EVASION AND PREDICTION II

JÖRG BRENDLE AND SAHARON SHELAH

ABSTRACT

A subgroup $G \leqslant \mathbb{Z}^\omega$ exhibits the Specker phenomenon if every homomorphism $G \to \mathbb{Z}$ maps almost all unit vectors to 0. We give several combinatorial characterizations of the cardinal $\mathfrak{se}$, the size of the smallest $G \leqslant \mathbb{Z}^\omega$ exhibiting the Specker phenomenon. We also prove the consistency of $\mathfrak{b} < \mathfrak{e}$, where $\mathfrak{b}$ is the unbounding number and $\mathfrak{e}$ the evasion number. Our results answer several questions addressed by Blass.

Introduction

Specker [8] proved that given a homomorphism h from $\mathbb{Z}^\omega$ to the infinite cyclic group $\mathbb{Z}$, where $\mathbb{Z}^\omega$ denotes the direct product of countably many copies of $\mathbb{Z}$, we have $h(e_n) = 0$ for all but finitely many unit vectors $e_n \in \mathbb{Z}^\omega$ (in other words, the nth component of e_n is 1, and its other components are 0). Blass [3] studied the *Specker–Eda number* $\mathfrak{se}$, the size of the smallest subgroup $G \leqslant \mathbb{Z}^\omega$ containing all unit vectors which still has the property that every homomorphism $h\colon G \to \mathbb{Z}$ annihilates almost all unit vectors. We shall give various (mostly less algebraic) characterizations of $\mathfrak{se}$ (some of which already play a prominent role in Blass' work); we shall also study some related cardinal invariants of the continuum.

To be more explicit, let $\leqslant^*$ denote the *eventual domination order* on the Baire space ω^ω; that is, $f \leqslant^* g$ if and only if $f(n) \leqslant g(n)$ *for all but finitely many n*. We shall usually abbreviate the statement in italics by $\forall^\infty n$; similarly, we shall write $\exists^\infty n$ for *there are infinitely many n*. The *unbounding number* $\mathfrak{b}$ is the smallest size of a $\leqslant^*$-unbounded family $\mathcal{F}$ of functions in ω^ω (that is, given any $g \in \omega^\omega$, there is $f \in \mathcal{F}$ with $\exists^\infty n\ (f(n) > g(n))$). Given a σ-ideal $\mathcal{I}$ on ω^ω, the *additivity* $\mathrm{add}(\mathcal{I})$ is the least cardinality of a family $\mathcal{F}$ of members of $\mathcal{I}$ whose union is not in $\mathcal{I}$. We shall use this cardinal only in the cases $\mathcal{I} = \mathcal{M}$, the ideal of meagre sets, and $\mathcal{I} = \mathcal{L}$, the ideal of Lebesgue null sets. While the preceding invariants have been studied by a number of people in the last two decades, the following concept was introduced only recently by Blass [3]. Given an at most countable set S, an *S-valued predictor* is a pair $\pi = (D_\pi, \langle \pi_n; n \in D_\pi \rangle)$, where $D_\pi \subseteq \omega$ is infinite, and for each $n \in D_\pi$, π_n is a function from S^n to S. π *predicts* $f \in S^\omega$ if and only if for all but finitely many $n \in D_\pi$, we have $f(n) = \pi_n(f \restriction n)$; otherwise, f *evades* π. The *evasion number* $\mathfrak{e}$ is the smallest size of a family $\mathcal{F}$ of functions in ω^ω such that no ω-valued predictor predicts all $f \in \mathcal{F}$. A $\mathbb{Z}$-valued predictor is *linear* if and only if all $\pi_n\colon \mathbb{Z}^n \to \mathbb{Q}$ are $\mathbb{Q}$-linear maps. The corresponding

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linear evasion number will be denoted by $\mathfrak{e}_\ell$ (that is, $\mathfrak{e}_\ell = \min\{|\mathcal{F}|;\ \mathcal{F} \subseteq \mathbb{Z}^\omega$ and no linear $\mathbb{Z}$-valued predictor predicts all $f \in \mathcal{F}\}$). (Blass' definition of linear evading [3, Section 4] is slightly different; however, it gives rise to the same cardinal. We use the present definition because we shall work with functions in $\mathbb{Z}^\omega$ in Lemma 2.2.)

These notions enable us to state our main results.

THEOREM A. *It is consistent with ZFC to assume* $\mathfrak{b} < \mathfrak{e}$.

THEOREM B. $\mathfrak{se} = \mathfrak{e}_\ell = \min\{\mathfrak{e}, \mathfrak{b}\}$.

These results will be proved in Sections 1 and 2. Section 2 also contains a further purely combinatorial characterization of the cardinal $\mathfrak{se}$ (see Definition 2.4 and Lemma 2.5). To put our results into a somewhat larger context, we point out the following consequences which involve some earlier results, due mostly to Blass [3].

COROLLARY. (a) $\mathrm{add}(\mathcal{L}) \leqslant \mathfrak{se} \leqslant \mathrm{add}(\mathcal{M}) \leqslant \mathfrak{b}$.
(b) *Any of the inequalities in* (a) *can be consistently strict.*
(c) *It is consistent with ZFC to assume* $\mathfrak{e}_\ell < \mathfrak{e}$.

Theorems A and B together with the Corollary give a complete solution to Questions (1) to (3) in [3, Section 5]. Note, in particular, that the cardinals (2) to (5) in Corollary 8 in [3, Section 3] are indeed equal.

Proof of Corollary. (a) This follows from Theorem B and Blass' results [3, Theorems 12, 13]. The well-known inequality $\mathrm{add}(\mathcal{M}) \leqslant \mathfrak{b}$ is due to Miller [7].

(b) The consistency of $\mathrm{add}(\mathcal{M}) < \mathfrak{b}$ is well known (it holds, for example, in the Mathias or Laver real models). For the consistency of $\mathrm{add}(\mathcal{L}) < \mathfrak{se}$, see [3] (in particular, [3, Theorem 9]). The consistency of $\mathfrak{se} < \mathrm{add}(\mathcal{M})$ follows from Theorem B and [4, Theorem A].

(c) This is immediate from Theorems A and B.

A set of reals predicted by a single predictor is small in various senses; it belongs, in particular, to both $\mathcal{M}$ and $\mathcal{L}$. This motivates us to introduce the σ-ideal $\mathcal{J}$ on ω^ω generated by such sets of reals (see [4, Section 4] for more on this). Clearly, the uniformity of $\mathcal{J}$ (that is, the size of the smallest set of reals not in $\mathcal{J}$) is closely related to the evasion number. In fact, $\mathfrak{e} \leqslant \mathfrak{e}(\omega)$, where $\mathfrak{e}(\omega)$ denotes the former cardinal. We shall show in Section 3 that these two cardinals are equal under some additional assumption, thus giving a partial answer to [4, Section 6, Question (4)].

The results of this work are due to the second author. It was the first author's task to work them out and to write the paper.

NOTATIONAL REMARKS. A p.o. $\mathbb{P}$ is *σ-centred* if and only if there are $\mathbb{P}_n \subseteq \mathbb{P}$ $(n \in \omega)$ so that $\mathbb{P} = \bigcup_n \mathbb{P}_n$, and given $n \in \omega$, $F \subseteq \mathbb{P}_n$ finite, there is $p \in \mathbb{P}$ extending all $q \in F$. $\mathbb{P}$-names are denoted by symbols like $\dot{h}, \dot{\pi}, \dot{D}, \ldots$, $|$ stands for *divides*, and $\nmid$ means *does not divide.*

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1. *Proof of Theorem A*

1.1. We shall use a finite support iteration of *ccc* p.o.s of length κ (where $\kappa \geqslant \omega_2$ is a regular cardinal) over a model V for CH to prove the consistency of $\mathfrak{e} > \mathfrak{b}$. In fact, in the resulting model, $\mathfrak{b} = \omega_1$ and $\mathfrak{e} = \kappa$. We start with defining the p.o. $\mathbb{P}$ that we want to iterate. Notice that it is quite similar to the one used in [**4**, 4.3] for predicting below a given function.

$$\langle d, \pi, F\rangle \in \mathbb{P} \quad \Leftrightarrow \quad \begin{array}{l} d \in 2^{<\omega} \text{ is a finite partial function,} \\ \pi = \langle \pi_n; n \in d^{-1}(\{1\})\rangle \text{ and } \pi_n : \omega^n \to \omega \\ \text{is a finite partial function,} \\ F \subseteq \omega^\omega \text{ is finite and } (f \neq g \in F \to \max\{n; f\restriction n = g\restriction n\} < |d|). \end{array}$$

The order is given by

$$\langle d', \pi', F'\rangle \leqslant \langle d, \pi, F\rangle \quad \Leftrightarrow \quad \begin{array}{l} d' \supseteq d,\ \pi' \supseteq \pi,\ F' \supseteq F \text{ and} \\ (f \in F, n \in (d')^{-1}(\{1\})\backslash d^{-1}(\{1\}) \to \pi'_n(f\restriction n) = f(n)) \\ (\text{in particular, } \pi'_n(f\restriction n) \text{ is defined}). \end{array}$$

Notice that we use the convention that stronger conditions are smaller in the p.o. The first two coordinates of a condition are intended as a finite approximation to a generic predictor; the third coordinate then guarantees that functions are predicted from some point on. Thus is is straightforward that $\mathbb{P}$ adjoins a predictor which predicts all ground-model functions. Hence iterating $\mathbb{P}$ increases $\mathfrak{e}$.

Furthermore, $\mathbb{P}$ is σ-centred (and thus in particular *ccc*). To see this, simply notice that conditions with the same initial segment in the first two coordinates are compatible.

So it remains to show that $\mathfrak{b} = \omega_1$ after iterating $\mathbb{P}$. For this it suffices to show the following:

$$\begin{array}{c} \text{whenever } G \in W \text{ is an unbounded family of functions from } \omega \text{ to } \omega, \\ \text{and } \mathbb{P} \in W \text{ is the p.o. defined above, then} \\ \Vdash_{\mathbb{P}} \text{`}G \text{ is unbounded'.} \end{array} \qquad (*)$$

Using $(*)$ we can show that $\omega^\omega \cap V$ is still unbounded in the final model: $(*)$ guarantees that it stays unbounded in successor steps of the iteration, and one of the usual preservation results for finite support iterations (see, for example, [**6**, Theorem 2.2]) shows that it does so in limit steps of the iteration as well. Now $V \vDash CH$; hence $\omega^\omega \cap V$ is an unbounded family of size ω_1 in the final model.

To start with the proof of $(*)$, let $\dot{h}$ be a $\mathbb{P}$-name for a function in ω^ω. For each $d \in 2^{<\omega}$, $\pi = \langle \pi_n; n \in d^{-1}(\{1\})\rangle$ an initial segment of a predictor (as in the definition of $\mathbb{P}$), $k \in \omega$ and $\bar{f}^* = \langle f^*_\ell \in \omega^{|d|}; \ell < k\rangle$, we define $h = h_{d,\pi,\bar{f}^*} \in (\omega+1)^\omega$ by

$$h(n) := \min\{m \leqslant \omega;\ \text{for no } p \in \mathbb{P} \text{ with } p = \langle d, \pi, F\rangle, F = \{f_\ell; \ell < k\}, f_\ell\restriction|d| = f^*_\ell, \\ \text{do we have } p \Vdash_{\mathbb{P}} \text{`}\dot{h}(n) > m\text{'}\}.$$

1.2. Main Claim. $h \in \omega^\omega$.

1.3. *Proof of* $(*)$ *from the Main Claim.* Let $h^* \in \omega^\omega$ such that for all $d, \pi, \bar{f}^*$ as above, we have $h_{d,\pi,\bar{f}^*} \leqslant^* h^*$. As G is unbounded, we can find $f \in G$ such that there are infinitely many n with $f(n) > h^*(n)$. We claim that $\Vdash_{\mathbb{P}}$ '$\exists^\infty n(f(n) > \dot{h}(n))$'. This will show $(*)$.

Assume that $m \in \omega$ and $p \in \mathbb{P}$ are such that

$$p \Vdash_{\mathbb{P}} `\forall n \geqslant m(f(n) \leqslant \dot{h}(n))'.$$

Find $d, \pi, \bar{f}^*$ such that $p = \langle d, \pi, F \rangle$, where $F = \{f_\ell; \ell < k\}$ and $f_\ell \restriction |d| = f_\ell^*$. Find $n \geqslant m$ such that $f(n) > h^*(n)$ and $h^*(n) \geqslant h_{d,\pi,\bar{f}^*}(n)$. Then

$$p \Vdash_{\mathbb{P}} `h_{d,\pi,\bar{f}^*}(n) < f(n) \leqslant \dot{h}(n)',$$

contradicting the definition of $h_{d,\pi,\bar{f}^*}$.

1.4. *Proof of the Main Claim* (1.2). Let d, π, $k, \bar{f}^* = \langle f_\ell^*; \ell < k \rangle$ be as above and let $n \in \omega$ be fixed. Now assume that we have $p_i = \langle d, \pi, \{f_\ell^i; \ell < k\} \rangle$ with $f_\ell^i \restriction |d| = f_\ell^*$ and

$$p_i \Vdash_{\mathbb{P}} `\dot{h}(n) > i'.$$

We shall reach a contradiction. As we can replace $\langle p_i; i \in \omega \rangle$ by a subsequence, if necessary, we may assume that for all $\ell < k$:

either (a)$_\ell$ for some $g_\ell \in \omega^\omega$ for all i $(f_\ell^i \restriction i = g_\ell \restriction i)$;
or (b)$_\ell$ for some $i_\ell \in \omega$ and $\hat{g}_\ell \in \omega^{i_\ell}$ $(f_\ell^i \restriction i_\ell = \hat{g}_\ell \wedge f_\ell^i(i_\ell) > i)$.

Notice that $i_\ell \geqslant |d|$ in the latter case. Let $d^* := d \cup 0_{[|d|, \max(i_\ell;\, (b)_\ell \text{ holds})+1]}$; that is, the function d^* takes value 0 between $|d|$ and the maximum of the i_ℓ. Put $F^* := \{g_\ell; (a)_\ell$ holds$\}$. Then clearly $p^* = \langle d^*, \pi, F^* \rangle \in \mathbb{P}$. Now choose ℓ^* and $q \leqslant p^*$ such that

$$q \Vdash_{\mathbb{P}} `\dot{h}(n) = \ell^*'.$$

We shall find $i > \ell^*$ so that q and p_i are compatible; this is a contradiction, because q and p_i force contradictory statements.

Assume $q = \langle d^q, \pi^q, F^q \rangle$. Choose $i \geqslant \ell^*$ large enough such that:
(A) $i \geqslant |d^q|$;
(B) $i \geqslant \max\{\max\{\sigma(j); \sigma \in \operatorname{dom}(\pi_m^q) \wedge j \in m\}; m \in (d^q)^{-1}(\{1\})\}$.
Notice that (A) implies that $f_\ell^i \restriction |d^q| = g_\ell \restriction |d^q|$ whenever (a)$_\ell$ holds, while $f_\ell^i(i_\ell) > \max\{\max\{\sigma(j); \sigma \in \operatorname{dom}(\pi_m^q) \wedge j \in m\}; m \in (d^q)^{-1}(\{1\})\}$ by (B) in case (b)$_\ell$ holds. For such i, let $q^i = \langle d^i, \pi^i, F^i \rangle$, where

- $d^i = d^q \cup 0_{[|d^q|, a)}$, where a is large enough such that all functions in F^i disagree before a;
- $\pi^i \supseteq \pi^q$ such that for all $m \in (d^q)^{-1}(\{1\}) \setminus d^{-1}(\{1\})$ and all f_ℓ^i so that (b)$_\ell$ holds, we have
$$f_\ell^i(m) = \pi_m^i(f_\ell^i \restriction m) \qquad (\bigstar)$$
(this can be done because, by (B), π_m^q was not yet defined for sequences of the form $f_\ell^i \restriction m$);
- $F^i = F^q \cup \{f_\ell^i; \ell < k\}$.

Now we clearly have $q^i \in \mathbb{P}$ and $q^i \leqslant q$. So we are left with checking $q^i \leqslant p_i$. The inclusion relations are all satisfied. Hence it suffices to see that for all $\ell < k$ and $m \in (d^i)^{-1}(\{1\}) \setminus d^{-1}(\{1\})$, we have

$$f_\ell^i(m) = \pi_m^i(f_\ell^i \restriction m). \qquad (+)$$

In case (b)$_\ell$ holds, this is true by ($\bigstar$). In case (a)$_\ell$ holds, we have $f_\ell^i \restriction (m+1) = g_\ell \restriction (m+1)$ for all such m. As $q \leqslant p^*$, we have $\pi_m^i(g_\ell \restriction m) = \pi_m^q(g_\ell \restriction m) = g_\ell(m)$ for such m, and (+) holds again. This completes the proof of the Main Claim.

2. *Proof of Theorem B*

2.1. THEOREM. $\mathfrak{se} \leqslant \mathfrak{e}$.

Proof. Let $\mathscr{F} \subseteq \omega^\omega$, $|\mathscr{F}| < \mathfrak{se}$. By Blass' result $\mathfrak{se} \leqslant \mathfrak{b}$ [3, Theorem 2], there is $g \in \omega^\omega$ such that for all $f \in \mathscr{F}$, $\forall^\infty n\ (f(n) < g(n))$. Without loss, g is strictly increasing. We let $\langle p_n; n \in \omega \rangle$ be a sequence of distinct primes such that $p_n \gg g(n)$ and $p_n \gg \prod_{\ell < n} p_\ell$. For $f \in \mathscr{F}$, let $a_f \in \omega^\omega$ be defined by

$$a_f(n) := f(n) \cdot \prod_{\ell \leqslant n} p_\ell.$$

Let $G \leqslant \mathbb{Z}^\omega$ be the pure closure of the subgroup generated by the unit vectors e_n, $n \in \omega$, and the a_f, $f \in \mathscr{F}$. Clearly, $|G| < \mathfrak{se}$. Hence there is $h\colon G \to \mathbb{Z}$, a homomorphism such that $W := \{n; h(e_n) \neq 0\}$ is infinite.

Let us define

$$W^* := \{n \in \omega; \exists i > n\ (p_i \mid h(e_m) \text{ whenever } m \in \{n+1, \ldots, i-1\} \text{ but } p_i \nmid h(e_n))\}.$$

We claim that W^* is an infinite subset of W. To see this, first note that trivially $W^* \subseteq W$, by the clause $p_i \nmid h(e_n)$. Next, given $n_0 \in W$, find $i > n_0$ so that $p_i \nmid h(e_{n_0})$. Then clearly there is $n \geqslant n_0$ so that $n \in W$ and $p_i \nmid h(e_n)$ and for all $m \in \{n+1, \ldots, i-1\}$, $p_i \mid h(e_m)$. Thus $n \in W^*$. This shows that W^* is infinite.

We introduce a predictor $\pi = (W^*, \langle \pi_n; n \in W^* \rangle)$ as follows. Given $n \in W^*$ and $s \in \omega^n$ so that $\max rng(s) < g(n-1)$, if there is $f \in \mathscr{F}$ with $s \subseteq f$ and $f(n) < g(n)$ and $|h(a_f)| < p_{n-1}$, then let $\pi_n(s) = f(n)$ for some f with the above property. Otherwise, $\pi_n(s)$ is arbitrary.

We claim that π predicts all $f \in \mathscr{F}$. This clearly finishes the proof. Assume that this is false, that is, there is $f \in \mathscr{F}$ which evades π. Let $n \in W^*$ be large enough such that $\max rng(f \restriction n) < g(n-1)$, $f(n) < g(n)$, $|h(a_f)| < p_{n-1}$ and $\pi_n(f \restriction n) \neq f(n)$. Then, by the definition of π, there must be $f' \in \mathscr{F}$ with $f' \restriction n = f \restriction n$, $f'(n) < g(n)$, $|h(a_{f'})| < p_{n-1}$ and $\pi_n(f' \restriction n) = f'(n) \neq f(n)$. Now for $k \in \{f, f'\}$, we let

$$\begin{aligned}
a_k^0 &= (a_k(0), \ldots, a_k(n-1), 0, \ldots),\\
a_k^1 &= (0, \ldots, 0, a_k(n), 0, \ldots),\\
a_k^2 &= (0, \ldots, 0, a_k(n+1), \ldots, a_k(i-1), 0, \ldots),\\
a_k^3 &= (0, \ldots, 0, a_k(i), a_k(i+1), \ldots),
\end{aligned}$$

where i witnesses that $n \in W^*$. So we have $a_k = a_k^0 + a_k^1 + a_k^2 + a_k^3$. Thus

$$h(a_{f'} - a_f) = h(a_{f'}^0 - a_f^0) + h(a_{f'}^1 - a_f^1) + h(a_{f'}^2 - a_f^2) + h(a_{f'}^3 - a_f^3). \tag{$*$}$$

Clearly, $h(a_{f'}^0 - a_f^0) = h(0) = 0$. Next, $p_i \cdot \prod_{\ell \leqslant n} p_\ell$ divides $h(a_{f'}^3 - a_f^3)$ by definition of the a_k; it also divides $h(a_{f'}^2 - a_f^2)$ by definition of the a_k and because $p_i \mid h(e_m)$ for $m \in \{n+1, \ldots, i-1\}$ as i witnesses $n \in W^*$. Thus $(*)$ yields the equation

$$h(a_{f'} - a_f) = h(a_{f'}^1 - a_f^1) \quad \text{in } \mathbb{Z}/(p_i \cdot \prod_{\ell \leqslant n} p_\ell)\mathbb{Z}. \tag{$**$}$$

The right-hand side in $(**)$ must be non-zero, because $p_i \nmid h(e_n)$ (as i witnesses $n \in W^*$) and $p_i \nmid (a_{f'}(n) - a_f(n)) = \prod_{\ell \leqslant n} p_\ell \cdot (f'(n) - f(n))$ (as $f'(n), f(n) < g(n) \ll p_n \ll p_i$). However, it certainly is divisible by $\prod_{\ell \leqslant n} p_n$, whereas the left-hand side in $(**)$ is not unless it is zero (as $|h(a_f)|, |h(a_{f'})| < p_{n-1} \ll p_n$). This shows that the equation $(**)$ cannot hold, the final contradiction.

Note that this result improves [**4**, Theorem 3.2].

2.2. LEMMA. $\mathfrak{e}_\ell \geqslant \min\{\mathfrak{e}, \mathfrak{b}\}$.

Proof. Let $\mathscr{F} \subseteq \mathbb{Z}^\omega$, $|\mathscr{F}| < \min\{\mathfrak{e}, \mathfrak{b}\}$. Find $g \in \omega^\omega$ strictly increasing so that for all $f \in \mathscr{F}$, we have $|f| <^* g$, where $|f|(n) = |f(n)|$. We partition ω into intervals I_n, $n \in \omega$, so that $\max(I_n)+1 = \min(I_{n+1})$, as follows. $I_0 = \{0\}$. Assume I_n is defined; choose I_{n+1} so that $|I_{n+1}| > [2 \cdot g(\max(I_n))]^{\sum_{i \leqslant n}|I_i|}$. For $f \in \mathscr{F}$, define $\bar{f}$ by $\bar{f}(n) := f \restriction I_n$, and let $\bar{\mathscr{F}} = \{\bar{f}; f \in \mathscr{F}\}$. Use $|\bar{\mathscr{F}}| < \mathfrak{e}$ to obtain a single predictor $\bar{\pi} = (\bar{D}, \langle \bar{\pi}_n; n \in \bar{D}\rangle)$ predicting all the $\bar{f} \in \bar{\mathscr{F}}$. For $n \in \bar{D}$, let $\Gamma_n := rng(\bar{\pi}_n \restriction (-g(\max(I_{n-1})), g(\max(I_{n-1})))^{\bigcup_{i<n} I_i}) \cap \mathbb{Z}^{I_n}$. So $|\Gamma_n| < |I_n|$; hence for some $i_n \in I_n$, the vector $\bar{x}_{i_n} = \langle t(i_n); t \in \Gamma_n\rangle$ depends on the vectors $\{\bar{x}_i = \langle t(i); t \in \Gamma_n\rangle; \min(I_n) \leqslant i < i_n\}$. Say $\bar{x}_{i_n} = \sum_{\min(I_n) \leqslant i < i_n} q_i^n \bar{x}_i$, where $q_i^n \in \mathbb{Q}$. In particular, for fixed $t \in \Gamma_n$, we have $t(i_n) = \sum_{\min(I_n) \leqslant i < i_n} q_i^n t(i)$. This allows us to define a linear predictor $\pi = (D, \langle \pi_n; n \in D\rangle)$ with $D = \{i_n; n \in \omega\}$ and $\pi_{i_n}(s) = \sum_{\min(I_n) \leqslant i < i_n} q_i^n s(i)$. Note that if $n \in \omega$ is such that $\max rng(|f| \restriction \bigcup_{i<n} I_i) < g(\max(I_{n-1}))$ and $\bar{\pi}_n(\bar{f} \restriction n) = \bar{f}(n)$, then $\pi_{i_n}(f \restriction i_n) = f(i_n)$. Hence, as $\bar{\pi}$ predicts all $\bar{f} \in \bar{\mathscr{F}}$, π predicts all $f \in \mathscr{F}$.

2.3. Clearly, Theorem B follows from Theorem 2.1, Lemma 2.2 and Blass' results $\mathfrak{e}_\ell \leqslant \mathfrak{se} \leqslant \mathfrak{b}$ [**3**, Theorem 2, Corollary 8, Theorem 10].

2.4. DEFINITION. Given $D \subseteq \omega$ infinite and $\bar{a} = \langle a_n \in [\omega]^{\leqslant n}; n \in D\rangle$, the *slalom* $S_D^{\bar{a}}$ is the set of all functions f in ω^ω with $f(n) \in a_n$ for almost all $n \in D$.

Using this notion we can give a combinatorial characterization of the cardinal $\mathfrak{e}_\ell = \mathfrak{se}$.

2.5. LEMMA. $\min\{\mathfrak{e}, \mathfrak{b}\} = \min\{|\mathscr{F}|;\ \mathscr{F} \subseteq \omega^\omega$ *and for all* $D \subseteq \omega$ *and* $\bar{a} = \langle a_n \in [\omega]^{\leqslant n}; n \in D\rangle$ *there is* $f \in \mathscr{F} \setminus S_D^{\bar{a}}\}$.

NOTE. It is immediate that the cardinal on the right-hand side is larger than or equal to the additivity of Lebesgue measure $\mathrm{add}(\mathscr{L})$, by Bartoszyński's characterization of that cardinal [**1, 2**]. We also note that the original proof of $\mathrm{add}(\mathscr{L}) \leqslant \mathrm{add}(\mathscr{M})$ [**1**] shows, in fact, that this cardinal is $\leqslant \mathrm{add}(\mathscr{M})$ as well. This gives an alternative proof of Blass' $\min\{\mathfrak{e}, \mathfrak{b}\} \leqslant \mathrm{add}(\mathscr{M})$ [**3**, Theorem 13].

Proof. '$\geqslant$'. By Theorem B, it suffices to show that $\mathfrak{e}_\ell$ is larger than or equal to the cardinal on the right-hand side. However, this is exactly like Blass' original proof of $\mathrm{add}(\mathscr{L}) \leqslant \mathfrak{e}_\ell$ [**3**, Theorem 12], and we therefore leave details to the reader.

'$\leqslant$'. This argument is very similar to the one in Lemma 2.2. So we just stress the differences.

Take $\mathscr{F} \subseteq \omega^\omega$, $|\mathscr{F}| < \min\{\mathfrak{e}, \mathfrak{b}\}$. Find g strictly increasing and eventually dominating all functions from $\mathscr{F}$. As before, partition ω into intervals I_n, $n \in \omega$; this time we require that $i_{n+1} := g(\max(I_n))^{\sum_{i \leqslant n}|I_i|} \in I_{n+1}$. $\bar{f}$, $\bar{\mathscr{F}}$ and $\bar{\pi}$, $\bar{D}$ are defined as before.

We put $D := \{i_n; n \in \bar{D}\}$ and $a_{i_n} = \{\bar{\pi}_n(s)(i_n); s \in g(\max(I_{n-1}))^{\bigcup_{i<n} I_i}\} \in [\omega]^{\leqslant i_n}$, and leave it to the reader to check that $\mathscr{F} \subseteq S_D^{\bar{a}}$.

2.6. The notion of linear predicting can be generalized as follows (see [4, Section 4] for details). Let $\mathbb{K}$ be an at most countable field. A $\mathbb{K}$-valued predictor $\pi = (D_\pi, \langle \pi_n; n \in D_\pi \rangle)$ is *linear* if and only if all $\pi_n: \mathbb{K}^n \to \mathbb{K}$ are linear. $\mathfrak{e}_{\mathbb{K}}$ is the corresponding *linear evasion number*. We easily see $\mathfrak{e}_{\mathbb{Q}} = \mathfrak{e}_\ell$. Rewriting the proof of Lemma 2.2 in this more general context gives $\mathfrak{e}_{\mathbb{K}} \geqslant \min\{\mathfrak{e}, \mathfrak{b}\}$ for arbitrary $\mathbb{K}$ and $\mathfrak{e}_{\mathbb{K}} \geqslant \mathfrak{e}$ in case $\mathbb{K}$ is finite. As $\mathfrak{e}_{\mathbb{K}} \leqslant \mathfrak{b}$ for infinite $\mathbb{K}$ [4, 5.4], we obtain $\mathfrak{e}_{\mathbb{K}} = \min\{\mathfrak{e}, \mathfrak{b}\}$ for such fields—in particular, all $\mathfrak{e}_{\mathbb{K}}$ for $\mathbb{K}$ a countable field are equal. We do not know whether this is true for finite $\mathbb{K}$. Note that $\mathfrak{e}_{\mathbb{K}} > \mathfrak{e}, \mathfrak{b}$ is consistent for such fields [4, Section 4].

3. *Some results on evasion ideals*

3.1. Definition. We say that a predictor $\pi = (D, \langle \pi_n; n \in D \rangle)$ *predicts* a function $f \in \omega^\omega$ *everywhere* if $\pi_n(f \upharpoonright n) = f(n)$ holds for all $n \in D$. We put $\mathfrak{e}(\omega) :=$ $\min\{|\mathscr{F}|; \mathscr{F} \subseteq \omega^\omega \wedge$ for all countable families of predictors Π there is $f \in \mathscr{F}$ evading all $\pi \in \Pi\}$, the *uniformity of the evasion ideal* $\mathscr{J}$. As usual, $\mathrm{cov}(\mathscr{M})$ denotes the *covering number* of the ideal $\mathscr{M}$, that is, the smallest size of a family $\mathscr{F} \subseteq \mathscr{M}$ so that $\bigcup \mathscr{F} = \omega^\omega$.

3.2. Observation. *Assume that $\langle D^n; n \in \omega \rangle$ is a decreasing sequence of infinite subsets of ω, and $\langle \pi^n = (D^n, \langle \pi^n_k; k \in D^n \rangle); n \in \omega \rangle$ is a sequence of predictors. Then there are a set $D \subseteq \omega$, almost included in all D^n, and a predictor $\pi = (D, \langle \pi_k; k \in D \rangle)$ predicting all functions which are predicted by one of the π^n.*

Proof. We can assume that each function which is predicted by some π^n is predicted everywhere by some π^m—otherwise, go over to sequences $\langle E^n; n \in \omega \rangle$ and $\langle \bar{\pi}^n = (E^n, \langle \bar{\pi}^n_k; k \in E^n \rangle); n \in \omega \rangle$ such that (i) for all $n \in \omega$ there is $m \in \omega$ so that $E^m \subseteq D^n$ and $\bar{\pi}^m_k = \pi^n_k$ for $k \in E^m$, and (ii) for all $n, m \in \omega$ there is $\ell \in \omega$ so that $E^\ell \subseteq E^n \backslash m$ and $\bar{\pi}^\ell_k = \bar{\pi}^n_k$ for $k \in E^\ell$.

Choose $d^n \in D^n$ minimal with $d^n > d^{n-1}$, and put $D = \{d^n; n \in \omega\}$. Fix $n \in \omega$ and $s \in \omega^{d^n}$. To define $\pi_{d^n}(s)$, choose, if possible, $i \leqslant n$ minimal so that for all $k \in D^i \cap d^n$, we have $\pi^i_k(s \upharpoonright k) = s(k)$, and let $\pi_{d^n}(s) = \pi^i_{d^n}(s)$. If this is impossible, let $\pi_{d^n}(s)$ be arbitrary.

To see that this works, take $f \in \omega^\omega$ and $i \in \omega$ minimal so that π^i predicts f everywhere. As the set of functions predicted everywhere by a single predictor is closed, there are $n \geqslant i$ and $s \in \omega^{d^n}$ so that $s \subseteq f$ and s is not predicted everywhere by any of the π^j where $j < i$. Then $\pi_{d^m}(f \upharpoonright d^m) = \pi^i_{d^m}(f \upharpoonright d^m)$ for all $m \geqslant n$, as required.

3.3. Theorem. $\mathfrak{e} \geqslant \min\{\mathfrak{e}(\omega), \mathrm{cov}(\mathscr{M})\}$*; thus either* $\mathfrak{e} < \mathrm{cov}(\mathscr{M})$ *or* $\mathfrak{e}(\omega) \leqslant \mathrm{cov}(\mathscr{M})$ *imply* $\mathfrak{e} = \mathfrak{e}(\omega)$.

Remark. The statement is very similar to a recent result of Kamburelis, who proved $\mathfrak{s} \geqslant \min\{\mathfrak{s}(\omega), \mathrm{cov}(\mathscr{M})\}$, where $\mathfrak{s}$ is the splitting number and $\mathfrak{s}(\omega)$ the $\aleph_0$-splitting number.

Proof. The second statement easily follows from the first. To prove the latter, let $\mathscr{F} \subseteq \omega^\omega$, $|\mathscr{F}| < \min\{\mathfrak{e}(\omega), \mathrm{cov}(\mathscr{M})\}$. We shall show $|\mathscr{F}| < \mathfrak{e}$. For $\sigma \in \omega^{<\omega} \backslash \{\langle\rangle\}$, we construct recursively sets $D^\sigma \subseteq \omega$ and predictors $\pi^\sigma = (D^\sigma, \langle \pi^\sigma_n; n \in D^\sigma \rangle)$ such that:

(i) $D^{\sigma \upharpoonright i} \supseteq D^\sigma$ for $i \in |\sigma|$;

(ii) for all $f \in \mathscr{F}$ and all $\sigma \in \omega^{<\omega}$, there is $i \in \omega$ so that f is predicted by $\pi^{\sigma^\frown\langle i \rangle}$.

First construct $\pi^{\langle i\rangle} = (D^{\langle i\rangle}, \langle \pi_n^{\langle i\rangle}; n \in D^{\langle i\rangle}\rangle)$ satisfying (ii) by applying $|\mathscr{F}| < \mathfrak{e}(\omega)$.

To do the recursion, assume that $\pi^\sigma = (D^\sigma, \langle \pi_n^\sigma; n \in D^\sigma\rangle)$ is constructed for some fixed $\sigma \in \omega^{<\omega}$. Given $f \in \omega^\omega$, define f^σ by

$$f^\sigma(i) := f(k_i^\sigma),$$

where $\{k_i^\sigma; i \in \omega\}$ is the increasing enumeration of the set D^σ. Let $\mathscr{F}^\sigma = \{f^\sigma; f \in \mathscr{F}\}$. Again we obtain ω many predictors $\bar{\pi}^{\sigma^\frown\langle i\rangle} = (\bar{D}^{\sigma^\frown\langle i\rangle}, \langle \bar{\pi}_n^{\sigma^\frown\langle i\rangle}; n \in \bar{D}^{\sigma^\frown\langle i\rangle}\rangle)$, $i \in \omega$, so that every $f^\sigma \in \mathscr{F}^\sigma$ is predicted by some $\bar{\pi}^{\sigma^\frown\langle i\rangle}$. Let $D^{\sigma^\frown\langle i\rangle} = \{k_j^\sigma; j \in \bar{D}^{\sigma^\frown\langle i\rangle}\}$. Fix $j \in \bar{D}^{\sigma^\frown\langle i\rangle}$ and $s \in \omega^{k_j^\sigma}$. Let $\bar{s} \in \omega^j$ be defined by $\bar{s}(\ell) = s(k_\ell^\sigma)$. Put $\pi_{k_j^\sigma}^{\sigma^\frown\langle i\rangle}(s) := \bar{\pi}_j^{\sigma^\frown\langle i\rangle}(\bar{s})$. Now it is easy to see that $\pi^{\sigma^\frown\langle i\rangle}$ predicts f whenever $\bar{\pi}^{\sigma^\frown\langle i\rangle}$ predicts f^σ. Thus (i) and (ii) hold. This completes the recursive construction.

Given $f \in \omega^\omega$, let $T_f = \{\sigma \in \omega^{<\omega};$ for all $i \leqslant |\sigma|$ $(\pi^{\sigma\restriction i}$ does not predict $f)\}$. By the above construction, the sets $[T_f]$ are nowhere dense for $f \in \mathscr{F}$. As $|\mathscr{F}| < \mathrm{cov}(\mathscr{M})$, there must be $g \in \omega^\omega \setminus \bigcup_{f \in \mathscr{F}} [T_f]$. Now use Observation 3.2 to construct a new predictor from the $\langle \pi^{g\restriction n}; n \in \omega\rangle$ which will predict all $f \in \mathscr{F}$.

3.4. It is unclear whether $\mathfrak{e} = \mathfrak{e}(\omega)$ can be proved in ZFC. In view of Theorem 3.3, it seems reasonable to ask first the following.

QUESTION. Is $\mathfrak{e} > \mathrm{cov}(\mathscr{M})$ consistent?

Of course, we may also consider the cardinal $\mathfrak{e}_\ell(\omega)$, the smallest size of a family $\mathscr{F}$ of functions from ω to ω such that no countable family of linear predictors predicts all $f \in \mathscr{F}$. However, it is now easy to see that $\mathfrak{e}_\ell(\omega) = \mathfrak{e}_\ell$. This is so because $\mathfrak{e}_\ell(\omega) \leqslant \min\{\mathfrak{e}(\omega), \mathfrak{b}\} \leqslant \min\{\mathfrak{e}, \mathfrak{b}\} \leqslant \mathfrak{e}_\ell$. To see the first inequality, note that the argument for $\mathfrak{e}_\ell \leqslant \mathfrak{b}$ gives $\mathfrak{e}_\ell(\omega) \leqslant \mathfrak{b}$ as well (see [4, Section 5.4] for a stronger result); for the second inequality, $\min\{\mathfrak{e}(\omega), \mathfrak{b}\} \leqslant \mathrm{cov}(\mathscr{M})$ by rewriting Blass' $\min\{\mathfrak{e}, \mathfrak{b}\} \leqslant \mathrm{cov}(\mathscr{M})$ [3, Theorem 13] and thus $\min\{\mathfrak{e}(\omega), \mathfrak{b}\} = \min\{\mathfrak{e}(\omega), \mathrm{cov}(\mathscr{M}), \mathfrak{b}\} \leqslant \min\{\mathfrak{e}, \mathfrak{b}\}$ by Theorem 3.3; the third inequality is Lemma 2.2.

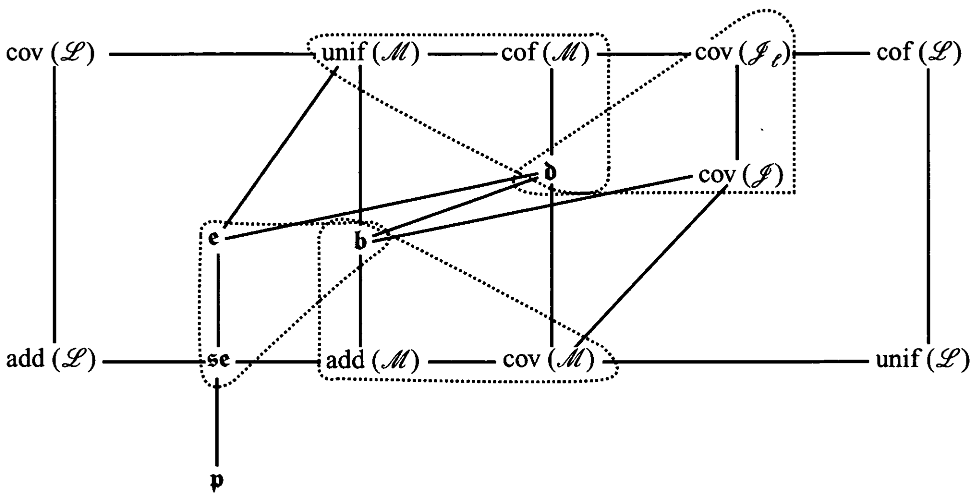
3.5. DUALITY. Most of the cardinal invariants of the continuum come in pairs, and results about them usually can be dualized (see [4, Section 4.5] for details). In our situation, the dual cardinals are: the *dominating number* $\mathfrak{d}$ (dual to $\mathfrak{b}$), the smallest size of a family $\mathscr{F} \subseteq \omega^\omega$ such that given any $g \in \omega^\omega$ there is $f \in \mathscr{F}$ with $g \leqslant^* f$; the (*linear*) *covering number* $\mathrm{cov}(\mathscr{J})$ ($\mathrm{cov}(\mathscr{J}_\ell)$) of the ideal $\mathscr{J}$ ($\mathscr{J}_\ell$) (the first being dual to both $\mathfrak{e}$ and $\mathfrak{e}(\omega)$, the second being dual to $\mathfrak{e}_\ell$), the least cardinality of a family of (linear) predictors Π such that every function $f \in \omega^\omega$ ($\mathbb{Z}^\omega$) is predicted by some $\pi \in \Pi$. Then we obtain the following.

THEOREM. (a) *It is consistent with ZFC to assume* $\mathfrak{d} > \mathrm{cov}(\mathscr{J})$.

(b) $\mathrm{cov}(\mathscr{J}_\ell) = \max\{\mathrm{cov}(\mathscr{J}), \mathfrak{d}\} = \min\{|\mathscr{S}|;$ *$\mathscr{S}$ consists of slaloms* $S_D^{\bar{a}}$ *where* $\bar{a} = \langle a_n \in [\omega]^{\leqslant n}; n \in D\rangle$ *and* $D \subseteq \omega$ *is infinite and* $\forall f \in \omega^\omega\ \exists S_D^{\bar{a}} \in \mathscr{S}\ \forall^\infty n \in D\ (f(n) \in a_n)\}$.

Proof. These dualizations are standard, and we therefore refrain from giving detailed proofs. The model for (a) is obtained by iterating the p.o. $\mathbb{P}$ from Section 1 ω_1 times with finite support over a model for $MA + \neg CH$. (b) is the dual version of Theorem B and Lemma 2.5.

We close our work with a diagram showing the relations between the cardinal invariants considered in this work (in particular, the Specker–Eda number $\mathfrak{se}$ and the evasion number $\mathfrak{e}$) and some other cardinal invariants of the continuum (in particular, those of Cichoń's diagram). We refer the reader to [3], [4] or [5] for the cardinals not defined here. A similar diagram was drawn in [4, Section 4].



In the diagram, cardinals increase as one moves up and to the right. To enhance readability, we omitted the relations $\mathfrak{e} \leqslant \mathrm{unif}(\mathscr{L})$, and its dual $\mathrm{cov}(\mathscr{L}) \leqslant \mathrm{cov}(\mathscr{I})$. The dotted lines give the relations $\mathrm{add}(\mathscr{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathscr{M})\}$, $\mathfrak{se} = \min\{\mathfrak{e}, \mathfrak{b}\}$, and their dual versions.

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Mathematisches Institut der Universität
Auf der Morgenstelle 10
72076 Tübingen
Germany

Current Address
Department of Mathematics
Dartmouth College
Hanover, NH 03755
USA

Department of Mathematics
Hebrew University
Givat Ram
91904 Jerusalem
Israel

and

Department of Mathematics
Rutgers University
New Brunswick, NJ 08854
USA