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NON-SPECIAL ARONSZAJN TREES ON $\aleph_{\omega+1}$

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ABSTRACT

We continue our research on the relative strength of L-like combinatorial principles for successors of singular cardinals. In [3] we have shown that the existence of a λ^+-special Aronszajn tree does not follow from that of a λ^+-Souslin tree. It follows from [5], [4] and [6] that under G.C.H. $\square_\lambda$ does imply the existence of a λ^+-Souslin tree. In [2] we show that $\square_\lambda$ does not follow from the existence of a λ^+-special Aronszajn tree. Here we show that the existence of such a tree implies that of an 'almost Souslin' λ^+-tree. It follows that the statement "All λ^+-Aronszajn trees are special" implies that there are no λ^+-Aronszajn trees.

THEOREM 1. *If there is a λ^+-special Aronszajn tree and λ is a singular strong limit cardinal $2^\lambda = \lambda^+$, then there is a (λ^+, ∞) distributive Aronszajn tree on λ^+.*

COROLLARY. *If there are λ^+-Aronszajn trees, λ as above, then there are non-special λ^+-Aronszajn trees.*

PROOF OF THE COROLLARY. Just note that a (λ^+, ∞) distributive tree cannot be special, forcing with such a tree (as a partial order) adds no sets of size $\leqq \lambda$ to the universe, so such a forcing does not collapse λ^+. On the other hand, if T is special and $f: T \to \lambda$ one-to-one on each branch, the specializing function and η is a generic branch through T, then $|\eta| = \lambda^+$ and $f \restriction \eta$ is a one-to-one function to λ. Thus forcing with a λ^+-special tree collapses λ^+.

Let $\boxtimes_\lambda$ (a square with a built-in diamond) denote the following combinatorial principle: There exists a $\square_\lambda$ sequence $\langle C_\alpha : \alpha \in \lim \lambda^+ \rangle$ and a $\diamondsuit_{\lambda^+}$ sequence $\langle S_\alpha : \alpha \in \lim \lambda^+ \rangle$ s.t. for any $X \subseteq \lambda^+$ for every closed unbounded $C \subseteq X^+$ and for every $\delta < \lambda$ there is some $\alpha < \lambda^+$ s.t. $\mathrm{otp}(C_\alpha) \geqq \delta C_\alpha \subseteq C$ and for every $\beta \in C'_\alpha \cup \{\alpha\}$, $X \cap \beta = S_\beta$.

Shelah has proved that for a strong limit singular λ, if $2^\lambda = \lambda^+$ then $\square_\lambda \to \boxtimes_\lambda$ [1].

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We shall use a modification of $\boxtimes_\lambda$. Let $\boxtimes^*_\lambda$ denote the existence of a weak square sequence $\langle A_\alpha : \alpha \in \lim(\lambda^+)\rangle$ and a $\diamondsuit'_{\lambda^+}$ sequence $\langle B_\alpha : \alpha \in \lim(\lambda^+)\rangle$ with enumerations

$$A_\alpha = \{a^i_\alpha : i < \lambda\}, \qquad B_\alpha = \{b^i_\alpha : i < \lambda\}$$

s.t. for all i, α $\mathrm{otp}(a^i_\alpha) < \lambda$, a^i_α cofinal in α, $b^i_\alpha \subseteq \alpha$ and for any $X \subseteq \lambda^+$ for every c.u.b. $C \subseteq \lambda^+$ and every $\delta < \lambda$ there is some $a^i_\alpha \subseteq C$ $\mathrm{otp}(a^i_\alpha) > \delta$ and for all $\beta \in (a^i_\alpha) \cup \{\alpha\}$, $a^i_\alpha \cap \beta \in A_\beta$ and $X \cap \beta \in B_\beta$.

LEMMA 1. *Let λ be a strong limit singular cardinal $2^\lambda = \lambda^+$ then $\boxtimes^*_\lambda$ follows from the existence of a λ^+ special Aronszajn tree.*

PROOF. By Jensen [5] the existence of such a tree is equivalent to $\square^*_\lambda$. Imitating the proof of $\square_\lambda \to \boxtimes_\lambda$ (th. 2.3 of [1]) one can easily get $\square^*_\lambda \to \boxtimes^*_\lambda$ (for λ as assumed by the lemma).

PROOF OF THE THEOREM. Assume $\boxtimes^*_\lambda$ and let us construct a (λ^+, ∞) distributive Aronszajn tree.

By Lemma 1 this will establish our theorem.

DEFINITION OF THE TREE. We define $T \upharpoonright (\alpha + 1)$ by induction on $\alpha < \lambda^+$.

α successor: For any node $X \in (T \upharpoonright \alpha)_{\alpha-1}$ (the last level of $T \upharpoonright \alpha$) add λ many immediate successors.

α limit: (i) We fix a one-one mapping of $\lambda^+ \times \lambda^+$ onto λ^+, through this mapping we regard each member of our $\diamondsuit$ part of the $\boxtimes^*$ sequence as a set of pairs $b^i_\beta \subseteq \beta \times \beta$, define $b^i_{\beta j}$ to be its projection on j, $b^i_{\beta j} = \{\gamma : \langle j\gamma \rangle \in b^i_\beta\}$. W.l.o.g. the nodes of T are ordinals in λ^+ and $T \upharpoonright \alpha \subseteq \alpha$ (where $T \upharpoonright \alpha = \bigcup_{\beta<\alpha} T \upharpoonright \beta$) for each $x \in T \upharpoonright \alpha$, $\delta < \lambda$ and $\langle ij \rangle \in \lambda \times \lambda$ we define a branch in $T \upharpoonright \alpha$ extending x, $\eta^{\langle ij \rangle}_{x,\delta}$ by induction. $\eta^{\langle ij \rangle}_{x,\delta}(0) =$ the δ's immediate successor of x. $\eta^{\langle ij \rangle}_{x,\delta}(\xi + 1) =$ the first ordinal that is above $\eta^{\langle ij \rangle}_{x,\delta}(\xi)$ (in the order of $T \upharpoonright \alpha$) s.t. its level is above $a^i_\alpha(\xi)$ (the ξ member of the $\square^*$ seq. a^i_α) and it belongs to $b^j_{\alpha\xi}$ (the ξth projection of the jth member of β_α).

If there is no such node we terminate the branch. At a limit ξ we pick the first node above $\bigcup_{\rho<\xi} \eta^{\langle ij \rangle}_{x,\delta}(\rho)$, if there is such a node, otherwise we terminate the branch.

(ii) We fix throughout the construction of T a $\diamondsuit_{\lambda^+}$ seq. $\langle S_\alpha : \alpha \in \lambda^+ \rangle$ (the existence of such a diamond seq. is guaranteed by our assumptions on λ).

Now we define the α's level of $T \upharpoonright (\alpha + 1)$ by adding a node on top of each $\eta^{\langle ij \rangle}_{x,\delta}$ that is cofinal in $T \upharpoonright \alpha$ iff $\eta^{\langle ij \rangle}_{x,\delta} \neq S_\alpha$ (as sets of ordinals).

This completes the definition of T. Let us show that it realizes our intentions.

LEMMA 2. *The construction can be carried on for all $\alpha < \lambda^+$. We prove by induction on α for every $x \in T \restriction \alpha$ there are λ-many members of $T_\alpha = (T \restriction (\alpha+1))_\alpha$ above it.*

If α is a successor it follows immediately from the definition of $(T \restriction (\alpha+1))_\alpha$. For a limit α pick any $a^i_\alpha \in A_\alpha$ s.t. $a^i_\alpha \cap \beta \in A_\beta$ for all $\beta \in (a^i_\alpha)'$, w.l.o.g. we can assume that for every $\alpha < \lambda^+$, $b^0_{\alpha\xi} = \alpha$ for all $\xi < \mathrm{otp}(a^i_\alpha)$.

For each $x \in T \restriction \alpha$ the set $\{\eta^{\langle i0 \rangle}_{x,\delta} : \delta < \lambda\}$ has size λ. The only possible reason for a termination of any branch there before it reaches α, is if for some β, a limit point of a^i_α, $\eta^{\langle i0 \rangle}_{x,\delta} \restriction \beta = S_\beta$; as $|a^i_\alpha| < \lambda$ this may happen for less than λ of these branches.

LEMMA 3. *T is (λ^+, ∞) distributive.*

As λ is singular it is enough to show (λ, ∞) distributively. Let $\langle D_\alpha : \alpha < \mu < \lambda \rangle$ be a list of dense open subsets of T. For each $\alpha < \mu$ there is a c.u.b. $C_\alpha \subseteq \lambda^+$ s.t. $\beta \in C_\alpha \rightarrow D_\alpha \cap \beta$ is dense in $T \restriction \beta$. Let $C = \bigcap_{\alpha<\mu} C_\alpha$.

By the properties of $\boxtimes$, for every $x \in T$ we can find $\alpha < \lambda^+$ s.t. $x \in T \restriction \alpha$ and: for some $a^i_\alpha \in A_\alpha$, $a^i_\alpha \subseteq C$, $\mathrm{otp}(a^i_\alpha) > \mu$ and for all $\delta \in (a^i_\alpha)' \cup \{\alpha\}$, $a^i_\alpha \cap \delta \in A_\delta$ and $X \cap \langle \delta \times \delta \rangle \in B_\delta$ where $X = \{\langle \gamma, \xi \rangle : \gamma < \mu,\ \xi \in D_\gamma\}$.

Let j be s.t. $X \cap \alpha = b^j_\alpha$. As $b^j_\alpha = X \cap \alpha$ we get for all $\xi < \mu$ $b^j_{\alpha\xi} = D_\xi \cap \alpha$. As $\mathrm{otp}(a^i_\alpha) > \mu$, if there is a branch of the form ${}^\alpha\eta^{\langle ij \rangle}_{x,\delta}$ cofinal in $T \restriction \alpha$ this branch intersects each of the D_ξ's. In the definition of $T \restriction (\alpha+1)$ we have added a node y on top of this branch so $x < y \in \bigcap_{\xi<\mu} D_\xi$. Let us check that such a cofinal branch does exist.

Our definition of the η's was uniform enough to guarantee that for $\beta \in a^i_\alpha$ if $a^{i'}_\beta = a^i_\alpha \cap \beta$ and $b^{j'}_\beta = b^j_\alpha \cap \beta$ then ${}^\beta\eta^{\langle i'j' \rangle}_{x,\delta} = {}^\alpha\eta^{\langle ij \rangle}_{x,\delta} \cap \beta$. (Note that as $a^i_\alpha \subseteq C$ each $D_\xi \cap \beta$ is dense in $T \restriction \beta$.) We will use double induction. By induction on $\beta \in a^i_\alpha$ we prove that all but $\leqq |\mathrm{otp}(a^i_\alpha | \beta)|$ of the ${}^\beta\eta^{\langle i'j' \rangle}_{x,\delta}$ are cofinal in $T | \beta$ for $\langle i', j' \rangle$ s.t. $a^i_\alpha \cap \beta = a^{i'}_\beta$ and $b^j_\alpha \cap \beta = b^{j'}_\beta$. This is proven by showing that ${}^\beta\eta^{\langle i'j' \rangle}_{x,\delta}(\xi)$ is defined for all $\xi < \mathrm{otp}\, a^{i'}_\beta$ and this by induction on ξ.

β limit point in a^i_α: Pick $\langle i'j' \rangle$ such that $a^i_\alpha \cap \beta = a^{i'}_\beta$, $b^i_\alpha \cap \beta = b^{j'}_\beta$ use the first induction hypothesis and the definition of the $(\beta+1)$'s level of T.

β successor in a^i_α: Here we use induction on $\xi < \mathrm{otp}\, a^{i'}_\beta$. As $\beta \in a^i_\alpha \subseteq C$ each $D_\xi \cap \beta$ is dense in $T \restriction \beta$ so the only obstacle that may stop ${}^\beta\eta^{\langle i'j' \rangle}_{x,\delta}$ from being cofinal in $T \restriction \beta$ are the demands of the diamond seq. S_γ. S_γ terminates, at stage γ, at most one branch; as $\mathrm{otp}(a^{i'}_\beta) < \lambda$ almost all of our branches reach their full length and are confinal in $T \restriction \beta$.

LEMMA 4. *T is a λ^+-Aronszajn tree.*

PROOF. It is clear by the definition of T that the cardinality of each level is at most λ.

By Lemma 2 the height of T is λ^+. It remains to show that there is no cofinal branch in T.

Assume that η is such a branch; as $|T| = \lambda^+$ we can regard T as a subset of λ^+ so η is a subset of λ^+. There is a closed unbounded subset of λ^+, C, s.t. for $\alpha \in C$, $\eta \restriction \alpha$ (the first α members of η in the order of T) equals $\eta \cap \alpha$ (as subsets of λ^+). $\langle S_\alpha : \alpha < \lambda^+ \rangle$ is a $\diamondsuit_{\lambda^+}$ seq. so for some stationary $S \subseteq \lambda^+$, $\eta \cap \alpha = S_\alpha$ for all $\alpha \in S$. Pick $\alpha \in S \cap C$; for such an α, $\eta \restriction \alpha = S_\alpha$ so by the definition of the $(\alpha + 1)$th level of T, $\eta \restriction \alpha$ has no extension in T, contradicting the assumption that η was unbounded in T.

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