq \mathrm{cl}^k(g(B), \mathscr{M}_n)$ with $\le k$ elements such that $y_g \in C^*$ and $C^* \cap g(B) \le_i C^*$. Let $B^* = g(B) \cup C^* \le \mathscr{M}_n$. Let B^+, g^+ be such that $B \le B^+ \in \mathscr{K}$, $g \subseteq g^+$, g^+ an isomorphism from B^+ onto B^*, and let $C = g^{-1}(C^*)$. Lastly, choose A^+ such that $A' \le_i A^+ \le_s B^+$; clearly it exists by 4.17(2). Now $|A^+| \le |B| + |C| \le t$ by the assumptions on A, B, k, t, hence $g^+(A^+) \subseteq \mathrm{cl}^t(f(A), \mathscr{M}_n)$; but as $g \in \mathscr{G}^3$ we have $g \notin \mathscr{G}^1$, hence $A = g(B) \cap \mathrm{cl}^t(f(A), \mathscr{M}_n)$, so we have $A^+ \cap B = A$. Also $C \nsubseteq B \cup A^+$, otherwise, as $g \notin \mathscr{G}^2$ and $g \notin \mathscr{G}^1$ we have $B \biguplus_A^{B^+} A^+$, hence $C \cap B \biguplus_{C \cap A} C \cap A^+$; but as $C \cap B <^*_i C$, by smoothness (e.g. 4.17(5)) we get $C \cap A <^*_i C \cap A^+$, so $C \cap A^+ \subseteq \mathrm{cl}^k(A, B^+)$, so $C^* \setminus g(B) = g^+(C \setminus B) \subseteq g^+(C \cap A^+) \subseteq \mathrm{cl}^k(f(A), \mathscr{M}_n)$, and thus $y_g \in \mathrm{cl}^k(f(A), \mathscr{M}_n)$, contradiction. So $\otimes_1$ holds.]

$\otimes_2$ in $\otimes_1$ for some λ' and m we have:
$(A^+, B^+, \lambda^+) \in \mathcal{T}$, $m \in \{1, \dots, n\}$, $\lambda' = \{(x, y) : x, y \in B^+ \setminus A$, $|g^+(x) - g^+(y)| < m\varepsilon/k(*)\}$ and $\mathbf{w}_{\lambda'}(A^+, B^+) < \xi - \varepsilon$.

[Why? We can find $m \in \{1, \dots, k\}$ such that for $\zeta = l\varepsilon/k(*)$ the set

$$\lambda' := \{(x,y) : x, y \in B^+ \text{ and } |x-y| \le n^\zeta\}$$

is an equivalence relation on $B^+ \setminus A^+$.

As $g \in \mathcal{G}$, clearly $\lambda' \restriction B$ is equal to λ. As $g \in \mathcal{G}$, necessarily some λ'-equivalence class is disjoint from B, but $B <_i B^+$, hence easily $\mathbf{w}_{\lambda'}(A^+, B^+) < \xi$ so by the choice of $\varepsilon, \mathbf{w}_{\lambda'}(A^+, B^+) < \xi - \varepsilon$.]

Let $\{(A_j^+, B_j^+, \lambda', m) : j < j_3^*\rangle$ list the possible (A^+, B^+, λ, m) up to isomorphism over B as described above. Let

$$\mathscr{G}_{j,h}^3 := \{g \in \mathscr{G} : g \text{ embeds } B_j^+ \text{ into } \mathscr{M}_n \text{ extending } f \text{ and moreover } h\}$$

for any $h \in \mathscr{H}_j^3 := \{h : h : A_j^+ \hookrightarrow \mathscr{M}_n \text{ extending } f\}$, so h necessarily satisfies $h(A_j^+) \subseteq \mathrm{cl}^k(f(A), \mathscr{M}_n) \subseteq \mathrm{cl}^t(f(A), \mathscr{M}_n)$. Now it follows easily (for random enough $\mathscr{M}_n$) by $\otimes_1$, $\otimes_2$ above, by 5.14(2), and by computation respectively that

$(*)_6$ if $g \in \mathscr{G}^3$ then

$$g \in \bigcup_{j<j_3^*} \bigcup_{h \in \mathscr{H}_j^3} \{g' \restriction B : g' \in \mathscr{G}_{j,h}^3\},$$

$(*)_7$ $|\mathscr{G}_{j,h}^3| < n^{\xi - 2\varepsilon}$ for each $h \in \mathscr{H}_j^3$,

$(*)_8$ $|\mathscr{H}_j^3| < |\mathrm{cl}^k(f(A), \mathscr{M}_n)|^k \le |\mathrm{cl}^t(f(A), \mathscr{M}_n)|^k < n^{\varepsilon, k}$.

Altogether,

$(*)_9$ the number of $g \in \mathscr{G}^3$ is $< n^{\xi - \varepsilon}$. $\blacksquare_{6.5}$

6.7. Conclusion. *If $A <_s^* B$ and $B_0 \subseteq B$ and $k \in \mathbb{N}$ then the tuple (B, A, B_0, k) is simply good* (see [I, Definition 2.12(1)]).

Proof. Read 6.5 and [I, Definition 2.12(1)]. $\blacksquare_{6.7}$

* * *

Toward simple niceness the "only" thing left is the universal part, i.e., [I, Definition 2.13(1)(A)].

The following Claims 6.8, 6.10 do not use §5 and have nothing to do with probability; they are the crucial step for proving the satisfaction of [I, Definition 2.13(1)(A)] in our case; Claim 6.8 is a sufficient condition for goodness (by 6.7). Our preceding the actual proof (of 6.11) by the two claims (6.8, 6.10) and separating them is for clarity, though it has a bad effect on the bound; also 6.8 using $\mathrm{cl}^{k,m}(\bar{a}b, M)$ instead of $\mathrm{cl}^k(\bar{a}b, M)$ when $k' < k$ may improve the bound.

6.8. Claim. *For every k and l (from $\mathbb{N}$) there are natural numbers $t = t(k,l)$ and $k^*(k,l) \ge t, k$ such that for any $k^* \ge k^*(k,l)$ we have:*

(∗) *if* $m^\otimes \in \mathbb{N}$ *and* $M \in \mathscr{K}$, $\bar a \in {}^{l\geq}M$, $b \in M$ *then*

⊗ *the set*

$$\mathscr{R} := \{(c,d) : d \in \mathrm{cl}^k(\bar a b, M) \setminus \mathrm{cl}^{k^*,m^\otimes+k}(\bar a, M) \text{ and } c \in \mathrm{cl}^{k^*,m^\otimes}(\bar a, M) \text{ and } \{c,d\} \text{ is an edge of } M\}$$

has less than t *members.*

Proof. If $k=0$ this is trivial so assume $k>0$. Choose $\varepsilon \in \mathbb{R}^+$ small enough such that

$(*)_1$ $C_0 <^* C_1$ & $(C_0, C_1, \lambda) \in \mathscr{T}$ & $|C_1| \leq k \Rightarrow \mathbf{w}_\lambda(C_0, C_1) \notin [-\varepsilon, \varepsilon]$

(in fact we can restrict ourselves to the case $C_0 <^*_i C_1$). Choose $\mathbf{c} \in \mathbb{R}^+$ large enough such that

$(*)_2$ $(C_0, C_1, \lambda) \in \mathscr{T}, |(C_1 \setminus C_0)/\lambda| \leq k \Rightarrow \mathbf{w}_\lambda(C_0, C_1) \leq \mathbf{c}$

(so actually $\mathbf{c} = k$ is enough). Choose $t_1 > 0$ such that $t_1 > \mathbf{c}/\varepsilon$ and $t_1 > 2$. Choose $t_2 \geq 2^{2^{t_1+k+l}}$ (overkill, we mainly need to apply twice the Δ-system lemma; but note that in the proof of 6.10 below we will use the Ramsey theorem). Lastly, choose $t > k^2 t_2$ and let $k^* \in \mathbb{N}$ be large enough, which actually means that $k^* > k$ & $k^* \geq (k+1)t_2$ so $k^*(k,l) := (k+1)t_2$ is O.K.

Suppose we have $m^\otimes$, M, $\bar a$, b as in (∗) but such that the set $\mathscr{R}$ has at least t members. Let $(c_i, d_i) \in \mathscr{R}$ for $i < t$ be pairwise distinct ([6]).

As $d_i \in \mathrm{cl}^k(\bar a b, \mathscr{M}_n)$, we can choose for each $i < t$ a set $C_i \leq M$ such that:

(i) $C_i \leq M$,
(ii) $|C_i| \leq k$,
(iii) $d_i \in C_i$,
(iv) $C_i \restriction (C_i \cap (\bar a b)) <_i C_i$.

For each $i < t$, as $C_i \cap \mathrm{cl}^{k^*,m^\otimes+k}(\bar a, M)$ is a proper subset of C_i (this is witnessed by d_i, i.e., as $d_i \in C_i \setminus (C_i \cap \mathrm{cl}^{k^*,m^\otimes+k}(\bar a, M_n)))$, clearly this set has $< k$ elements and hence for some $k[i] < k$ we have

(v) $C_i \cap \mathrm{cl}^{k^*,m^\otimes+k[i]+1}(\bar a, M) \subseteq \mathrm{cl}^{k^*,m^\otimes+k[i]}(\bar a, M)$.

So without loss of generality

(vi) $i < t/k^2 \Rightarrow k[i] = k[0]$ & $|C_i| = |C_0| = k' \leq k$

(remember $t_2 < t/k^2$); also

(vii) $b \in C_i$.

([6]) Note: we do not require the d_i's to be distinct; though if $w = \{i : d_i = d^*\}$ has $\geq k' > 1/\alpha$ elements then $d^* \in \mathrm{cl}^{k'}(\mathrm{cl}^{k^*,m^\otimes+k}(\bar a, M))$.

[Why? If not then by clause (iv) we have $(C_i \cap \bar{a}) <_i C_i$, hence $d_i \in C_i \subseteq \mathrm{cl}^k(\bar{a}, M) \subseteq \mathrm{cl}^{k^*, m^\otimes + k}(\bar{a}, M)$, contradiction.]

As $k^* \geq k^*(k, l) \geq t_2(k+1)$ (by the assumption on k^*), clearly

$$\Big| \bigcup_{i<t_2} C_i \cup \{c_i : i < t_2\} \Big| \leq \sum_{i<t_2} |C_i| + t_2 \leq \sum_{i<t_2} k + t_2 \leq t_2(k+1) \leq k^*$$

and we define

$$D = \bigcup_{i<t_2} C_i \cup \{c_i : i < t_2\}, \qquad D' = D \cap \mathrm{cl}^{k^*, m^\otimes + k[0]}(\bar{a}, M).$$

So by the previous sentence we have $|D'| \leq |D| \leq k^*$. Now

$\otimes_0$ $D' <_s D$.

[Why? As otherwise there is D'' such that $D' <_i D'' \leq_s D$, so as $|D''| \leq |D| \leq k^*$, clearly $D'' \subseteq \mathrm{cl}^{k^*, m^\otimes + k[0]+1}(\bar{a}b, M)$; contradiction.]

So we can choose $\lambda \in \Xi(D', D)$ (see Definition 4.8(2)). Let $C_i = \{d_{i,s} : s < k'\}$, with $d_{i,0} = d_i$, and recalling (vii) also $b \neq d_i \Rightarrow b = d_{i,1}$, and with no repetitions.

Clearly $d_{i,0} = d_i \notin D'$. By the finite Δ-system lemma for some $S_0, S_1, S_2 \subseteq \{0, \ldots, k'-1\}$ and $u \subseteq \{0, \ldots, t_2 - 1\}$ with $\geq t_1$ elements we have:

$\oplus_1$(a) $\lambda' := \{(s_1, s_2) : d_{i,s_1} \lambda d_{i,s_2}\}$ is the same for all $i \in u$ and $S_0 = \{0, \ldots, k'-1\} \setminus \mathrm{Dom}(\lambda')$, so $d_{i,s} \in D' \Leftrightarrow i \in S_0$,
(b) for each $j < \lg(\bar{a}) + 1$, and $s < k'$, the truth value of $d_{i,s} = (\bar{a}b)_j$ is the same for all $i \in u$ for each $s \in S_0 = \{0, \ldots, k'-1\} \setminus \mathrm{Dom}(\lambda')$,
(c) $d_{i_1, s_1} = d_{i_2, s_2} \Rightarrow s_1 = s_2$ for $i_1, i_2 \in u$,
(d) $d_{i_1, s} = d_{i_2, s} \Leftrightarrow s \in S_1$ for $i_1 \neq i_2 \in u$,
(e) $d_{i_1, s_1} \lambda d_{i_2, s_2} \Rightarrow d_{i_1, s_1} \lambda d_{i_1, s_2}$ & $d_{i_1, s_2} \lambda d_{i_2, s_2}$ for $i_1 \neq i_2 \in u$,
(f) $d_{i_1, s} \lambda d_{i_2, s} \Leftrightarrow s \in S_2$ for $i_1 \neq i_2 \in u$; so $i \in u \,\&\, s \in S_2 \Rightarrow d_{i,s} \notin D'$,
(g) the statement $b = d_{i,0}$ has the same truth value for all $i \in u$.

Now we necessarily have

$\oplus_2$ $0 \notin S_2$ (i.e., $\lambda \restriction \{d_i : i \in u\}$ is equality).

[Why? Otherwise, let $X = d_i / \lambda$ for any $i \in u$; then the triple $(D', D' \cup X, \lambda \restriction X) \in \mathscr{T}$ has weight

$$\begin{aligned}\mathbf{w}(D', D' \cup X, \lambda \restriction X) &= \mathbf{v}(D', D' \cup X, \lambda \restriction X) - \alpha \mathbf{e}(D', D' \cup X, \lambda \restriction X) \\ &= 1 - \alpha \cdot |\{e : e \text{ an edge of } M \text{ with one end in} \\ &\qquad D' \text{ and the other in } X\}|.\end{aligned}$$

Now as $c_i \in \mathrm{cl}^{k^*, m^\otimes}(\bar{a}, M)$, clearly $c_i \in D'$ and the pairs $\{c_i, d_i\} \in \mathrm{edge}(M)$ are distinct for different i; clearly the number above is $\leq 1 - \alpha |\{(c_i, d_i) : i \in u\}| = 1 - \alpha |u| = 1 - \alpha t_1 < 0$; contradiction to $\lambda \in \Xi(D', D)$.]

Let $D_0 = \bar{a} \cup \bigcup\{d_{i,s}/\lambda : s \in S_2 \text{ and } i \in u\}$; clearly D_0 is a λ-closed subset of D though not necessarily $\subseteq \mathrm{Dom}(\lambda) = D \setminus D'$ because of $\bar{a}$. We have:

$\oplus_3$ $b = d_{i,1}$ and $1 \in S_1 \setminus S_0$ and $0 \notin S_0 \cup S_1 \cup S_2$ and $S_1 \setminus S_0 \subseteq S_2$ (hence $b \in D_0$).

[Why? The first two clauses hold as $b \in C_i$, $b \in \{d_{i,0}, d_{i,1}\}$ and by $\oplus_2$ and (g) of $\oplus_1$. The last clause holds by $\oplus_1$(d),(f), and the "hence $b \in D_0$" by the definition of $D_0, S_1 \setminus \mathrm{Dom}(\lambda') \subseteq S_2$ and the first clause. Also $0 \notin S_0 \cup S_1 \cup S_2$ should be clear.]

$\oplus_4$ For each $i \in u$ we have $\mathbf{w}_\lambda(C_i \cap D_0, C_i) < 0$.

[Why? As $C_i \cap (\bar{a}b) \subseteq C_i \cap D_0$ by clauses (b) + (f) of $\oplus_1$ and by monotonicity of $<_i$ we have $C_i \restriction (C_i \cap \bar{a}b) <_i C_i \Rightarrow C_i \cap D_0 \leq_i C_i$, but $d_{i,0} = d_i \in C_i \setminus C_i \cap D_0$.] Hence

$\oplus_5$ $\mathbf{w}_\lambda(C_i \cap D_0, C_i) \leq -\varepsilon$ for $i \in u$.

[Why? See the choice of ε.] Let

$$\begin{aligned} D_1 &:= D' \cup \bigcup\{d_{i,s}/\lambda : i \in u, s < k'\} \\ &= D' \cup D_0 \cup \{d_{i,s}/\lambda : i \in u, s < k' \;\&\; s \notin S_2\}. \end{aligned}$$

Then clearly D_1 is a λ-closed subset of D including D' but $D_1 \neq D'$ as $i \in u \Rightarrow d_i \in D_1$ by $\oplus_2$. Also clearly

$\oplus_6$ $D' \subseteq D' \cup D_0 \subseteq D_1 \subseteq D$ and D_0, D_1 are λ-closed.

So, as we know $\lambda \in \Xi(D', D)$, we have

$\oplus_7$ $\mathbf{w}_\lambda(D', D_1) > 0$.

Now

$$\begin{aligned} \mathbf{w}_\lambda(D', D_1) &= \mathbf{w}_\lambda\Big(D', \bigcup\Big\{x/\lambda : x \in \bigcup_{i \in u} C_i \setminus D'\Big\} \cup D'\Big) \\ &= \mathbf{w}_\lambda(D', D' \cup D_0) + \mathbf{w}_\lambda\Big(D' \cup D_0, D' \cup D_0 \\ &\qquad \cup \bigcup\{d_{i,s}/\lambda : i \in u, s < k', s \notin S_0 \cup S_2\}\Big) \end{aligned}$$

[so by 6.9 below with $B_i = \{d_{i,s} : s < k', s \notin S_2 \cup S_0\}$ and $B_i^+ = \bigcup\{d_{i,s}/\lambda : s < k', s \notin S_2\}$]

$$\begin{aligned} &\leq \mathbf{w}_\lambda(D', D' \cup D_0) \\ &\quad + \sum_{i \in u} \mathbf{w}_\lambda(D' \cup D_0, D' \cup D_0 \cup \{d_{i,s} : s < k', s \notin S_2 \cup S_0\}) \end{aligned}$$

[as $C_i = \{d_{i,s} : s < k'\}$ and $d_{i,s} \in D' \cup D_0$ if $s < k', s \in S_0 \cup S_2, i \in u$]

$$\leq \mathbf{w}_\lambda(D', D' \cup D_0) + \sum_{i \in u} \mathbf{w}_\lambda(D' \cup D_0, D' \cup D_0 \cup C_i)$$

[so as $\mathbf{w}_\lambda(A_1, B_1) \leq \mathbf{w}_\lambda(A, B)$ when $A \leq A_1 \leq B_1, A \leq B \leq B_1, B_1 \setminus A_1 = B \setminus A$ by 4.15(3)]

$$\leq \mathbf{w}_\lambda(D' \cap D_0, D_0) + \sum_{i \in u} \mathbf{w}_\lambda(C_i \cap D_0, C_i)$$

[so by the choice of $\mathbf{c}, D_0$, i.e., $(*)_2$ and the choice of $\varepsilon, u + \oplus_5$ respectively]

$$\leq \mathbf{c} + |u|(-\varepsilon) = \mathbf{c} - t_1\varepsilon < 0,$$

contradicting the choice of λ, i.e., $\oplus_7$. $\blacksquare_{6.8}$

6.9. Observation. *Assume*

(a) $A \leq^* A \cup B_i \leq^* A \cup B_i^+ \leq^* B$ *for* $i \in u$,
(b) $B \setminus A$ *is the disjoint union of* $\langle B_i^+ : i \in u \rangle$,
(c) λ *is an equivalence relation on* $B \setminus A$,
(d) *each* B_i^+ *is* λ*-closed*,
(e) $B_i^+ = \bigcup\{x/\lambda : x \in B_i \setminus A\}$,

Then $\mathbf{w}_\lambda(A, B) \geq \sum\{\mathbf{w}_\lambda(A, B_i) : i \in u\}$.

Proof. By (b) + (d),

$$\mathbf{v}_\lambda(A, B) = \sum\{\mathbf{v}_\lambda(A, A \cup B_i^+) : i \in u\} = \sum\{\mathbf{v}_\lambda(A, A \cup B_i) : i \in u\}$$

and by clause (b) the set $e_\lambda(A, B)$ contains the disjoint union of $\langle e_\lambda(A, B_i) : i \in u \rangle$. Altogether, the result follows. $\blacksquare_{6.9}$

6.10. Claim. *For every* k, m *and* l *from* $\mathbb{N}$ *and some* $m^* = m^*(k, l, m)$, *for any* $k^* \geq k^*(k, l)$ (*the function* $k^*(k, l)$ *is the one from Claim* 6.8) *satisfying* $k^* \geq km^*$ *we have*

(*) *if* $M \in \mathscr{K}$, $\bar{a} \in {}^{l\geq}M$ *and* $b \in M \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, M)$ *then for some* $m^\otimes \leq m^* - m$ *we have*

$$\mathrm{cl}^k(\bar{a}b, M) \cap \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M) \subseteq \mathrm{cl}^{k^*,m^\otimes}(\bar{a}, M).$$

Proof. For $k = 0$ this is trivial so assume $k > 0$. Let $t = t(k, l)$ be as in Claim 6.8. Choose m^* such that, e.g., $\lfloor m^*/(km) \rfloor \to (t+5)^2_{2^{k!+l}}$ in the usual notation in Ramsey theory. We could get more reasonable bounds but there is no need now. Remember that $k^*(k, l)$ is from 6.8 and k^* is any natural number $\geq k^*(k, l)$ such that $k^* \geq km^*$.

If the conclusion fails, then the set

$$Z := \{j \leq m^* - k : \mathrm{cl}^k(\bar{a}b, M) \cap \mathrm{cl}^{k^*,j+1}(\bar{a}, M) \not\subseteq \mathrm{cl}^{k^*,j}(\bar{a}, M)\}$$

satisfies

$$j \le m^* - m - k \;\Rightarrow\; Z \cap [j, j+m) \ne \emptyset.$$

Hence $|Z| \ge (m^* - m - k)/m$. For $j \in Z$ there are $C_j \le M$ and d_j such that

$$|C_j| \le k, \qquad (C_j \cap (\bar{a}b)) <^*_i C_j, \qquad d_j \in C_j \cap \mathrm{cl}^{k^*, j+1}(\bar{a}, M) \setminus \mathrm{cl}^{k^*, j}(\bar{a}, M).$$

Now we use the same argument as in the proof of 6.8. As $d_j \in C_j \cap \mathrm{cl}^{k^*,j+1}(\bar{a}, M) \setminus \mathrm{cl}^{k^*,j}(\bar{a}, M)$ we find that $C_j \cap \mathrm{cl}^{k^*,j}(\bar{a}, M)$ is a proper subset of $C_j \cap \mathrm{cl}^{k^*,j+1}(\bar{a}, M)$ (witnessed by d_j), so $|C_j \cap \mathrm{cl}^{k^*,j}(\bar{a}, M)| < |C_j \cap \mathrm{cl}^{k^*,j+1}(\bar{a}, M)| \le k$, so $|C_j \cap \mathrm{cl}^{k^*,j}(\bar{a}, M)| < k$. Hence for some $k_j \in \{1, \ldots, k\}$ we have $C_j \cap \mathrm{cl}^{k^*, m^* - k_j + 1}(\bar{a}, M) \subseteq \mathrm{cl}^{k^*, m^* - k_j}(\bar{a}, M)$, hence for some $k' \in \{1, \ldots, k\}$ we have $|Z'| \ge (m^* - m - k)/(mk)$, where $Z' = \{j \in Z : k_j = k'\}$.

Let $C_j = \{d_{j,s} : s < s_j \le k\}$ with $d_{j,0} = d_j$ and no repetitions. We can find $s^* \le k$ and $S_1, S_0 \subseteq \{0, \ldots, s^* - 1\}$ and $u \subseteq Z'$ satisfying $|u| = t + 5$ such that (because of the partition relation):

- (a) $i \in u \Rightarrow s_j = s^*$,
- (b) for each $j < \lg(\bar{a}) + 1$ and $s < s^*$ the truth value of $d_{i,s} = (\bar{a}b)_j$ is the same for all $i \in u$,
- (c) if $i \ne j$ are from u then $|i - j| > k + 1$, i.e., the C_i's for $i \in u$ are quite far from each other,
- (d) the truth value of "$\{d_{i,s_1}, d_{i,s_2}\}$ is an edge" is the same for all $i \in u$,
- (e) for all $i_0 < i_1$ from u:

$$d_{i_0,s} \in \mathrm{cl}^{k^*, i_1}(\bar{a}, M) \;\Leftrightarrow\; s \in S_0,$$

- (f) for all $i_0 < i_1$ from u:

$$d_{i_1,s} \in \mathrm{cl}^{k^*, i_0}(\bar{a}, M) \;\Leftrightarrow\; s \in S_1,$$

- (g) for each $s < s^*$, the sequence $\langle d_{i,s} : i \in u \rangle$ is constant or with no repetitions,
- (h) if $d_{i_1,s_1} = d_{i_2,s_2}$ then $d_{i_1,s_1} = d_{i_1,s_2} = d_{i_2,s_2}$, moreover, $s_1 = s_2$ (recalling that $\langle d_{i,s} : s < s_j \rangle$ is with no repetitions).

Now let $i(*)$ be, e.g., the third element of the set u and

$$B_1 := C_{i(*)} \cap \mathrm{cl}^{k^*, \min(u)}(\bar{a}, M), \qquad B_2 := C_{i(*)} \cap \mathrm{cl}^{k^*, \max(u)}(\bar{a}, M).$$

So

- $\circledast_1$ $B_1 <^* B_2 \le^* C_{i(*)}$ (note: $B_1 \ne B_2$ because $d_{i(*)} \in B_2 \setminus B_1$),
- $\circledast_2$ $(\bar{a}b) \cap B_2 \subseteq B_1$ by clause (b),
- $\circledast_3$ there is no edge in $(C_{i(*)} \setminus B_2) \times (B_2 \setminus B_1)$.

Why? Assume that this fails. Let the edge be $\{d_{i(*),s_1}, d_{i(*),s_2}\}$ with $d_{i(*),s_1} \in C_{i(*)} \setminus B_2$ and $d_{i(*),s_2} \in B_2 \setminus B_1$; hence

$(*)_1$ $d_{i(*),s_1} \in C_{i(*)} \setminus \mathrm{cl}^{k^*,\max(u)}(\bar{a}, M)$ and $d_{i(*),s_2} \in \mathrm{cl}^{k^*,\max(u)}(\bar{a}, M) \setminus \mathrm{cl}^{k^*,\min(u)}(\bar{a}, M)$ and $\{d_{i(*),s_1}, d_{i(*),s_2}\}$ is an edge.

Hence by clause (d),

$(*)_2$ $\{d_{i,s_1}, d_{i,s_2}\}$ is an edge for every $i \in u$

and by clauses (e), (f) we have

$(*)_3$ if $i_0 < i_1 < i_2$ are in u then $d_{i_1,s_2} \notin \mathrm{cl}^{k^*,i_0}(\bar{a}, M)$ and $d_{i_1,s_2} \in \mathrm{cl}^{k^*,i_2}(\bar{a}, M)$ and $d_{i_1,s_1} \notin \mathrm{cl}^{k^*,i_2}(\bar{a}, M)$,

and so necessarily

$(*)_4$ $\langle d_{i,s_2} : i \in u\rangle$ is with no repetitions.

[Why? By clause (g) and $(*)_3$.]

So the set of edges $\{\{d_{i,s_1}, d_{i,s_2}\} : i \in u$ but $|u \cap i| \geq 2$ and $|u \setminus i| \geq 2\}$ contradicts 6.8 using $m^{\otimes} = \max(u) - k$ there (and our choice of parameters and $C_i \subseteq \mathrm{cl}^k(\bar{a}b, M)$). Thus $\circledast_3$ holds.

As $C_{i(*)} \restriction (\bar{a}b) <_i C_{i(*)}$ and $B_2 \cap (\bar{a}b) \subseteq B_1$ (by $\circledast_2$), clearly $C_{i(*)} \cap \bar{a}b \subseteq C_{i(*)} \setminus (B_2 \setminus B_1) \subset C_{i(*)}$, the strict $\subset$ as

$$d_{i(*)} \in C_{i(*)} \cap (\mathrm{cl}^{k^*,i(*)+1}(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,i(*)}(\bar{a}b, M)) \subseteq B_2 \setminus B_1.$$

But, as stated above, $C_{i(*)} \setminus (B_2 \setminus B_1) \bigcup_{B_1} B_2$, hence by the previous sentence (and smoothness, see 4.17(5)) we get $B_1 <_i^* B_2$; also $|B_2| \leq |C_{i(*)}| \leq k \leq k^*$. By their definitions, $B_1 \subseteq \mathrm{cl}^{k^*,\min(u)}(\bar{a}, M)$, but $B_1 \leq_i^* B_2$, $|B_2| \leq k \leq k^*$ and hence $B_2 \subseteq \mathrm{cl}^{k^*,2^{\text{nd}} \text{ member of } u}(\bar{a}, M)$. Contradiction to the choice of $d_{i(*)}$. $\blacksquare_{6.10}$

6.11. Lemma. *For every k, m and l (from $\mathbb{N}$), for some m^*, k^* and t^* we have*:

$(*)$ *if $M \in \mathscr{K}, \bar{a} \in {}^{l\geq}M$ and $b \in M \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, M)$ then for some $m^{\otimes} \leq m^* - m$ and B we have*

- (i) $|B| \leq t^*$,
- (ii) $\bar{a} \subseteq B \subseteq \mathrm{cl}^k(B, M) \subseteq \mathrm{cl}^{k^*,m^{\otimes}}(\bar{a}, M)$,
- (iii) $\mathrm{cl}^{k^*,m^{\otimes}+m}(\bar{a}, M), (\mathrm{cl}^k(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,m^{\otimes}+m}(\bar{a}, M)) \cup B$ *are free over B inside M*,
- (iv) $B \leq_s^* B^* := M \restriction ((\mathrm{cl}^{k^*}(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,m^{\otimes}+m}(B, M)) \cup B)$.

6.12. Remark. Clearly this will finish the proof of simply nice.

6.13. Comments. Let us describe the proof below.

(1) In the proof we apply the last two claims. By them we arrive at the following situation: inside $\mathrm{cl}^k(\bar{a}b, M)$ we have $B \leq B^*, |B| \leq t^*$ and there is no "small" D such that $B <_i^* D \leq B^*$ and we have to show that $B <_s^* B^*$, a kind of compactness lemma.

(2) Note that for each $d \in \mathrm{cl}^k(\bar{a}b, M)$ there is $C_d \subseteq \mathrm{cl}^k(\bar{a}b, M)$ witnessing it, i.e., $C_d \cap (\bar{a}b) \leq_i C_d$, $d \in C_d, |C_d| \leq k$. To prove the statement above we choose an increasing sequence $\langle D_i : i \leq i(*)\rangle$ of subsets of B^*, $D_0 = B \cup \{b\}$, $|D_i|$ has an a priori bound, D_{i+1} "large" enough. So by our assumption toward contradiction $B <_s^* D_{i(*)}$, hence there is $\lambda \in \Xi(B, D_{i(*)})$; without loss of generality, $B^* = B \cup \bigcup\{C_d : d \in D_{i(*)}\}$. For each $i < i(*)$ we try to "lift" $\lambda\restriction(D_i \setminus B)$ to $\lambda^+ \in \Xi(B, B^*)$; a failure will show that we could have put elements satisfying some conditions in D_{i+1} so we had done so. As this occurs for every $i < i(*)$, by weight computations we get a contradiction.

Proof of Lemma 6.11. Without loss of generality $k > 0$. Let $t = t(k, l)$ and $k^*(k, l)$ be as required in 6.8 (for our given k, l).

Choose $m(1) = t(m + 1) + k + 2$ and let $t^* = t + l + k$.

Choose m^* as in 6.10 for k (given in 6.11), $m(1)$ (chosen above) and l (given in 6.11), i.e., $m^* = m^*(k, m(1), l)$. Let $\varepsilon^* \in \mathbb{R}^{>0}$ be such that

$$(A', B', \lambda) \in \mathscr{T} \;\&\; |B'| \leq k \;\&\; A' \neq B' \;\Rightarrow\; \mathbf{w}_\lambda(A', B') \notin (-\varepsilon^*, \varepsilon^*).$$

Let $i(*) > 1/\varepsilon^*$. Define inductively k_i^* for $i \leq i(*)$ as follows:

$$k_0^* = \max\{k^*(k, l), mk, m^*t^* + 1\}, \qquad k_{i+1}^* = 2^{2^{k_i^*}},$$

and lastly let

$$k^* = kk_{i(*)}^*.$$

We shall prove that m^*, k^*, t^* are as required in 6.11. Let M, $\bar{a}$, b be as in the assumption of $(*)$ of 6.11. So $M \in \mathscr{K}$, $\bar{a} \in {}^{l\geq}M$ and $b \in M \setminus \mathrm{cl}^{k^*,m^*}(\bar{a}, M)$; but this means that the assumption of $(*)$ in 6.10 holds for k, $m(1)$, l, so we can apply it (i.e., as $m^* = m^*(k, m(1), l), k^* \geq k^*(k, l)$, where $k^*(k, l)$ is from 6.8 and $k^* \geq km^*$ as $k^* \geq k_{i(*)}^* > k_0^* \geq m^*k$). Hence for some $r \leq m^* - m(1)$ we have

$\oplus_1$ $\mathrm{cl}^k(\bar{a}b, M) \cap \mathrm{cl}^{k^*,r+m(1)}(\bar{a}, M) \subseteq \mathrm{cl}^{k^*,r}(\bar{a}, M)$.

Let us define

$$\begin{aligned} \mathscr{R} = \{(c, d) : \; & d \in \mathrm{cl}^k(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,r+m(1)}(\bar{a}, M) \text{ and} \\ & c \in \mathrm{cl}^{k^*,r+m(1)-k}(\bar{a}, M) \text{ and} \\ & \{c, d\} \text{ is an edge of } M\}. \end{aligned}$$

How many members does $\mathscr{R}$ have? By 6.8 (with $r + m(1) - k$ here standing for $m^\otimes$ there as $k^* \geq k^*(k,l))$ at most t members. But by $\oplus_1$ above

$$\begin{aligned}\mathscr{R} = \{(c,d) : {} & d \in \mathrm{cl}^k(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,r}(\bar{a}, M) \text{ and} \\ & c \in \mathrm{cl}^{k^*,r+m(1)-k}(\bar{a}, M) \text{ and} \\ & \{c,d\} \text{ is an edge of } M\}.\end{aligned}$$

But $t(m+1)+1 < m(1)-k$ by the choice of $m(1)$ (and, of course, $\mathrm{cl}^{k^*,i}(\bar{a}, M)$ increase with i), hence for some $m^\otimes \in \{r+1, \dots, r+m(1)-k-m\}$ we have

$$\oplus_2\ (c,d) \in \mathscr{R} \Rightarrow c \notin \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M) \setminus \mathrm{cl}^{k^*,m^\otimes-1}(\bar{a}, M).$$

So

$$\oplus_3\ r \leq m^\otimes - 1 < m^\otimes + m \leq r + m(1) - k.$$

Let

$$B := \{c \in \mathrm{cl}^{k^*,m^\otimes-1}(\bar{a}, M) : \text{for some } d \text{ we have } (c,d) \in \mathscr{R}\} \cup \bar{a}.$$

So by the above $B = \{c \in \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M) : (\exists d)((c,d) \in \mathscr{R})\} \cup \bar{a}$.

Let us check the demands (i)–(iv) of $(*)$ of 6.11; remember that we are defining $B^* = (\mathrm{cl}^k(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M)) \cup B$, that is, the submodel of M with this set of elements.

Clause (i): $|B| \leq t^*$. As said above, $|\mathscr{R}| \leq t$, hence clearly $|B| \leq t + \lg(\bar{a}) \leq t + l \leq t^*$.

Clause (ii): $\bar{a} \subseteq B \subseteq \mathrm{cl}^k(B, M) \subseteq \mathrm{cl}^{k^*,m^\otimes}(\bar{a}, M)$. As by its definition $B \subseteq \mathrm{cl}^{k^*,m^\otimes-1}(\bar{a}, M)$, and $k \leq k^*$, clearly $\mathrm{cl}^k(B, M) \subseteq \mathrm{cl}^{k^*,m^\otimes}(\bar{a}, M)$, and $B \subseteq \mathrm{cl}^k(B, M)$ always and $\bar{a} \subseteq B$ by its definition.

Clause (iii): Clearly

$$\begin{aligned}B &= \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M) \cap ((\mathrm{cl}^k(\bar{a}b, M) \setminus \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M)) \cup B) \\ &= (\mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M)) \cap B^*.\end{aligned}$$

Now the "no edges" holds by the definitions of B and $\mathscr{R}$.

Clause (iv): $B \leq^*_s B^*$. Clearly $B \subseteq B^*$ by the definition of B^* before the proof of clause (i). Toward contradiction assume $\neg(B \leq^*_s B^*)$; then 4.17(2) holds for some D with $B <_i D \leq B^*$; choose such a D with a minimal number of elements. Note that as $B \subseteq \mathrm{cl}^{k^*,m^\otimes-1}(\bar{a}, M)$ and $B^* \cap \mathrm{cl}^{k^*,m^\otimes+m}(\bar{a}, M) = B$, necessarily $|D| > k^*$ (and $B <^* D \leq B^*$). For every $d \in D \backslash B$, as $d \in B^*$, clearly $d \in \mathrm{cl}^k(\bar{a}b, M)$, hence there is a set $C_d \leq M$ with $|C_d| \leq k$ such that $C_d \restriction (\bar{a}b) \leq_i C_d$ and $d \in C_d$; note that $C_d \subseteq \mathrm{cl}^k(\bar{a}b, M)$ by the definition of cl^k, hence by the choice of B^* and $m^\otimes$ and $\oplus_1$ we have $C_d \subseteq B^* \cup \mathrm{cl}^{k^*,m^\otimes-1}(\bar{a}, M)$. Let $C'_d = C_d \cap (B \cup \{b\})$ and $C''_d = C_d \cap B^*$. Clearly $C_d \cap (\bar{a}b) \leq C'_d \leq C''_d \leq C_d$, hence $C'_d \leq_i C_d$. Now by clause (iii),

$C''_d \overset{M}{\underset{C'_d}{\biguplus}} C'_d \cup (C_d \setminus C''_d)$, hence (by smoothness) we have $C'_{d_i} \leq_i C''_{d_i}$. Of course, $|C''_d| \leq |C_d| \leq k$. For $d \in B$ let $C_d = C'_d = C''_d = \{d\}$.

We now choose a set D_i, by induction on $i \leq i(*)$, such that (letting $C^{**}_i = \bigcup_{d \in D_i} C''_d$):

(a) $D_0 = B \cup \{b\}$,

(b) $j < i \Rightarrow D_j \subseteq D_i \subseteq D$,

(c) $|D_i| \leq k^*_i$,

(d) if λ is an equivalence relation on $C^{**}_i \setminus B$ and for some $d \in D \setminus D_i$ one of the clauses below holds then there is such $d \in D_{i+1}$, where

$\otimes^1_{\lambda,d}$ for some $x \in C''_d \setminus C^{**}_i$, there are no $y \in C''_d \cap C^{**}_i$, $j^* \in \mathbb{N}$ and $\langle y_j : j \leq j^* \rangle$ such that $y_j \in C''_d$, $y_{j^*} = x$, $y_0 = y, \{y_j, y_{j+1}\}$ an edge of M (actually an empty case, i.e., never occurs; see $(*)_{14}$ below),

$\otimes^2_{\lambda,d}$ there are $x \in C''_d \setminus C^{**}_i$, $y \in (C^{**}_i \setminus C''_d) \cup B$ and $y' \in C''_d \cap C^{**}_i$ such that $\{x, y\}$ is an edge of M and y is connected by a path $\langle y_0, \ldots, y_j \rangle$ inside C''_d to x so $x = y_j$, $y = y_0$ and $[y_i \in C^{**}_i \equiv i = 0]$ and $\neg(y' \lambda y)$,

$\otimes^3_{\lambda,d}$ there is an edge $\{x_1, x_2\}$ of M such that we have:

(A) $\{x_1, x_2\} \subseteq C''_d$,

(B) $\{x_1, x_2\}$ is disjoint from C^{**}_i,

(C) for $s \in \{1, 2\}$ there is a path $\langle y_{s,0}, \ldots, y_{s,j_s} \rangle$ in C''_d, $y_{s,j_s} = x_s$, $[y_{s,j} \in C^{**}_i \equiv j = 0]$ and $\neg(y_{1,0} \lambda y_{2,0})$,

(e) if λ is an equivalence relation on $C^{**}_i \setminus B$ to which clause (d) does not apply but there are d_1, $d_2 \in D$ satisfying one of the following, then we can find such d_1, $d_2 \in D_{i+1}$:

$\otimes^4_{\lambda,d_1,d_2}$ for some $x_1 \in C''_{d_1} \setminus C^{**}_i$, $x_2 \in C''_{d_2} \setminus C^{**}_i$ and $y_1 \in C''_{d_1} \cap C^{**}_i$, $y_2 \in C''_{d_2} \cap C^{**}_i$ we have: for $s = 1, 2$ there is a path $\langle y_{s,0}, \ldots, y_{s,j_s} \rangle$ in C''_{d_s}, $y_{s,j_s} = x$, $y_{s,0} = y_s$, $[y_{s,j} \in C^{**}_i \Leftrightarrow j = 0]$ and $x_1 = x_2 \mathbin{\&} \neg(y_1 \lambda y_2)$,

$\otimes^5_{\lambda,d_1,d_2}$ for some x_1, x_2, y_1, y_2 as in $\otimes^4_{\lambda,d_1,d_2}$ we have $\neg(y_1 \lambda y_2)$ and $\{x_1, x_2\}$ is an edge.

So $|D_{i(*)}| \leq k^*/k$ (by the choice of k^*, $i(*)$ and clause (c)), hence $C^{**}_{i(*)} := \bigcup_{d \in D_{i(*)}} C''_d$ has $\leq k^*$ members, $\bar{a}b \subseteq B \cup \{b\} \subseteq D_0 \subseteq C^{**}_{i(*)} \subseteq \mathrm{cl}^k(\bar{a}b, M)$ and $C^{**}_{i(*)} \cap \mathrm{cl}^{k^*, m^\otimes + m}(\bar{a}, M) = B \subseteq \mathrm{cl}^{k^*, m^\otimes - 1}(\bar{a}, M)$. Hence necessarily $B \leq_s C^{**}_{i(*)}$, so there is $\lambda \in \Xi(B, C^{**}_{i(*)})$. Let $\lambda_i = \lambda \restriction (C^{**}_i \setminus B)$. Now

$\boxdot$ $(B, C^{**}_i, \lambda_i) \in \Xi(B, C^{**}_i)$.

[Why? Easy.]

Case 1: For some i and an equivalence relation λ_i on $D_i \setminus B$, clauses (d) and (e) are vacuous for λ_i. Let λ_i^* be the set of pairs (x,y) from $C^{**} \setminus B$, where $C^{**} = \bigcup_{d \in D} C''_d$, which satisfy (α) or (β), where

(α) $x, y \in C_i^{**} \setminus B$ and $x \lambda_i y$,

(β) for some $d \in D$ we have $x \in C^{**} \setminus C_i^{**}, x \in C''_d$, $y \in C_i^{**} \cap C''_d$ and there is a sequence $\langle y_j : j \leq j^* \rangle$, $j^* \geq 1$, such that $y_{j^*} = x$, $y_j \in C''_d$, $y_0 = y$, $\{y_j, y_{j+1}\}$ is an edge of M and $[j > 0 \Rightarrow y_j \notin C_i^{**}]$.

This in general is not an equivalence relation. Let

$$C^{\otimes} = \{x : \text{for some } (x_1, x_2) \in \lambda_i^* \text{ we have } x \in \{x_1, x_2\}\},$$

$$\lambda_i^+ = \{(x_1, x_2) : \text{for some } y_1, y_2 \in D_i \text{ we have } y_1 \lambda y_2, (x_1, y_1) \in \lambda_i^*, (x_2, y_2) \in \lambda_i^*\}.$$

Now

$(*)_1$ λ_i^+ is a set of pairs from $C^{\otimes}$ with $\lambda_i^+ \restriction D_i = \lambda_i$.

$(*)_2$ $x \in C^{\otimes} \Rightarrow (x,x) \in \lambda_i^+$.

[Why? Read (α) or (β) and the choice of λ_i^+.]

$(*)_3$ For every $x \in C^{\otimes}$, for some $y \in C_i^{**}$ we have $x \lambda_i^* y$.

[Why? Read the choice of λ_i^+, λ_i^*.]

$(*)_4$ λ_i^+ is a symmetric relation on $C^{\otimes}$.

[Why? Read the definition of λ_i^+ recalling λ is symmetric.]

$(*)_5$ λ_i^+ is transitive.

[Why? Looking at the choice of λ_i^* this is reduced to the case excluded in $(*)_6$ below.]

$(*)_6$ If $(x, y_1), (x, y_2) \in \lambda_i^*, \{y_1, y_2\} \subseteq D_i, x \notin D_i$, then $y_1 \lambda y_2$.

[Why? Because clause (e) in the choice of D_{i+1} is vacuous. More fully, otherwise possibility $\otimes^4_{\lambda, d_1, d_2}$ holds for λ_i.]

$(*)_7$ For every $x \in C^{**} \setminus C_i^{**}$, clause (β) applies to $x \in C^{\otimes}$, that is, $C^{\otimes} = C^{**}$.

[Why? As $x \in C^{**}$ there is $d \in D$ such that $x \in C''_d$, hence by $\otimes^1_{\lambda,d}$ of clause (d) of the choice of D_{i+1} holds for x, so is not vacuous, contradicting the assumption on i in the present case.]

$(*)_8$ λ_i^+ is an equivalence relation on $C^{**} \setminus B$.

[Why? Its domain is $C^{**} \setminus B$ by $(*)_7$, it is an equivalence relation on its domain by $(*)_1 + (*)_2 + (*)_4 + (*)_5$.]

$(*)_9$ $\lambda_i^+ \restriction C_i^{**} = \lambda_i$.

[Why? By the choice of λ_i^+, that is, by $(*)_1$.]

$(*)_{10}$ Every λ_i^+-equivalence class is represented in C_i^{**}.

[Why? By the choice of λ_i^+ and λ_i^*.]

$(*)_{11}$ If $x_1, x_2 \in C^{**} \setminus B$ and $\neg(x_1 \lambda_i^+ x_2)$ but $\{x_1, x_2\}$ is an edge then $\{x_1, x_2\} \subseteq C_i^{**}$.

[Why? Assume $\{x_1, x_2\}$ is a counterexample, so $\{x_1, x_2\} \nsubseteq C_i^{**}$; assume $x_1 \notin C_i^{**}$. Now for $l = 1, 2$ if $x_l \notin C_i^{**}$ then we can choose $d_l \in D_i$ and $y_l \in C''_{d_l} \cap C_i^{**}$ such that d witnesses that $(x_l, y_l) \in \lambda_i^*$, that is, as in clause (β) there is a path $\langle y_{l,0}, \ldots, y_{l,j_l} \rangle$ such that $y_{l,0} = y_l, y_{l,j_l} = x_l$ and $(j > 0 \Rightarrow y_{l,j} \notin C_i^{**})$.

We separate into cases:

(A) $x_1, x_2 \notin C_i^{**}$, $d_1 = d_2$. This case cannot happen as $\otimes^3_{\lambda,d_1}$ of clause (d) is vacuous.

(B) $x_1, x_2 \notin C_i^{**}, d_1 \neq d_2$. In this case by the vacuousness of $\otimes^5_{\lambda_i,d_1,d_2}$ of clause (e) we get a contradiction.

(C) $x_1 \in C''_{d_l}$ and $x_2 \in C_i^{**}$. By the vacuousness of $\otimes^2_{\lambda_i,d_1}$ of clause (d) we get a contradiction.

Altogether we have proved $(*)_{11}$.]

As $\lambda_i \in \Xi(B, C_i^{**})$, by $(*)_8 + (*)_9 + (*)_{10} + (*)_{11}$ and $\boxdot$, it follows easily that $\lambda_i^+ \in \Xi(B, C^{**})$, hence (see 4.16) $B <_s^* C^{**}$, so as $B \subseteq D \subseteq C^{**}$ we have $B <_s^* D$, the desired contradiction.

Case 2: For every $i < i(*)$, at least one of the clauses (d), (e) is non-vacuous for λ_i. Let $\mathbf{w}_i = \mathbf{w}_{\lambda_i}(B, C_i^{**})$. For each i let $\langle d_{i,j} : j < j_i \rangle$ list $D_{i+1} \setminus D_i$, such that: if clause (d) applies to λ_i then $d_{i,0}$ form a witness and if clause (e) applies to λ_i then $d_{i,0}$, $d_{i,1}$ form a witness. For $j \leq j_i$ let $C_{i,j}^{**} = C_i^{**} \cup \bigcup_{s<j} C''_{d_{i,s}}$, so $C_{i,0}^{**} = C_i^{**}$ and $C_{i,j_i}^{**} = C_{i+1}^{**}$. Let $\mathbf{w}_{i,j} = \mathbf{w}_{\lambda_i}(B, C_{i,j}^{**})$.

So it suffices to prove:

(A) $\mathbf{w}_{i,j} \geq \mathbf{w}_{i,j+1}$,

(B) $\mathbf{w}_{i,0} - \varepsilon^* \geq \mathbf{w}_{i,1}$ or $\mathbf{w}_{i,1} - \varepsilon^* \geq \mathbf{w}_{i,2}$.

Let $i < i(*)$ and $j < j_i$. Clearly $C_{i,j+1}^{**} \setminus C_{i,j}^{**} \subseteq C''_{d_{i,j}} \subseteq C_{i,j+1}^{**}$. Let

$$A_{i,j} = \{x \in C''_{d_{i,j}} : x \in B \text{ or } x/\lambda \text{ is not disjoint from } C_{i,j}^{**}\}.$$

Clearly $A_{i,j} \setminus B$ is $(\lambda \restriction C''_{d_{i,j}})$-closed, hence $A_{i,j} \leq^* C''_{d_{i,j}}$, $C''_{d_{i,j}} \setminus A_{i,j}$ is disjoint from $C_{i,j}^{**}$ and $C'_{d_{i,j}} = C_{d_{i,j}} \cap (B \cup \{b\}) \subseteq C_{i,j}^{**}$, and $C'_{d_{i,j}} \subseteq C''_{d_{i,j}}$. Hence $C'_{d_{i,j}} \subseteq A_{i,j}$ and $A_{i,j} \leq^* C''_{d_{i,j}}$, but $C'_{d_{i,j}} \leq_i C''_{d_{i,j}}$, so $A_{i,j} \leq_i^* C''_{d_{i,j}}$.

Clearly

$(*)_{12}$ $\mathbf{w}_{i,j+1} = \mathbf{w}_{i,j} + \mathbf{w}_\lambda(A_{i,j}, C''_{d_{i,j}}) - \alpha \mathbf{e}^1_{i,j} - \alpha \mathbf{e}^2_{i,j}$, where

$$\mathbf{e}^1_{i,j} = |\{\{x,y\} : \{x,y\} \text{ an edge of } M, \{x,y\} \subseteq A_{i,j}, \neg(x\lambda y) \text{ but } \{x,y\} \nsubseteq C^{**}_{i,j}\}|,$$

$$\mathbf{e}^2_{i,j} = |\{\{x,y\} : \{x,y\} \text{ an edge of } M,\ x \in C''_{d_{i,j}} \setminus C^{**}_{i,j}, y \in C^{**}_{i,j} \setminus C''_{d_{i,j}} \text{ but } \neg(x\lambda y)\}|.$$

Note

$(*)_{13}$ $\mathbf{w}_\lambda(A_{i,j}, C''_{d_{i,j}})$ can be zero if $A_{i,j} = C''_{d_{i,j}}$ and is $\le -\varepsilon^*$ otherwise.

[Why? As $A_{i,j} \le^*_i C''_{d_{i,j}}$.]

$(*)_{14}$ In clause (d), $\otimes^1_{\lambda,d}$ never occurs.

[Why? If $x \in C''_d$ is as there, let $Y = \{y \in C''_d : y, x$ are connected in $M{\restriction}C''_d\}$. So $x \in Y \subseteq C''_d$ and $Y \cap C^{**}_i = \emptyset$, and $C'_d = C''_d \cap (B \cup \{b\}) = C''_d \cap C^{**}_0 \subseteq C^{**}_i$. Hence $(C''_d \setminus Y) <^*_i C''_d$, but the equivalence relation $\{(y', y'') : y', y'' \in Y\}$ exemplifies that this fails.]

Proof of (A). Easy by $(*)_{12}$, because $\mathbf{w}_\lambda(A_{i,j}, C''_{d_{i,j}}) \le 0$ holds by $(*)_{13}$, $-\alpha\mathbf{e}^1_{i,j} \le 0$, and $-\alpha\mathbf{e}^2_{i,j} \le 0$ as $\mathbf{e}^1_{i,j}$, $\mathbf{e}^2_{i,j}$ are natural numbers.

Proof of (B). It suffices to prove that $\mathbf{w}_{i,0} \ne \mathbf{w}_{i,1}$ or $\mathbf{w}_{i,1} \ne \mathbf{w}_{i,2}$ (as inequality implies the right order (by clause (A)) and the difference is $\ge \varepsilon^*$ by definition of ε^* (if $\mathbf{w}_\lambda(A_{i,1}, C''_{d_{i,j}}) \ne 0$) and $\ge \alpha$ (if $\mathbf{e}^1_{i,j} \ne 0$ or $\mathbf{e}^2_{i,j} \ne 0$). But if $\mathbf{w}_{i,0} = \mathbf{w}_{i,1}$, recalling $(*)_{14}$ it follows easily that clause (d) does not apply to λ_i, and if $\mathbf{w}_{i,0} = \mathbf{w}_{i,1} = \mathbf{w}_{i,2}$ also clause (e) does not apply.

So (A), (B) hold, so does Case 2 and hence the claim. $\blacksquare_{6.11}$

6.14. Remark. (a) We could use smaller k^* by building a tree $\langle (D_t, D^+_t, C_t, \lambda_t) : t \in T\rangle$, where T is a finite tree with a root Λ, $D_\Lambda = \emptyset$, $D^+_\Lambda = B \cup \{b\}$, each λ_t is an equivalence relation on $C_t \setminus B$ and $C_t = \bigcup\{C''_d : d \in D_t\} \cup B$, $s \in \mathrm{suc}_T(t) \Rightarrow D^+_t = D_s$ and $D^+_t \setminus D_t$ is $\{d\}$ or $\{d_1, d_2\}$, witnessing clause (d) or clause (e) for (D_t, λ_t) when $t \ne \Lambda$ and

$$\{(D_s, \lambda_s) : s \in \mathrm{suc}_T(t)\} = \{(D^+_t, \lambda) : \lambda{\restriction}D_t = \lambda_t,\ \lambda \text{ an equivalence relation on } D^+_t \setminus B\}.$$

(b) We can make the argument separated, that is, prove as a separate claim that for any k and l there is k^* such that: if $A, B \subseteq M \in \mathscr{K}$, $|B|, |A| \le l$, $B \subseteq B^*$, $\mathrm{cl}^k(A, M) \setminus \mathrm{cl}^k(B, M) \subseteq B^* \setminus B \subseteq \mathrm{cl}^k(A, M)$ and $(\forall C)(B \subseteq C \subseteq B^* \wedge |C| \le k^* \Rightarrow B <_s C)$ then $B <_s B^*$. This is a kind of compactness.

6.15. Conclusion. *Requirements* (A) *of* [I, 2.13(1)] *and even* (B) + (C) *of* [I, 2.13(3)] *hold.*

Proof. Requirement (B) of [I, 2.13(3)] holds by 6.7. Requirement (A) of [I, 2.13(2)] holds by 6.11 (and the previous sentence). $\blacksquare_{6.15}$

6.16. Conclusion. (a) *$\mathfrak{K}$ is smooth and transitive and local and transparent.*

(b) *$\mathfrak{K}$ is simply nice (hence simply almost nice).*

(c) *$\mathfrak{K}$ satisfies the 0-1 law.*

Proof. (a) By 6.4.

(b) By 6.15 we know that $\mathfrak{K}$ is simply nice.

(c) By 4.2 we know that for each k, for every random enough $\mathscr{M}_n$, $\mathrm{cl}^k(\emptyset, \mathscr{M}_n)$ is empty. Hence by [I, 2.19(1)] we get the desired conclusion. $\blacksquare_{6.16}$

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