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SOUSLIN FORCING

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Abstract. We define the notion of Souslin forcing, and we prove that some properties are preserved under iteration. We define a weaker form of Martin's axiom, namely $\mathrm{MA}(\Gamma^+_{\aleph_0})$, and using the results on Souslin forcing we show that $\mathrm{MA}(\Gamma^+_{\aleph_0})$ is consistent with the existence of a Souslin tree and with the splitting number $s = \aleph_1$. We prove that $\mathrm{MA}(\Gamma^+_{\aleph_0})$ proves the additivity of measure. Also we introduce the notion of proper Souslin forcing, and we prove that this property is preserved under countable support iterated forcing. We use these results to show that ZFC + there is an inaccessible cardinal is equiconsistent with ZFC + the Borel conjecture + Σ^1_2-measurability.

Annotated table of contents.

§0. Introduction. In this work we will present a systematic treatment of the forcing notion which has a Souslin definition. We define explicitly when a forcing

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notion has a Souslin definition, and we will center our attention on iteration of such forcing notions. It is well known that the Cohen real forcing has a very simple definition, but this is not the unique partial order which satisfies these requirements; for example in Shelah [Sh1, §6] one uses the simplicity of the definitions of the random real forcing and of the amoeba forcing in order to give a model for "every Δ^1_3-subset of reals is Lebesgue measurable" from a model of ZFC. Also in Ihoda [Ih1] these facts are used in order to show that Cons(ZFC) implies Cons(ZFC + "every Δ^1_3-subset of reals is Lebesgue measurable, has the property of Baire and is Ramsey"). Also in Shelah [Sh3] there appears the notion of Borel forcing, in order to build a model for the splitting number $s = \aleph_1$ and the dominating number $d = 2^{\aleph_0} > \aleph_1$. Also in Miller [Mi] this type of argument was considered. In §1 we generalize these ideas, and we introduce the definition of Souslin forcing, proving a series of lemmas for finite support iterated forcing of Souslin forcing satisfying the countable chain condition. We prove that a very nice property, which is satisfied for c.c.c. Souslin forcing, is preserved under the finite support iteration. This property is used in §3 to show that an apparently weaker form of Martin's Axiom does not decide the Souslin problem and the minimal cardinal for which there exists a splitting family of this cardinality. Using this, and the fact that this part of Martin's Axiom implies the additivity of measure, Ihoda [Ih2] proved that "every Σ^1_2-subset of reals is Ramsey" is not a consequence of "every Σ^1_2-subset of reals is Lebesgue measurable".

In the Ph.D. research of the first author there appears the problem of building a model for the Borel conjecture in which it is possible to find a rapid filter on ω. Models for the Borel conjecture were well known from the works of Laver [La] and Baumgartner [Ba], and using the ideas of Shelah [Sh1], Raisonier [Ra] built a rapid filter from the following hypothesis:

(i) $\omega_1^L = \omega_1$ and

(ii) every Σ^1_2-set of reals is Lebesgue measurable.

Clearly (i) holds in the Laver model, but in Ihoda [Ih3] it was proved that (i) + (ii) + "Borel's conjecture" is inconsistent.

Therefore the following question arises: is Borel's conjecture + "every Σ^1_2-set of reals is Lebesgue measurable" consistent?

In §4 we prove that this last question is equivalent to asking: is "ZFC + there exists an inaccessible cardinal" consistent? In this proof we used the result given in §2 on proper Souslin forcing, and countable support of proper Souslin forcing.

0.1. Definition. A *tree* is a partially ordered set $(T, \leq_T)$ such that for every $t \in T$ the set $\{s \in T: s <_T t\}$ is well-ordered.

A *branch* of a tree T is a maximal chain of T, where a subset $A \subseteq T$ is a chain if and only if for every $t, s \in A$ we have $t \leq_T s$ or $s \leq_T t$. If $s \in T$, then

$$T_s = \{t \in T: s \leq_T t \vee t \leq_T s\}.$$

Clearly if T is a tree then T_s is a tree.

If T is a tree, then

$$[T] = \{A: A \text{ is a branch of } T\}.$$

0.2. Definition. We say that $T \subseteq \lambda^{<\omega} \times \cdots \times \lambda^{<\omega} \times \omega^{<\omega}$ (λ an ordinal) is a *tree* if and only if $\langle T, \subseteq \rangle$ is a tree. If $X \in \lambda^{\omega} \times \cdots \times \lambda^{\omega} \times \omega^{\omega}$ and $T \subseteq \lambda^{<\omega} \times \cdots \times \lambda^{<\omega} \times \omega^{<\omega}$ (the number of coordinates of X is less than the number of coordinates of some member of T), then

$$T(X) = \{s: \langle s, X \mid |s| \rangle \in T\}.$$

Clearly, if T is a tree, then $T(X)$ is a tree. If $T \subseteq \lambda^{<\omega} \times \cdots \times \lambda^{<\omega} \times \omega^{<\omega}$ is a tree then we say that T is a *λ-tree*.

0.3. Definition. If T is a λ-tree, then the *λ-Souslin set defined by T* is

$$D(T) = \bigcup \{[T(X)]: X \in \omega^{\omega}\}.$$

0.4. Theorem (Well known). *Let $V_1 \subseteq V_2$ be a model of some part of ZFC, and suppose that $T \in V_1$ and $\langle \lambda_i: i < \omega \rangle$ belong to $\lambda^{\omega} \cap V_1$. Then $V_1 \models$ "$\langle \lambda_i: i < \omega \rangle \in D(T)$" if and only if $V_2 \models$ "$\langle \lambda_i: i < \omega \rangle \in D(T)$".*

For the proof, see Jech [Je]. □

0.5. Definition. Let P be a notion of forcing.

(i) We say that P *belongs to* Γ_λ if and only if there exists trees T_1 and T_2 such that $T_1 \subseteq \lambda^{<\omega} \times \omega^{<\omega}$, $T_2 \subseteq \lambda^{<\omega} \times \lambda^{<\omega} \times \omega^{<\omega}$, and $P = D(T_1)$ and $\leq_P = D(T_2)$.

(ii) We say that P *belongs to* Γ_λ^+ if and only if P belongs to Γ_λ and there exists a tree $T_3 \subseteq \lambda^{<\omega} \times \lambda^{<\omega} \times \omega^{<\omega}$ such that

$$\{\langle p, q \rangle: p \in P \text{ and } q \in P \text{ and } p, q \text{ are incompatible in } \leq_P\} = D(T_3).$$

In this case we say that $\langle T_1, T_2, T_3 \rangle$ *witnesses* $P \in \Gamma_\lambda^+$.

0.6. Definition. We say that P is a Souslin forcing notion if and only if there exists an ordinal λ such that $P \in \Gamma_\lambda^+$.

0.7. Definition. If P and Q are forcing notions, then we say that $P \lessdot Q$ if and only if $P \subseteq Q$ and every maximal antichain of P is a maximal antichain of Q.

0.8. Definition. Let $V_1 \subseteq V_2$ be models of ZFC* and let T_1, T_2, T_3 be λ-trees, $\langle T_1, T_2, T_3 \rangle$ witnessing $P \in \Gamma_\lambda^+$. Then, for $i = 1, 2$,

$$P^{V_i} = \{p: V_i \models \text{"} p \in D(T_1) \text{"}\}, \qquad \leq_P^{V_i} = \{\langle p, q \rangle: V_i \models \text{"} \langle p, q \rangle \in D(T_2) \text{"}\}.$$

$\langle p, q \rangle$ incompatible in V_i if and only if $V_i \models$ "$\langle p, q \rangle \in D(T_3)$". (From this, in §1 we will ask when $P^{V_1} \lessdot P^{V_2}$, i.e., does every maximal antichain of P^{V_1} which lies in V_1 have to be a maximal antichain of P^{V_2}?)

0.9 Remark. Suppose that $P \lessdot Q$ and τ is a P-name. Then, without loss of generality, τ is a Q-name and if $G \subseteq Q$ is a generic filter then $\tau[G] = \tau[G \cap P]$, where $G \cap P$ is a generic filter for P. This fact will be continually used without special remark.

0.10. Definition. (i) Let P be a forcing notion and let τ be a P-name for an α-sequence of ordinals. We say τ is a *simple P-name* if and only if for every $i < \alpha$ there exists a maximal antichain $A_i \subseteq P$ and $\langle \alpha_p: p \in A_i \rangle$ such that

$$\langle q, \boldsymbol{\beta}, \mathbf{i} \rangle \in \tau \text{ if and only if } (\exists i < \alpha \exists p \in A_i)(\langle q, \boldsymbol{\beta}, \mathbf{i} \rangle = \langle p, \dot{\alpha}_p, \dot{i} \rangle).$$

(ii) Let $\bar{Q} = \langle P_\alpha; \mathbf{Q}_\alpha: \alpha < \beta \rangle$ be a β-system of finite support iterated forcing satisfying, for every $\alpha < \beta$,

$$P_\alpha \Vdash \text{"}\mathbf{Q}_\beta \text{ is a Souslin forcing notion"}.$$

We will define by induction on β when $p \in P_\beta = \lim(\bar{Q})$ is a simple member of P_β, and we show that

$$\underline{P}_\beta = \{p: p \text{ is a simple member of } P_\beta\}$$

is a dense subset of P_β.

If $\beta = 0$, this is clear.

If $\beta = \bigcup \beta \neq 0$ then $p \in P_\beta$ is a simple member of P_β if and only if for every $\alpha < \beta$, $p | \alpha$ is a simple member of P_β. It is clear that $\underline{P}_\beta$ is dense in P_β.

If $\beta = \alpha + 1$ then $p \in P_\beta$ is a simple member of P_β if and only if $p | \alpha$ is a simple member of P_α and $p(\alpha)$ is a simple $\underline{P}_\alpha$-name of a member of $\mathbf{Q}_\alpha$. It is not difficult to show that $\underline{P}_\beta$ is dense in P_β. □

In our argument we will need to work with countable transitive models of some rich part of set theory, a fragment of ZFC sufficiently rich in order to develop the method of forcing without problems. We call this part ZFC*. In all partial orders we assume that 0 is the minimal element. We presume that the reader has a good knowledge of iterated forcing as it is presented in [Ba] or [Sh2].

§1. Γ_λ^+ with countable chain condition.

1.1. Let $\bar{Q} = \langle P_i; \mathbf{Q}_i : i < \alpha \rangle$ be an α-system of finite support iterated forcing satisfying the following conditions:

(i) For each $i < \alpha$, there are P_i-simple names $\mathbf{T}_1^i, \mathbf{T}_2^i, \mathbf{T}_3^i$ of trees such that

$$\Vdash_{P_i} \text{“}\langle \mathbf{T}_1^i, \mathbf{T}_2^i, \mathbf{T}_3^i \rangle \text{ witnesses } \mathbf{Q}_i \text{ belongs to } \Gamma_\lambda^+\text{”}.$$

(ii) $\Vdash_{P_i}$ “$\mathbf{Q}_i$ satisfies countable chain condition”.

(iii) The elements of P_i are finite functions of simple names.

In this case we say that $\bar{Q}$ is a *Γ_λ^+-c.c.c-α-iteration.*

1.2. Let $\bar{Q} = \langle P_i; \mathbf{Q}_i : i < \alpha \rangle$ be a Γ_λ^+-c.c.c-α-iteration, and let $S \subseteq \alpha$ be closed. By induction on α we will define and prove, simultaneously, the following

1.3. Definition. S is closed for $\bar{Q}$, and $\bar{Q} | S$.

1.4. Lemma. *If S is closed for $\bar{Q}$, then*:

(i) $\bar{Q} | S \subseteq \bar{Q}$;

(ii) *for* $p, q \in \bar{Q} | S$

(a) $p \leq q$ *in* $\lim \bar{Q} | S$ *if and only if* $p \leq q$ *in* $\lim \bar{Q}$, *and*

(b) *p and q are compatible in* $\lim \bar{Q} | S$ *if and only if they are compatible in* $\lim \bar{Q}$;

(iii) *$Q | S$ satisfies the countable chain condition.*

1.5. Lemma. *If $G_\alpha \subseteq P_\alpha$ is generic over V for $P_\alpha = \lim \bar{Q}$, then $G_\alpha \cap \lim \bar{Q} | S$ is generic over V for $\lim \bar{Q} | S$.*

Proof and Definition.

Case 1. $\alpha = 0$, clear.

Case 2. $\alpha = \beta + 1$: $S \cap \beta$ is closed for $\bar{Q} | \beta$. Let $G_\beta \subseteq P_\beta$ be generic over V. Let $V_1 = V[G_\beta]$ and $V_2 = V[G_\beta \cap \lim \bar{Q} | S \cap \beta]$; by the induction hypothesis $G_\beta \cap \lim \bar{Q} | S \cap \beta = G'_\beta$ is generic over V for $\lim \bar{Q} | S \cap \beta$.

1.3. S will be closed for $\bar{Q}$ if and only if $S \cap \beta$ is closed for $\bar{Q} | \beta$ and if $\beta \in S$ then

$$\Vdash_{P_\beta} \text{“}\langle \mathbf{T}_1^\beta, \mathbf{T}_2^\beta, \mathbf{T}_3^\beta \rangle \text{ belongs to } \mathbf{V}_2\text{”}$$

(equivalent $\mathbf{T}_i^\beta$ is a $P_\beta | S \cap \beta$-name).

Now if $S \cap \beta = S \cap \alpha$ then $\bar{Q} | S = \bar{Q} | S \cap \beta$, and if $\beta \in S$ then $p \in \bar{Q} | S$ if and only if $p \in \bar{Q}$ and $p | \beta \in \bar{Q} | S \cap \beta$ and $\Vdash_{\lim Q|S \cap \beta}$ "$p(\beta) \in \mathbf{Q}_\beta$" (in other words $V_1 \models$ "$\mathbf{p}(\beta) \in V_2$").

1.4. (i) Let $p \in \bar{Q} | S$; by definition

$$\Vdash_{\lim Q|S \cap \beta} \text{“}p(\beta) \in \mathbf{Q}_\beta\text{”},$$

therefore $V_2 \models (\exists x)(p(\beta)[G'_\beta] \in [\mathbf{T}_1^\beta[G'_\beta](x)])$, and this expression is absolute for models of dependent choice, so we have

(i) $V_1 \models$ "$(\exists x)(p(\beta)[G'_\beta] \in [T_1^\beta[G'_\beta](x)])$",

(ii) $p(\beta)[G'_\beta] = p(\beta)[G_\beta]$, and

(iii) $T_1^\beta[G'_\beta] = T_1^\beta[G_\beta]$;

thus

$$V_1 \models \text{“}(\exists x)(p(\beta)[G_\beta] \in [T_1^\beta[G_\beta](x)])\text{”},$$

and this implies that $V_1 \models$ "$p(\beta) \in \mathbf{Q}_\beta$". Because G_β was arbitrary, this proves that $p \in \lim \bar{Q}$. Similar arguments show that 1.4(ii) holds.

1.4. (iii) It is sufficient to prove that $V_2 \models$ "$\mathbf{Q}_\beta[G'_\beta]$" satisfies the countable chain condition. This is a consequence of 1.4(i), (ii) and the fact that $V_1 \models$ "$\mathbf{Q}_\beta[G_\beta]$ satisfies c.c.c".

1.5. Let $G_\beta = G_\alpha | \beta$. Clearly G_β is generic over V for P_β and G'_β is generic over V for $\lim \bar{Q} | S \cap \beta$. Let $G(\beta)$ be the generic object for $\mathbf{Q}_\beta[G_\beta]$ over V_1 defined by G_α. We need prove that $G(\beta)' = G(\beta) \cap V_2$ is a generic object for $Q_\beta[G'_\beta]$ over V_2. By Lemma 1.4, without loss of generality, $G(\beta)'$ is directed; therefore it is sufficient to show that $G(\beta)'$ intersects every maximal antichain of $Q_\beta[G'_\beta]$ that belongs to V_2. All the parameters in the definition of $\mathbf{Q}_\beta[G'_\beta]$ are in V_2, and it is standard work to build in V_2 a tree $T \subseteq \lambda^{<\omega} \times \lambda^{<\omega} \times \omega^{<\omega}$ such that if $\Gamma: \omega \times \omega \to \omega$ is one-to-one and onto, fixed in V_2, and for every $f: \omega \to \lambda$ we define $f_i: \omega \to \lambda$ by setting $f_i(n) = f(\Gamma(i, n))$, then, in V_2, for every $g: \omega \to \lambda$ and $h: \omega \to \omega$ we have that $\langle f, g, h \rangle \in [T]$ if and only if for every $i \in \omega$ there exist x, $\langle f_i \rangle \in [T_1^\beta(x)]$, and there exists x, $\langle g \rangle \in [T_1^\beta(x)]$, and there exists x, $\langle f_i, g \rangle \in [T_3^\beta(x)]$. Thus $\langle p_i: i < \omega \rangle$ is a maximal antichain of $Q_\beta[G'_\beta]$ in V_2 if and only if, if $f: \omega \to \lambda$ is such that $\langle f_i \rangle = p_i$, then $T(\langle f \rangle)$ is not well founded. This relation also holds in V_1, and as every antichain of $Q_\beta[G'_\beta]$ in V_2 is countable, we have proved that every antichain of $Q_\beta[G'_\beta]$ in V_2 is a maximal antichain of $Q_\beta[G_\beta]$ in V_1. And this shows that $G(\beta)'$ is generic over V_2.

This concludes Case 2.

Case 3. $\alpha = \bigcup \alpha \neq 0$.

1.3. S will be closed for $\bar{Q}$ if and only if $S \cap \beta$ is closed for $\bar{Q} | \beta$ for every $\beta < \alpha$ and $p \in \lim Q | S$ if and only if there exists $\beta < \alpha$ such that $p \in \lim Q | S \cap \beta$.

1.4(i), (ii), (iii) are clear from the definition of directed limit.

1.5. Let $\langle p_i: i < \omega \rangle \subseteq \lim \bar{Q} | S$ be a maximal antichain in V; clearly $\langle p_i: i < \omega \rangle \subseteq P_\alpha$. In order to reach a contradiction, let $q \in P_\alpha$ be such that q is incompatible with every p_i, $i < \omega$. By definition there is $\gamma < \alpha$ such that $q \in P_\gamma$. We know that $\langle p_i | \gamma: i < \omega \rangle$ is a maximal antichain of $\lim(\bar{Q} | S \cap \gamma)$, and by the induction hypothesis this set is a maximal antichain in P_γ. Therefore there exists $r \in P$ and $i \in \omega$ such that $r \geq p_i | \gamma$ and $r \geq q$. Let r' be a member of P_α defined by $r' = r \cup p_i | [\gamma, \alpha)$. Clearly

$r' \geq p_i$ and $r' \geq q$, and this is a contradiction. This concludes Definition 1.3 and the proofs of 1.4 and 1.5. $\square$

REMARK. A particular case of Lemma 1.5 is used, and proved, in Ihoda [Ih1] in order to give a model of "Every Δ^1_3-set of reals is Lebesgue measurable, has the property of Baire and is Ramsey" from a model of ZFC. In this case a $\Gamma^+_{\omega_1}$-c.c.c.-ω_1-iteration was constructed.

1.6. LEMMA. *Let $\bar{Q} = \langle P_i; \mathbf{Q}_i : i < \alpha \rangle$ be a Γ^+_λ-c.c.c.-α-iteration. $\bar{Q} \in V_1 \subseteq V_2$, where V_1 and V_2 are models of ZFC. Then for every $G_\alpha \subseteq P^{V_2}_\alpha = \lim \bar{Q}^{V_2}$ generic over V_2, $G_\alpha \cap V_1 \subseteq P^{V_1}_\alpha = \lim \bar{Q}^{V_1}$ is generic over V_1.*

PROOF. The proof is by induction on α. The unique difference from 1.5 is that we need prove that if $\alpha = \beta + 1$ then

$$\langle \mathbf{T}^1_\beta, \mathbf{T}^2_\beta, \mathbf{T}^3_\beta \rangle [G_\beta] = \langle \mathbf{T}^1_\beta, \mathbf{T}^2_\beta, \mathbf{T}^3_\beta \rangle [G_\beta \cap V_1]$$

are the same object in V_2, and this holds because $\langle \mathbf{T}^1_\beta, \mathbf{T}^2_\beta, \mathbf{T}^3_\beta \rangle$ is a P_β-name in V_1. $\square$

REMARK. 1.6 is a generalization of §6.3 in Shelah [Sh1]. Also §6.4 of Shelah [Sh1] is a corollary of our 1.6 and a density argument. Here P_{ω_1} is $\Gamma^+_{\omega_1}$-c.c.c-ω_1-iteration.

1.7. We will restrict our attention to the $\Gamma^+_{\aleph_0}$-c.c.c forcing notion. In other words, membership, the order and incompatibility are Σ^1_1. It is well known that, in this case, the notion of maximal antichain is Σ^1_1 and these relations are absolute for countable transitive models of a part of the set theory. We will make further use of these remarks.

1.8. LEMMA. *Let V be a model of ZFC, and let $N \models ZFC^*$ be a submodel of V (we do not require $N < H(\lambda)$). Let $\bar{Q} = \langle P_i, \mathbf{Q}_i : i < \alpha \rangle$ be an $\Gamma^+_{\aleph_0}$-c.c.c-α-iteration. Suppose that $\alpha \in N$, $\alpha \cap N$ is closed for $\bar{Q}$, and that for every $i \in \alpha \cap N$, $\langle \mathbf{T}^i_1, \mathbf{T}^i_2, \mathbf{T}^i_3 \rangle \in N^{\bar{Q}|S \cap i}$. Then the following assertions hold*:

(i) *P^N_α is a subordering of P_α, and incompatibility is preserved.*

(ii) *$\Vdash_{P_\alpha}$ "$N \cap G_\alpha$ is generic for P^N_α over N".*

(iii) *$N \models$ "$P^N_\alpha \models$ "$\mathbf{Q}_\alpha$ satisfies c.c.c.""*

PROOF. By induction on α we prove (i) and (ii); (iii) is a consequence of Theorem 3.14.

$\alpha = 0$ is clear.

$\alpha = \beta + 1$. Let $G_\beta \subseteq P_\beta$ be generic. Clearly $\beta \in N$ and $\bar{Q} \mid \beta \in N$, and we know that $N^1 = N[G_\beta \cap N]$ is a model of ZFC*. Let $V^1 = V[G_\beta]$. Clearly $N^1 \subseteq V^1$. We need to prove that $Q^{N^1}_\beta$ is a submodel of $Q^{V^1}_\beta$.

Because $\langle \mathbf{T}^\beta_1, \mathbf{T}^\beta_2, \mathbf{T}^\beta_3 \rangle \in N^{\bar{Q}|S \cap i}$ we know that $\mathbf{T}^\beta_i[G_\beta] = \mathbf{T}^\beta_i[G_\beta \cap N]$ belongs to N_i for $i = 1, 2, 3$. Therefore by absoluteness arguments we obtain that $Q^{N^1}_\beta$ is a submodel of $Q^{V^1}_\beta$ and that incompatibility is preserved. By absoluteness of the Π^1_1-relation, we have that every maximal antichain of $Q^{N^1}_\beta$ in N^1 is a maximal antichain of $Q^{V^1}_\beta$ in V^1; and this implies that for every $G(\beta) \subseteq Q^{V^1}_\beta$ generic over V^1, $G(\beta) \cap N^1$ is generic for $Q^{N^1}_\beta$ over N^1.

This concludes the case $\alpha = \beta + 1$. If $\alpha = \bigcup \alpha \neq 0$, then $\alpha \in N$ so $N \models \alpha = \bigcup \alpha \neq 0$. Let $q \in P_\alpha$. Let $\alpha' \in \alpha \cap N$ be such that $\beta \in \sup(q) \cap N \Rightarrow \beta < \alpha'$. Let $\langle p_i : i < i^* \rangle$ be a maximal antichain of P^N_α belonging to N. Clearly $\langle p_i \mid \alpha' : i < i^* \rangle$ is a maximal antichain of $P^N_{\alpha'}$, and lies in N. By the induction hypothesis there exist $r \in P_{\alpha'}$ and $i < i^*$ such that $r \geq_{P_{\alpha'}} p_i \mid \alpha'$ and $r \geq q \mid \alpha'$.

Now let $r' \in P_\alpha$ be defined as follows:

$$r'(\beta) = r(\beta) \quad \text{if } \beta \in \text{dom}(r);$$
$$r'(\beta) = p_i(\beta) \quad \text{if } \beta \in \alpha \cap N - \text{dom}(r);$$
$$r'(\beta) = q(\beta) \quad \text{if } \beta \in \alpha - \alpha \cap N.$$

Then r' is well defined, and $r' \geq_{P_\alpha} p_i$ and $r' \geq_{P_\alpha} q$. Therefore $\langle P_i : i < i^* \rangle$ is a maximal antichain of P_α in V. This concludes the proof of the lemma. □

§2. Proper Souslin forcing. We want to obtain the results of §1 but for iteration of forcing notions in Γ_λ^+ not necessarily satisfying countable chain condition. For example if N is a countable model of ZFC*, and $N \in V$, then we can ask when a Mathias real over V is a Mathias real over N. In general this is false, but for every Mathias real condition $p \in N$, there exists a Mathias real condition $q \in V$ such that $p \leq q$ and every Mathias real over V obtained by a generic object containing q is a Mathias real over N (for further discussion of this, see §4). Following this, we will generalize the above property and will prove that it is preserved under countable support iterated forcing.

2.1. Let $\bar{Q} = \langle P_i; \mathbf{Q}_i : i < \alpha \rangle$ be an α-system of countable support iterated forcing satisfying the following conditions:

(i) For each $i < \alpha$ there are P_i-simple names $\mathbf{T}_1^i$, $\mathbf{T}_2^i$, $\mathbf{T}_3^i$ of trees such that $\Vdash_{P_i}$ "$\langle \mathbf{T}_1^i, \mathbf{T}_2^i, \mathbf{T}_3^i \rangle$ witnesses $\mathbf{Q}_i$ belong to Γ_λ^+".

(ii) The elements of P_i are countable functions of simple names.

In this case we say that $\bar{Q}$ is a *Γ_λ^+-countable-α-iteration.*

2.2. Definition. Let V_1 be a model of set theory and let $V_1 \subseteq V_2$, a model of ZFC*. Let P be a forcing notion, $P \in V_1$. Let $p \in P \cap V_1$; we say that $q \in P \cap V_2$ is a *(p, V_1)-good-generic condition* if $P \vDash$ "$p \leq q$" and for every $G \subseteq P \cap V_2$ generic over V_2 (including q), $G \cap V_1$ is generic over V_1 (including p).

2.3. Examples. (i) If P is the directed limit of a Γ_λ^+-c.c.c.-α-iteration (see 1.1) with definition in V_1, then, for every $p \in P^{V_1}$, p is a (p, V_1)-good-generic condition.

(ii) If P is Mathias forcing then $P \in \Gamma_{\aleph_0}^+$, and if $(2^c)^{V_1}$ is countable in V_2 then for every $p \in P^{V_1}$ there exists a (p, V_1)-good-generic condition $q \in P^{V_2}$ (for a proof of this see §4).

2.4. Let $V_1 \subseteq V_2$ be as in 2.2. Let P in Γ_λ^+ be such that the parameters of the definition of P belong to V_1. Let $P^{V_1} = \{p \in V_1 : V_1 \vDash \text{"}p \in P\text{"}\}$ and $P^{V_2} = \{p \in V_2 : V_2 \vDash \text{"}p \in P\text{"}\}$. By absoluteness considerations it is possible to prove the following:

(i) $P^{V_1} \subseteq P^{V_2}$.

(ii) $(p \leq q)^{V_1}$ if and only if $(p \leq q)^{V_2}$ for p, q in V_1.

(iii) (p, q are compatible)^{V_1} if and only if (p, q are compatible)^{V_2} for p, q in V_1.

2.5a. Definition. Let P be in Γ_λ^+; we say that P is a *proper Souslin forcing* if and only if for every $N \in V$, $N \vDash$ ZFC*, containing the definition of P and containing countable many antichains of P, we have that for every $p \in N \cap P$ there exists a (p, N)-good-generic condition $q \in P$.

2.5b. Definition. We say that P is *proper $\Gamma_{\aleph_0}^+$ forcing* if there exists a $\Gamma_{\aleph_0}^+$-countable-α-iteration $\bar{Q} = \langle P_i; \mathbf{Q}_i : i < \alpha \rangle$ such that $P = \lim \bar{Q}$ and for every $\beta < \alpha$ we have that $\Vdash_{P_\beta}$ "(*)", where

(*) For every countable $N \models \mathrm{ZFC}^*$, if $\mathbf{Q}_\beta \in N$, then for every $p \in \mathbf{Q}_\beta^N$ there exists a(p, N)-good-generic condition $q \in \mathbf{Q}_\beta$.

Let $\bar{Q} = \langle P_i, \mathbf{Q}_i : i < \alpha \rangle$ be such that $P_\alpha = \lim \bar{Q}$ is a proper $\Gamma^+_{\aleph_0}$ forcing, and let $S \subseteq \alpha$ be closed. Let $N \in V$ be a model of ZFC*, N countable in V. By induction on α we will define and prove the following:

2.6. DEFINITION. (i) *S is closed for $(\bar{Q}, N)$.*

(ii) $\bar{Q} \,|\, S^N$ and $\lim^N Q \,|\, S^N = P_\alpha \,|\, S^N$.

(iii) $P_\alpha \,|\, S$ and for $p \in P_\alpha \,|\, S$ we define $p^N \in P_\alpha \,|\, S^N$.

(iv) If $G_\alpha \subseteq P_\alpha$ is generic over V, then we define $G_\alpha \cap P_\alpha \,|\, S^N = G_\alpha \,|\, S^N$.

2.7. LEMMA. *If S is closed for $(\bar{Q}, N)$, then*

(i) *$P_\alpha \,|\, S \subseteq P_\alpha$, and*

(ii) *for $p, q \in P_\alpha \,|\, S$ we have that*

$$(p^N \leq q^N)^N \textit{ if and only if } (P_\alpha \models p \leq q)^V,$$
$$(p^N, q^N \textit{ are compatible})^N \textit{ if and only if } (p, q \textit{ are compatible})^V.$$

2.8. LEMMA. *If S is closed for $(\bar{Q}, N)$ then for every $p^N \in P_\alpha \,|\, S^N$ there exists $q \in P_\alpha$ such that*

(i) *$P_\alpha \models p \leq q$ (note that this p corresponds to p^N), and*

(ii) *for every $G_\alpha \subseteq P_\alpha$ generic over V containing q, we have that $G_\alpha \,|\, S^N$ is generic over N for $P_\alpha \,|\, S^N$ (containing p^N).*

PROOF AND DEFINITION. In all cases S is countable in N, and $S \in N$ and $S \subseteq N$.

Case 1. $\alpha = 0$ is clear.

Case 2. $\alpha = \beta + 1$. 2.6.(i) S is closed for $(\bar{Q}, N)$ if $S \cap \beta$ is closed for $(\bar{Q} \,|\, \beta, N)$ and if $\beta \in S$ then there exists $\langle T_1, T_2, T_3 \rangle \in N$, such that

$$\Vdash_{P_\beta} \text{“}\langle \mathbf{T}_1^\beta, \mathbf{T}_2^\beta, \mathbf{T}_3^\beta \rangle = \langle \dot{T}_1, \dot{T}_2, \dot{T}_3 \rangle\text{”}$$

(here we could be more general, but the notation is more complicated).

(ii) $\bar{Q} \,|\, S^N$ is the definition in N of

$$\bar{Q} \,|\, (S \cap \beta)^N \cup \{\langle P_\beta \,|\, (S \cap \beta)^N, \mathbf{Q}_\beta^N \rangle\}$$

where $\mathbf{Q}_\beta^N$ is in N a $P_\beta \,|\, S \cap \beta$-name of a forcing notion such that

$$\Vdash_{P_\beta | (S \cap \beta)^N} \text{“}\mathbf{Q}_\beta^N \text{ is obtained from } \langle \dot{T}_1, \dot{T}_2, \dot{T}_3 \rangle\text{”}$$

holds in N.

Now $P_\alpha \,|\, S^N$ is $P_\beta \,|\, S \cap \beta^N * \mathbf{Q}_\beta^N$ defined in N.

(iii) $p \in P_\alpha \,|\, S$ if and only if $p \,|\, \beta \in P_\beta \,|\, S \cap \beta$ and $p \in P_\alpha$, and if $\gamma \notin S$ then $p(\gamma) = 0$ and there exists $r \in P_\alpha \,|\, S^N$ (unique) such that $r \,|\, S \cap \beta = p \,|\, \beta^N$ and there exists $\langle q_i : i < i^* \rangle \cup \langle t_j : j < j^* \rangle$, a maximal antichain of P_β such that $\langle q_i : i < i^* \rangle$ is a maximal antichain such that for every $i < i^*$, q_i satisfies 2.8(i), (ii) for $p \,|\, \beta$, and for every $i < i^*$

$$q_i \Vdash_{P_\alpha} \text{“}\mathbf{p}(\beta) = \mathbf{r}(\beta)[G_\beta \,|\, S \cap \beta^N]\text{”}$$

and for every $j < j^*$

$$t_j \Vdash \text{“}\mathbf{p}(\beta) = 0\text{”}.$$

This definition also says that for every $r \in P_\alpha \,|\, S^N$ there exists a unique $p \in P_\alpha \,|\, S$ such that $p^N = r$ and for every $p \in P_\alpha \,|\, S$, $p^N \in P_\alpha \,|\, S^N$ is well defined. This uniqueness is modulo equivalency of names, i.e. $\tau_1 \equiv \tau_2$ if and only if $0 \Vdash$"$\tau_1 = \tau_2$". (Note that $P_\alpha | S$ depends on N.)

(iv) Let $G_\alpha \subseteq P_\alpha$ be generic over V. Then

$$G_\alpha | S^N = \{p^N \in P_\alpha | S^N : p \in G_\alpha \cap P_\alpha | S\}.$$

2.7(i) and (ii) are easy using the absoluteness of Σ^1_1-relations and the induction hypothesis.

2.8. We shall show by induction on $\beta \le \alpha$ that every $P_\beta | S \cap \beta^N$ has the somewhat stronger property:

(*) For all $\gamma \in \beta \cap S$, and for all $p \in P_\beta | S \cap \beta$ and $q \in P_\gamma$, if q satisfies 2.8(i) and (ii) for $p | \gamma$ then there is an $r \in P_\alpha$ such that r satisfies 2.8(i) and (ii) for p and $r | \gamma = q$. Therefore, in the case $\alpha = \beta + 1$, since (*) holds for β we may assume, without loss of generality, that $\gamma = \beta$.

Let $p \in P_\alpha | S$, and let $q \in P_\beta$ satisfy 2.8(i), (ii); for $p | \beta$ let $G_\beta \subseteq P_\beta$ be generic over V containing q. By hypothesis $G_\beta | S \cap \beta$ is generic over N containing $p | \beta^N$, and $N[G_\beta | S \cap \beta]$ is a countable model of some part of ZFC* in $V[G_\beta]$; also the parameters of the definition of $\mathbf{Q}_\beta[G_\beta]$ are in $N \subseteq N[G_\beta | S \cap \beta]$. Therefore, for $p(\beta)[G_\beta | S \cap \beta] \in \mathbf{Q}_\beta[G_\beta] \cap N[G_\beta | S \cap \beta]$ there exists $q(\beta) \in \mathbf{Q}_\beta[G_\beta]$ such that $q(\beta)$ is a $(p(\beta)[G_\beta | S \cap \beta], N[G_\beta | S \cap \beta])$ good-generic condition. Because G_β is arbitrary containing q, there exists a P_β-name of a condition $\mathbf{q}(\beta)$ for $\mathbf{Q}_\beta$ such that

$$q \Vdash \text{"}\mathbf{q}(\beta) \text{ is a } (p(\beta)[G_\beta | S \cap \beta], N[G_\beta | S \cap \beta])\text{-good-generic condition".}$$

Now let $r = q \cup \langle \beta, \mathbf{q}(\beta) \rangle$. Clearly r satisfies (*) for this q and p. Clearly (*) implies 2.8(i) and (ii).

Case 3. $\alpha = \bigcup \alpha \neq 0$.

2.6.(i) S is closed for $(\bar{Q}, N)$ if $S \cap \beta$ is closed for $(\bar{Q} | \beta, N)$ for every $\beta < \alpha$, and there exists $f \in N$ such that $\operatorname{dom} f = S$ and for every $\beta \in S$, $f(\beta) = \langle T^\beta_1, T^\beta_2, T^\beta_3 \rangle$ (i.e. in N we can define the directed limit of $\bigcup (Q | S \cap \beta^N)$).

(ii) $\bar{Q} | S^N$ is $\bigcup_{\beta \in S} \bar{Q} | S \cap \beta^N$ taken in N and $P_\alpha | S^N$ is the countable support iteration of $\bar{Q} | S^N$ taken in N.

(iii) $P_\alpha | S$ and p^N are defined analogously.

(iv) The same for $G_\alpha | S^N$.

2.7 (i), (ii). Easy (using the inductive hypothesis).

2.8. Let $\langle \alpha_i : i < \omega \rangle$ be cofinal in α, $\langle \alpha_i : i < \omega \rangle \subseteq S$, and let $\langle D_i : i < \omega \rangle$ be an enumeration of the dense subsets of $P_\alpha | S^N$ which belong to N.

Let $p \in P_\alpha | S$ and $q_0 \in P_{\alpha_0}$ be given satisfying the requirement of (*). We will construct two sequences $\langle p_i : i < \omega \rangle$ and $\langle q_i : i < \omega \rangle$ such that

(i) $p_0 = p$;

(ii) $q_n \in P_{\alpha_n}$ and q_n satisfies 2.8(i), (ii) for $p_n | \alpha_n$;

(iii) $q_{n+1} | \alpha_n = q_n$;

(iv) $p_n \in P_\alpha \cap S$;

(v) $p_{n+1} | \alpha_n = p_n | \alpha_n$ and $q_{n+1} \mathbin{\bar{\cup}} p_{n+1} \ge p_n$ (where $q_{n+1} \mathbin{\bar{\cup}} p_{n+1} = q_{n+1} \cup p_{n+1} | \alpha - \alpha_{n+1}$); and

(vi) $D_n | p_{n+1}$ is predense above p_{n+1}.

Suppose then we have $p_0, \ldots, p_n$ and $q_0, \ldots, q_n$ satisfying (i)–(vi).

Let $G \subseteq P_{\alpha_n} | S \cap \alpha_n^N$ be generic over N, $p_n | \alpha_n^N \in G$. Let $D'_n \in N[G]$ be the corresponding dense subset of $P_\alpha | S \cap (\alpha - \alpha_n)^{N[G]}$, generated by D_n in $N[G]$. As $p_n | (\alpha - \alpha_n)^{N[G]}$ belongs to $P_\alpha | S \cap (\alpha - \alpha_n)^{N[G]}$ we have that in $N[G]$ there exists $r \in P | S \cap (\alpha - \alpha_n)^{N[G]}$ such that $p_n | (\alpha - \alpha_n)^{N[G]} \le r \in D'_n$. As G is arbitrary containing $p_n | \alpha_n$, there exists $\mathbf{r} \in N$, a $P_{\alpha_n} | S \cap \alpha_n^N$-name, such that

$$(p_n | \alpha_n^N \Vdash p_n | (\alpha - \alpha_n) \le \mathbf{r} \in D'_n)^N.$$

Thus there is $t \in P_\alpha \cap S^N$ such that $t | \alpha_n = p_n | \alpha_n$ and $(p_n | \alpha_n \Vdash t | (\alpha - \alpha_n) \ge \mathbf{r})^N$ satisfying, for every $G \subseteq P_{\alpha_n} | S \cap \alpha_n^N$ containing $p_n | \alpha_n^N$ there exists $p \in D_n$ such that

$$N[G] \vDash p_n | [\alpha_n, \alpha)[G] \le \mathbf{r}[G] = p | [\alpha_n, \alpha][G] \in D'_n.$$

Let $p_{n+1} \in P_\alpha | S$ be such that

$$p_{n+1}^N | \alpha_n = p_n | \alpha_n, \qquad p_{n+1}^N | [\alpha_n, \alpha] = t | [\alpha_n, \alpha].$$

Now q_n satisfies 2.8(i), (ii) for $p_n | \alpha_n$; hence by the induction hypothesis there exists $q_{n+1} \in P_{\alpha_{n+1}}$ such that $q_{n+1} | \alpha_n = q_n$ and q_{n+1} satisfies 2.8(i) and (ii) for $p_{n+1} | \alpha_{n+1}$. There is no problem in seeing that p_{n+1}, q_{n+1} satisfies the inductive hypothesis. Now we define $\bar{q} \in P_\alpha$ by $\bar{q} | \alpha_n = q_n$. By the construction we have that for every n

$$P_\alpha \vDash p_n \le \bar{q}$$

and $\bar{q}$ satisfies (*) for p and q.

§3. A weaker form of Martin's axiom. Many problems related to sets of reals are decidable from the assumption of Martin's axiom, introduced by Martin and Solovay [MS]. For example, MA implies that the union of less than continuum many measure zero sets is a measure zero set, and that there is no Souslin tree. In this section we will present a weaker form of Martin's axiom which decides some specific problems concerning the real line.

3.1. Definition. For a family Γ of partial ordered sets we say that MA(Γ) holds if and only if for every partial order $P \in \Gamma$ satisfying c.c.c. and for every family $\langle D_i : i < \kappa \rangle$ of dense open subsets of P, of cardinality less than $2^{\aleph_0}$, there exists $G \subseteq P$ directed such that for every $i < \kappa$, $G \cap D_i$ is not empty.

3.2. Theorem. MA($\Gamma^+_{\aleph_0}$) *implies the following*:

(i) *The union of less than continuum many measure zero sets is a measure zero set.*

(ii) *The union of less than continuum many meager sets is a meager set.*

(iii) *The union of less than continuum many strong measure zero sets is a strong measure zero set.*

(iv) $2^{\aleph_0}$ *is a regular cardinal.*

(v) *Every family of maximal almost disjoint subsets of* ω *has cardinality* $2^{\aleph_0}$.

(vi) *For every family* $F \subseteq \omega^\omega$ *of cardinality less than* $2^{\aleph_0}$ *there exists* $f \in \omega^\omega$ *such that for every* $g \in F$ *there exists* $n \in \omega$ *such that for every* $m \ge n$, $g(m) < f(n)$.

(vii) *There is no real valued measurable cardinal* $\le 2^{\aleph_0}$.

Proof. (i) It will be sufficient to show that amoeba* forcing $\langle \{T \subseteq 2^\omega : T \text{ a closed tree}, \mathrm{LbMs}(\lim T) > \frac{1}{2}, \supseteq\} \rangle$ is in $\Gamma^+_{\aleph_0}$. That amoeba* forcing satisfies the countable chain condition is well known, and that amoeba* belong to $\Gamma^+_{\aleph_0}$ is clear from the definition of amoeba* forcing (more information about amoeba* can be found in [Sh1]).

(ii) This is a consequence of (i). The Bartoszyński-Raisonnier-Stern theorem says exactly "(i) ⇒ (ii)"; see Raisonnier and Stern [RS].

(iii) This is a consequence of (i); see Fremlin [Fr1].

(iv) This is a consequence of (i).

(v) This is a consequence of (vi); see Shelah [Sh3].

(vi) This is a consequence of (i) (or of (ii)).

(vii) follows from (ii).

3.3. THEOREM. $\mathrm{MA}(\Gamma^+_{\aleph_0})$ *does not imply any of the following*:

(i) *Every tower has cardinality* $2^{\aleph_0}$.

(ii) *Every splitting family has cardinality* $2^{\aleph_0}$.

(iii) *There is no Souslin tree.*

(iv) *For every* $\kappa < 2^{\aleph_0}$, $2^\kappa = 2^{\aleph_0}$.

(v) *There exists a Q-set.*

PROOF. We begin by building a model for $\mathrm{MA}(\Gamma^+_{\aleph_0})$, and after we will see that in this model the negation of (i), (ii), and (iii) holds.

Let $V = L$ be the constructible universe. In V we define the following partial order P. The members of P are $\Gamma^+_{\aleph_0}$-c.c.c.-α-iterations satisfying

(i) $\alpha < \omega_2$ *(or* κ*), and*

(ii) if $\bar{Q} \in P$ and $\alpha < \mathrm{length}(\bar{Q})$, then

$$P_\alpha \Vdash \text{"}\mathbf{Q}_\alpha \text{ satisfies the countable chain condition"}.$$

The order on P is inclusion.

Clearly P is $\aleph_2$-closed forcing and if $G \leq P$ is generic over V, then $\bar{R} = \bigcup\{\bar{Q} \in G\}$ is a $\Gamma^+_{\aleph_0}$-c.c.c.-ω_2-iteration.

Let P_{ω_2} be the directed limit of $\bar{R}$. Let $G_{\omega_2} \subseteq P_{\omega_2}$ be generic over $V[G] = V_1$. Then by the genericity of G we have

$$V_1[G_{\omega_2}] \vDash \text{"}\mathrm{MA}(\Gamma^+_{\aleph_0}) + 2^{\aleph_0} = \aleph_2\text{"}.$$

Now we will use the result given in §1 in order to show that in this model (i), (ii) and (iii) fail. Set $V_2 = V_1[G_{\omega_2}]$.

3.4. DEFINITION. A family S of infinite subsets of ω is a *splitting family* if for every infinite $x \subseteq \omega$ there exists $y \in S$ satisfying

$$|y \cap x| = |\sim y \cap x| = \aleph_0.$$

The *splitting number* s is the minimal cardinality of a splitting family. We will show that in V_2 the splitting number s is equal to $\aleph_1$.

3.5. *Claim. Let* $\mathbf{r}$ *be a* P_{ω_2}*-name of an infinite subset of* ω*, and let* $\langle a_i : i < \omega_1 \rangle$ *be a family of almost adjoint subsets of* ω*. Suppose that* $0 \Vdash_{P_{\omega_2}}$ "$(\exists i < \omega_1)(\mathbf{r} \subseteq^* a_i)$." *Then there exists* $i < \omega$ *satisfying* $0 \Vdash_{P_{\omega_2}}$ "$\mathbf{r} \cap \dot{a}_i$ *is finite*".

Proof. Suppose this does not hold. Then for every $i < \omega_1$, there exists $p_i \in P_{\omega_2}$ satisfying $p_i \Vdash_{P_{\omega_2}}$ "$\mathbf{r} \cap \dot{a}_i$ is infinite". By the countable chain condition there exists $i \neq j < \omega_1$ such that p_i and p_j are compatiable. Let q extend both, and let q_1 extending q and $k < \omega_1$ be such that

(1) $$q_1 \Vdash_{P_{\omega_2}} \text{"}\mathbf{r} \subseteq^* a_k\text{"}.$$

Therefore

(2) $$q_1 \Vdash \text{“}|\mathbf{r} \cap \dot{a}_i| = |\mathbf{r} \cap \dot{a}_j| = \aleph_0\text{”}.$$

But (1) and (2) are contradictory to the assumption on $\langle a_i : i < \omega_1 \rangle$. □

Note that Claim 3.6 is provable if we replace P_{ω_2} by P satisfying the countable chain condition.

3.6. *Claim. There is no* $r \in V_2 \cap [\omega]^\omega$ *satisfying, for every* $y \in [\omega]^\omega \cap V_1$,

$$r \subseteq^* y \quad or \quad r \subseteq^* \sim y.$$

Proof. Suppose this does not hold, and let $\mathbf{r}$ be a P_{ω_2}-name of an infinite subset of ω satisfying, for every $y \in V_1 \cap [\omega]^\omega$,

$$0 \Vdash_{P_{\omega_1}} \text{“}\mathbf{r} \subseteq^* \dot{y} \text{ or } \mathbf{r} \subseteq^* \sim \dot{y}\text{”}.$$

Without loss of generality the transitive closure of $\mathbf{r}$, which we denote by tc($\mathbf{r}$), is countable, and there is a countable $S \subseteq \omega_2$ such that $\mathbf{r}$ is a $P_{\omega_2} | S$-name of a subset of ω. Set $P_{\omega_2} | S = P_S$. All parameters used in order to define P_S are countable; therefore there exists $N < H(\lambda, \varepsilon)$, λ large, such that tc($\mathbf{r}$) $\in N$ and S is closed for $(N, \bar{R})$. Clearly $N \models$ “$\mathbf{r}$ is a P_S-name”.

We proved in §1 that $P_S \lessdot P_{\omega_2}$, and therefore we would remain working only with P_S. In V_1 there are $\langle c_i : i < \omega_1 \rangle$ such that $c_i \subseteq \omega$ for every $i < \omega_1$ and if $i \neq j < \omega_1$ then $c_i \cap c_j$ is finite, and, for every $i < \omega_1$, c_i is a Cohen real over N.

LEMMA. *Let N be a countable model of some rich part of ZFC. Then there exists* $\langle c_i : i < \omega_i \rangle$ *such that*

(i) *for each* $i < \omega_i$, char(c_i) *is a Cohen real over N, and*

(ii) *for each* $i < j$, $|c_i \cap c_j| < \aleph_0$.

PROOF. Let $\langle I_n : n < \omega \rangle$ be an enumeration of the maximal antichain of $2^{<\omega}$ which belongs to N. We will build $\langle t_s : s \in 2^{<\omega} \rangle$ satisfying the following requirements:

(i) For every $s \in 2^{<\omega}$, $t_s \subseteq t_{s^\frown\langle i \rangle}$ $(i = 0, 1)$.

(ii) For every $s \in 2^{<\omega}$, $t_s \in 2^{<\omega}$.

(iii) For every $s \in \omega^{<\omega}$, if $|s| = m$ then there exists $t \in I_m$ such that $t \subseteq t_s$.

(iv) For every $s_1, s_2 \in 2^m$, if n is the first natural number such that $s_1(n) \neq s_2(n)$, and $k \geq |t_{s_1 \restriction n}|$, then

$$t_{s_1}(k) = 1 \rightarrow t_{s_2}(h) = 0 \quad \text{and} \quad t_{s_2}(h) = 1 \rightarrow t_{s_1}(h) = 0.$$

Proof. We proceed by induction on m = length(s).

$m = 0$. Let $t_{\langle\,\rangle}$ be such that there exists $t \in I_0$ satisfying $t \subseteq t_{\langle\,\rangle}$.

$m \rightarrow m + 1$. Let $\langle s^i : i < 2^m \rangle$ be an enumeration of 2^m. Without loss of generality, for every $i, j < 2^m$ we assume that $|t_{s^i}| = |t_{s^j}|$.

For $i = 0$, let $t_{s^0\frown\langle 0 \rangle}$ be such that there exists $t \in I_{m+1}$ with $t \subseteq t_{s^0\frown\langle 0 \rangle}$. Let t'_{s^0} be such that $t_{s^0} \subseteq t'_{s^0}$ and for every $k \geq |t_{s^0}|$ if $t_{s^0\frown\langle 0 \rangle}(k) = 1$ then $t'_{s^0}(k) = 0$.

Let $t_{s^0\frown\langle 1 \rangle}$ extend t'_{s^0} and be such that there exists $t \in I_{m+1}$ with $t \subseteq t_{s^0\frown\langle 1 \rangle}$.

For $i \neq 0$ the proof is similar.

Now for every $f \in {}^\omega 2$, if t_f is such that, for every $m \in \omega$, $t_{f \restriction m} \subseteq t_f$, then t_f is a Cohen real over N and if $f_1 \neq f_2$ then $|t_{f_1} \cap t_{f_2}| < \aleph_0$. This completes the proof of the lemma.

By Claim 3.5, there exists a Cohen real over N and c satisfying

$$0 \Vdash_{P_s} \text{“}\mathbf{r} \cap \dot{c} \text{ is finite”}.$$

Set $N_1 = N[c]$. Also N_1 is a model for ZFC* and S is closed for (N_1, P_s). By §1.8, if $G \subseteq P_s$ is generic over V_1 then $G \cap N_1$ is generic for $P_s^{N_1}$ over N_1.

Set $N_2 = N_1[G \cap N_1]$.

Because the computation of $\mathbf{r}$ using $G \cap N_1$ and using G are the same, we obtain that

$$N_2 \vDash \text{“}\mathbf{r}[G \cap N_1] \cap c \text{ is finite”}.$$

Also no restrictions on the members of G are imposed, so

$$N_1 \vDash 0 \Vdash_{P_s} \text{“}\mathbf{r} \cap \dot{c} \text{ is finite”}.$$

Therefore there exists $s \in [\omega]^{<\omega}$, $s \subseteq c$ such that

$$N \vDash_s \Vdash_{\text{Cohen}} \text{“}0 \Vdash_{\mathbf{P}_s} \mathbf{r} \cap \dot{c} \text{ is finite”}.$$

But c is in V_1; therefore $c' \in V_1$, where c' is such that $s \subseteq c'$ and $c' \cap c = s$ and $c' \cup c = \omega$. Clearly c' is a Cohen real over N extending s. So $N[c'] \vDash$ “$0 \Vdash_{P_s} \mathbf{r} \cap \dot{c}$ is finite.” (Note that in all this argument we worked with a standard name for $\mathbf{r}$ in the Cohen forcing language.)

Again S is closed for $(N[c'], P_S)$, and using §1.8 we obtain

$$N[c'][G \cap N[c']] \vDash \text{“}\mathbf{r}[G \cap N[c']] \cap c' \text{ is finite.”}$$

And this is absolute; therefore

$$V_2 \vDash \text{“}\mathbf{r}[G \cap N[c']] \cap c' \text{ is infinite”}.$$

But in V_2, $\mathbf{r}[G \cap N[c']] = \mathbf{r}[G']$, where $G' = G_{\omega_2} | S \cap V_1$, and this implies that

$$V_2 \vDash \text{“}\mathbf{r} \cap c' \text{ is finite and } r \cap c \text{ is finite”},$$

and this is a contradiction because $c' \cup c = \omega$. $\square$

3.7. COROLLARY. *$[\omega]^\omega \cap L$ is a splitting family in V_2.* $\square$

A *tower* is a family $\langle a_i : i < \kappa \rangle$ of infinite subsets of ω satisfying (i) $a_i \subseteq^* a_j$ for $i > j$ and $a_j \not\subseteq^* a_i$, and (ii) for every infinite $x \subseteq \omega$, there exists $i < \kappa$ such that $x \not\subseteq^* a_i$.

3.8. COROLLARY. *In V_2 there is a tower of cardinality $\aleph_1$.*

PROOF. Let $\langle a_i : i < \omega_1 \rangle$ be a tower on L such that in L

$$U = \{x \in [\omega]^\omega : (\exists i < \omega_1)(a_i \subseteq^* x)\}$$

is a ultrafilter over ω. By Claim 3.7, for every $x \in [\omega]^\omega$ there exists $y \in U$ such that $(x \cap y) = (x \cap \sim y) = \aleph_0$. Let $i < \omega_1$ be such that $a_i \subseteq^* y$; then clearly $x \not\subseteq^* a_i$. $\square$

3.9. LEMMA. *Let P be a forcing notion, let T be a Souslin tree, and suppose that*

$$\langle T, \leq \rangle \Vdash \text{“}P \text{ satisfies c.c.c.”}.$$

Then

$$\Vdash_P \dot{T} \text{ is a Souslin tree.}$$

PROOF. If this does not hold, there exist $\mathbf{I}$ (a P-name) and $p \in P$ such that

$$p \Vdash_P \text{“}\mathbf{I} \text{ is an uncountable maximal antichain of } \dot{T}\text{”}.$$

Set $J = \{t \in T: (\exists q \geq p)(q \Vdash_P \dot{t} \in \mathbf{I})\}$.

3.10. *Claim. If $p_1, p_2 \in P$ and $t_1, t_2 \in T$ and*

(i) $$p_1 \Vdash_P \text{“}\dot{t}_1 \in \mathbf{I}\text{”}, \qquad p_2 \Vdash_P \text{“}\dot{t}_2 \in \mathbf{I}\text{”},$$

(ii) $$p \leq p_1, p_2 \text{ and } T \vDash t_1 \leq t_2.$$

Then p_1 and p_2 are incompatible in P. □

3.11. *Claim. J is uncountable.*

PROOF. $p \Vdash_P$ “$\mathbf{I}$ is uncountable”. □

3.12. *Claim. There exists $t^* \in T$ such that*

$$(\forall s)(t^* \leq s \in T \Rightarrow (\exists t \in T)(s \leq t \in J)).$$

PROOF. Suppose this does not hold, and set

$$K = \{s \in T: T_s \cap J = \varnothing \text{ and if } t \leq s, \text{ then } T_t \cap J \neq \varnothing\}.$$

Let s_1 and s_2 be members of K. Then, if $s_1 \leq_T s_2$, we have that $T_{s_1} \cap J \neq \varnothing$ by the minimality of s_2; and this implies that $s_1 \notin K$. Thus every pair of members of K is incompatible. Let $t \in T$; by assumption there exists $s \geq_T t$ satisfying $T_s \cap J = \varnothing$, and thus there exists $s \in K$ with $t \leq s$. This implies that K is a maximal antichain. Therefore K is countable, and

$$J \subseteq \{t \in T: (\exists s \in K)(t \leq_T s)\}.$$

This implies that J is countable. □

Let $t^* \in T$ satisfy the condition in Claim 3.12. Let $G \subseteq T_{t^*}$ be generic over V for $\langle T_{t^*}, \leq \rangle$. Then $\aleph_1^{V[G]} = \aleph_1^V$. By hypothesis, P satisfies the countable chain condition in $V[G]$. Now G is a branch of T_{t^*} and, by a density argument using 3.12, $G \cap J$ has cardinality $\aleph_1$. Therefore there exists $\langle \langle t_i, p_i \rangle : i < \omega_1 \rangle$ such that $G \cap J = \{t_i : i < \omega_1\}$ and $p \leq p_i \Vdash$ “$t_i \in \mathbf{I}$”.

By the countable chain condition of P in $V[G]$ there exists $i \neq j < \omega_1$ such that p_i and p_j are compatible; and by 3.11, this implies that t_1 and t_2 are incomparable in T. This contradicts the fact that G is a branch of T. □

3.13. LEMMA. *Suppose that $\bar{Q} = \langle P_i; \mathbf{Q}_i : i < \alpha \rangle$ is a $\Gamma^+_{\aleph_0}$-c.c.c.-α-iteration, let T be a Souslin tree, and suppose that, for every $\beta < \alpha$,*

$$P_\beta * \dot{T} \Vdash \text{“}\mathbf{Q}_\beta \text{ satisfies c.c.c.”}.$$

Then $\Vdash_{(T, \leq)}$ “P_α satisfies c.c.c.”.

PROOF. We argue by induction over α (note that $V^{P_\beta} \vDash$ “$\langle T, \leq \rangle$ is a Souslin tree”). The case $\alpha = 0$ is clear.

For $\alpha = \beta + 1$, without loss of generality we can take $P^V_\beta = P^{V^{\langle T, \leq \rangle}}_\beta$. Forcing with $T * P_\beta$ is the same as forcing with $T \times P_\beta$, which is the same as forcing with $P_\beta \times T$, and this in turn is the same as $P_\beta * T$, and as $P_\beta * T \Vdash$ “$\mathbf{Q}_\beta$ satisfies c.c.c.” we have that

$$(T, \leq) \Vdash \text{“}P_\beta * \mathbf{Q}_\beta \text{ satisfies c.c.c.”}.$$

Case 3. For $\alpha = \bigcup \alpha \neq 0$, it is well known that the directed limit of c.c.c. forcing is c.c.c. $\square$

3.14. THEOREM. *Let $V_1 \subseteq V_2$ be models of ZFC*, and suppose that P is in $\Gamma^+_{\aleph_0}$ with parameters in V_1. Then $V_1 \models$ "P satisfies c.c.c." if and only if $V_2 \models$ "P satisfies c.c.c."*

PROOF. Let $\varphi(x)$ and $\psi(x, y)$ be Σ^1_1-formulas with parameters in $V_1 \cap R$, and such that (i) $x \in P$ if and only if $\varphi(x)$, while (ii) $\langle x, y \rangle$ are incompatible if and only if $\psi(x, y)$. Without loss of generality we suppose that the parameters of φ and ψ are the real number b.

The following argument will use techniques of model theory.

We will construct a formula $\theta \in L_{\omega_1\omega}(Q)$ as follows.

The relation symbols of $L_{\omega_1\omega}(Q)$ will be $\{N, R, E, \leq, =, +, \times, 0, 1, a\}$ and the logical symbols for first order logic.

Let θ be the conjunction of the following formulas of $L_{\omega_1\omega}(Q)$.

(1) $$(\forall x)(N(x) \vee R(x)),$$

(2) $$(\forall x)(\neg(N(x) \wedge R(x))),$$

(3) $$(\forall x \forall y)(x \in y \rightarrow N(x) \wedge R(y)),$$

(4) $$(\forall x \forall y)(R(x) \wedge R(y) \wedge (\forall z)(N(z) \rightarrow (z \in x \leftrightarrow z \in y)) \rightarrow x = y),$$

(5) The conjunction of formulas saying that $\langle N, 0, 1, +, \times \rangle$ is a standard model of Peano arithmetic

(here we use $L_{\omega_1\omega}$ in order to write $(\forall x \in N)(x = 0 \vee x = 1 \vee x = 1 + 1 \vee \cdots)$),

(6) $$R(a),$$

(7) $$\bar{m} \in a \text{ for every } m \in b,$$

(8) $$\bar{m} \notin a \text{ for every } m \notin b$$

(i.e., a represents b in a model for θ),

(9) $$\leq \text{ is a well order over } \{x : x \leq x\} \subseteq R,$$

(10) $$(x \leq x \rightarrow \underline{\varphi}(x))$$

(where $\underline{\varphi}(x)$ is the same as $\varphi(x)$ but b is replaced by a),

(11) $$(x \leq y \wedge x \neq y \rightarrow \underline{\psi}(x, y))$$

(where $\underline{\psi}(x, y)$ is the same as $\psi(x, y)$ but b is replaced by a), and

(12) $$(Qx)(x \leq x).$$

3.15. *Fact. Let V be a model of ZFC* and suppose that a belongs to V. Then the following are equivalent*:

(a) *θ has a model.*

(b) *There exists $\langle r_i : i < \omega_1 \rangle \subseteq \mathfrak{R}$ such that, for every $i < \omega_1$, $\varphi(r_i)$, and, for every $i < j < \omega_1$, $\psi(r_i, r_j)$.*

Proof. (a) $\Rightarrow$ (b). Let M be a model for θ. Then, without loss of generality,

(i) $N^M = N$, the natural numbers;
(ii) $R^M \subseteq \mathfrak{R}$, the set of subsets of natural numbers;
(iii) $\in^M = \in | N^M \times R^M$;
(iv) "$+^M = +$" and "$\times^M = \times$"; and
(v) $\langle \text{dom} \leq^M, \leq \rangle$, a well ordering of type ω_1 of the real numbers.
(vi) Therefore there exists $\langle a_i : i < \omega_1 \rangle$ such that $a_i \leq a_i$ for every $i < \omega_1$, and $a_i < a_j$ for every $i < j < \omega_1$.
(vii) For every $i < \omega_1$

$$M \models \underline{\varphi}[a_i] \quad \text{and} \quad \varphi[a_i] = \exists x \varphi_1(x, a, a_i),$$

where φ_1 is an arithmetic formula.

Therefore there exists $c \in R^M$ such that $M \models \varphi_1(c, a, a_i)$. Now, inductively and using the formulas in φ_1, we can prove that $V \models \varphi_1(c, b, a_i)$; and this implies $V \models \varphi[a_i]$.

(viii) Analogously we can prove that $V \models \psi[a_i, a_j]$. And this is the proof of (b) from (a).

(b) $\Rightarrow$ (a). Let $\langle a_i : i < \omega_1 \rangle$ be as in (b). We define

$$M = \langle N \cup \mathfrak{R}, +, =, \times, \leq, \in, 0, 1, a \rangle$$

with the obvious interpretation of the symbols, and $\leq = \{\langle a_i, a_j \rangle : i \leq j\}$. Clearly $M \models \theta$. $\square$

Continuing the proof of 3.14, if, in $V_2 \models$ "P does not satisfy c.c.c.", then, in V_2, $\models$ "θ has a model"; therefore, by Keisler [Ke], $V_2 \models$ "θ is consistent". Therefore $V_1 \models$ "θ is consistent", and, by Keisler [Ke], $V_1 \models \theta$ has a model. So we have proved $V_1 \models$ "θ has a model" if and only if $V_2 \models$ "θ has a model"; and this implies that

$$V_1 \models \text{"}P \text{ satisfies c.c.c." if and only if } V_2 \models \text{"}P \text{ satisfies c.c.c."} \quad \square$$

3.15. COROLLARY. *In V_2 there is a Souslin tree.*

PROOF. Let T be a Souslin tree of L. As P is $\aleph_2$-closed, T is a Souslin tree in V_1. By 3.14, for every $\beta < \omega_2$

$$P^*_\beta \langle \dot{T}_1 \leq \rangle \Vdash \text{"}\mathbf{Q}_\beta \text{ satisfies c.c.c."}$$

Then, by 3.10 and 3.13, T is a Souslin tree in V_2. $\square$

Now we will show 3.4(iv) and (v).

3.16. LEMMA. *Let V be a model for $ZFC + CH + 2^{\aleph_1} = \aleph_3$. Let P be as in the proof of* 3.4. *Let $G \subseteq P$ be generic over V.* Set $V_1 = V[G]$. *Then*

$$V_1 \models CH + 2^{\aleph_1} = \aleph_3.$$

Let P_{ω_2} be defined from G as in the proof of 3.4. Let $G_{\omega_2} \subseteq P_{\omega_2}$ be generic over V_1, and set $V_2 = V_1[G_{\omega_2}]$. Then, as every real number appears in some intermediate stage of the iteration, we have that

$$V_1 \models \text{MA}(\Gamma^+_{\aleph_0}) + 2^{\aleph_0} = \aleph_2 + 2^{\aleph_1} = \aleph_3.$$

This proves 3.4(iv), and 3.5(v) is a consequence of 3.4(iv). $\square$

More results on $\mathrm{MA}(\Gamma^+_{\aleph_0})$ will appear in a subsequent paper. For example, we have proved that the following theories are equiconsistent.

(i) ZFC + there exists an inaccessible cardinal.

(ii) ZFC + $\mathrm{MA}(\Gamma^+_{\aleph_0}) + (\forall r \in \mathfrak{R})(\omega_1^{L[r]} < \omega_1)$.

(iii) ZFC + $\mathrm{MA}(\Gamma^+_{\aleph_0})$ + "every projective set of reals is Lebesgue measurable, etc."

But the following is an open question: Are $\mathrm{MA}(\Gamma^+_{\aleph_0})$ and "additivity of measure" equivalent?

§4. Inaccessible cardinals and the Borel conjecture. Here we will use the techniques introduced in §§1 and 2 in order to show the following.

4.1. THEOREM. *The following theories are equiconsistent*:

(i) *ZFC + there exists an inaccessible cardinal.*

(ii) *ZFC + the Borel conjecture + every Σ^1_2-set of reals is Baire.*

(iii) *ZFC + the dual Borel conjecture + every Σ^1_2-set of reals is Lebesgue measurable.*

4.2. By Galvin, Prikry and Solovay [GPS] a set of reals X has strong measure zero if and only if for every meager set M there exists a real x such that $x + X \cap M = \varnothing$. From this, we define a set of reals X to be *strongly meager* if and only if for every measure zero set M there exists a real x such that $x + X \cap M = \varnothing$. The Borel conjecture is the assertion that every strong measure zero set is countable, and the dual Borel conjecture is the assertion that every strongly meager set is countable. R. Laver [La] has proved that the Borel conjecture is consistent, and T. Carlson [Ca] has proved that the dual Borel conjecture is consistent. We do not know if there is a model of ZFC + Borel conjecture + dual Borel conjecture.

PROOF OF THEOREM 4.1.

4.3. (ii) implies (i). Let V be a model for (ii) and suppose that $\aleph_1$ is not an inaccessible cardinal in L. Therefore there exists a real number a such that $\omega_1^{L[a]} = \omega_1$. In other words, $X = L[a] \cap \mathfrak{R}$ is an uncountable set of reals in V. We will show that X has strong measure zero. In fact, it is sufficient to prove that for every Borel-meager set M there exists x such that $x + X \cap M = \varnothing$. Let m be a Borel-code for M. By the Solovay characterization of "every Σ^1_2-set of reals is Baire", there exists a Cohen real x over $L[a][m]$. As $X \subseteq L[a][m]$ we have that for every $y \in X$, $x + y$ is a Cohen real over $L[a][m]$, and this implies that $x + X \cap M = \varnothing$. Thus, $\aleph_1$ is an inaccessible cardinal in L.

The proof that (iii) $\Rightarrow$ (i) is similar, using random reals.

4.4. (i) $\Rightarrow$ (iii). Let V be a model of ZFC + there exists an inaccessible cardinal κ in V. Let $\mathrm{coll}(\omega, <\kappa)$ be the Levy collapse of all cardinals less than κ to ω. Let $G_\kappa \subseteq \mathrm{coll}(\omega, <\kappa)$ be a generic filter over V; let $V_1 = V[G_\kappa]$. It is well known that in V_1 for every real number a, $\omega_1^{L[a]}$ is a countable ordinal in V_1. Now let P be the product of $\aleph_2$-Cohen reals and let $G \subseteq P$ be generic over V_1; then by Carlson [Ca], $V_1[G]$ satisfies the dual Borel conjecture. We will finish if we show that for every real number $a \in V_1[G]$, $\aleph_1$ is an inaccessible cardinal in $L[a]$.

Suppose that this does not hold; then there exists $\mathbf{a}$, a Cohen-name of a real number, and a Cohen real c in $V[G_1]$ such that $V_1[c] \models$ "$\aleph_1$ is accessible in $L[\mathbf{a}[c]]$". So by the countable chain condition of $2^{<\omega}$ and by the κ-chain condition of coll(ω, $<\kappa$) there exists $\alpha < \kappa$ such that $\mathbf{a} \in V[G_\alpha]$, where G_α is the restriction of G_κ to coll(ω, $<\alpha$). Therefore

$$V[G_\alpha][c] \models \text{“}\kappa \text{ is an accessible cardinal in } L[\mathbf{a}[c]]\text{”}$$

because the computation of $\mathbf{a}$ in $V_1[c]$ and in $V[G_\alpha][C]$ is the same. By 1.6, c is a Cohen real over $V[G_\alpha]$, and thus there exists $s \in 2^{<\omega}$ such that

$$V[G_\alpha] \models_S \Vdash_{2^{<\omega}} \text{“}\kappa \text{ is an accessible cardinal in } L[\mathbf{a}]\text{”}.$$

It is well known that $\omega_1^{V[G_\alpha]}$ is countable in V_1; hence there exists $c_1 \in V_1 \cap 2^\omega$, $s \subseteq c_1$, such that c_1 is a Cohen real over $V[G_\alpha]$, and this implies that

$$V[G_\alpha][c_1] \models \text{“}\kappa \text{ is an accessible cardinal in } L[\mathbf{a}[c_1]]\text{”}.$$

But this implies that

$$V_1 \models (\exists a \in \mathfrak{R})(\aleph_1 \text{ is accessible in } L[a]),$$

contradicting the above remark.

4.5. (i) $\Rightarrow$ (ii). Let V_1 be as in 4.4, only we require that $V_1 \models$ CH. R. Laver (see [Ba]) proved that if P_{ω_2} is an ω_2-iteration of Mathias reals and $G_{\omega_2} \subseteq P_{\omega_2}$ is generic over V_1, then $V_1[G_{\omega_2}] \models$ “Borel conjecture”. So we need prove that for every real number $a \in V_1[G_{\omega_2}]$, κ is an inaccessible cardinal in $L[a]$. For this we remember the

Mathias real forcing M. $(s, A) \in M$ if and only if $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$ and $\sup s < \inf A$; the order is given by setting $(s_1, A_1) \le (s_2, A_2)$ if and only if $s_1 \subseteq s_2$ and $A_2 \subseteq A_1$ and $s_2 - s_1 \subseteq A_1$. Clearly $M \in \Gamma^-_{\aleph_0}$.

Claim. M is a proper $\Gamma^+_{\aleph_0}$-forcing.

Proof. Let N be a countable model for some part of set theory and let $(s, A) \in M^N$; let a be a Mathias real over N extending (s, A). So $s \cup A \supseteq a$ and, if $B = a - \sup s$, we have that $(s, B) \in M$, $(s, A) \le (s, B)$ and (s, B) is an $((s, A), N)$-good generic condition for (s, A). Therefore P_{ω_2} is proper $\Gamma^+_{\aleph_0}$-forcing.

Returning to the proof of 4.5, let $\boldsymbol{a}$ be a P_{ω_2}-name of a real such that

$$V_1[G_{\omega_2}] \models \kappa \text{ is accessible in } L[\boldsymbol{a}[G_{\omega_2}]].$$

As P_{ω_2} is proper, there exists $S \subseteq \omega_2$, S closed, $|S| = \aleph_0$, and $p \in P_S$ (where P_S is the iteration of M only using indexes in S) and $\boldsymbol{a}$ a P_S-name of a real such that

$$p \Vdash_{P_{\omega_2}} \kappa \text{ is accessible in } L[\boldsymbol{a}]$$

(in this case, over p, $\boldsymbol{a}$ is a P_{ω_2}-name of a real). As before, let $\alpha < \kappa$ be such that S is countable in $V[G_\alpha]$ and $p \in P_S^{V[G_\alpha]}$ and $\boldsymbol{a}$ is a $P_S^{V[G_\alpha]}$-name of a real. Clearly if β = order type (S), then $P_S^{V[G_\alpha]}$ is isomorphic to $P_\beta^{V[G_\alpha]}$ in $V[G_\alpha]$, and as κ is an inaccessible cardinal in $V[G_\alpha]$, we have that the number of maximal antichains of $P_S^{V[G_\alpha]}$ is less than κ in $V[G_\alpha]$. Therefore if $\lambda = 2^{2^\kappa}$, working in $V[G_\alpha]$ we can find $N < H(\lambda, \in)$ satisfying

(i) $|N| < \kappa$,
(ii) $P_S^{V[G_\alpha]} \subseteq N$,
(iii) for every maximal antichain $D \subseteq P_S^{V[G_\alpha]}$, $D \in N$, and

(iv) $\kappa \in N$, $\boldsymbol{a} \in N$, $2^\omega \cap V[G_\alpha] \subseteq N$, etc.

Then if $G \subseteq P_S^{V[G_\alpha]}$ is generic over N, we have that G is generic over $V[G_\alpha]$. But N is countable in V_1, and clearly S is closed for (N, P_{ω_2}) and $P_{\omega_2} | S^N$ is exactly $P_S^{V[G_\alpha]}$. Using 2.8 we can find $q \in P_{\omega_2}$, $p \leq q$, such that if $q \in G_{\omega_2}$ then $G_{\omega_2} | S^N$ is generic over N. Clearly $\mathbf{a}[G_{\omega_2}] = \mathbf{a}[G_{\omega_2} | S^N]$, and thus

$$V[G_\alpha][G_{\omega_2} | S^N] \models \kappa \text{ is accessible in } L[\mathbf{a}[G_{\omega_2} | S^N]].$$

By the above remark, $G_{\omega_2} | S^N$ is generic over $V[G_\alpha]$, and therefore there exists $p_1 \in P_S^{V[G_\alpha]}$ such that

$$V[G_\alpha] \models p_1 \Vdash_{P_S} \text{"}\kappa \text{ is accessible in } L[\mathbf{a}]\text{"}.$$

In V_1 there exists $G \subseteq P_S^{V[G_\alpha]}$ containing p_1 and generic over $V[G_\alpha]$ (remember that N is countable in V_1). Therefore

$$V[G_\alpha][G] \models \text{"}\kappa \text{ is accessible in } L[\mathbf{a}[G]]\text{"}.$$

As $V[G_\alpha][G] \subseteq V_1$, we have that $V_1 \models$ "κ is accessible in $L[\mathbf{a}[G]]$", and this is a contradiction. □

4.6. From the Bartoszyński-Raisonnier-Stern theorem, which says that if every Σ^1_2-set of reals is Lebesgue measurable, then every Σ^1_2-set of reals is Baire, it is easy to show that

(iv) ZFC + Borel conjecture + every Σ^1_2-set of reals is Lebesgue measurable is equiconsistent with 4.1(i). In a subsequent paper, we will prove the following result:

THEOREM. *The following theories are equiconsistent*:

(i) *ZFC.*

(ii) *ZFC + MA (σ-centered) + dual Borel conjecture.*

Therefore, in 4.1(iii) it is not possible to replace Σ^1_2-measurability by Σ^1_2-categoricity. But, about this, the following questions are open.

(1) Add(measure) implies Add(strongly meager)

(2) Add(category) implies Add(strong measure zero)

4.6. S. Todorčević has remarked that if we add a Cohen real over a model to MA, then MA (σ-linked) fails in the generic extension. Using the Carlson theorem [Ca] we can prove this by establishing the following fact: *Suppose that on adding a Cohen real over a model to MA then MA (σ-linked) holds in the generic extension. Then, adding ω_2-Cohen reals over a model to MA, we have that every ω_1-sequence of reals which lies in the ground model is a strongly meager set.*

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