STABLE THEORIES

BY

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ABSTRACT

We study $K_T(\lambda) = \sup \{|S(A)| : |A| \leqq \lambda\}$ and extend some results for totally transcendental theroies to the case of stable theories. We then investigate categoricity of elementary and pseudo-elementary classes.

0. **Introduction** In this article we shall generalize Morley's theorems in [2] to more general languages.

In Section 1 we define our notations.

In Theorems 2.1, 2.2. we in essence prove the following theorem: every first-order theory T of arbitrary infinite cardinality satisfies one of the possibilities:

1) for all χ, $|A| = \chi \Rightarrow |S(A)| \leqq \chi + 2^{|T|}$, (where $S(A)$ is the set of complete consistent types over a subset A of a model of T).

2) for all χ, $|A| = \chi \Rightarrow |S(A)| \leqq \chi^{|T|}$, and there exists A such that $|A| = \chi$, $|S(A)| \geqq \chi^{\aleph_0}$.

3) for all χ there exists A, such that $|A| = \chi$, $|S(A)| > |A|$.

Theories which satisfy 1 or 2 are called *stable* and are similar in some respects to totally transcendental theories. In the rest of Section 2 we define a generalization of Morley's *rank of transcendence*, and prove some theorems about it. Theorems whose proofs are similar to the proofs of the analogous theorems in Morley [2], are not proven here, and instead the number of the analogous theorem in Morley [2] is mentioned.

In Section 3, theorems about the existence of sets of *indiscernibles* and *prime models* on sets are proved.

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In Section 4, a two-cardinal Skolem-Löwenhiem theorem is given without proof, and is followed by some theorems about *categorical* elementary and pseudo-elementary classes.

Among them appear:

THEOREM. *If T is categorical in λ, $\lambda > |T| + \aleph_0$, $\lambda \neq \inf\{\mu : \mu^{\aleph_0} > \mu + |T|\}$ then T is categorical in every cardinal $\geqq \lambda$, and in some cardinal $< \mu(|T|) < \beth((2^{|T|})^+)$.*

THEOREM. *If the class of reducts of models of T to the language L is categorical in λ, $\lambda > |T|$, $\beth_\gamma > |T|$ and the ordinal γ is divided by $(2^{|T|})^+$, then the class of reducts of models of T to the language L is categorical in $\beth_\gamma$.*

Some of the results of this article appear in my notices [8], [7].

After proving the theorems in this article, an unpublished article of J. P. Ressayre [5] came to my attention. It deals with categorical theories and includes results previously obtained by F. Rowbottom. Among the results in Ressayre's article are a weaker version of Theorems 2.1 and 2.2, a partial version of 3.5, and a somewhat weaker version of 4.6.

1. **Notations.** M will denote a model, $|M|$ is the set of its elements, $|A|$ is the cardinality of A, and $\|M\|$ is the cardinality of the model M. We shall write $a \in M$ instead $a \in |M|$. $\alpha, \beta, \gamma, i, j, k, l$, will denote ordinale, δ a limit ordinal and n, m natural numbers.

λ, χ, μ will denote infinite cardinals. λ^+ is the first cardinal greater than λ. $\beth(\chi,\alpha)$ is defined by induction: $\beth(\chi,0) = \chi, \beth(\chi,\alpha+1) = 2^{\beth(\chi,\alpha)}$, and $\beth(\chi,\delta) = \bigcup_{\alpha<\delta}\beth(\chi,\alpha)$; $\beth(\alpha) = \beth_\alpha = \beth(\aleph_0,\alpha)$. If $\chi = \aleph_\alpha$ then $\aleph(\chi,\beta) = \alpha_{\alpha+\beta}$, where $\aleph_\alpha = \omega_\alpha$ is the α'th infinite cardinal.

T will denote a fixed first-order theory with equality. If $\psi(x)$ is a formula in the language of T with one variable., $\psi(M)$ is the set of elements satisfying ψ. $M \vDash \psi[a]$ if $\psi[a]$ is satisfied in M. Without loss of generality we assume that for every formula $\psi(x_1,\cdots,x_h)$ there is a predicate $R(x_1,\cdots,x_n)$ such that $(\forall \bar{x})(\psi(x_1,\cdots,x_n) \equiv R(x_1,\cdots,x_n)) \in T$ and that there are no function symbols in the language. Morley [2] explains why there is no loss of generality here. The language of T will be denoted by $L(T)$. The predicates in $L(T)$ will be $\{R_i : i < |T|\}$. T is complete unless stated otherwise. Usually x, y, z will be individual variables, $\bar{x}, \bar{y}, \bar{z}$— finite sequences of variables, a, b, c will denote elements of models, and $\bar{a}, \bar{b}, \bar{c}$ will denote finite sequences of elements of models. It is implicitly assumed that different sequences of variables contain no common variables. $\langle\,\rangle$ will be the empty sequence. $\bar{a}_i$ or $\bar{a}(i)$ will be the i'th element of the sequence $\bar{a}$. Instead of writing $(\forall n<\omega)(\bar{a}_n \in A)$ we shall write $\bar{a}_n \in A$ or $\bar{a} \in A$. A, B, C will denote substructures of T-models, and when we speak about a set A, or define A, we speak about its relations as well. That is we do not distinguish between the substructure A and the set A. By $A \subset M$ we mean that $A \subset |M|$, and the relations on A are the relations on M restricted to A. $T(A)$ is the theory T together with all the true sentences $R[\bar{a}]$, $\bar{a} \in A$, and $T(A)$ is a complete theory. When writing $R[\bar{a}]$ we assure implicitly that the length of the sequence $\bar{a}$ is equal to the number of places in the predicate R.

We define p to be a *type* on A iff p is a set whose elements are of the form $\psi(\bar{x}, \bar{a})$, where $\bar{a} \in A$, and ψ is an arbitrary formula in L. q, r will also denote types. If for every ψ, $\bar{a} \in A \rightarrow \psi(\bar{x}, \bar{a}) \in p$ or $\psi(\bar{a}, \bar{a}) \in p$, p is called a *complete type* on A. If A is not mentioned, then it is assumed A is the empty set. When we speak about a type we implicitly assume that $T(A) \cup p$ is a consistent set. We define $p \mid A = \{\psi(\bar{x}, \bar{a}) \in p : \bar{a} \in A\}$. If not otherwise assumed $\bar{x} = x$ in p.

$S^T(A)$ is the set of complete types on A. As T is fixed we write $S(A)$. If I is a set of predicates then $p \mid I = \{\psi \in p : \psi = R(x, \bar{a})$ or $\psi = \neg R(x,, \bar{a})$ and $R \in I\}$, $S_I(A) = \{p \mid I : p \in S(A)\}$, $p \mid R = p \mid \{R\}$, and $S_R(A) = S_{\{R\}}(A)$. By our notations we can distinguish easily between $p \mid I$ and $p \mid A$. On $S(A)$ $(S_I(A))$ a compact topology is defined by the sub-base which has the following sets as elements: for every $\phi = \psi(x, \bar{a})$, $V_\phi = \{p : \psi(x, \bar{a}) \in p\}$. M *realizes* a type p on $|M|$, if there is an element b of M such that for every $\psi(x, \bar{a}) \in p$ $M \models \psi[b, \bar{a}]$ (that is : $\psi(b, \bar{a})$ is satisfied in M). M *omits* p if it does not realize p. M is called λ-*saturated* if every type on A with $A \subset M$, $|p| < \lambda$, is realized in M. If M is $\|M\|$-saturated it is called *saturated.*

$\mu(\chi)$ is the smallest cardinal such that if T with $|T| = \chi$, has a model omitting a type p in every cardinal smaller than $\mu(\chi)$ and not smaller than $|T|$, then it has such a model in every cardinal $\geqq |T|$. In Vaught [9] the following results are mentioned:

$$\mu(\chi) < \beth_\gamma \text{ where } \gamma = (2^\chi)^+;\ \mu(\aleph_0) = \beth_{\omega_1};\ \mu(\beth_\delta) = \beth(\beth_{\delta+1}) \text{ when } \operatorname{cf}\delta = \omega.$$

T is *categorical* in λ if all models of T of cardinality λ are isomorphic. *pc* (T_1, T) is the class of reducts of models of T_1 to $L(T)$. (We assume implicitly that

$T = T_1 \cap L(T)$.) $pc(T_1, T)$ is *categorical* in λ if all models in $pc(T_1, T)$ of cardinality λ are isomorphic.

2. On possible cardinalities of $S(A)$

DEFINITION 2.1: $K_T(\lambda) = \sup\{|S(A)| : |A| \leqq \lambda\} = \inf\{\mu : |A| \leqq \lambda \Rightarrow |S(A)| < \mu\}$.

NOTATIONS: η, τ will denote *ordinal sequences* of zeroes and ones. For $0 \leqq i < l(\eta)$ η_i is the i'th element of the sequence, where $l(\eta)$ is the length of the sequence. $\psi^{\eta(i)}$ will denote ψ if $\eta(i) = 0$, and $\neg\psi$ if $\eta(i) = 1$. $\eta | \alpha$ is the sequence of the first α elements of η.

THEOREM 2.1. 1) *If there exists* A, $|A|^{|T|} = |A|$, $|S(A)| > |A|$. *Then for every* λ, $K_T(\lambda) \geqq \inf\{(2^\chi)^+ : 2^\chi > \lambda\}$.

2) There exists A as mentioned in 1, iff tthere exists a predicate R such that: $\Gamma_R = \{(\exists x)(\bigwedge_{0 \leqq i < l(\eta)} R(x, \bar{y}^{\eta|i})^{\eta(i)}) : l(\eta) < \omega\} \cup T$ is consistent.

REMARKS. The same argument will show that if there exists an A such that $|A|^{|T|} < |S(A)|$, then Γ_R is consistent.

Proof. Let us assume that A satisfies $|A|^{|T|} = |A|$, $|S(A)| > |A|$. Then we shall show that there exists a consistent Γ_R as mtntioned in 2, and that the consistency of Γ_R implies the conclusion of 1. This will prove the theorem.

Now for every R, we define $p_1 \sim p_2 \pmod R$ iff $p_1 | R = p_2 | R$. This is an equivalence relation on $S(A)$, which divides it into $|S_R(A)|$ equivalence classes. Since, for every $p_1, p_2 \in S(A)$, $p_1 \neq p_2$, there is an R such that $P_1 \nsim p_2 \pmod R$, $|S(A)| \leqq |\prod_R S_R(A)| = \prod_R |S_R(A)|$. If for every R, $|S_R(A)| \leqq |A|$, then $|S(A)| \leqq |A|^{|T|} = |A|$, a contradiction. Hence, there esxits an R such that $|S_R(A)| > |A| \geqq \aleph_0$. We shall prove that Γ_R is consistent.

For every $\bar{a}$ such that $\bar{a} \in A$, $R(x, \bar{a})$ divides $S_R(A)$ into two sets: the types p such that $R(x, \bar{a}) \in p$, and the types p such that $\neg R(x, \bar{a}) \in p$. If in every such division one of the sets is of cardinality $\leqq |A|$, for example the set $\{p \in S_R(A) : R(x, \bar{a}^{\tau(\bar{a})}) \in p\}$ then,

$$|S_R(A)| = |\bigcup_{\bar{a}} \{p \in S_R(A) : R(x, \bar{a})^{\tau(\bar{a})} \in p\} \cup \{p \in S_R(A) : \text{ for all } \bar{a}$$

$$R(x, \bar{a})^{\tau(\bar{a})} \notin p\}| \leqq \sum_{\bar{a}} |\{p \in S_R(A) : R(x, \bar{a})^{\tau(\bar{a})} \in p\}| + 1 = |A|, \quad \text{a contradiction.}$$

So there exists $\bar{a} = \bar{a}^{\langle\rangle}$ such that $R(x, \bar{a}^{\langle\rangle})$ divides $S_R(A)$ into two sets of cardinality $> |A|$. For every one of them we can repeat the above discussion and find. $\bar{a}^{\langle 0\rangle}$, $\bar{a}^{\langle 1\rangle}$ such that there exists $>|A|$ types p with either $R(x, \bar{a}^{\langle\rangle})$, $R(x, \bar{a}^{\langle 0\rangle}) \in p$; $R(x, \bar{a}^{\langle\rangle})$, $\neg R(x, \bar{a}^{\langle 0\rangle}) \in p$; $\neg R(x, \bar{a}^{\langle\rangle})$, $R(x, \bar{a}^{\langle 1\rangle}) \in p$; or $\neg R(x, \bar{a}^{\langle\rangle})$, $\neg R(x, \bar{a}^{\langle 1\rangle}) \in p$. We can continue defining $\bar{a}^{\eta}$, and proving by it the consistency of Γ_R. And so we have shown one direction.

Let $\chi = \inf\{\mu : 2^{\mu} > \lambda\}$. We define

$$\Gamma = \{R(\chi_{\eta} y^{\eta|\gamma})^{\eta(\gamma)} : l(\eta) = \chi, \gamma < \chi\} \cup T.$$

It is easy to see that if Γ is not consistent then Γ_R is not consistent. Let M be a model of Γ, and A_1 the set of elements which realize the variables $\{(\bar{y}^{\eta|\gamma})_n : l(\eta) = \chi, \gamma < \chi, \eta < l(\bar{y}^{\eta|\gamma})\}$. The cardinality of A_1 is $\leqq \sum_{\gamma<\chi} 2^{|\gamma|} \leqq \lambda$, and in M 2^{χ} different complete types on A_1 are realized. (The types realized by elements which realizes the variables χ_{η}, $l(\eta) = \chi$). So $|A_1| \leqq \lambda$. $|S(A_1)| \geqq 2^{\chi} > \lambda$, and so $K_T(\lambda) \geqq (2^{\chi})^{+} > \lambda^{+}$.

Definition 2.2. If in T there is no predicate R such that Γ_R is consistent, T is called *stable*.

Definition 2.3. If for every λ, $K_T(\lambda) \leqq \lambda^{+} + (2^{|T|})^{+}$ then T is called *super stable*.

Theorem 2.2. 1) *If T is stable and there exists A, $|A| \geqq 2^{|T|}$ such that $S(A)| > |A|$, then for every λ, $K_T(\lambda) > \lambda^{\chi_0}$. So there exists arbitrarily large powers for which $K_T(\lambda) > \lambda^{+} + (2^{|T|})^{+}$.*

2) *There exists A as mentioned in 1 iff there exists a sequence of ω predicates $\langle R^n : n < \omega\rangle$ such that*

$$\Gamma\langle R^n : n\omega <\rangle = \{R^m(x^f, \bar{y}^{g,h}) \equiv \neg R^m(x^{f'}, \bar{y}^{g,h}) : \text{for all}$$

$$f = \langle i_0, \cdots, i_{m-1}, i_m, \cdots, i_l \cdots : l < \omega\rangle, f' = \langle i_0, \cdots, i_{m-1}, i_m, \cdots', i_l, \cdots' : l < \omega\rangle,$$

$$i'_m \neq i_m,\ g = \langle i_0, \cdots, i_{m-1}\rangle,\ h = \{i_m, i'_m\} \text{ and}$$

$$i_l, i'_l < \omega \text{ for all } l < \omega\}$$

is consistent.

3) *If T is super stable and there exists A with $|S(A)| > |A|, |T|$ and if $\lambda > |A| + |T|$, $\lambda \leqq S(A)$ is regular then there exists $B \subset A$, $|B| = |T|$* such that $||S(B)| \geqq \lambda$. We can conclude that, for super stable T, if $K_T(\lambda) > \lambda^{+} > |T|$ then $K_T(|T|) > |T|^{+}$.

Proof. The way we prove 1 and 2 will be similar to that of Theorem 2.1. First, we shall prove from the assumption of 1 that there exists $\langle R^n : n<\omega\rangle$ such that $\Gamma\langle R^n : n<\omega\rangle$ is consistent, and then that if $\Gamma\langle R^n : n<\omega\rangle$ is consistent then for every λ there exists A, such that $|A| = \lambda$, $|S(A)| \geqq \lambda^{\aleph_0}$. Then choosing such an A for $\lambda = \aleph(2^{|T|}, \omega)$, we close the circle.

Let A be as in the assumption of 1.

LEMMA 2.3. *There exists R^0, a predicate of $L(T)$, such that the partition of $S(A)$ by the equivalence relation $(\mathrm{mod}\, R^0)$ contains at least $|T|^+$ classes of cardinality $> |A|$.*

Proof of the lemma. If not $-|S(A)| \leqq \sum_R |S_R(A)| + |T|^{|T|} = |A|$, a contradiction.

For every one of the $|T|^+$ classes there exists R_i that divides it in a similar manner. But there are only $|T|$ predicates. So there exists R^1 such that there are $|T|^+$ classes $(\mathrm{mod}\, R^0)$ such that in each of their partitions by R^1 there are $|T|^+$ classes of cardinality $> |A|$. It is easy to see that we can continue to define R_n for $n < \omega$.

Now $\langle R^n : n<\omega\rangle$ is defined. By the construction just mentioned there exists for every n $\{p(j; i_0, \cdots, i_{m-1}) : j < |T|^+,\ i_l < |T|^+, m < n\}$ such that the following three conditions are satisfied:

$p(j; i_0, \cdots, i_{m-1}) \in S_R m(A)$; if $j \neq j'$ then $p(j : i_0, \cdots, i_{m-1})$ and $p(j'; i_0, \cdots, i_{m-1})$ are contradictory; and $p(i_1) \cup P(i_2; i_1) \cup p(i_3, i_1, i_2) \cup \cdots \cup p(i_m;\ i_0, \cdots, i_{m-1})$ is consistent.

From this it can be easily seen that $\Gamma\langle R^n : n<\omega\rangle$ is consistent. Now we shall prove that if $\Gamma\langle R^n : n\langle\omega\rangle$ is consistent, then for every λ there exists an A such that $|A| = \lambda$, $|S(A)| \geqq \lambda^{\aleph_0}$. Let $\Gamma = T \cup \{R^m(x^f, \bar{y}^{g,h}) \equiv R^m(x^{f'}, \bar{y}^{g,h})$: for, all $m < \omega$, $f = \langle i_0, \cdots, i_{m-1}, i_m, \cdots, i_l, \cdots : l < \omega\rangle$, $g = \langle i_0, \cdots, i_m\rangle$, $h = \{i'_m, i_m\}$, and $f' = \langle i_0, \cdots, i_{m-1}, i'_m, \cdots, i'_l \cdots : l<\omega\rangle$ such that $(\forall j < \omega)\ (i_j < \lambda \Lambda i'j < \lambda)\}$.

If Γ is inconsistent, then a finite subset of Γ is inconsistent and so $\Gamma < R^n : n<\omega\rangle$ is inconsistent, a contradiction. Therefore Γ has a model. Let A be the set of elements realizing the variables appearing in $\bar{y}^{g,h}$. Then elements realizing different variables from $\{x^f : f = <i_0, \cdots, i_l, \cdots : l<\omega\rangle, i_l<\lambda\}$ realizes different types on A.

So $|A| \leqq \sum_{m<\omega} \lambda^m = \lambda$, $|S(A)| \geqq \lambda^{\aleph_0}$.

Now it remains to prove part 3. We can try again to build the construction that appears in the beginning of the proof replacing "more than $|A|$" by "at least λ", As that attempt must fail by our assumption, we get a set S of $\geqq \lambda$ types in $S(A)$. such that for every R there are no more than $|T|$ equivalence classes of power $\geqq \lambda$,

$\{S_i(R) : i < j_R \leqq |T|\}$. Now $|S - \bigcup_i S_i(R)| < \lambda$ and $|S - \cap_R \bigcup_i S_i(R)| \leqq \Sigma_R |S - \bigcup_i S_i(R)| < \lambda$ and this implies that $|\cap_R \bigcup_i S_i(R)| \geqq \lambda > |A|$. If $p_1, p_2 \in \cap_R \bigcup_i S_i(R)$, $p_1 \neq p_2$ there is an R such that $p_1 | R \neq p_2 | R$; but $p_1 | R$ is one of $|T|$ elements of $\{p | R : p \in \bigcup_i S_i(R)\}$ (by the definition of $S_i(R)$), and so there is $A(R) \subset A$, $|A(R)| = |T|$ such that for every $p_1, p_2 \in \cap_R \bigcup_i S_i(R)$ if $p_{1R} | \neq p_2 | R$ then $p_1 | A(R) \neq p_2 | A(R)$. It follows that $|S(\bigcup_R A(R)| \geqq |\cap_R \bigcup_i S_i(R) \geqq \lambda$, and $|\bigcup_R A(R)| \leqq |T|$.

Remark. By a more refined proof we can replace $\Gamma\langle R^n : n < \omega \rangle$ by the more elegant set

$$\Gamma'\langle R^n : n < \omega \rangle = T \bigcup \{(\exists x) \bigwedge_{j=0}^{m} [R^j(x, \bar{y}^g) \wedge \bigwedge_{h=0}^{i_{j-1}} \cdots R^j(x, \bar{y}^f)] : m < \omega,$$

$$g = \langle i_0, \cdots, i_j \rangle,\ f = \langle i_0, \cdots, i_{j-1}, h \rangle,\ i_0, \cdots, i_m < \omega\}$$

Definition 2.4. We shall define $S_I^\alpha(A)$ and $TR_I^\alpha(A)$ by induction on α, where I is a set of predicates in $L(T)$. $S_I^0(A) = S_I(A)$. $TR_I^\alpha(A)$ will be the set of types in $S_I^\alpha(A)$, which have, in every extension B of A, at most one extension which is an element of $S_I^\alpha(B)$. $S_I^\alpha(A) = S_I(A) - \bigcup_{i<\alpha} TR_I^i(A)$.

Remark. An analogous definition appears in Morley [1], 2.2 and footnote 13.

Theorem 2.4. *If R is a predicate of $L(T)$, Γ_R is consistent iff $S_R^\alpha(A) \neq 0$ for every α and A. If for some α and A $S_R^\alpha(A) = 0$, then there exists $\beta < \omega_1$ such that for every A, $S_R^\beta(A) = 0$.*

Proof. As in Morley [1], 2.7, 2.8.

Remark. In fact, $\beta < \omega$.

Definition 2.5. 1) If Γ_R is not consistent, then to every type $p \in S(A)$, we define Rank(R, p) as the first α such that $p | R \in TR_R^\alpha(A)$.

2) If T is stable then Rank$(p) = \langle$ Rank$(R_i, p) : i < |T| \rangle$.

Lemma 2.5. *It is possible to define a lexicographic order on Rank(p), such that there is no monotonically decreasing sequence of type $|T|^+$.*

Proof. Immediate.

Theorem 2.6. 1) *If $B \subset A$, and $p \in S(A)$, then* Rank$(R, p) \leqq$ Rank$(R, p) | B)$ *and* Rank$(p) \leqq$ Rank $(p | B)$, *and there is no more than one extension q of $p | B$, $q \in S(A)$, such that* Rank$(q) =$ Rank$(p | B)$.

2) *For all A, and $p \in S(A)$, and for every R, there exists a finite set $B \subset A$, such that* $\mathrm{Rank}(R, p) = \mathrm{Rank}(R, p \mid B)$.

Proof. See Morley [2] 2.4, 2.6. Notice the difference in terminology. (Rank here is rank and degree there.)

3. On some properties of stable theories.

THEOREM. 3,1. *If M is a model of a stable theory T, $|T| < \lambda = |A| < \|M\|$, $K_T(\lambda) = \lambda^+$ and A a substructure of M, then there exists a set Y in M, $|Y| = \lambda^+$, which is indiscernible on A (that is, for all $y_1, \cdots, y_h$; $z_1, \cdots, z_n \in Y$. $a_1, \cdots, a_m \in A$, $M \vDash R(y_1, \cdots, y_n, a_1, \cdots, a_m) \equiv R(z_1, \cdots, z_n, a_1, \cdots, a_m)$ if for every $i \neq j$, $y_i \neq y_j$ and $z_i \neq z_j$).*

REMARK 1. A similar theorem, for totally transcendental theories appears in Morley [2] 4.6. Rowbottom has a weaker unpublished theorem.

REMARK 2. In fact we can prove more: in every $B \subset M$, $|B| > \lambda$, and for every regular $\chi \leqq |B|$, $\chi > \lambda$, there is such a Y, provided $|B| < \chi \Rightarrow |\{p \in S(A)\colon p$ is realized in $M\}| < \chi$.

Proof. In $S(A)$ there are λ types, and so at least one of them, p, is realized at least $|A|^+$ times. Let the set of elements of M realizing p be B.

LEMMA 3.2. *There exists A_1, $|A_1| = \lambda$ $A \subset A_1 \subset M$, and $p_1 \in S(A_1)$ $p_1 \supset p$, such that, if $M \supset B_1 \supset A_1$, $|B_1| = \lambda$, p_1 has one and only one extension of the same rank in $S(B_1)$ and the extension is realized $\geqq \lambda^+$ times in M.*

Proof. of 3.2. Let us assume the lemma is not correct. We shall define by induction C_i which fulfills the following conditions:

1) $C_i = \{\langle A(k,j), p(k,j)\rangle : j;\ k \leqq i\}$ where $p(i,j) \in S(A(i,j))$, $A(i,j) \supset A$, $A(ij)| = \lambda$.

2) If $p(i,j) \subsetneqq p(i',j')$ then $i < i'$ and there exists $p(i+1, j'')$ such that $p(i,j) \subsetneqq p(i+1,j'') \subseteq p(i',j')$.

3) If $p(i,j) \subsetneqq p(i+1,j')$ then $\mathrm{Rank}(p(i,j)) < \mathrm{Rank}(p(i+1,j'))$ or $|B(i,j)| > \lambda \geqq |B(i+1,j')|$, where $B(i,j)$ is the set of elements of M realizing $p(i,j)$.

4) For every i,j,i',j', $p(i,j) \subset p(i',j')$ or $p(i,j) \supset p(i',j')$ or they are contradictory (that is, $T \cup p(i,j) \cup p(i',j')$ is inconsistent);

5) $C_i \subset C_j$ for $i < j$.

We shall not prove the conditions explicitly as they are obvious from the construction.

Let $C_0 = \{\langle A, p\rangle\} = \{\langle A(0,0), p(0,0)\rangle\}$.

Let us define C_{i+1}. If $|B(i,j)| > \lambda$ then by our assumption there exists $A_1 \subset M$, $A(ij) \subset A_1$, $|A_1| = \lambda$ such that every extension of $p(i,j)$ to A_1 has a smaller rank or is realized at most λ times. Then we add to C_i $\langle A(i+1,k), p(i+1,k)\rangle$ (where $A(i+1,k) = A_1$) for every extension of $p(i,j)$, $p(i+1,k)$, which belongs to $S(A(i+1,k))$ and is realized in M. Their number is $\leqq \lambda$ as $|A_1| = \lambda$ implies $|S(A_1)| \leqq \lambda$. We do so to every $\langle A(i,j), p(i,j)\rangle \in C_i$, and we get C_{i+1}. (We have enough indices so that there will be no confusion.) It is easily seen that $|C_{i+1}| \leqq |C_i| + \lambda |C_i|$, and for every j, $|A(i+1,j)| \leqq |A(i,j')| + \lambda \leqq \lambda$ for some j'.

Now we define C_δ. Let $\langle A^1, p^1\rangle < \langle A^2, p^2\rangle$ if $A^1 \subset A^2$ and $p^1 \subset p^2$. I $\langle A^i, p^i\rangle$ $i < j$ is an increasing sequence, then $\bigcup_{i<j} \langle A^i, p^i\rangle = \langle \bigcup_{i<j} A^i, \bigcup_{i<j} p^i\rangle$. The elements of C_δ will be the elements of $\bigcup_{i<\delta} C_i$, and unions of increasing sequences in $\bigcup_{i<\delta} C_i$, $\langle A^1, p^1\rangle$, such that p^1 is realized in M.

It will now be proved that $C_{|T|^+} = C_{|T|^+ +1} = C_{|T|_+ +2} = \cdots$. It is sufficient to show that $\bigcup_i \{C_i : i < |T|^+\} = C_{|T|^+}$. That comes from the construction, for if it is not correct, there is an increasing sequence $\langle\langle A^i, p^i\rangle : i < |T|^+\rangle$. Then $\operatorname{Rank}(p_i)$ is decreasing sequence, and by Lemma 2.4 that sequence cannot be strictly decreasing, so there exists an i such that $\operatorname{Rank}(p_i) = \operatorname{Rank}(p_{i+2}) = \cdots$. By condition 3 $|\{a \in M\colon$ a realizes $p_{i+1}\}| \leqq \lambda$ (as $\operatorname{Rank}(p_i) = \operatorname{Rank}(p_{i+1})$ and $p_i \subsetneqq p_{i+1}$) and similarly $|\{a \in M\colon$ a realizes $p_{i+1}\}| > \lambda$ (as $\operatorname{Rank}(p_{i+1}) = \operatorname{Rank}({}_{pi+2})$ and $p_{i+1} \subsetneqq p_{i+2}$), a contradiction.

We shall now show that $|C_i| \leqq \lambda$ and $|A(i,j)| \leqq \lambda$ for $i \leqq |T|^+$. If not, let k be the first ordinal that contradicts our assertion. If $k = i+1$ then $|C_k| \leqq |C_i| + \lambda |C_i| \leqq \lambda$ and for every j, for some j' $|A(k,j)| \leqq |A(i,j')| + \lambda = \lambda$ (as remarked in the definition of C_{i+1}), so that k has to be a limit ordinal, and $k \leqq |T|^+$. Let $A^i = \bigcup \{A(l,j) : j; l \leqq i\}$. Now it can be seen easily that $|A^i| \leqq |C_i| \cdot \max_j |A(i,j)| \leqq \lambda$ for $i < k$, and from that, and the construction, it can be easily seen that $|A^k| \leqq \lambda$, and therefore $|S(A_k)| = \lambda$. Now the $\{B(k,j) : j\}$ are disjoint sets, and every one of them is the union of sets realizing some complete types on A_k, and by the construction $B(k,j) \neq 0$, and so the number of $B(k,j)$ is no more than λ. Thus $|C_k - \bigcup_{i<k} C_i| \leqq \lambda$. We can conclude that $|C_k| \leqq |C_k - \bigcup_{i<k} C_i| + \sum_{i<k} |C_i| \leqq \lambda + k\ \lambda = \lambda$, a contradiction; and so $|C_{|T|^+}| \leqq \lambda$, $|A^{|T|^+}| \leqq \lambda$.

For every b with $b \in B(0,0)$, the set of $\langle A(i,j), p(i,j)\rangle$ in $C_{|T|^+}$ such that $b \in B(i,j)$ is an increasing sequence in $C_{|T|^+}$. The union of the sequence is also in $C_{|T|^+}$, and so there is a last such element in $C_{|T|^+}$, $\langle A^b, p^b\rangle$. The set of elements of M realizing

p^b will be denoted by B^b. Now if there is an element of $C_{|T|+}$ greater than $\langle A^b, p^b \rangle$, then by the construction of C_i there is such an element $\langle A', p' \rangle$ such that b realizes p', in contradiction to the definition of $\langle A^b, p^b \rangle$. Therefore $|B^b| \leqq \lambda$.] Now $B = B(0,0) \subset \bigcup\{B_b : b \in B\} = \bigcup\{B(i, j) : | B(i,j) | \leqq \lambda; j; i < |T|^+\}$ $\lambda < |B| \leqq |C_{|T|+}| \cdot \lambda = \lambda$ — contradiction.

So we have proved Lemma 3.2.

It follows that without loss of generality we can assume that for every C, such that $A \subset C \subset M$ and $|C| = \lambda$, p has one and only one extension in $S(C)$ which is of the same rank, and that extension is realized at least λ^+ times. Let the set of elements of M realizing p be B.

We define by induction the sequence $\{y_i : i < \lambda^+\}$. y_0 is an arbitrary element of B. If we define y_i for every $i < j < \lambda^+$, then $y_j = y(j)$ will be an element of M that realizes the only extension of p to a type q in $S(A \bigcup \{y_i : i < j\})$ such that $\mathrm{Rank}(p) = \mathrm{Rank}(q)$. By the definitions of B and p, there is such a y_j.

LEMMA 3.3. *If* $i_1 < i_2 < \cdots < i_n < \lambda^+$, $j_1 < \cdots < j_n < \lambda^+$ *then for every predicate* R *in* T *and every* $\bar{a}$, $\bar{a} \in A$,

$$M \vDash R[y(i_1), \cdots, y(i_n), \bar{a}] \equiv R[y(j_1), \cdots, y(j_n), \bar{a}].$$

Proof of Lemma 3.3. Without loss of generality $i_k = k$.

Now, in the construction of the y_i, in every stage in $S(A \bigcup \{y_i : i < j\})$ there is only one extension p_j of p such that $\mathrm{Rank}(p_1) = \mathrm{Rank}(p)$, so the type which y_j realizes on $A \bigcup \{y_i : i < j\}$ is independent of the choice of y_j. If $\{z_i : i < j\}$ satisfies: for every i, z_i realizes a type q_i on $A \bigcup \{z_k : k < i\}$ such that $q_i \supset p$, $\mathrm{Rank}(q_i) = \mathrm{Rank}(p)$, then it can be easily proved by induction that $\langle y_{i_1}, \cdots, y_{i_n} \rangle$ satisfies the same type on A as $\langle z_{i_1}, \cdots, \cdots, z_{i_n} \rangle$. Now, if we choose y_{j_1} as the first y, and y_{j_2} as the second, etc., they will satisfy the same formulaes as $y_1, \cdots, y_n$. It remains to prove that after choosing $y_{j_1}, \cdots, y_{j_k}$ as the first k y's we can choose $y_{j_{k+1}}$ as the next y. That is, perhaps $y_{j_{k+1}}$ realized a type p on $A \bigcup \{y_{j_1}, \cdots, y_{j_k}\}$, such that $\mathrm{Rank}(\bar{p}) < \mathrm{Rank}(p)$. But if q is the type of $y_{j_{k+1}}$ on $A \bigcup \{y_1, \cdots, y_l\}$ $(l = j_{k+1} - 1)$ then $\mathrm{Rank}(p) = \mathrm{Rank}(q) \leqq \mathrm{Rank}(\bar{p}) < \mathrm{Rank}(p)$, contradiction. So Lemma 3.3 is proved.

LEMMA 3.4. *Y is indiscernible on A.*

Proof. The proof is the same as in Morley [2] 4.6, since in every cardinal χ there is an ordered set that has more than χ Dedekind cuts.

So Theorem 3.1 is proved.

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DEFINITION 3.1. Let K be a class of models, A a substructure of such models, $M \in K$ is called K-prime on A, if for every $M_1 \supset A$, $M_1 \in K$, there exists an isomorphism from M into M_1 that is the identity on A.

THEOREM 3.5. *If T is a stable theory, and $|A| \leqq \sum_{\chi<\lambda} 2^{\chi} \Rightarrow |S(A)| < 2^{\lambda}$, or $\lambda \geqq |T|^{+}$, then among the λ-saturated models of T, there is a prime model on every substructure A of a model of T.*

REMARK. An analogous theorem appears in Morley [2] 4.3.

DEFINITION 3.2. $p \in S(A)$ is called λ-isolated if there is a type $p_1 \subset p$, $|p_1| < \lambda$, such that p is the only element in $S(A)$ that includes p_1.

Proof of 3.5. In order that the model we will build on A be λ-saturated, we should realize every type of cardinality $< \lambda$, and in order that it be a prime we should realize only types which are realized in every λ-saturated model including A. So it is sufficient to show that if p is type on a set A, $|p| < \lambda$, then there exists an extension p_1 of p, $p_1 \in S(A)$, and p_1 is λ-isolated. For if it is right, we can add an element to A for every λ-isolated type. And if we continue adding such elements for every type p, $|p| < \lambda$ (by adding an element which realizes a λ-isolated complete type containing it) we shall get the wanted prime model.

Now let $\lambda \geqq |T|^{+}$ and $|p| < \lambda$ where p is a type on A. Among the elements of $S(A)$ containing p, there is a q with minimal $\mathrm{Rank}(R_0, q)$, so there are a finite number of formulaes which define the type completely with regard to R_0 (among the extension of p). We adjoin these formulaes to p, and continue with $R_1, R_2, \cdots$. Because of the compactness theorem, this operation does not lead to a contradiction at the limit. So after $|T|$ steps we get the required type — a type of power $\leqq |p| + |T| < \lambda$, which has only one extension in $S(A)$.

It remains to deal with the case $|A| \leqq \sum_{\chi<\lambda} 2^{\chi} \Rightarrow |S(A)| < 2^{\lambda}$. Let p be a type on A, $|p| < \lambda$, which contradicts our conjecture. Let $p = p_{\langle\rangle}$. If $p_{\langle\rangle}$ has more than one extension to a type in $S(A)$, then there is a formula $R(x, \bar{a})$, such that $p_{\langle 0\rangle} = p \bigcup \{R(x, \bar{a})\}$, and $p_{\langle 1\rangle} = p \bigcup \{-R(x, \bar{a})\}$ are consistent. We continue with $p_{\langle 1\rangle}$ and $p_{\langle 0\rangle}$ as with $p_{\langle\rangle}$ and can define p_η for every sequence η of ones and zeroes, $l(\eta) \leqq \lambda$, $|p_\eta| \leqq \lambda$, such that:

1) if η_1 is not is not an initial segment of η_2 or conversely, then $p_{\eta_1} \cup p_{\eta_2}$ is not consistent;

2) if η_1 is an initial segment of η_2, then $p_{\eta_1} \subset p_{\eta_2}$; and

3) if $l(\eta)$ is a limit ordinal then $p_\eta = \bigcup_{i<l(\eta)} p_{\eta|i}$. Then $\{p_\eta : l(\eta) \leqq \lambda\}$ are 2^λ contradictory types on a set of cardinality $\leqq \lambda + \sum_{\chi<\lambda} 2^\chi = \sum_{<\lambda} 2^\chi < 2^\lambda$, a contradiction.

4. On categorical elementary and pseudo elementary classes.

THEOREM 4.1. *Let M be a model of a not necessarily complete theory T, Q predicate in $L(T)$, p a type. Let $(2^{|T|})^+ = \gamma$.*

1) *If M omits the type p, and $\beth(|Q(M)|, \gamma) \leqq \|M\|$, then in every cardinal $\geqq |T|$, there is a model M_1 of T which omits p and such that $|Q(M_1)| \leqq |T|$.*

2) *If M omits the type p, and $\beth(|Q(M)|, \gamma) \leqq \|M\|$, $|Q(M)| \geqq \beth_\gamma$ then for all cardinals $\chi \geqq \lambda \geqq |T|$, there is a model M_1 of T which omits p and such that $|Q(M_1)| = \lambda$, $\|M_1\| = \chi$.*

Proof. The proof is by the methods of Morley [3] and is not given here. (Also see Vaught [9].)

REMARKS. The theorem can be slightly improved as don by Morley [2], in analogous theorems.

THEOREM 4.2. *If $p\mathrm{C}(T_1, T)$ is categorical in a cardinal $\lambda > |T_1|$, then for every χ, $|T_1| \leqq \chi < \lambda$, $K_T(\chi) = \chi^+$, and so T is stable.*

Proof. By Morley [1], 3.7 (the proof for the non-denumerable case in the same) there exists a model M of T_1, $\|M\| = \lambda$, such that for every $A \subset M$, at most $|A| + |T_1|$ types on A are realized in M, and it follows from this that the same holds for the reduct of M to $L(T)$. If $K_T(\chi) = \chi^+$, $|T_1| \leqq \chi < \lambda$, there is a reduct to $L(T)$ of a model of T_1 of cardinality λ, for which there exists $A \subset M$ satisfying $|A| = \chi$, and $> \chi$ types of $S(A)$ are realized on A in the model. This contradicts the categoricity.

THEOREM 4.3. *If $p\mathrm{C}\ (T_1, T)$ is not categorical in $\lambda_1 = \beth(\gamma \cdot \alpha) > |T_1|$* (where $\gamma = (2^{|T|})^+$, $\alpha > 0$), *then it has a non-$|T|^+$-saturated model in every cardinality. This is also true if we replace the assumption by: "$p\mathrm{C}\ T_1, T)$has a non-saturated model in λ_1".*

Proof. As any two saturated models of the same cardinality > are isomorphic (see Morley and Vaught [1]), the second assumption follows from the first.

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Let M be a non-saturated model such that $\| M \| = \lambda_1$ and M is the reduct to $L(T)$ of M_1. Then there exists $A \subset M$ $|A| < \| M \|$, and $p \in S(A)$, such that p is omitted in M. When we adjoin to M_1 the relations $Q(M) = A$ and to every predicate R of $L(T)$ $\psi_R(M_2) = \{\bar{a}: R(x, \bar{a}) \in p\}$, we get a model M_2. Now $|Q(M_2)| < \| M_2 \|$ and M_2 omits $p_1 = \{(\forall \bar{y})(\bigwedge_i Q(\bar{y}_i) \rightarrow R(x, \bar{y}) \equiv \psi_R(\bar{y})) : R$ a predicate in $L(T)\}$.

By 4.1 in every cardinality there is a model M_3 of the theory of M_2 such that $|Q(M_3)| \leq |T_1|$, and M_3 omits the type $\{R(x, \bar{a}): \bar{a}_i \in Q(M_3), M_3 \models \psi_R[a]$, R a predicate of $L(T)\}$ which is a type on a set of cardinality $\leqq |T_1|$ (its consistency follows from the theory of M_2), and this proves the theorem.

LEMMA 4.4. 1) *If* $|T_1| \leqq \lambda$, λ *is regular, and* $|A| < \lambda \Rightarrow |S^T(A)| \leqq \lambda$, *then* $p \subset T_1, T)$ *has a saturated model in* λ.

2) *If* $|T_1| \leqq \lambda$, $\mu \leqq \lambda$, μ *is regular and* $|A| \leqq \lambda \Rightarrow |S^T(A)| \leqq \lambda$, *then* $p \subset (T_2, T)$ *has a* λ*-saturated model in* λ.

Proof. Since the proofs are essentially similar, we prove only 1). Let $T_1 = \bigcup_{i < |T_1|} T_1^i$ where $T_1^i \subset T_1^j$ if $i < j$ and $T_1^i = T_1$ for $i \geqq |T_1|$ and $|T_1^i| < \lambda$. By the conditions in 1 we can easily define a sequence $\langle M^i : i \leqq \lambda \rangle$ such that: $|i| \leqq \| M^i \| < \lambda$; M^i as a model of T_1^i; if $i < j$ then the reduct of M^j to $L(T_1^i)$ is an elementary extension of M^i; the $L(T)$-types on M^i are $< p_j^i : j < j_0 \leqq \lambda \rangle$ and p_j^i is realized in M^j; $M^\delta = \bigcup_{i < \delta} M^i$. M_λ is the required model.

COROLLARY. 1) *If* T *is not stable then* $p \subset (T_1, T)$ *has a saturated model in a regular cardinal* $\lambda \geqq |T_1|$ *iff* $\chi < \lambda = 2^\chi \leqq \lambda$.

Proof. Suppose there exists $\chi_1 < \lambda < 2^{\chi_1}$. Let $\chi = \inf\{\chi : 2^\chi > \lambda\}$. As T is not stable, by Theorem 2.1, there exists A, $|A| \leqq \sum_{\mu < \chi} 2^\mu \leqq \lambda$ such that there exists $2^\chi > \lambda$ contradicting types of power $\chi < \lambda$ on A. If T has a saturated model M of power λ, then there is $A' \subset M$, with A' isomorphic to A. Thus in M more than $\| M \|$ contradicting types have to be realized, a contradiction. The opposite direction in trivial by 4.4.1, since always, $S(A) \leqq 2^{|A|| + |T|}$.

THEOREM 4.5. 1) *If* $|T_1| = \aleph_\alpha$ *and* T *is not stable then the number of isomorphism types of* $p \subset (T_1, T)$ *in* $\aleph_\beta$ *is at least* $|\beta - \alpha|$.

2) *If* $|T_1| = \aleph_\alpha$ *and* T *is not super stable, then the number of isomrophism types of* $p \subset T_1,(T)$ *in* $\aleph_\beta$ *is at least* $|(\beta - \alpha)/\omega|$.

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3) *If* $p \subset (T_1, T)$ *is categorical in a cardinal* $> |T_1|$, *different from* $\inf\{\chi: \chi \geqq T, \chi^{\aleph_0} > \chi + |T|\}$, *then a)* T *is superstable, b)* $K_T(\lambda) = \lambda^+$ *for* $\lambda \geqq |T|$. c) $p \subset (T_1, T)$ *is categorical in a cardinal* $> |T_1|$ *iff all models in it are saturated, and* d) $p \subset (T_1, T)$ *is caterogical in* $\beth(\gamma \cdot \alpha)$ *for* $\gamma = (2^{|T|})^+$, $\alpha > 0$, $\beth(\gamma.\alpha) > |T_1|$.

Proof. 1, 2) If $K_T(\lambda) > \lambda^+$, then for every $\chi \geqq \lambda^+$ there is a model M in $p\subset(T_1, T)$ such that there exists a set A with more than $|A|$ types realized on it in M, $|A| = \lambda$, and there is no such set of greater cardinality. (The existence is proved as in 4.2.)

3) By 4.2, for every χ with $|T_1| \leqq \chi < \lambda$, $K_T(\chi) = \chi^+$. So, if λ is regular, then there is a model in $p \subset (T_1, T)$ of cardinality λ which is saturated by Lemma 4.1.2. If λ is singular, then $\lambda > \chi = \inf\{\chi: \chi \geqq |T|, \chi^{\aleph_0} > \chi\}$, and so $K_T(\chi) = \chi^+$, and as $\chi^{\aleph_0} > \chi$, this implies that T is super stable. As $K_T(|T_1|) = |T_1|^+$, by 2.2 $K_T(\lambda) = \lambda^+$, and so by Lemma 4.4.1 $p \subset (T_1, T)$ has a $|T_1|^+$ -saturated model in λ. Therefore by 4.3, $p \subset (T_1, T)$ is categorical in $\beth(\gamma \cdot \alpha)$ ($\gamma = (2^{|T|})^+$, $\alpha > 0$), and so $K_T(\mu) = \mu^+$ for every $\mu \geqq |T_1|$. That implies by Lemma 4.4, that in every power $\mu > |T_1|$ and regular $\chi \leqq \mu$, there exists a model of power μ in $p\subset(T_1, T)$, which is χ-saturated. So if $p\subset(T_1\ T)$ is categorical in μ, its only model in μ is saturated. It is clear that if $p\subset(T_1, T)$ has only saturated models in $\mu > |T|$, then it is categorical in μ.

Remark. In 4.3 and 4.5.3 we apply a two-cardinal theorem to a categoricity theorem. In fact, a more general connection exists among the following conditions on χ, λ, μ ($\chi \leqq \lambda, \mu$):

1) If $|T| < \chi$ and T has a model which omits a type p and such that $\|M\| = \lambda$, $|Q(M)| < \lambda$, then T has a model M' which omits p such that $\mu = \|M'\| > |Q(M')|$.

2) If $|T_1| < \chi$ and $p\subset(T_1\ T)$ is categorical in μ, then it is categorical in λ.

3) If $|T_1| < \chi$ and every model of power μ in $p \subset (T_1, T)$ is homogeneous, then the same holds for λ.

1 implies 3. (Keisler proves this in [1].) 1 implies 2 if $\mu \neq \inf\{\lambda_1 : \lambda_1^+ \geqq \chi, \lambda_1^{\aleph_0} \geqq \lambda_1\}$ or if $\chi_1 < \chi \Rightarrow \aleph(\chi_1, \omega) < \chi$. 3 implies 1 if χ is not greater than the first measurable cardinal, and there is no weakly compact χ_1 such that $\chi_1 < \chi \leqq (2^{\chi_1})^+$. 2 implies 1 if in addition $\mu \neq \inf\{\lambda_1 : \lambda_1^+ \geqq \chi, \lambda_1^{\aleph_0} \geqq \lambda\}$ or $\chi_1 < \chi \Rightarrow \aleph(\chi_1, \omega) < \chi$.

Theorem 4.6. *If T is categorical in a power λ, $\lambda > |T|$, $\lambda \neq \inf\{\chi: \chi^{\aleph_0} > \chi + |T|\}$, then there exists a cardinal λ_0, such that T is categorical in every cardinal*

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$\geqq \lambda_0$, *and is not categorical in any power* χ, $|T| < \chi < \lambda_0$. *Furthermore* λ_0 *is such that* $\lambda_0 < \mu(|T|) < \beth((2^{|T|})^+)$.

Proof. If for every $\chi < \mu(|T|)$ T has a model M which is not $|T|^+$-saturated, $\|M\| \geqq \chi$, then it has such a model in every cardinal $> |T|$ a contradiction by 4.4. (For if $A \subset M$, $|A| \leqq |T|$, $p \in S(A)$, and p is omitted, we adjoin to M the constants $\{c_i : i < |T|\}$ a names for the element of A, and relations as in 4.3, and the result follows by the definition of $\mu(|T|)$.) Now if M is a $|T|^+$-saturated but not saturated model of T, then there exists $A \subset M$, $|A| < \|M\|$, $|A| > |T|$, $p \in S(A)$, such that p is omitted. As $K_T(|A|) = |A|^+$ and $\|M\| > |A| > |T| \Rightarrow \|M\| > |T|^+$, there exists an indiscernible set Y over A, $|Y| = |A|^+$, by Theorem 3.1. If $Y = \{y_i : i < |A|^+\}$, let $B = A \cup \{y_i : i < \chi\}\}$, where $\{y_i : i < \chi\}$ is indiscernible over A, and M_1 be a prime model over B among the $|T|^+$-saturated models, which exists by 3.2. Now it will be proved that p is not realized in M_1. In the construction of M_1, we adjoined to B the elements of $\{c_i : i < |B|\}$ one after another, such that c_j realizes a $|T|^+$-isolated type on $B \cup_i \{c_i : i < j\}$, defined by p_j, $|p_j| < |T|^+$. If c_k realizes p, let $B_1 = \{c_k\}$, and $B_{i+1} = B_i \cup \{b : b$ is mentioned in p_l, and $c_l \in B_i\}$. Now $|\bigcup_i B_i| \leqq |T|$, and it can be easily seen that in a prime model over $A \cup (\{y_i : i < \chi\} \cap \bigcup_{i<\omega} B_i)$, p is realized, and so it is realized in M a contradiction, so p is not realized in M_1. As we can take $\chi = \beth_\gamma$, $(2^{|T|})^+ | \gamma, \chi > |A|$, it follows that T has a non-saturated model in χ, in contradiction to 4.3, 4.4.3. So every $|T|^+$-saturated model is saturated. If T is not categorical in λ_1, then it has a non-$|T|^+$-saturated model of cardinality λ_1, and so T is not categorical in any cardinal λ_2. $|T| < \lambda_2 \leqq \lambda_1$. As we have shown that there exits a cardinal $\lambda < \mu(|T|)$ in which every model of T is $|T|^+$-saturated, the theorem follows.

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