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MONADIC LOGIC AND LÖWENHEIM NUMBERS

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We investigate the monadic logic of trees with $\omega+1$ levels, the monadic topology of the product space ${}^{\omega}\lambda$ and a strengthening of monadic logic for trees with ω levels.

0. Introduction

This work continues two lines of research. The first is of the analysis of the complexity of the monadic theory of linear orders, the monadic theory of trees and the monadic topology (see Shelah [8, §7], Gurevich [4], Gurevich and Shelah [5, 6, 7], and Gurevich [3]).

Here we use similar methods and prove analogous results for more cases (first-countable topological spaces like the product topology on ${}^{\omega}\lambda$ and the tree ${}^{\omega>}\lambda$ with a quantification that is slightly stronger than the monadic one). We shall explain this in the introduction to each section. Now we shall deal with the second line. In Baldwin–Shelah [1] we analyzed the complexity of the $L(Q)$-theory of the class of models of T, where Q is a first-order definable second-order quantifier and T is first-order. The classification showed the naturality of monadic logic, and in particular naturality of the monadic theories of linear orders and trees. It was almost proved there that those are the only interesting cases, i.e. any essentially different case is either too complicated (i.e., we can interpret in the $L(Q)$-theory of T second-order logic in a model-theoretic way) or the model theory of the $L(Q)$-theory of T is trivial. However, there were few gaps in this picture. One of them was about unstable theories is dealt with in [9].

We deal here with the two other gaps. First for some theory T, in some monadic expansion we can interpret the tree ${}^{\omega\geq}\lambda$; but T is stable not superstable hence by [1] we cannot interpret general trees. So the monadic theory of the class of models of T is approximately as complicated as that of the monadic theory of $\{{}^{\omega\geq}\lambda : \lambda \text{ is a cardinal}\}$. In the second section we show that this class has a complicated monadic theory: if $V=L$ it is as complicated as second-order logic.

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So the Löwenheim number of the monadic theory of T is large, but not the Hanf number. Second, for some superstable T in the monadic theory of models of T we can interpret fragments of the $L_{\omega_1,\omega}$(Mon)-theory of the tree ${}^{\omega>}\lambda$. We prove in Section 1 that under $V=L$ we can interpret there the second-order theory of λ. In fact, instead of dealing with infinitary languages we deal with the following quantifier on trees ($Q^{\text{pd}}f$) meaning "there is a function f, $f(x)\leqslant x$ such that ...". As in our trees f is determined by $\langle P_l^f : l< \rangle$, where $P_l^f=\{x : f(x)$ is in the lth level$\}$, for infinitary languages there is no difference between monadic logic and the logic with the new quantifiers (for suitable trees). Again we can deduce some results on the Löwenheim numbers.

1. Boolean interpretation in the tree ${}^{\omega>}\lambda$

The standard way to prove that a theory T_1 is at least as complicated as T_2 (i.e., the Turing degrees of T_1 is $\geqslant$ the Turing degree of T_2) is to find a first-order interpretation of the models of T_2 in models of T_1.

In [8] we have proved that the monadic theory of the real order is complicated but our interpretation was not standard. (We used there CH but this was eliminated in Gurevich and Shelah [5].) Gurevich [4] saw that really only the topology was used. In Gurevich and Shelah [6] the meaning of our interpretation was clarified. We interpreted not a usual model, but a Boolean-valued model where the Boolean algebra was the completion of the Boolean algebra of regular open subsets of the real line. In particular it is possible to interpret in the monadic theory of the real line the second-order theory of $\aleph_0$ in the world resulting from Cohen forcing.

It is known that the monadic theory of the tree ${}^{\omega>}\lambda$ is decidable (see a generalization of Rabin's theorem due to Shelah and Stup, see [8, Theorems 0.4, 0.5, 0.6]). However we consider here a stronger quantifier ($Q^{\text{pd}}f$) ranging over one-place functions f satisfying $f(x)\leqslant x$. We generalize easily the results of [6] for the new quantifier. Here the Boolean algebra involved is that of the standard collapse of λ. Now if $V=L$ is the universe, then after the collapse we can reconstruct easily the old universe. In this case in the $L(Q^{\text{pd}})$-theory of ${}^{\omega>}\lambda$ we can interpret the second-order theory of λ.

1.1. Definition. (1) Let K_{tr} be the class of trees $T=(T,\leqslant)$, i.e., partial order, so that for each $x\in T$, $\{y : y<x\}$ is linearly well ordered, with order-type being called the level of x, lev(x). Let

$$T_\alpha=\{x\in T : \text{lev}(x)=\alpha\},\qquad K_{\text{tr}}^{\alpha}=\left\{T : T=\bigcup_{\beta<\alpha} T_\beta\right\}.$$

(2) We look at ${}^{\beta>}\alpha$ as a tree, by the natural ordering: being an initial segment.

(3) We call T endless if $(\forall x\in T)(\exists y\in T)[x<y]$.

1.2. Definition. For trees T, the quantifier $(Q^{\text{pd}}f)$ ranges over partial functions f such that $f(x) \leqslant x$ for $x \in \operatorname{Dom} f$. We call such f a p.d. function.

Note that by their domain they give monadic quantification. Let W, X, Y, Z be monadic variables.

1.3. Notation. (1) A basic open subset of T is $T_{\geqslant x} = \{y : y \geqslant x\}$ for $x \in T$.

(2) An open set is a union of basic open sets (so the topology is not Hausdorff).

(3) A subset of T is dense if it is not disjoint to any basic open set.

(4) A subset A of T is a *full subtree* if $A \neq \emptyset$, A is downward closed $(y < x \wedge x \in A \rightarrow y \in A)$, for every $x \in A$, $|\mathrm{Suc}_T(x) \cap A| = 1$ or $\mathrm{Suc}_T(x) \subseteq A$, (where $\mathrm{Suc}_T(x)$ is the set of immediate successors of x) and for every $x \in A$ for some y, $x < y \in A$, $\mathrm{Suc}_T(y) \subseteq A$.

1.3A. Convention. A model in this section is an *endless* $(T, \leqslant) \in K^{\omega}_{\text{tr}}$ expanded by some p.d. functions (hence monadic predicates).

1.4. Claim. *The $L(Q^{\text{pd}})$-theories of K^{ω}_{tr} and $\{{}^{\omega>}\lambda : \lambda\}$ are recursive one in the other.*

1.5. Lemma. *There is a formula $\phi(X, f) \in L_{\omega,\omega}$ such that if $T = (T, \leqslant) \in K_{\text{tr}}$, and P_i $(i < \alpha)$ are pairwise disjoint dense subsets of T, then for some p.d. function f, and for every $X \subseteq T$,*

$$T \models \phi[X, f] \quad \textit{iff} \quad \bigcup\{T_{\geqslant x} : (\exists i < \alpha)((T_{\geqslant x} \cap X = T_{\geqslant x} \cap P_i)\} \quad \textit{is dense and open.}$$

Proof. Define f: if $x \in P_i$, then $f(x)$ is the minimal $y \leqslant x$ such that $y \in P_i$; if $x \notin \bigcup_{i<\alpha} P_i$, $f(x)$ is not defined.

$\phi(X, f)$ says: for every x there is $y > x$, such that for every $z \geqslant y$, $[f(z) = f(y) \Leftrightarrow z \in X]$.

1.6. Lemma. *For $k < \omega$ there is a formula $\psi(X_0, \ldots, X_{k-1}, \bar{f}) \in L(Q^{\text{pd}})$ (or even in $L(\text{Mon})$) such that: for every $T \in K^{\omega}_{\text{tr}}$ and $\{\bar{P}^i : i < \alpha\}$ where $(\forall x \in T)$ $(|T_{\geqslant x}| \geqslant |\alpha|)$, $\bar{P}^i = \langle P^i_l : l < k \rangle$, P^i_l a subset of T, there is $\bar{f}'$ (a sequence of p.d. functions for T of the length of $\bar{f}$) such that: for any $A_0, \ldots, A_{k-1} \subseteq T$, $T \models \psi[A_0, \ldots, A_{k-1}, \bar{f}']$ iff $\bigcup\{T_{\geqslant x}$: for some $i < \alpha$ for each $l < k$, $A_l \cap T_{\geqslant x} = P^i_l \cap T_{\geqslant x}\}$ is dense.*

Proof. We can find functions $g^i_l : T \rightarrow T$ (for $i < \alpha, l < k$) such that $(\forall x \in T)(g^i_l(x) > x)$, each g^i_l is one-to-one, $g^i_l(x) \leqslant g^i_m(x)$ for $l < m < k$ and their ranges are pairwise disjoint. [Let $A' = \{x \in T$: $(\forall y)(x \leq y \in T \rightarrow |T_{\leqslant x}| = |T_{\leqslant y}|\}$, $A'' = \{x \in A' : (\forall y < x)(y \notin A')\}$, now define the g^i_l on each $T_{\geqslant x}$ $(x \in A'')$ separately so that infinitely many pairwise comparable elements of $T_{\geqslant x}$ are in the range of no g^i_l.]

We now define a partial p.d. function h_l, h^m (for $l < k$, $m < k$): $h^m(g^i_{m+1}(x)) = g^i_m(x)$ and $h_l(g^i_l(x)) = x$ (as the ranges of the g^i_l $(i < \alpha)$ are pairwise disjoint, h_l and

h^m are well-defined functions). Let $Q^i_l = \{g^i_l(x) : x \in P^i_l\}$. Let $\bar{h} = \langle h_l, h^l : l < k\rangle$, and let g_l be such that for every $X \subseteq T$: $T \models \varphi[X, g_l]$ iff $\bigcup \{T_{\geq x} : (\exists i < \alpha)(T_{\geq x} \cap X = T_{\geq x} \cap Q^i_l)\}$ is dense and open. (The formula φ is from 1.5, and g_l exists by 1.5.)

Let $\psi(X_0, \ldots, X_{k-1}, h_0, \ldots, h_{k-1}, g_0, \ldots, g_{k+1}, h^0, \ldots, h^{k-2})$ say: there are $Y_0, \ldots, Y_{k-1}$ s.t. $\bigwedge_{l<k} \varphi(Y_l, g_l)$ and

$$\bigwedge_{l<k} (\forall z)(z \in Y_l \equiv h_l(x) \in X_l) \wedge \bigwedge_{l<k-1} (\forall z)(z \in Y_{l+1} \equiv h^l(z) \in Y_l)$$

$$\wedge \bigwedge_{l<k-1} (\forall z)(\exists z_1)(z \in Y_l \to z_1 \in Y_{l+1} \wedge h^l(z_1) = z).$$

1.7. Notation. (1) u will vary on basic open sets (for a relevant T).

(2) $\mathrm{Val}_T \theta(u, X_1, \ldots, f_1, \ldots) = \bigcup \{u : T \models \theta(u, X_1, \ldots, f_1, \ldots)\}$; we omit the subscript T when its identity is clear.

(3) We say $X \equiv Y$ if $(X - Y) \cup (Y - X)$ is a nowhere dense set (this is an equivalence relation).

1.8. Lemma. *There are $\partial(u, X, \bar{f})$, $\partial_+(u, X, Y, Z, \bar{f})$, $\partial_\times(u, X, Y, Z, \bar{f})$ such that for any T and pairwise disjoint dense $P_n \subseteq T$ $(n < \omega)$ there is an $\bar{f}$ such that*:

(a) *For $X \subseteq T$*

$$\mathrm{val}\, \partial(u, X, \bar{f}) \equiv \mathrm{val}\Big(\bigvee_n [P_n \cap u = X \cap u]\Big).$$

(b) *For $X, Y, Z \subseteq T$*

$$\mathrm{val}\, \partial_+(u, X, Y, Z, \bar{f})$$
$$\equiv \mathrm{val}\Big(\bigvee_{n+m=k} [P_n \cap u = X \cap u \wedge P_m \cap u = Y \cap u \wedge P_k \cap u = Z \cap u]\Big).$$

(c) *For $X, Y, Z \subseteq T$*

$$\mathrm{val}\, \partial_\times(u, X, Y, Z, \bar{f})$$
$$\equiv \mathrm{val}\Big(\bigvee_{nm=k} [P_n \cap u = X \cap u \wedge P_m \cap u = Y \cap u \wedge P_k \cap u = Z \cap u)\Big).$$

1.8A Remark. In (a), (b), (c) inside the right-side formulas we can replace = by ≡.

Proof. By 1.6 (as T is endless).

1.9 Lemma. *There is a formula $\partial^*(\bar{f}) \in L(Q^{\mathrm{pd}})$ such that: $T \models \partial^*(\bar{f})$ iff for some P_n $(n < \omega)$ the conclusion of 1.8 is satisfied.*

Proof. Straightforward. Let ∂^*_1 tell the obvious properties and then say $\{X/\equiv :$ val $\partial(u, X, \bar{f}) \equiv T\}$ is minimal among $\{\{X/\equiv : \mathrm{val}\, \partial(u, X, \bar{g}) = T\} : T \models \partial^*_1(\bar{g})$, for $\bar{g}$ and $\bar{f}$ we have the same "one", "zero" and successor operations when defined$\}$.

1.10. Definition. Let P_n $(n<\omega)$, $\bar{f}$ be as in 1.8. We can consider T as a forcing notion.

(1) If $\mathbf{A}$ is a T-name of a set of natural numbers, let

$$R[\mathbf{A}]=\{x\in T: \text{for some } y\leqslant x \text{ and } n,\ y\Vdash_T \text{“}n\in\mathbf{A}\text{” and } x\in P_n\}.$$

(2) If $X\subseteq T$ let $N(X)$ be the following T-name of a set of natural numbers: for $x\in T$,

$$x\Vdash \text{“}n\in N(X)\text{”} \quad \text{iff} \quad (T_{\geqslant x})\text{-val}(P_n\cap u\subseteq X) \text{ is nowhere dense.}$$

1.11. Fact. *In the context of* 1.10:

(1) $\Vdash$“$\mathbf{A}=N(R[\mathbf{A}])$”, *and if* $\Vdash_T$“$\mathbf{A}_1=\mathbf{A}_2$” *then* $R(\mathbf{A}_1)=R(\mathbf{A}_2)$.

(2) $R[\mathbf{A}]\equiv R[N(R[\mathbf{A}])]$, *and if* $X_1\equiv X_2$ $(X_1,X_2\subseteq T)$ *then* $\Vdash_T$“$N(X_1)=N(X_2)$”.

1.12. Lemma. *For every formula* $\phi(X_1,\ldots,X_n,y_1,\ldots,y_m)$ *in second-order number theory we can recursively compute a formula*

$$\phi^+(u,X_1,\ldots,X_n,Y_1,\ldots,Y_m)$$

such that the following holds: *(for* $P_n(n<\omega)$, $\bar{f}$ *as in* 1.8)

(*) *For any* T*-names* $\mathbf{A}_1,\ldots,\mathbf{A}_n$ *of sets of natural numbers, and natural numbers* $k_1,\ldots,k_m$:

$$\bigcup\{T_{\geqslant x}: x\Vdash \text{“}\phi(\mathbf{A}_1,\ldots,\mathbf{A}_n,k_1,\ldots,k_n)\text{”}\}$$
$$\equiv \text{val}\ \phi^+(u,R[\mathbf{A}_1],\ldots,R[\mathbf{A}_n],P_{k_1},\ldots,P_{k_m}).$$

Proof. By straightforward induction on ϕ.

From 1.12 and 1.8, 1.9 we get:

1.13. Conclusion. *For every sentence* ∂ *in second-order logic we can compute a sentence* ∂^* *in* $L_{\omega,\omega}(Q^{\text{pd}})$ *(tree's language) such that*:

$$({}^{\omega>}\lambda,<)\Vdash\partial^* \quad \textit{iff} \quad \Vdash_{\text{col}(\aleph_0,\lambda)}\text{“}(\alpha,<)\models\partial \textit{ for some } \alpha<\lambda^+.\text{”}$$

1.14. Conclusion. $(V=L)$. *Second-order logic and* $L(Q^{\text{pd}})$*-theory of* K^{ω}_{tr} *are bi-interpretable and have the same Löwenheim number.*

1.15. Conclusion. *If* T *has finite language,* T *is stable but not superstable, then* $K(^{\omega}_{\text{tr}},Q^{\text{pd}})\leqslant(\text{T},\text{mon})$ *(hence the conclusions on Löwenheim numbers).*

Proof. See [1, VII 2.1, 2.2].

1.16. Claim. *There is a superstable* T *with finite language,* $(T_\infty,\text{2nd})\not\leqslant(T,\text{mon})$, T *has nice decomposition but* $(K^{\omega}_{\text{tr}},Q^{\text{pd}})\leqslant(T,\text{mon})$.

Proof. Let

$|M| = \{(\eta, k)$: for some $m < \omega$, $\eta \in {}^m\lambda$, and $(\exists l)\ k < l^2 \leq m\}$.

G^M a one-place partial function, $G((\eta, k)) = (\eta, k+1)$.

F^M a one-place partial function,

$$F((\eta,k)) = \begin{cases} (\eta, k) & \text{if } l(\eta) = 0, \\ (\eta \restriction (m-1), k) & \text{if } l(n) = m,\ k < m,\ \exists l\ (k = l^2), \\ (\eta \restriction (m-1), k-1) & \text{if } l(\eta) = m,\ k = m,\ \exists l\ (k = l^2). \end{cases}$$

T will be Th(M).

1.17. Claim. If $\lambda \geq \aleph_1$, $\mu \geq \aleph_0$, then every formula in $L_{\lambda,\mu}(Q^{pd})$ is equivalent to some formula in $L_{\lambda,\mu}(\text{Mon})$ (if we restrict ourselves to e.g. endless trees in K_{tr}^{ω}). So the Löwenheim number of those logics are equal.

2. Boolean interpretation in the trees or topologies ${}^{\omega}\lambda$

We know that the class of linear orders has a very complicated monadic theory: by Gurevich and Shelah [7] under a weak set theoretic hypothesis (there are arbitrarily large cardinals λ with $\lambda = \lambda^{<\lambda}$), the monadic theory of linear orders and second-order logic are bi-interpretable (hence have the same Turing degree). We also know, by similar methods, that this holds for the class of trees and for the class of topologies (see below), (see [7]). As the results for topologies imply easily the rest, we shall concentrate on the monadic topology for a topological space X in the following first-order structure M_X. The universe is the family of all subsets of X, and the relations are the inclusion, and a unary relation for being a closed set. We shall write $X \vDash \phi$ instead $M_x \vDash \phi$. Let K denote a class of topological spaces.

Now in [7], the topological spaces, that played the central role in the proof, were those of a quite saturated linear orders. However, many interesting classes of topological spaces consist of first-countable spaces only, hence the proof of [7] is not applicable. We shall interpret second-order logic for certain classes of that sort, but at some price, e.g., assuming $V = L$. Note that set-theoretic hypothesis occur twice. First, as in Section 1, we interpret in monadic theory of ${}^{\omega}\lambda$ second-order logic of $\aleph_0$ of the universe after the standard collapse of λ. Secondly we need the continuum hypothesis to carry the combinatorial argument. In our proof the central role, fulfilled by saturated linear orders in [7], is fulfilled here by the product topology on ${}^{\omega}\lambda$.

A reader may ask why we should concentrate on the case ${}^{\omega}\lambda$ and not on ${}^{\kappa}\lambda$ for, say, $\kappa = \omega_1$. Our only reason is that this case has applications whereas for other cases there are no obvious applications. If we want to deal with the case of ${}^{\kappa}\lambda$ (and (Pr_κ^1), (Pr_κ^2), see 2.5) we should consider κ-distributively as in [7].

For history of the subject (Boolean interpretations, modesty etc.) see [3]. We could use other topologies, e.g., the space ${}^{\omega\geq}\lambda$, where the basic open sets are $A^k_\eta=\{\eta\restriction\alpha : k<\alpha\leq\omega\}$. Our methods apply with minor changes to these topologies.

2.1. Definition. (1) We consider λ as the discrete topological space with $\lambda=\{\alpha:\alpha<\lambda\}$ as set of points.

(2) $\prod_{n<\omega}\lambda_n=\{\eta : l(\eta)=\omega,\ \eta(n)<\lambda_n\}$ is considered as the product topological space (the basic open sets are $B_\nu=\{\eta\in\prod_{n<\omega}\lambda_n : \nu\triangleleft\eta\}$ for $\nu\in{}^{\omega>}\lambda$). Note ${}^\omega\lambda=\prod_{n<\omega}\lambda$; similarly for ${}^\kappa\lambda$.

(3) K^H_λ is the class of topological spaces, with no isolated points satisfying: *if* $\alpha\leq\lambda$, and for $l=1,2$, $\langle\mathcal{D}^l_i : i<\alpha\rangle$ is a sequence of pairwise disjoint dense subsets of X, then for some autohomomorphism G of X, for every i, $G(\mathcal{D}^1_i)\cap\mathcal{D}^2_i$ is a dense subset of X.

(4) For a topological space X let $Q[X]$ be the following forcing notion: the open subsets of X ordered by inverse inclusion.

(5) A set $\mathcal{D}\subseteq X$ is p-modest in X, if for any $Z_1,\ldots,Z_p\subseteq\mathcal{D}$ s.t. each Z_m is dense in $\bigcup^p_{l=1}Z_l$, there is a perfect subset P of X, $P\cap\mathcal{D}\subseteq\bigcup^p_{l=1}Z_l$ and $P\cap Z_m$ is dense in P for $m=1,\ldots,p$.

(6) We let u vary on open sets, and define (as in 1.7) $\mathrm{val}_X\,\theta(u, Y_1, Y_2,\ldots)=\bigcup\{u : X\models\theta[u, Y_1,\ldots]\}$,

$$Y\equiv Z \quad\text{iff}\quad (Y-Z)\cup(Z-Y) \quad\text{is nowhere dense.}$$

(7) We call $P\subseteq{}^\kappa\lambda$ *perfect* if

(a) P is not empty.

(b) For every $\eta\in P$ and $\alpha<\kappa$, there is $\nu\in P$, $\nu\neq\eta$, $\nu\restriction\alpha=\eta\restriction\alpha$.

(c) For every limit $\delta<\kappa$, and $\nu\in{}^\delta\lambda$ if $(\forall i<\delta)(\exists\eta\in P)(\nu\restriction i=\eta\restriction i)$, then $(\exists\eta\in P)(\nu=\eta\restriction\delta)$.

Note that for $\kappa=\omega$, (c) says nothing and then this is the usual notion of perfect; but even for $\kappa>\aleph_0$, our "perfect" is definable (in M_X, $X={}^\kappa\lambda$) if we expand the model by the lexicographic order on ${}^\kappa\lambda$; for this it suffices to replace X by its completion Y (as a linear order) and expand M_Y by a predicate for X.

2.2. Remark. We shall show that for some suitable X's, we can interpret what we want. However, this is enough to prove undecidability of the monadic topological theory of K, but not the stronger results we want. So we shall need to show that the subclass of suitable X's is definable, or restrict ourselves to this class.

Note that in the first section, this point is trivial: We can easily define the class of endless trees with $\leq\omega$ levels, but such problem was dealt with in [7] (and was a major obstacle).

2.3. Fact. (1) *For any closed* $C\subseteq{}^\omega\lambda$ *satisfying*:

$$(*)\qquad [\nu\in{}^{\omega>}\lambda\wedge C\cap B_\nu\neq\emptyset\rightarrow C\cap B_\nu \textit{ has density character } \mu],$$

C is homeomorphism to ${}^\omega\mu$.

(2) *For any closed* $C \subseteq {}^{\omega}\lambda$, $\bigcup\{u: u$ *an open subset of* C, $C \cap u$ *is homeomorphic to some* ${}^{\omega}\mu\}$ *is a dense subset of* C.

(3) ${}^{\omega}\lambda \in K_{\lambda}^{H}$, *and* ${}^{\kappa}\lambda \in K_{\lambda}^{H}$.

(4) $\{\eta \in {}^{\kappa}\lambda : \eta$ *is eventually constant*$\}$ *is* p*-modest in* ${}^{\kappa}\lambda$ (*for* $p < \omega$).

Proof. Easy.

2.4. Definition. Let the formula $\phi(u, Y, \mathscr{D}, W)$ say:

(a) $\mathscr{D} \cap u$ is a dense subset of u which contains no perfect set and is 1-modest in X; and $Y \subseteq \mathscr{D}$.

(b) If P is perfect, $P \cap \mathscr{D}$ dense in P, $P\text{-}\mathscr{D} \subseteq W$, then there is an open $u' \subseteq u$, $u' \cap P \neq \emptyset$ and $u' \cap P \subseteq Y$ or $u' \cap P \cap Y = \emptyset$.

2.5. Definition. (1) We say X has $(\mathrm{Pr}_{\lambda}^{1})$ if: *for every* $\mathscr{D}$, $\alpha \leqslant \lambda$, $\mathscr{D}$ a dense 1-modest subset of X containing no perfect subset of X and dense pairwise disjoint $\mathscr{D}_i \subseteq \mathscr{D}$ for $i < \alpha$ *for some* W, for every $Y \subseteq X$

$$\mathrm{val}(\phi(u, Y, \mathscr{D}, W)) \equiv \mathrm{val}\Big(\bigvee_{i<\alpha} [Y \cap u = \mathscr{D}_i \cap u]\Big).$$

(2) We say X has $(\mathrm{Pr}_{\lambda}^{2})$ if for every $\mathscr{D}$, $\mathscr{D}_i$ as above, $\mathscr{D}_i = \mathscr{D}_i^1 \cup \mathscr{D}_i^2$, each $\mathscr{D}_i^l$ dense, $\mathscr{D}_i^1 \cap \mathscr{D}_i^2 = \emptyset$ there are dense $E_i^l \subseteq \mathscr{D}_i^l$ and W s.t., for every $Y \subseteq X$

$$\mathrm{val}(\phi(u, Y, \mathscr{D}, W) \equiv \mathrm{val}\Big(\bigvee_{i<\alpha} [Y \cap u = (E_i^1 \cup E_i^2) \cap u]\Big).$$

Remark. For some of the uses of $(\mathrm{Pr}_{\lambda}^{2})$ we can weaken it (by not using the E_i^l): in 2.7 we need the stronger version, but not in the new use in 2.4: eliminating non-standard integers.

2.6. Fact. (1) *If* X *has* $(\mathrm{Pr}_{\lambda}^{1})$, *then it has* $(\mathrm{Pr}_{\lambda}^{2})$.

(2) *If* $X \in K_{\lambda}^{H}$, *then in the definition of* $(\mathrm{Pr}_{\lambda}^{2})$ *we can replace "for every* $\mathscr{D}$, $\alpha \leqslant \lambda$, $\mathscr{D}_i$ $(i < \alpha)$" *by "for every* $\alpha \leqslant \lambda$ *for some* $\mathscr{D}$, $\mathscr{D}_i$, $\mathscr{D}_i^1$, $\mathscr{D}_i^2$ $(i < \alpha)$".

Proof. Immediate.

2.7. Fact. *Suppose* X *has* $(\mathrm{Pr}_{\lambda}^{2})$, $\mathscr{D}_i \subseteq X$ $(i < \lambda)$ *are dense and pairwise disjoint* $W \subseteq X$, $\bigcup_{i<\lambda} \mathscr{D}_i = \mathscr{D}$, $\mathscr{D}$ *contains no perfect subset of* X *and is* 1*-modest and for every* $Y \subseteq \mathscr{D}$

$$\mathrm{val}(\phi(u, Y, \mathscr{D}, W)) \equiv \mathrm{val}\Big(\bigvee_{i<\lambda} [Y \cap u = \mathscr{D}_i \cap u]\Big).$$

Then for every n-place relation R on λ for some $W_R \subseteq X$, for every $Y_1, \dots, Y_n \subseteq X$

$$(*) \quad \operatorname{val}(\psi_n(u, Y_1, \dots, Y_n, \mathscr{D}, W, W_R)) \equiv \operatorname{val}\Big(\bigvee_{\langle i_1,\dots,i_n\rangle \in R} [Y_1 \cap u = \mathscr{D}_{i_1} \cap u \wedge \cdots \wedge Y_n \cap u = \mathscr{D}_{i_n} \cap u]\Big)$$

(the ψ_n depend on n only).

Proof. Easy. We can code any relation using equivalence relations. Let $\{\langle \iota_1^\xi, \dots, \iota_n^\xi\rangle : \xi < \lambda\}$ be w.l.o.g., an enumeration of R such that: $\iota_1^\xi, \dots, i_n^\xi \neq \xi$ mod$(n+1)$. Let $\mathscr{D}^l = \bigcup \{\mathscr{D}_\xi : \xi = l \text{ mod}(n+1)\}$ for $l \leq n$, and E_m^l be the equivalence relation on λ defined by: $\alpha E_m^l \gamma$ iff for some $\xi = l\,\text{mod}(n+1)$, $\{\alpha, \gamma\} = \{\xi, i_m^\xi\}$, or $\alpha = \gamma$.

We can apply Definition 2.5(2) to the family $\{\mathscr{D}_{\{\alpha,\gamma\}} : \alpha, \gamma < \lambda,\ \{\alpha, \gamma\}$ is an E_m^l-equivalence class$\}$ where $\mathscr{D}_{\{\alpha,\gamma\}} = \mathscr{D}_\alpha \cup \mathscr{D}_\gamma$ [standing instead $\{\mathscr{D}_i : i < \lambda\}$] and with $\mathscr{D}^1_{\{\alpha,\gamma\}}, \mathscr{D}^2_{\{\alpha,\gamma\}}$ being $\mathscr{D}_\alpha$, $\mathscr{D}_\gamma$ when $\alpha < \gamma$ (quite arbitrary otherwise); we get W_m^l. By W_m^l, $\mathscr{D}^l$ $(l < n+1,\ m < m+1)$ we can get that $(*)$ holds. (The phrasing of ψ_n is left as an exercise to the reader.)

2.8. Lemma. *For every sentence θ from second-order logic, we can compute θ^* s.t., for any X which have* (Pr_λ^2)

$$X \models \theta^* \quad \text{iff} \quad \Vdash_{Q[X]} \text{“}\lambda \restriction \theta\text{”}.$$

Proof. As in 1.12, 1.13 (on $Q[x]$ – see Definition 2.1(4)).

The previous discussion is somewhat empty; as, concerning our main aim, does ${}^{\omega}\lambda$ have $(\mathrm{Pr}^2_{\aleph_0})$? So now comes the main point.

2.9. The Main Lemma. *Suppose $\lambda > \kappa$ are regular cardinals and* $((\mathrm{a}) \vee (\mathrm{b})) \wedge (\mathrm{c})$ *where*

(a) GCH *and* $\lambda = \mu^+$; $\operatorname{cf} \mu \neq \kappa$,

(b) $\diamondsuit_S$ *holds where* $S = \{\delta < \lambda : \operatorname{cf} \delta = \kappa\}$,

(c) (1) $\kappa = \kappa^{<\kappa}$ *or* (2) *for every* $\eta \in \mathscr{D}$, $\eta(i) = \sup\{\eta(\gamma) : \gamma < \kappa\}$ *for arbitrarily large* $i < \kappa$.

Suppose:

(i) $p^* \leq \omega$.

(ii) $\mathscr{D} \subseteq {}^\kappa\lambda$ *contains no perfect subset of* X.

(iii) *For* $i < \lambda$, $\mathscr{D}_i \subseteq \mathscr{D}$ *is a dense subset of* X.

(iv) *For* $p < p^*$, $\mathscr{D}$ *is* p*-modest.*

Then there is a set $W \subseteq {}^\kappa\lambda - \mathscr{D}$ *such that*:

(α) *If* $i < \lambda$, $p < p^*$, *u open and* $Y_1, \dots, Y_p$ *are dense subsets of* $\mathscr{D}_i \cap u$ *(hence of* ${}^\kappa\lambda$*), then for some perfect* $P \subseteq u$, $P - \mathscr{D} \subseteq W$, *and* $Y_l \cap P$ *is dense in* P *for* $l = 1, \dots, P$.

(β) *If* $P \subseteq {}^\kappa\lambda$ *is perfect,* $P \cap \mathscr{D}$ *is dense in* P, *and* $P - \mathscr{D} \subseteq W$, *then for some* $i < \lambda$ *and* u, $u \cap P \cap \mathscr{D} \subseteq \mathscr{D}_i$ *and* $u \cap P \neq \emptyset$.

Remarks. By Gregory [2] and Shelah [10], we know that if GCH holds, $\lambda = \mu^+$, cf $\mu \neq \kappa$, then $\diamondsuit_S$ holds (in fact it suffices that $\lambda = 2^\mu$ and $\mu^\kappa = \mu \vee [\mu > \kappa \wedge (\forall \chi < \mu)\chi^\kappa \leq \mu]$. Hence we shall use (b) only.

Proof. As $\diamondsuit_S$ holds, there are a natural number p_δ, a basic open set u_δ and subsets $Y_1^\delta, \ldots, Y_{p_\delta}^\delta$ of $\bigcup_{i<\delta}({}^\kappa i) \cap u$ such that for every basic open u, natural number $p < p^*$ and subsets $Y_1, \ldots, Y_p$ of u for stationarily many $\delta \in S$, $u_\delta = u$, $p_\delta = p$ and $Y_l^\delta = Y_l \cap (\bigcup_{i<\delta} {}^\kappa i)$ for $l = 1, \ldots, p_\delta$. (We assume for simplicity $|i|^\kappa < \lambda$ for $i < \lambda$.)

We shall define for some $\delta \in S$, a perfect subset P_δ of ${}^\kappa\lambda$ (for the others $P_\delta = \emptyset$).

Case A. For each $l = 1, \ldots, p_\delta$, Y_l^δ is a dense subset of $\bigcup_{i<\delta}({}^\kappa i) \cap u_\delta$. Moreover, for a closed unbounded subset C of δ, for every $i \in C$, $l = 1, \ldots, p_\delta$, $Y_l^\delta \cap \bigcup_{j<i} {}^\kappa j$ is dense in $\bigcup_{j<i}({}^\kappa j) \cap u$ and for some $i(\delta) < \lambda$: $\bigcup_{l=1}^\delta Y_l^\delta \subseteq \mathscr{D}_{i(\delta)}$, and $p_\delta < p^*$. We now choose a perfect $P_\delta \subseteq {}^\kappa\delta \cap u_\delta$, such that:

(1) $P_\delta \cap Y_l^\delta$ is dense in P_δ for each $l = 1, \ldots, p_\delta$.

(2) $P_\delta \cap (\bigcup_{i<\delta} {}^\kappa i)$ is included in $\bigcup_{l=1}^{p_\delta} Y_l^\delta$.

(3) $P_\delta \cap \mathscr{D} \subseteq \mathscr{D}_{i(\delta)} \cap (\bigcup_{i<\delta} {}^\kappa i)$.

Let $i(\gamma)$ $(\gamma < \kappa)$ be increasing continuous, $\delta = \bigcup_{\gamma<\kappa} i(\gamma)$, such that $Y_l^\delta \cap (\bigcup_{j<i(\gamma)} {}^\kappa j)$ is dense in $\bigcup_{j<i(\gamma)}({}^\kappa j)$. We define by induction on $\gamma < \kappa$, a set $A_\gamma \subseteq {}^{\kappa \geq}(i(\gamma))$ as follows: for $\gamma = 0$, choose $\eta_l^0 \in Y_l^\delta \cap (\bigcup_{j<i(0)} {}^\kappa j)$, and let $A_0 = \{\eta_l^0 \upharpoonright \beta : \beta \leq \kappa,\ l = 1, \ldots, n_p\}$. For $\gamma < \kappa$ limit, let A_γ be $\{\eta \in {}^{\kappa\geq} i(\gamma)$: for every $\beta < l(\eta)$; $\eta \upharpoonright (\beta+1) \in \bigcup_{\beta<\gamma} A_\beta\}$. For γ successer choose for every $\nu \in A_{\gamma-1} - \bigcup_{\beta<(\gamma-1)} A_\beta, l(\nu) < \kappa$ and $l = 1, \ldots, p_\delta$ a sequence $\eta_{\nu,l}^\gamma \in Y_l^\delta$, s.t. $\nu \leq \eta_{\nu,l}^\gamma$, $(\forall \beta)[l(v) = \beta \rightarrow \eta_{\nu,l}^\gamma(\beta) > i(\gamma-1)]$ and $\eta_{\nu,l}^\gamma \in \bigcup_{j<i(\gamma)}({}^\kappa j)$. By the p-modesty of $\mathscr{D}$ in X applied to $Y_l^\delta \cap (\cup_{\gamma<\kappa} A_\gamma)$ we get P_δ as required, because

(*) if $\eta \in {}^\kappa\lambda - \mathscr{D}$ is in the closure of $\bigcup_{\gamma<\kappa} A_\gamma \cap {}^\kappa\lambda$, then $\sup\{\eta(\gamma) : \gamma < \kappa\}$ is δ (in fact, for each $i < \delta$ for a closed unbounded set of $\gamma < \kappa$, $\eta(\gamma) > i(\gamma)$).

Case B. Note Case A. Let $P_\delta = \emptyset$.

Now let $W = \bigcup\{P_\delta - \mathscr{D} : \delta \in S\}$. Let us check (α) and (β). As for (α) – this is directly guaranteed by the choice of the Y_l^δ's and that of W. So we shall work on (β). So suppose $P \subset {}^\kappa\lambda$ is perfect, $P \cap \mathscr{D}$ dense in P and $P - \mathscr{D} \subseteq W$.

Clearly $P - \mathscr{D}$ is dense in P (as $\mathscr{D}$ does not contain a perfect set). So there is a minimal i such that $\bigcup\{P_\delta \cap P - \mathscr{D} : \delta \in S, \delta < i\}$ is somewhere dense in P. Now i cannot be limit of cofinality $<\kappa$ (as for any perfect $P \subseteq {}^\kappa\lambda$, the union of $<\kappa$ nowhere dense subsets is nowhere dense).

If i is a successor, necessarily $(i-1) \in S$, $P_{i-1} \neq 0$ and $P_{i-1} \cap P - \mathscr{D}$ is somewhere dense in P, so as both are closed for some i, $P_{i-1} \cap u = P \cap u \neq \emptyset$ as required.

Let i be limit, cf $i = \kappa$. Now if $\eta \in P_\delta - \mathscr{D}$, $\mathrm{Sup}\{\eta(\gamma) : \gamma < \kappa\} = \delta$, hence if $\eta \in P - \mathscr{D}$ is in the closure of $\bigcup\{P_\delta \cap P - \mathscr{D} : \delta < i,\ \delta \in S\}$, but not in the closure of $\bigcup\{P_\delta \cap P - \mathscr{D} : \delta < j,\ \delta \in S\}$ for $j < i$, then $\xi_\eta = \mathrm{supp}\{\eta(\gamma) : \gamma < \kappa\}$ satisfies by the first, $\xi_\eta \leq i$, and by the second, η is not in $\bigcup_{\delta<i} P_\delta$, hence is in $\bigcup_{\delta \geq i} P_\delta$, hence $\xi_\eta \geq i$; so $\xi_\eta = i$, hence η belongs to P_i. But for some u, $P \cap u \neq 0$, and $\bigcup\{P_\delta \cap P \cap u - \mathscr{D} : \delta < i,\ \delta \in S\}$ is dense in $P \cap u$, and the set of $\eta \in P \cap u - P_i$ is

$\subseteq \bigcup_{\delta<i}(P_\delta \cap P)$, hence the union of κ nowhere dense sets in P, and by (ii) $P-\mathcal{D}$ cannot be included in such a union, hence $(P\cap u-\mathcal{D})\cap P_i$ is dense in $P\cap u$, hence $P\cap u = P_\delta \cap u$, and we finish as before.

We are left with the case cf $i>\kappa$. Let us assume (c)(2). Clearly w.l.o.g. $P-\mathcal{D}\subseteq \bigcup_{\delta<i} P_\delta$. Choose $\eta\in(P-\mathcal{D})\cap u$, so $\eta\in P_{\delta(0)}$ for some $\delta(0)<i$. For each $j\in S-(\delta(0)+1)$ for some $\alpha=\alpha_j<\kappa$, $\{\nu\in{}^\kappa\lambda:\nu\restriction\alpha=\eta\restriction\alpha\}\cap P_j=\emptyset$. We now define by induction on $\xi<\kappa$, $\langle\eta^\xi_\alpha:\alpha<\kappa\rangle$, $\beta_\xi<i$ and $\langle\gamma^\xi_\alpha:\alpha<\kappa\rangle$ such that:

(a) $\eta^\xi_\alpha\in P$, $\eta^0_\alpha=\eta$.

(b) $\gamma^\zeta_\alpha\leqslant\gamma^\xi_\alpha<\kappa$ for $\zeta<\xi$ and $\gamma^0_\alpha=\alpha$ and for limit ξ, $\gamma^\xi_\alpha=\bigcup_{\zeta<\xi}\gamma^\zeta_\alpha$.

(c) $\beta_\xi=\sup\{\eta^\zeta_\alpha(j)+1:\zeta\leqslant\xi, \alpha<\kappa, j<\kappa\}$ which is $<i$.

(d) $\eta^\xi_\alpha\restriction\gamma^\zeta_\alpha=\eta^\zeta_\alpha\restriction\gamma^\zeta_\alpha$ for $\zeta<\xi$.

(e) For each α $(\exists j)[\gamma^\xi_\alpha<j<\gamma^{\xi+1}_\alpha\wedge\eta^{\xi+1}_\alpha(j)>\beta_\xi]$

There is no problem to do this, and then define it for $\alpha<\kappa$; let η^*_α be the unique member of ${}^\kappa\lambda$ such that $(\forall\xi<\kappa)[\eta^*_\alpha\restriction\gamma^\xi_\alpha=\eta^\xi_\alpha\restriction\gamma^\xi_\alpha]$. By (c) (2), $\eta^*_\alpha\notin\mathcal{D}$, but η^*_α is in the closure of P, hence is $P-\mathcal{D}$. As $\sup\{\eta^*_\alpha(j):j<\kappa\}$ is $\beta(*)\overset{\text{def}}{=}\bigcup_{\xi<\kappa}\beta_\xi$, $\eta^*_\alpha(j)<\beta(*)$, $P-\mathcal{D}\subseteq W$, necessarily $\eta^*_\alpha\in P_{\beta(*)}$. But $\eta^*_{\alpha_{\beta(*)}}$ contradicts the choice of $\alpha_{\beta(*)}$ above.

If (c) (1) occurs, the proof is similar, choosing many η's as in the definition of the P_δ's.

2.10. Definition. For ultrafilters E_1, E_2 on a regular cardinal κ, we say E_1 is orthogonal to E_2 if in the following play, player I has no winning strategy.

The play lasts κ moves, in the ith move player I chooses an ordinal α_{2i}, $\sup_{j<2i}\alpha_j<\alpha_{2i}<\kappa$, and then player II chooses an ordinal α_{2i-1}, $\alpha_{2i}<\alpha_{2i+1}<\kappa$.

Player II wins in the play if $\bigcup_{i<\kappa}[\alpha_{4i},\alpha_{4i+1})$ belongs to E_1, and $\bigcup_{i<\kappa}[\alpha_{4i+2},\alpha_{4i+3})$ belongs to E_2.

Remark. For $\kappa=\aleph_0$, the case we shall be interested in, this relation is symmetric.

2.11. Lemma. *Suppose E_1, E_2 are non-principal orthogonal ultrafilters on ω. Let*

$$\mathcal{D}_i=\{\eta\in{}^\omega\lambda:\text{ for every large enough } n,\ \eta(n)=i\},$$

$$W_i=\{\eta\in{}^\omega\lambda:\{n:\eta(n)=i\}\in E_1\cap E_2\}.$$

$$\mathcal{D}=\bigcup_{i<\alpha}\mathcal{D}_i,\qquad W=\bigcup_{i<\alpha}W_i.$$

Then

(a) *$\mathcal{D}$ is dense (in ${}^\omega\lambda$) and p-modest (for every p).*

(b) *If $i<\lambda$, $p<\omega$, u open and $Y_1,\ldots,Y_p$ are dense subsets of $\mathcal{D}_i\cap u$, then for some perfect $P\subseteq u$; $P-\mathcal{D}\subseteq W$, and $Y_l\cap P$ is dense in P for $l=1,\ldots,p$.*

(c) *If $P\subseteq{}^\omega\lambda$ is perfect, $P\cap\mathcal{D}$ dense in P and $P-\mathcal{D}\subseteq W$, then for some $i<\lambda$, and u, $u\cap P\cap\mathcal{D}\subseteq\mathcal{D}_i$ (and $u\cap P\neq\emptyset$).*

Remark. (1) We can of course get such results for more complicated families of $\mathcal{D}_i$'s, and for $\kappa>\aleph_0$.

(2) See [5] for a closely related proof.

Proof. (a) This is easy.

(b) Let $i<\lambda$, $p<\omega$, open u and dense subsets $Y_1,\dots,Y_p$ of $\mathscr{D}_i\cap u$. We define by induction on n, a $k_n<\omega$ and a finite subset s_n of $Y_1\cup\cdots\cup Y_p$ such that:

(1) $|s_n|=n+1$,

(2) $s_n\subseteq s_{n+1}$,

(3) $k_n<k_{n+1}$,

(4) for every $\eta\in s_n$, $[k_n<m<\omega\Rightarrow\eta(m)=i]$,

(5) for every $\eta\neq\nu\in s_n$, $\eta\restriction k_n\neq\nu\restriction k_n$,

(6) for some $\eta_n\in s_n$ and $m_n\in\{1,\dots,p\}$, $s_{n+1}-s_n=\{\nu_n\}$, $\nu_n\in Y_{m_n}$, $\nu_n\restriction k_m=\eta_n\restriction k_n$, $\nu_n(k_n)\neq i$,

(7) if $n=(p+1)^l$, then $\{(\eta_q,m_q):(p+1)^l\leqslant q<(p+1)^{l+1}\}=\{(\eta,m):\eta\in s_n, m=1,\dots,p\}$.

There is no problem to do this. Now P, the closure of $\bigcup_{n<\omega}s_n$, is perfect, $P-\bigcup_{n<\omega}s_n$ is disjoint to $\mathscr{D}$ (as $\nu_n(k_n)\neq i$), and $P\cap Y_l$ is dense in P. Is $P-\mathscr{D}\subseteq W$? It is easy to prove that if ν_1, $\nu_2\in P-\mathscr{D}$, $\nu_1\restriction n\neq\nu_2\restriction n$, then $\{m:m\geqslant n, \nu_1(m)\neq i\}$ and $\{m:m\geqslant i, \nu_2(m)\neq i\}$ are disjoint.

Hence $\{\nu\in P-\mathscr{D}:\{m:\nu(m)\neq i\}\in E_i\}$ has at most one member; hence for all $\nu\in P-\mathscr{D}$ except possibly two, $\{m:\nu(m)=i\}\in E_1\cap E_2$. So for some u, $P-u$ is as required.

(c) We suppose P is a counterexample, and we shall construct from this a winning strategy for player I in the game from Definition 2.10. If possible choose u and $\gamma(*)$ such that $P\cap u\neq 0$, $P\cap W_{\gamma(*)}\cap u$ is dense in $P\cap u$.

Player I, in the ith move, chooses also $\eta_i\in P\cap\mathscr{D}$ such that the following holds:

(1) If u, $\gamma(*)$ are defined, then $\eta_i\in u$.

(2) If $\gamma(*)$ is defined, for even i, then $\eta_i\in W_{\gamma(*)}$, for odd i, $\eta_i\notin W_{\gamma(*)}$.

(3) If $\gamma(*)$ is not defined, let $\eta_i\in W_{\gamma(i)}$ ($\gamma(i)$ is uniquely determined) and $\gamma(i)\notin\{\gamma(j):j<i\}$.

(4) $\eta_i\restriction(\bigcup_{j<2i}\alpha_j)=\eta_{i-1}\restriction(\bigcup_{j<2i}\alpha_j)$.

(5) $\langle\eta_i(m):\alpha_{2i}\leqslant m<\omega\rangle$ is constant ($\gamma(i)$, in fact).

There is no problem for player I to carry the strategy. If player II wins a play in which player I uses this strategy; then $\eta\overset{\text{def}}{=}\bigcup_i[\eta_i\restriction(\bigcup_{j<2i}\alpha_j)]$ is in $P-\mathscr{D}$ but not in any W_γ, contradiction.

2.12. Claim. (1) *Any closed subset of* ${}^\omega\lambda$ (*or* $\prod_{n<\omega}\lambda_n$) *is a completely metrizable space.*

(2) *Suppose X is a completely metrizable space, $\mathscr{D}_i$ ($i<\kappa$) are dense subsets of X.*

Then $\bigcup\{u$: for some μ there is a homeomorphism from ${}^\omega\mu$ onto a co-meagre (dense) subset of $X\restriction u$; and for $i<\mathrm{Min}\{\mu,\kappa\}$ $\{\eta\in{}^\omega\lambda:\eta(n)=i$ for every large enough $n\}$ is mapped into $\mathscr{D}_i\}$ is a dense subset of X.

Proof. (1) Let $d(\eta,\nu)=\mathrm{Inf}\{2^{-n}:\eta\restriction\eta=\nu\restriction n\}$.

(2) Let λ be the number of open subsets of X. We shall define by induction on $n<\omega$, I_n, x_η, u_η, ($\eta\in I_n$) s.t.:

(i) $I_n \subseteq {}^n\lambda$, $I_0 = \{\langle\rangle\}$, and $\eta \in I_n \equiv (\exists\alpha)(\eta^\frown\langle\alpha\rangle \in I_{n+1})$ and $\{\alpha : \eta^\frown\langle\alpha\rangle \in I_{n+1}\}$ is an initial segment of the ordinals or the union of an initial segment and a singleton.

(ii) $u_{\langle\rangle} = X$, u_η an open subset of u.

(iii) If $\eta \in I_{n+1}$, then $u_\eta \subset u_{\eta \restriction n}$, and even the closures of u_η is $\subset u_{n \restriction m}$.

(iv) $\bigcup\{u_{\eta^\frown\langle\alpha\rangle} : \eta^\frown\langle\alpha\rangle \in I_{n+1}\}$ is a dense subset of u_η for $\eta \in I_n$.

(v) u_η has diameter $< 2^{-l(\eta)}$ when $\eta \neq \langle\rangle$.

(vi) $x_\eta \in u_\eta$: if $\eta(l(\eta)-1) = \alpha < \kappa$, then $x_\eta \in \mathcal{D}_\alpha$, and if $\eta(f(\eta)-1) = \eta(l(\eta)-2) = \alpha$, then $x_\eta = x_{\eta \restriction (l(\eta)-1)}$.

(vii) $|\{\alpha : \eta^\frown\langle\alpha\rangle \in I_{n+1}\}|$ is the maximal number of pairwise disjoint open non-empty subsets of u_η (the supremum is obtained for metrizable spaces).

There are no problems to carry the definition. Let $I_\omega = \{\eta \in {}^\omega\lambda : \eta \restriction n \in I_n$ for every $n < \omega\}$. By (v) above, for every $\eta \in I_\omega$; $\bigcap_{k<\omega} u_{\eta \restriction k}$ is non-empty, and choose $x_\eta \in \bigcap_{k<\omega} u_{\eta \restriction k}$, and if $\langle\eta(n) : n < \omega\rangle$ is eventually constant, $x_\eta = x_{\eta \restriction n}$ for n large enough. Let $C = \{x_\eta : \eta \in I_\omega\}$, it is a dense subset of X: if u is any open subset of X, we shall choose by induction on n, $\nu_n \in I_n$ s.t., $u \cap u_{\nu_n} \neq \emptyset$ and $\nu_n = \nu_{n+1} \restriction n$; for $n = 0$ no problem, for $n+1$: as $\bigcup\{u_{\nu_n^\frown\langle\alpha\rangle} : v_{\nu^\frown\langle\alpha\rangle} \in I_{n+1}\}$ is a dense subset of u_{ν_n}, ν_{n+1} exists, so the diameter of u_{ν_n} which is $2^{-l(\nu_n)} \leqslant 2^{-n}$; as this holds for every n, $d(x_n, u) = 0$; this proves that C is a dense subset of X, because X is a regular space.

It is also clear that the mapping $x_\eta \to \eta$ is a one-to-one homeomorphism from $X \restriction C$ onto ${}^\omega\lambda \restriction I_\omega$.

We still have one minor problem: we have I_ω rather than ${}^\omega\mu$ for some μ. As we want the homeomorphism locally, the following suffices. Let $I = \bigcup_{n<\omega} I_n$ and for $\nu \in I$ let $\lambda_\nu = |\{\eta \in I : \nu \leqslant \eta\}|$, and let $A = \{\nu \in I$: for every ρ, η, if $\nu \leqslant \rho \leqslant \eta \in I$ then $\lambda_\nu \leqslant \lambda_\rho$ (but always $\lambda_\nu \geqslant \lambda_\rho$)$\}$. Clearly it suffices to prove that for $\nu \in A$, $B_\nu \cap C$ is homeomorphism to ${}^\omega(\lambda_\nu)$, but this follows by 2.3.

2.13. Conclusion. *Suppose there are orthogonal non-principal ultrafilters on* ω. *Then*

(1) *Every completely metrizable space has* $(\mathrm{Pr}^2_{\aleph_0})$.

(2) *In the monadic topological theory of a class K of completely metrizable spaces we can interpret* $\{\theta$: *θ a second-order sentence and for every* $X \in K$, $\Vdash_{Q[X]}$"$\omega \Vdash \theta$"$\}$.

2.14. Fact. *If* CH (*i.e.*, $2^{\aleph_0} = \aleph_1$), *then there are orthogonal* (*non-principal*) *ultrafilters on* ω.

2.14A Remark. We think that CH is not necessary. Note that this is equivalent to: E_1, E_2 being non-principal ultrafilters on ω, and for every increasing $\langle n_i : i < \omega\rangle$, $E_1/E \neq E_2/E$ where E is the equivalence relation on ω defined by

$$k\,E\,e \stackrel{\text{def}}{=} (\forall i)[k \geqslant n_i \equiv l \geqslant n_i].$$

2.15 Lemma. *Suppose E is a filter on* ω *such that*:

(i) *All co-finite sets belong to E.*

(ii) *$\mathcal{P}(\omega)/E$ (as a Boolean algebra) has no $2^{\aleph_0}$ pairwise disjoint elements.*

(iii) *In the following variant of the game from Definition* 2.10 *player* I *has no winning strategy: the only change is that player* II *wins the play iff $\bigcup_i [\alpha_{y_i}, \alpha_{y_{i+1}})$, and $\bigcup_i [\alpha_{y_{i+2}}, \alpha_{y_{i+3}})$ are $\neq \emptyset \bmod E$.*

Then: (1) *Lemma* 2.11 *holds when we redefine*

$$W_i = \{\eta \in {}^{\omega}\lambda : \{n : \eta(n) = i\} \in E\}.$$

(2) *The conclusion of* 2.13 *holds.*

Proof. As before.

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