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On the Arrow property

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Abstract

We deal with a finite combinatorial problem arising for a question on generalizing Arrow theorem on social choices.

Keywords: Arrow theorem; Social choice functions; Finite combinatorics

0. Introduction

Let X be a finite set of alternatives. A choice function c is a mapping which assigns to nonempty subsets S of X an element $c(S)$ of S. A rational choice function is one for which there is a linear ordering on the alternatives such that $c(S)$ is the maximal element of S according to that ordering. (We will concentrate on choice functions which are defined on subsets of X of fixed cardinality k and this will be enough.)

Arrow's impossibility theorem [1] asserts that under certain natural conditions, if there are at least three alternatives then every non-dictatorial social choice gives rise to a non-rational choice function, i.e., there exist profiles such that the social choice is not rational. A profile is a finite list of linear orders on the alternatives which represent the individual choices. For general references on Arrow's theorem and social choice functions see [2,5,7].

Non-rational classes of choice functions which may represent individual behavior where considered in [3,4]. For example: $c(S)$ is the second largest element in S according to

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some ordering, or $c(S)$ is the median element of S (assume $|S|$ is odd) according to some ordering. Note that the classes of choice functions in these classes are symmetric, namely are invariant under permutations of the alternatives. Gil Kalai asked if Arrow's theorem can be extended to the case when the individual choices are not rational but rather belong to an arbitrary non-trivial symmetric class of choice functions. (A class is non-trivial if it does not contain all choice functions.) The main theorem of this paper gives an affirmative answer in a very general setting. See also [6] for general forms of Arrow's and related theorem.

The part of the proof which deals with the simple case is related to clones which are studied in universal algebras (but we do not use this theory). On clones see [8,9].

Notation:
(1) $n, m, k, \ell, r, s, t, i, j$ natural numbers; always k, many times r are constant (there may be some misuses of k).
(2) X a finite set.
(3) $\mathfrak{C}$ a family of choice function on $\binom{X}{k} = \{Y\colon\ Y \subseteq X,\ |Y| = k\}$.
(4) $\mathcal{F}$ is a clone on X (see Definition 2.3(2)).
(5) $a, b, e \in X$.
(6) $c, d \in \mathfrak{C}$.
(7) $f, g \in \mathcal{F}$.

Annotated content

Section 1: Framework

[What are X, $\mathfrak{C}$, $\mathcal{F} = \mathrm{Av}(\mathfrak{C})$, the Arrow property restricted to $\binom{X}{k}$, $\mathfrak{C}$ is $(X, k) =$ FCF (note: no connection for different $k - s$) and the Main Theorem. For $\mathfrak{C}$, $\mathcal{F}$, $r = r(\mathcal{F})$.]

Part A: The simple case

Section 2: Context and on nice f's

[Define a clone, $r(\mathcal{F})$. If $f \in \mathcal{F}_{(r)}$ is not a monarchy, $r \geqslant 4$ on the family of not one-to-one sequences $\bar{a} \in {}^r X$ then f is a projection, Claim 2.5.
Define $f_{r;\ell,k}$, basic implications on $f_{r;\ell,k} \in \mathcal{F}$, Definition 2.6, Claim 2.7.
If $r = 3$, $f \in \mathcal{F}_{[s]}$ is not a monarchy on one-to-one triples, then f without loss of generality, is $f_{r;1,2}$ or $g_{r;1,2}$ on a relevant set, Claim 2.8.
If $r = 3$, f is not a semi monarchy on permutations of $\bar{a}$.
If $r = 3$, there are some "useful" f, Claim 2.11. Implications on $f_{r;\ell,k} \in \mathcal{F}$.]

Section 3: Getting $\mathfrak{C}$ is full

[Sufficient condition for $r \geqslant 4$ with $f_{r;1,2}$ or so (Lemma 3.1), similarly when $r = 3$.
Sufficient condition for $r = 3$ with $g_{r;1,2}$ or so (Claim 3.3).
A pure sufficient condition for $\mathfrak{C}$ full, Claim 3.4.
Subset $\binom{X}{3}$, closed under a distance, Claim 3.5.
Getting the final conclusion (relying on Section 4).]

Section 4: The $r = 2$ case
[By stages we get a $f \in \mathcal{F}_{[r]}$ which is a monarchy with exactly one exceptional pair, Claims 4.2–4.4. Then by composition we get $g \in \mathcal{F}_2$ similar to $f_{r;1,2}$.]
***Part B*:** Non-simple case
Section 5: Fullness – the non-simple case
[We derive "$\mathfrak{C}$ is full" from various assumptions, and then prove the main theorem.]
Section 6: The case $r = 2$
Section 7: The case $r \geqslant 4$

1. Framework

1.1. Context. We fix a finite set X and $r = \{0, \dots, r-1\}$.

1.2. Definition. (1) An (X, r)-election rule is a function c such that for every "vote" $\bar{t} = \langle t_a \colon a \in X \rangle \in {}^X r$ we have $c(\bar{t}) \in r = \{0, \dots, r-1\}$.

(2) c is a monarchy if $(\exists a \in X)(\forall \bar{t} \in {}^X \bar{r})[c(\bar{t}) = t_a]$.

(3) c is reasonable if $(\forall \bar{t})\ (c(t) \in \{t_a \colon a \in X\})$.

1.3. Definition. (1) We say $\mathfrak{C}$ is a family of choice functions for X (X-FCF in short) if

$$\mathfrak{C} \subseteq \big\{c\colon c \text{ is a function with } \mathrm{Dom}(c) = \mathcal{P}^-(X) \ (= \text{family of nonempty subsets of } X) \\ \text{and } \big(\forall Y \in \mathcal{P}^-(X)\big)\big(c(Y) \in Y\big)\big\}.$$

(2) $\mathfrak{C}$ is called symmetric if for every $\pi \in \mathrm{Per}(X) =$ group of permutations of X, we have

$$c \in \mathfrak{C} \quad \Rightarrow \quad \pi * c \in \mathfrak{C} \quad \text{where } \pi * c(Y) = \pi^{-1}\big(c\pi(Y)\big).$$

(3) $\mathcal{P}_{\mathfrak{C}} = \mathcal{P}^-(X)$.

1.4. Definition. (1) We say av is a r-averaging function for $\mathfrak{C}$ if

(a) av is a function written $\mathrm{av}_Y(a_1, \dots, a_r)$;
(b) for any $c_1, \dots, c_r \in \mathfrak{C}$, there is $c \in \mathfrak{C}$ such that

$$\big(\forall Y \in \mathcal{P}^-(X)\big) \quad \big(c(Y)\big) = \mathrm{av}_Y\big(c_1(Y), \dots, c_r(Y)\big);$$

(c) if $a \in Y \in \mathcal{P}^-(X)$ then $\mathrm{av}_Y(a, \dots, a) = a$.

(2) av is simple if $\mathrm{av}_Y(a_1, \dots, a_r)$ does not depend on Y, so we may omit Y.

(3) $\mathrm{AV}_r(\mathfrak{C}) = \{\text{av}\colon \text{av is an } r\text{-averaging function for } \mathfrak{C}\}$, similarly $\mathrm{AV}_r^s(\mathfrak{C}) = \{\text{av}\colon \text{av is a simple } r\text{-averaging function for } \mathfrak{C}\}$.

(4) $\mathrm{AV}(\mathfrak{C}) = \bigcup_r \mathrm{AV}_r(\mathfrak{C})$ and $\mathrm{AV}^s(\mathfrak{C}) = \bigcup_r \mathrm{AV}_r^s(\mathfrak{C})$.

1.5. Definition. (1) We say that $\mathfrak{C}$ which is an X-FCF, has the simple r-Arrow property if

$$\mathrm{av} \in \mathrm{AV}^s_r(\mathfrak{C}) \quad \Rightarrow \quad \bigvee_{t=1}^{r} (\forall a_1, \dots, a_r)\big(\mathrm{av}(a_1, \dots, a_r) = a_t\big);$$

such av is called monarchical.

(2) Similarly without simple (using $\mathrm{AV}_2(\mathfrak{C})$).

1.6. Question. (1) Under reasonable conditions does $\mathfrak{C}$ have the Arrow property?

(2) Does $|\mathfrak{C}| \leqslant \mathrm{poly}(|X|) \Rightarrow r$-Arrow property? This means, e.g., for every natural numbers r, t^n for every X large enough for every symmetric $\mathfrak{C}$; an X-FCF with $\leqslant |X|^n$ member, $\mathfrak{C}$ has the r-Arrow property.

1.7. Remark. The question was asked with $\mathfrak{C}_{(X)}$ defined for every X; but in the treatment here this does not influence.

We actually deal with:

1.8. Definition. If $1 \leqslant k \leqslant |X| - 1$ and we replace $\mathcal{P}^-(X)$ by $\binom{X}{k} := \{Y\colon\ Y \subseteq X,\ |Y| = k\}$, then $\mathfrak{C}$ is called (X, k)-FCF, $\mathcal{P}_{\mathfrak{C}} = \binom{X}{k}$, $k = k(\mathfrak{C})$, av is [simple] r-averaging function for $\mathfrak{C}$; let $k(\mathfrak{C}) = \infty$ if $\mathcal{P}_{\mathfrak{C}} = \mathcal{P}^-(X)$; let $\mathcal{F} = \mathcal{F}(\mathfrak{C}) = \mathrm{AV}^s(\mathfrak{C})$ and let $\mathcal{F}_{[r]} = \{f \in \mathcal{F}\colon\ f$ is r-place$\}$.

1.9. Discussion. This is justified because:

(1) For simple averaging function, $k \geqslant r$, the restriction to $\binom{X}{k}$ implies the full result.
(2) For the non-simple case, there is a little connection between the various $\mathfrak{C} \restriction \binom{X}{k}$ (exercise).

Our aim is (but we shall first prove the simple case) the following.

1.10. Main Theorem. *There are natural numbers $r_1^*, r_2^* < \omega$ (we shall be able to give explicit values, e.g. $r_1^* = r_2^* = 7$ are OK) such that*:

⊛ *if X is finite, $r_1^* \leqslant k$, $|X| - r_2^* \geqslant k$ and $\mathfrak{C}$ is a symmetrical (X, k)-FCF and some $\mathrm{av} \in \mathrm{AV}_r(\mathfrak{C})$ is not monarchical, then every choice function for $\binom{X}{k}$ belongs to $\mathfrak{C}$ (i.e., $\mathfrak{C}$ is full).*

Proof. By Claim 5.10. $\square$

1.11. Conclusion. Assume X is finite, $r_1^* \leqslant k \leqslant |X| - r_2^*$ (where r_1^*, r_2^* from Theorem 1.10).

(1) If $\mathfrak{C}$ is an (X, k)-FCF and some member of $\mathrm{Av}_r(\mathfrak{C})$ is not monarchical, then $|\mathfrak{C}| = k^{\binom{|X|}{k}}$.

Part A. The simple case

2. Context and on nice f's

Note. Sometimes Part B gives alternative ways.

2.1. Hypothesis (*for Part A*).

(a) X a finite set;
(b) $5 < k < |X| - 5$;
(c) $\mathfrak{C}$ a symmetric (X, k)-FCF and $\mathfrak{C} \neq \emptyset$;
(d) $\mathcal{F}_{[r]} = \{f\colon\ f$ an r-place function from X to X such that $\mathfrak{C}$ is closed under f, that is $f \in \mathrm{AV}^s_r(\mathfrak{C})\}$;
(e) $\mathcal{F} = \bigcup\{\mathcal{F}_{[r]}\colon\ r < \omega\}$;

2.2. Fact. $\mathcal{F}$ is a clone on X (see Definition 2.3) satisfying $f \in \mathcal{F}_{[r]} \Rightarrow f(x_1, \ldots, x_r) \in \{x_1, \ldots, x_r\}$ and $\mathcal{F}$ is symmetric, i.e. closed by conjugation by $\pi \in \mathrm{Per}(X)$.

2.3. Definition. (1) f is monarchical = is a projection, if f is an r-place function (from X to X) and for some t, $(\forall x_1, \ldots, x_n)\ f(x_1, \ldots, x_r) = x_t$.

(2) $\mathcal{F}$ is a clone on X if it is a family of functions from X to X (for all arities, i.e., number of places) including the projections and closed under composition.

2.4. Definition. For $\mathfrak{C}, \mathcal{F}$ as in Hypothesis 2.1:

$$r(\mathfrak{C}) = r(\mathcal{F}) := \min\{r\colon \text{ some } f \in \mathfrak{C}_r \text{ is not monarchical}\}$$

(let $r(\mathcal{F}) = \infty$ if $\mathfrak{C}$ is monarchical).

2.5. Claim. *Assume*

(a) $f \in \mathcal{F}_{[r]}$;
(b) $4 \leqslant r = r(\mathcal{F}) = \min\{r$: *some* $f \in \mathcal{F}$ *is not a monarchy*$\}$.

Then

(1) *for some* $\ell \in \{1, \ldots, r\}$ *we have* $f(x_1, \ldots, x_r) = x_\ell$ *if* $x_1, \ldots, x_r$ *has some repetition.*
(2) $r \leqslant k$.

Proof. (1) Clearly there is a two-place function h from $\{1, \ldots, r\}$ to $\{1, \ldots, r\}$ such that: if $y_\ell = y_k \wedge \ell \neq k$ then $f(y_1, \ldots, y_r) = y_{h(\ell,k)}$; we have some freedom, so without loss of generality:

$\boxtimes$ $\ell \neq k \Rightarrow h(\ell, k) \neq k$.

Sh:782

Assume toward contradiction that (1)'s conclusion fails, i.e.

$\circledast$ $h \restriction \{(\ell, k)\colon 1 \leqslant \ell < k \leqslant r\}$ is not constant.

Case 1. For some $\bar{x} \in {}^r X$ and $\ell_1 \neq k_1 \in \{1, \ldots, r\}$ we have

$$x_{\ell_1} = x_{k_1}, \qquad f(\bar{x}) \neq x_{\ell_1};$$

equivalently: $h\{\ell_1, k_1\} \notin \{\ell_1, k_1\}$, recalling $\boxtimes$.

Without loss of generality, $\ell_1 = r - 1$, $k_1 = r$, $f(\bar{x}) = x_1$ (as for a permutation σ of $\{1, \ldots, r\}$, we can replace f by f_σ, $f_\sigma(x_1, \ldots, x_r) = f(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$).

We can choose $x \neq y$ in X, so $h(x, y, \ldots, y) = x$ hence $\ell \neq k \in \{2, \ldots, r\}$ implies $h(\ell, k) = 1$.

Now for $\ell \in \{2, \ldots, r\}$ we have agreed $h(1, \ell) \neq \ell$, (see $\boxtimes$) so as $h \restriction \{(\ell, k)\colon \ell < k\}$ is not constantly 1 (by $\circledast$), without loss of generality $h(1, 2) = 3$. But as $r \geqslant 4$, letting $x \neq y \in X$ we have $f(x, x, y, y, \ldots)$ is y as $h(1, 2) = 3$ and is x as $h(3, 4) = 1$, contradiction.

Case 2. Not Case 1.

Let $x \neq y$, now consider $f(x, x, y, y, \ldots)$, it is x as $h(1, 2) \in \{1, 2\}$ and it is y as $h(3, 4) \in \{3, 4\}$, contradiction.

(2) follows as for $r > k$ we always have a repetition (see Definition 1.4(1), f plays the role of c). $\square$

2.6. Definition. $f_{r;\ell,k} = f_{r,\ell,k}$ is the r-place function on X defined by

$$f_{r;\ell,k}(\bar{x}) = \begin{cases} x_\ell, & \bar{x} \text{ is with repetition,} \\ x_k, & \text{otherwise.} \end{cases}$$

2.7. Claim. (1) *If $f_{r,1,2} \in \mathcal{F}$ then $f_{r,\ell,k} \in \mathfrak{C}$ for $\ell \neq k \in \{1, \ldots, r\}$.*

(2) *If $f_{r,1,2} \in \mathcal{F}$ and $r = r \geqslant 3$ then $f_{r+1,1,2} \in \mathcal{F}$.*

Proof. (1) Trivial (by Fact 2.2).

(2) First, assume $r \geqslant 5$. Let $g(x_1, \ldots, x_{r+1}) = f_{r,1,2}(x_1, x_2, \tau_3, \ldots, \tau_r)$ where $\tau_m \equiv f_{r,1,m}(x_1, \ldots, x_m, x_{m+2}, \ldots, x_{r+1})$; (that is x_{m+1} is omitted). So for any $\bar{a}$:

- if $\bar{a}$ has no repetitions then

$$\tau_3(\bar{a}) = a_3, \ldots, \tau_r(\bar{a}) = a_r, \qquad g(\bar{a}) = f(a_1, a_2, a_3, \ldots, a_r) = a_2;$$

- if $\bar{a}$ has repetitions, say $a_\ell = a_k$, then there is $m \in \{3, \ldots, r\} \setminus \{\ell - 1, k - 1\}$, hence $\langle a_1, \ldots, a_m, a_{m+2}, \ldots, a_{r+1}\rangle$ is with repetition; so $\tau_m(\bar{a}) = a_1$, so $(a_1, a_2, \ldots, \tau_m(\bar{a}), \ldots)$ has a repetition, so $g(\bar{a}) = a_1$.

Second, assume $r = 4$. Let g be the function of arity 5 defined by: for $\bar{x} = (x_1, \ldots, x_5)$ we let $g(\bar{x}) = f_{r,1,2}(\tau_1(\bar{x}), \ldots, \tau_4(\bar{x}))$ where

$(*)_1$ $\tau_1(\bar{x}) = x_1$;
$(*)_2$ $\tau_2(\bar{x}) = f_{r,1,2}(x_1, x_2, x_3, x_4)$;
$(*)_3$ $\tau_3(\bar{x}) = f_{r,1,3}(x_1, x_2, x_3, x_5)$;
$(*)_4$ $\tau_4(\bar{x}) = f_{r,1,4}(x_1, x_2, x_5, x_4)$.

Note that

$(*)_5$ for $\bar{x}$ with no repetition $\tau_\ell(\bar{x}) = x_\ell$.

Now check that g is as required.

Third, assume $r = 3$. Let $g(x_1, x_2, x_3, x_4) = f_{r,1,2}(\tau_1, \tau_2, \tau_3)$ where

$$\tau_1 = x_1, \qquad \tau_2 = f_{r,1,2}(x_1, x_2, x_4), \qquad \tau_3 = f_{r,1,2}(x_1, x_3, x_4).$$

Now check (or see the proof of Claim 4.7). $\square$

2.8. Claim. *Assume*:

(α) $\mathcal{F}$ *is as in Fact* 2.2;
(β) *every* $f \in \mathcal{F}_{[2]}$ *is a monarchy,* $r = r[\mathcal{F}] = 3$;
(γ) $f^* \in \mathcal{F}_{[3]}$ *and for no* $i \in \{1, 2, 3\}$ *do we have* $(\forall \bar{b} \in {}^3X)$ ($\bar{b}$ *not one-to-one* $\Rightarrow f^*(\bar{b}) = b_i$).

Then for some $g \in \mathcal{F}_{[3]}$ *not a monarchy we have:* (a) *or* (b) *where*

(a) *for* $\bar{b} \in {}^3X$ *which is not one-to-one* $g(\bar{b}) = f_{r;1,2}(\bar{b})$, *i.e.* $= b_1$;
(b) *for* $\bar{b} \in {}^3X$ *which is not one-to-one* $g(\bar{b}) = g_{r;1,2}(\bar{b})$, *see below.*

Where

2.9. Definition. $g_{r;1,2}$ is the following function[1] from X to X:

$$g_{r;1,2,}(x_1, x_2, \dots, x_r) = \begin{cases} x_2, & \text{if } x_2 = x_3 = \cdots = x_r, \\ x_1, & \text{otherwise.} \end{cases}$$

Similarly $g_{r;\ell,k}(x_1, \dots, x_r)$ is x_k if $|\{x_i\colon i \neq \ell\}| = 1$ and x_ℓ otherwise.

Proof of Claim 2.8. The same as the proof of the next claim ignoring the one-to-one sequences (i.e. $f(a_1, a_2, a_3)$), see more later.

2.10. Claim. *Assume* $\mathcal{F}$ *is as in Fact* 2.2, $r = r(\mathcal{F}) = 3$, $f^* \in \mathcal{F}$, f^* *is a* 3*-place function and not a monarchy and* $\bar{a} \in {}^3X$ *is with no repetition such that*: *if* $\bar{a}' = (a'_1, a'_2, a'_3)$ *is a permutation of* $\bar{a}$ *then* $f^*(\bar{a}') = a'_1$; *but* $\neg(\forall \bar{b} \in {}^3X)$ ($\bar{b}$ *not one-to-one* $\Rightarrow f^*(\bar{b}) = b_1$)).

[1] This is the majority function for $r = 3$.

Then for some $g \in \mathcal{F}_3$ *we have* (a) *or* (b) *where*:

(a) (i) *for* $\bar b \in {}^3X$ *with repetition*, $g(\bar b) = f_{r;1,2}(\bar b)$, *i.e.* $g(\bar b) = b_1$;
  (ii) $g(\bar a') = a_2'$ *for any permutation* $\bar a'$ *of* $\bar a$;
(b) (i) *for* $\bar b \in {}^3X$ *with repetition*, $g(\bar b) = g_{r;1,2}(\bar b)$;
  (ii) $g(\bar a') = a_1'$ *for any permutation* $\bar a'$ *of* $\bar a$ (*see on* $g_{r;1,2}$ *in Definition* 2.9).

Proof. Let $\bar a = (a_1, a_2, a_3)$; (a, b, c) denote any permutation of $\bar a$.

Let $W = \{\bar b \colon \bar b \in {}^3X$ and $[\bar b$ is a permutation of $\bar a$ or $\bar b$ not one-to-one$]\}$. Let $\mathcal{F}^- = \{f \restriction W \colon f \in \mathcal{F}\}$, $f = f^* \restriction W$.

Let for $\eta \in {}^3\{1,2\}$, f_η be the 3-place function with domain W, such that

$\boxtimes_0$ $f_\eta(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = a_{\sigma(1)}$ for $\sigma \in \mathrm{Per}\{1,2,3\}$;
$\boxtimes_1$ $f_\eta(a_1, a_2, a_2) = a_{\eta(1)}$;
$\boxtimes_2$ $f_\eta(a_1, a_2, a_1) = a_{\eta(2)}$;
$\boxtimes_3$ $f_\eta(a_1, a_1, a_2) = a_{\eta(3)}$.

Now

$(*)_0$ $f \in \{f_\eta \colon \eta \in {}^32\}$.

[Why? Just think: by the assumption on f^* and as $r(\mathcal{F}) = 3$, in details: for $\boxtimes_1, \boxtimes_2, \boxtimes_2$ remember that $f(x,y,y), f(x,y,x), f(x,x,y)$ are monarchies and for $\boxtimes_0$ remember the assumption on $\bar a$ and of course $f(x,x,x) = x$.]

$(*)_1$ if $\eta = \langle 1,1,1\rangle$ then $f_\eta \neq f$.

[Why? $f_\eta(x_1, x_2, x_3) = x_1$ on W, i.e. is a monarchy.]

$(*)_2$ if $\eta, \nu \in {}^3\{1,2\}$, $\eta(1) = \nu(1)$, $\eta(2) = \nu(3)$, $\eta(3) = \nu(2)$, then $f_\eta \in \mathcal{F}^- \Leftrightarrow f_\nu \in \mathcal{F}^-$.

[Why? In $f(x,y,z)$ we just exchange y and z.]

$(*)_3$ if $f_{\langle 2,2,2,\rangle} \in \mathcal{F}^-$ then $f_{\langle 1,2,2\rangle} \in \mathcal{F}^-$.

[Why? Define g by $g(x,y,z) = f_{\langle 2,2,2\rangle}(x, f_{\langle 2,2,2\rangle}(y,x,z), f_{\langle 2,2,2\rangle}(z,x,y))$ (so $g \in \mathcal{F}^-$) hence

$$g(a,b,c) = f_{\langle 2,2,2\rangle}(a,b,c) = a; \quad \text{hence } g \text{ satisfies } \boxtimes_0,$$
$$g(a,b,b) = f_{\langle 2,2,2\rangle}\big(a, f_{\langle 2,2,2\rangle}(b,a,b), f_{\langle 2,2,2\rangle}(b,a,b)\big) = f_{\langle 2,2,2\rangle}(a,a,a) = a,$$
$$g(a,b,a) = f_{\langle 2,2,2\rangle}\big(a, f_{\langle 2,2,2\rangle}(b,a,a), f_{\langle 2,2,2\rangle}(a,a,b)\big) = f_{\langle 2,2,2\rangle}(a,a,b) = b,$$
$$g(a,a,b) = f_{\langle 2,2,2\rangle}\big(a, f_{\langle 2,2,2\rangle}(a,a,b), f_{\langle 2,2,2\rangle}(b,a,a)\big) = f_{\langle 2,2,2\rangle}(a,b,a) = b.$$

So $g = f_{\langle 1,2,2\rangle}$ hence $f_{\langle 1,2,2\rangle} \in \mathcal{F}^-$ as promised.]

$(*)_4$ $f_{\langle 1,2,2\rangle} \in \mathcal{F}^- \Rightarrow f_{\langle 2,1,2\rangle} \in \mathcal{F}^-$.

[Why? Let

$$g(x,y,z) = f_{\langle 1,2,2\rangle}\big(x, y, f_{\langle 1,2,2\rangle}(z,x,y)\big),$$

so $g(a,b,c) = a$, hence g satisfies $\boxtimes_0$ and

$$\begin{aligned}
g(a,b,b) &= f_{\langle 1,2,2\rangle}\big(a,b,f_{\langle 1,2,2\rangle}(b,a,b)\big) = f_{\langle 1,2,2\rangle}(a,b,a) = b,\\
g(a,b,a) &= f_{\langle 1,2,2\rangle}\big(a,b,f_{\langle 1,2,2\rangle}(a,a,b)\big) = f_{\langle 1,2,2\rangle}(a,b,b) = a,\\
g(a,a,b) &= f_{\langle 1,2,2\rangle}\big(a,a,f_{\langle 1,2,2\rangle}(b,a,a)\big) = f_{\langle 1,2,2\rangle}(a,a,b) = b.
\end{aligned}$$

So $g = f_{\langle 2,1,2\rangle}$, hence $f_{\langle 2,1,2\rangle} \in \mathcal{F}^-$, as promised.]

$(*)_5$ $f_{\langle 2,1,2\rangle} = f_{3;3,1}$, i.e.

$$f_{\langle 2,1,2\rangle}(x_1,x_2,x_3) = \begin{cases} x_1, & \text{if } |\{x_1,x_2,x_3\}| = 3,\\ x_3, & \text{if } |\{x_1,x_2,x_3\}| \leqslant 2,\end{cases} \quad \text{when } (x_1,x_2,x_3) \in W.$$

[Why? Check.]

$(*)_6$ $f_{\langle 2,2,1\rangle}(x_1,x_2,x_3) = x_2$ if $2 \geqslant |\{x_1,x_2,x_3\}|$.

[Why? Check.]

$(*)_7$ $f_{\langle 2,1,2\rangle} \in \mathcal{F}^- \Leftrightarrow f_{\langle 2,2,1\rangle} \in \mathcal{F}^-$.

[Why? See $(*)_2$ in the beginning.]

$(*)_8$ $f_{\langle 1,2,1\rangle} \in \mathcal{F}^- \Leftrightarrow f_{\langle 1,1,2\rangle} \in \mathcal{F}^-$.

[Why? By $(*)_2$ in the beginning.]

$(*)_9$ $f_{\langle 1,2,1\rangle} \in \mathcal{F}^- \Rightarrow f_{\langle 2,2,1\rangle} \in \mathcal{F}^-$.

[Why? Let $g(x,y,z) = f_{\langle 1,2,1\rangle}(x, f_{\langle 1,2,1\rangle}(y,z,x), f_{\langle 1,2,1\rangle}(z,x,y))$; then

$$g(a,b,c) = f_{\langle 1,2,1\rangle}\big(a, f_{\langle 1,2,1\rangle}(b,c,a), f_{\langle 1,2,1\rangle}(c,a,b)\big) = f_{\langle 1,2,1\rangle}(a,b,c) = a,$$

and hence g satisfies $\boxtimes_0$,

$$
\begin{aligned}
g(a,b,b) &= f_{\langle 1,2,1\rangle}\bigl(a,\, f_{\langle 1,2,1\rangle}(b,b,a),\, f_{\langle 1,2,1\rangle}(b,a,b)\bigr) = f_{\langle 1,2,1\rangle}(a,b,a) = b,\\
g(a,b,a) &= f_{\langle 1,2,1\rangle}\bigl(a,\, f_{\langle 1,2,1\rangle}(b,a,a),\, f_{\langle 1,2,1\rangle}(a,a,b)\bigr) = f_{\langle 1,2,1\rangle}(a,b,a) = b,\\
g(a,a,b) &= f_{\langle 1,2,1\rangle}\bigl(a,\, f_{\langle 1,2,1\rangle}(a,b,a),\, f_{\langle 1,2,1\rangle}(b,a,a)\bigr) = f_{\langle 1,2,1\rangle}(a,b,b) = a.
\end{aligned}
$$

So $g = f_{\langle 2,2,1\rangle}$, hence $f_{\langle 2,2,1\rangle} \in \mathcal{F}^-$.]

Diagram (arrows mean belonging to $\mathcal{F}^-$ follows)

$$
\begin{tikzcd}
f_{\langle 2,2,2\rangle} \in \mathcal{F}^- \arrow[d, Rightarrow, "(*)_3"] & & \\
f_{\langle 1,2,2\rangle} \in \mathcal{F}^- \arrow[d, Rightarrow, "(*)_4"] & f_{\langle 1,2,1\rangle} \in \mathcal{F}^- \arrow[r, Leftrightarrow, "(*)_8"'] \arrow[d, Rightarrow, "(*)_9"] & f_{\langle 1,1,2\rangle} \in \mathcal{F}^- \\
f_{\langle 2,1,2\rangle} \in \mathcal{F}^- \arrow[r, Leftrightarrow, "(*)_7"] & f_{\langle 2,2,1\rangle} \in \mathcal{F}^- &
\end{tikzcd}
$$

among the 2^3 functions f_η; one, $f_{\langle 1,1,1\rangle}$, is discarded being a monarchy, see $(*)_1$, six appear in the diagram and imply $f_{r;3,1} \in \mathcal{F}^-$ by $(*)_5$; hence clause (a) of Claim 2.10 holds; and one is $g_{r;1,2}$ because

$(*)_{10}$ $g_{r;1,2} = f_{\langle 2,1,1\rangle}$ on W.

[Why? Check.] So clause (b) of Claim 2.10 holds. $\square$

Continuation of the proof of Claim 2.8. As $r(\mathcal{F}) = 3$ for some $\eta \in {}^3 2$, f^* agrees with f_η for all not one-to-one triples $\bar{b}$. If $\eta = \langle 1,1,1\rangle$, we contradict assumption (γ) as in $(*)_1$ of the proof of Claim 2.10, and if $\eta = \langle 2,1,1\rangle$, possibility (b) of Claim 2.8 holds as in $(*)_{10}$ in the proof of Claim 2.10. If $\eta = \langle 2,1,2\rangle$ then $f^*(\bar{b}) = b_3$ for $\bar{b} \in {}^3 X$ not one-to-one (see $(*)_5$) and this contradicts assumption (γ); similarly if $\eta = \langle 2,2,1\rangle$. In the remaining case (see the diagram in the proof of Claim 2.10), there is $f \in \mathcal{F}$ agreeing on $\{\bar{b} \in {}^3 X\colon\ \bar{b}$ is not one-to-one$\}$ with f_η for $\eta = \langle 1,2,2\rangle$ or $\eta = \langle 1,2,1\rangle$, without loss of generality $f^* = f$.

If $\eta = \langle 1,2,2\rangle$, define g as in $(*)_4$, i.e. $g(x,y,z) = f^*(x,y,f^*(z,x,y))$; so for a non-one-to-one sequence $\bar{b} \in {}^3 X$ we have $g(\bar{b}) = f_{\langle 2,1,2\rangle}(\bar{b}) = b_3$. If for some one-to-one $\bar{a} \in {}^3 X$ we have $f^*(a_3,a_1,a_2) \neq a_3$ then $g(a_1,a_2,a_3) = f^*(a_1,a_2,f^*(a_3,a_1,a_2)) \in \{a_1,a_2\}$; so permuting the variables we get possibility (a). So we are left with the case $\bar{a} \in {}^3 X$ is one-to-one $\Rightarrow f^*(\bar{a}) = a_1$.

Let us define $g \in \mathcal{F}_{[3]}$ by $g(x_1,x_2,x_3) = f^*(f^*(x_2,x_3,x_1),x_3,x_2)$. Let $\bar{b} \in {}^3 X$; if $\bar{b}$ is without repetitions then $g(\bar{b}) = f^*(b_2,b_3,b_2) = b_3$. In case $\bar{b} = (a,b,b)$, we have $g(\bar{b}) = f^*(f^*(b,b,a),b,b) = f^*(a,b,b) = a = b_1$; for $\bar{b} = (a,b,a)$, it follows that $g(\bar{b}) = f^*(f^*(b,a,a),a,b) = f^*(b,a,b) = a = b_1$; and for $\bar{b} = (a,a,b)$ we derive $g(\bar{b}) = f^*(f^*(a,b,a),b,a) = f^*(b,b,a) = a = b_1$; together for $\bar{b}$ non-one-to-one, $g(\bar{b}) = b_1$. So g is as required in clause (a).

Lastly, let $\eta = \langle 1,2,1\rangle$ and let $g(x,y,z) = f^*(x, f^*(y,z,x), f^*(z,x,y))$; now by $(*)_9$ of the proof of Claim 2.10, easily [$\bar b$ is non-one-to-one $\Rightarrow g(\bar b) = f_{\langle 2,2,1\rangle}(\bar b) = b_2$]. Now if (a_1,a_2,a_3) is without repetitions and $f^*(a_2,a_3,a_1) = a_1$ then $g(a_1,a_2,a_3) = a_1$ and possibility (a) holds for this g. Otherwise, we have [$\bar b \in {}^3X$ is one-to-one $\Rightarrow f^*(\bar b) \in \{b_1,b_2\}$]; so if $(a_1,a_2,a_3) \in {}^3X$ is one-to-one and $f^*(a_2,a_3,a_1) \neq a_2$ then $g(a_1,a_2,a_3) \neq a_2$ (as $f^*(a_3,a_1,a_2) \neq a_2$, hence $g(a_1,a_2,a_3) = g(a_1,a_2',a_3')$ for some $a_2', a_3' \neq a_2$); so g is not a monarchy, hence possibility (a) holds. Hence [$\bar b \in {}^3X$ is one-to-one $\Rightarrow f^*(\bar b) = b_2$]. Let $g^* \in \mathcal{F}$ be $g^*(x,y,z) = f^*(f^*(x,y,z), f^*(x,z,y), x)$. Now if $\bar b$ is one-to-one then $g^*(\bar b) = f^*(b_2,b_3,b_1) = b_3$. Also for $\bar b = (a,b,b)$ we have $g^*(\bar b) = f^*(f^*(a,b,b), f^*(a,b,b), a) = f^*(a,a,a) = a$; for $\bar b = (a,b,a)$ we derive $g^*(\bar b) = f^*(f^*(a,b,a), f^*(a,a,b), a) = f^*(b,a,a) = b$, and for $\bar b = (a,a,b)$ we obtain $g^*(\bar b) = f^*(f^*(a,a,b), f^*(a,b,a), a) = f^*(a,b,a) = b$. So g^* is as required in the case $\eta = \langle 1,2,2\rangle$; so we can return to the previous case. $\square$

2.11. Claim. *Assume*:

(α) *$\mathcal{F}$ is as in Fact* 2.2;
(β) *every $f \in \mathcal{F}_{[2]}$ is monarchical*;
(γ) *$f^* \in \mathcal{F}_{[3]}$ is not monarchical.*

Then one of the following holds:

(a) *for every one-to-one $\bar a \in {}^3X$ for some $f = f_{\bar a}$, we have*:
  (i) *$f_{\bar a}(\bar a) = a_2$,*
  (ii) *if $\bar b \in {}^3X$ is not one-to-one then $f_{\bar a}(\bar b) = b_1$*;
(b) *for every one-to-one $\bar a \in {}^3X$, for some $f = f_{\bar a} \in \mathcal{F}_{[3]}$, we have*:
  (i) *if $\bar b$ is a permutation of $\bar a$ then $f_{\bar a}(\bar b) = b_1$,*
  (ii) *if $\bar b \in {}^3X$ is not one-to-one then $f_{\bar a}(\bar b) = g_{r;1,2}(\bar b)$.*

Proof. As $\mathcal{F}$ is symmetric, it suffices to prove "for some $\bar a$" instead of "for every $\bar a$."

Case 1. For some $\ell(*)$ if $\bar b \in {}^3X$ is not one-to-one then $f^*(\bar b) = b_{\ell(*)}$.

As f^* is not monarchical for some one-to-one $\bar a \in {}^3X$, $f^*(\bar a) \neq a_{\ell(*)}$, say $f^*(\bar a) = a_{k(*)}$, $k(*) \neq \ell(*)$. As $\mathcal{F}$ is symmetrical; without loss of generality, $\ell(*) = 1$, $k(*) = 2$. So possibility (a) holds.

Case 2. Not Case 1.

By Claim 2.8, without loss of generality, f^* satisfies (a) or (b) of Claim 2.8 with f^* instead of g. But clause (a) of Claim 2.8 is Case 1 above. So we can assume that case (b) of Claim 2.8 holds, i.e.

$(*)$ if $\bar b \in {}^3X$ is not one-to-one then $f^*(\bar b) = g_{r;1,2}$, i.e.,

$$f^*(\bar b) = \begin{cases} b_2 & \text{if } b_2 = b_3,\\ b_1 & \text{if } b_2 \neq b_3.\end{cases}$$

If Claim 2.10 applies, we are done as then (a) or (b) of Claim 2.10 holds; hence (a) or (b) of Claim 2.11 respectively holds; so assume Claim 2.10 does not apply. So consider a one-to-one sequence $\bar{a} \in {}^3X$ and (recalling that for $\bar{b} \in {}^3X$ with repetitions $g_{r;1,2}(\bar{b})$ is preserved by permutations of $\bar{b}$) it follows that we have sequences $\bar{a}^1, \bar{a}^2$, both permutations of $\bar{a}$ such that

$$\bigvee_i \big[\big(f^*(\bar{a}^1) = a_i^1\big) \equiv \big(f^*(\bar{a}^2) \neq a_i^2\big)\big].$$

Using closure under composition of $\mathcal{F}$ and its being symmetric, for every permutation σ of $\{1, 2, 3\}$ (and as $g_{r;1,2}(\bar{b})$ is preserved by permuting the variables $\bar{b}$ when $\bar{b}$ is with repetition), for each $\sigma = \mathrm{Per}_{\{1,2,3\}}$ there is $f_\sigma \in \mathcal{F}_{[3]}$ such that

(i) $f_\sigma(a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = a_1$,
(ii) if $\bar{b} \in {}^3X$ not one-to-one then $f(\bar{b}) = g_{r;1,2}(\bar{b})$.

Let $\langle \sigma_\rho \colon \rho \in {}^32\rangle$ list the permutations of $\{1, 2, 3\}$, necessarily with repetitions. Now we define by downward induction of $k \leqslant 3$, $f_\rho \in \mathcal{F}$ for $\rho \in {}^k2$ (sequences of zeroes and ones of length k) as follows:

$$\begin{aligned}
\ell g(\rho) = 3 \quad &\Rightarrow \quad f_\rho = f_{\sigma_\rho},\\
\ell g(\rho) < 3 \quad &\Rightarrow \quad f_\rho(x_1, x_2, x_3) = f_\rho\big(x_1, f_{\rho \frown \langle 0\rangle}(x_1, x_2, x_3), f_{\rho \frown \langle 1\rangle}(x_1, x_2, x_3)\big).
\end{aligned}$$

Easily (by downward induction):

$(*)_1$ if $\bar{b} \in {}^3X$ is with repetitions and $\rho \in {}^k2$, $k \leqslant 3$, then $f_\rho(\bar{b}) = g_{r;1,2}(\bar{b})$ (as $g_{r;1,2}$ act as majority).

Now we prove by downward induction on $k \leqslant 3$:

$(*)_2$ if $\bar{b}$ is a permutation of $\bar{a}$, $\rho \in {}^k2$, $\rho \triangleleft \nu \in {}^32$ and $f_\nu(\bar{b}) = a_1$ then $f_\rho(\bar{b}) = a_1$.

This is straightforward and so $f_{\langle\rangle}$ is as required in clause (b). □

Similarly we derive

2.12. Claim. *If $g_{r;\ell,k} \in \mathcal{F}$ then*

$$g_{r;\ell_1,k_1} \in \mathcal{F} \quad \textit{when } \ell_1 \neq k_1 \in \{1, \ldots, r\}.$$

Proof. Trivial. □

3. Getting $\mathfrak{C}$ is full

3.1. Lemma. *Assume*:

(a) $r \geqslant 3$, $\mathcal{F}$ *is as in Fact* 2.2 (*or just is a clone on* X),
  $(*)$ $f_{r;1,2} \in \mathcal{F}$ *or just*
  $(*)^-$ *if* $\bar{a} \in {}^r X$ *is one-to-one then for some* $f = f_{\bar{a}} \in \mathcal{F}$, $f_{\bar{a}}(\bar{a}) = a_2$ *and* $[\bar{b} \in {}^r X$ *non-one-to-one* $\Rightarrow f_{\bar{a}}(\bar{b}) = b_1]$;
(b) $\mathfrak{C}$ *is a* (*non empty*) *family of choice functions for* $\binom{X}{k} = \{Y \subseteq X\colon |Y| = k\}$;
(c) $\mathfrak{C}$ *is closed under every* $f \in \mathcal{F}$;
(d) $\mathfrak{C}$ *is symmetric*;
(e) $k \geqslant r > 2$, $k \geqslant 7$, $|X| - k \geqslant 5, r$.

Then $\mathfrak{C}$ *is full* (*i.e. every choice function is in it*).

Proof. Without loss of generality, $r \geqslant 4$ (if $r = 3$ then clause (e) is fine also for $r = 4$; if in clause (a) the case $(*)$ holds, it is OK by Claim 2.7, and if $(*)^-$ then we repeat the proof of Claim 2.7 for the case $r = 3$, only with $g(x_1, x_2, x_3, x_4) = f_{\langle a_1,a_2,a_3\rangle}(x_1, \tau_2, \tau_3)$ where $\tau_2 = f_{\langle a_1,a_2,a_4\rangle}(x_1, x_2, x_4)$, $\tau_3 = f_{\langle a_1,a_3,a_4\rangle}(x_1, x_3, x_4)$ where for one-to-one $\bar{a} \in {}^3X$, $f_{\bar{a}}$ is defined by the symmetry; this is the proof of Claim 4.7). Assume

$\boxtimes$ $c_1^* \in \mathfrak{C}$, $Y^* \in \binom{X}{k}$, $c_1^*(Y^*) = a_1^*$ and $a_2^* \in Y^* \setminus \{a_1^*\}$.

Question. Is there $c \in \mathfrak{C}$ such that $c(Y^*) = a_2^*$ and $(\forall Y \in \binom{X}{k})$ $(Y \neq Y^* \Rightarrow c(Y) = c_1^*(Y))$?

Choose $c_2^* \in \mathfrak{C}$ such that

(a) $c_2^*(Y^*) = a_2^*$,
(b) $n(c_2^*) = |\{Y \in \binom{X}{k}\colon c_2^*(Y) = c_1^*(Y)\}|$ is maximal under (a).

Easily $\mathfrak{C}$ is not a singleton, so $n(c_2^*)$ is well defined.

3.2. Subfact. A positive answer to the question implies that $\mathfrak{C}$ is full.

[Why? Easy.]

Hence if $n(c_2^*) = \binom{|X|}{k} - 1$, we are done; so assume not and let $Z \in \binom{X}{k}$, $Z \neq Y^*$, $c_1^*(Z) \neq c_2^*(Z)$.

Case 1. For some Z as above and $c_3^* \in \mathfrak{C}$, we have

$$c_3^*(Y^*) \notin \{a_1^*, a_2^*\}, \qquad c_3^*(Z) \in \{c_1^*(Z), c_2^*(Z)\}.$$

If so, let $a_3^* = c_3^*(Y^*)$ and $a_4^* \in Y^* \setminus \{a_1^*, a_2^*, a_3^*\}$, etc.; so $\langle a_1^*, \ldots, a_r^*\rangle$ is one-to-one, $a_\ell^* \in Y^*$.

Let $c_\ell^* \in \mathfrak{C}$ for $\ell = 4, \ldots$ be such that $c_\ell^*(Y^*) = a_\ell$ exists as $\mathfrak{C}$ is symmetric. By assumption (a) we can choose $f \in \mathcal{F}_{[r]}$ such that

$$f(a_1^*, \dots, a_r^*) = a_2^*, \tag{1}$$

$$\bar{a} \in {}^r X \text{ has repetitions} \quad \Rightarrow \quad f(\bar{a}) = a_1. \tag{2}$$

Let $c = f(c_1^*, c_2^*, \dots, c_r^*)$, so $c \in \mathfrak{C}$ and

$$c(Y^*) = f\left(a_1^*, a_2^*, \dots, a_r^*\right) = a_2^*,$$

$$Y \in \binom{X}{k} \quad \& \quad c_1^*(Y) = c_2^*(Y)$$

$$\Rightarrow \quad c(Y) = f\left(c_1^*(Y), c_2^*(Y), \dots\right) = f\left(c_1^*(Y), c_1^*(Y), \dots\right) = c_1^*(Y),$$

$$c(Z) = f\left(c_1^*(Z), c_2^*(Z), c_3^*(Z), \dots\right) = c_1^*(Z) \quad \left(\text{as } \left|\left\{c_1^*(Z), c_2^*(Z), c_3^*(Z)\right\}\right| \leqslant 2\right).$$

So c contradicts the choice of c_2^*.

Case 2. There are $c_3^*, c_4^* \in \mathfrak{C}$ such that $c_3^*(Y^*) \neq c_4^*(Y^*)$ and $\neq a_1^*, a_2^*$, but $c_3^*(Z) = c_4^*(Z)$ or at least $|\{c_1^*(Z), c_2^*(Z), c_3^*(Z), c_4^*(Z)\}| < 4$.

Proof is similar.

Case 3. Neither Case 1 nor Case 2.

Let $\mathcal{P} = \{Z\colon\ Z \subseteq X,\ |Z| = k \text{ and } c_1^*(Z) \neq c_2^*(Z)\}$, so

$(*)_1$ $Y^* \in \mathcal{P}$ and $\mathcal{P} \neq \binom{X}{k}, \{Y^*\}$.

[Why? $\mathcal{P} \neq \{Y^*\}$ by Subfact 3.2. Also we can find $Z \in \binom{X}{k}$ such that $|Y^* \backslash Z| = 2$, $c_1^*(Y^*) \notin Z$. Let $\pi \in \operatorname{Per}(X)$ be the identity on Z, $\pi(c_1^*(Y^*)) \neq c_1^*(Y^*)$, $\pi(Y^*) = Y$. So conjugating c_1^* by π, we get c_2^* satisfying $n(c_2^*) > 0$.]

$(*)_2$ If $Z \in \mathcal{P}$, $c \in \mathfrak{C}$ and $c(Z) \in \{c_1^*(Z), c_2^*(Z)\}$ then $c(Y^*) \in \{c_1^*(Y^*), c_2^*(Y^*)\}$.

[Why? By negating Case 1 except for $Z = Y^*$ which is trivial.]

Subcase 3a. For some Z, we have $Z \in \mathcal{P}$ and

$$|Y^* \setminus Z| \geqslant 4 \quad \text{or just} \quad \left|Y^* \setminus Z \setminus \left\{a_1^*, a_2^*\right\}\right| \geqslant 2 \quad \text{and} \quad |Y^* \setminus Z| \geqslant 3.$$

Let $b_1, b_2, b_3 \in Y^* \setminus Z$ be pairwise distinct. As $\mathfrak{C}$ is symmetric, there are $d_1, d_2, d_3 \in \mathfrak{C}$ such that $d_\ell(Y^*) = b_\ell$ for $\ell = 1, 2, 3$. The number of possible truth values of $d_\ell(Z) \in Y^*$ is 2; so without loss of generality, $d_1(Z) \in Y^* \Leftrightarrow d_2(Z) \in Y^*$, and we can forget b_3, d_3.

So for some $\pi \in \operatorname{Per}(X)$ we have $\pi(Y^*) = Y^*$, $\pi(Z) = Z$, $\pi \restriction (Y^* \backslash Z) = \text{identity}$, hence $\pi(b_\ell) = b_\ell$ for $\ell = 1, 2$ and $\pi(d_1(Z)) = d_2(Z)$; note that $d_\ell(Z) \in Z$, so this is possible; so without loss of generality, $d_1(Z) = d_2(Z)$.

As $|Y^* \setminus Z \setminus \{a_2^*, a_2^*\}| \geqslant 2$, using another $\pi \in \operatorname{Per}(X)$ and without loss of generality, $\{b_1, b_2\} \cap \{a_1^*, a_2^*\} = \emptyset$. So d_1, d_2 gives a contradiction by our assumption “not Case 2.”

Remark. This is enough for non-polynomial $|\mathfrak{C}|$ as $|\{Y\colon\ |Y\setminus Z^*|\leqslant 3\}|\leqslant |Y|^6$.

Subcase 3b. Not Subcase 3a.

So $Z\in\mathcal{P}\setminus\{Y^*\}\Rightarrow |Z\setminus Y^*|\leqslant 3$, hence (recalling $|Z\setminus Y^*|=|Y^*\setminus Z|$) we have $Z\in\mathcal{P}\setminus\{Y^*\}\Rightarrow |Z\cap Y^*|\geqslant k-3\geqslant 1$. Now

$\boxtimes_0$ for $Z\in\mathcal{P}\setminus\{Y^*\}$ there is $c^*\in\mathfrak{C}$ such that $c^*(Y^*)\neq c^*(Z)$.

[Why? Otherwise "by $\mathfrak{C}$ is symmetric" for any $Z\in\mathcal{P}\setminus\{Y^*\}$ we have:

$$\circledast\ \ c\in\mathfrak{C}\wedge Y',\ Y''\in\binom{X}{k}\ \&\ |Y'\cap Y''|=|Z\cap Y^*|\ \Rightarrow\ c(Y')=c(Y'').$$

Define a graph $\mathfrak{G}=\mathfrak{G}_Z$: the set of nodes $\binom{X}{k}$, the set of edges $\{(Y',Y'')\colon\ |Y'\cap Y''|=|Y^*\cap Z|\}$. This graph is connected: if $\mathcal{P}_1,\mathcal{P}_2$ are nonempty disjoint set of nodes with union $\binom{X}{k}$, then there is a cross edge by Claim 3.5 below (why? clause (α) there is impossible by $(*)_1$ and clause (β) is impossible by the first sentence of Subcase 3b). This gives contradiction to $\circledast$. So $\boxtimes_0$ holds.]

We claim:

$\boxtimes_1$ for $Z\in\mathcal{P}$ and $d\in\mathfrak{C}$ we have $d(Y^*)\in Z\cap Y^*\Rightarrow d(Z)=d(Y^*)$.

[Why? Assume d,Z forms a counterexample; recall that $|Y^*\setminus Z|\leqslant 3$ and $k\geqslant 7$ (see Lemma 3.1(e)) so if $k\geqslant 8$ then $|Y^*\cap Z|\geqslant k-3\geqslant 5$ so $Y^*\cap Z\setminus\{a_1^*,a_2^*\}$ has $\geqslant 3$ members; looking again at Subcase 3a, this always holds. Now for some $\pi_1,\pi_2\in\mathrm{Per}(X)$ we have that $\pi_1(Y^*)=Y^*=\pi_2(Y^*)$, $\pi_1(Z)=Z=\pi_2(Z)$, $\pi_1(d(Z))=\pi_2(d(Z))$, $\pi_1(d(Y^*))\neq\pi_2(d(Y^*))$ are from $Z\cap Y^*\setminus\{a_1^*,a_2^*\}$; recall we are assuming that $d(Y^*)\in Z\cap Y^*$ and $d(Z)\neq d(Y^*)$. Let d_1,d_2 be gotten from d by conjugating by π_1,π_2, so we get Case 2, contradiction to the assumption of Case 3.]

$\boxtimes_2$ if $d\in\mathfrak{C}$, $Y\in\binom{X}{k}$ and $d(Y)=a$ then $(\forall Y')(a\in Y'\in\binom{X}{k}\Rightarrow d(Y')=a)$.

[Why? By $\boxtimes_1$ + "$\mathfrak{C}$ closed under permutations of X," we get: if $k^*\in N:=\{|Z\cap Y^*|\colon Z\in\mathcal{P}\setminus\{Y^*\}\}$ (which is not empty) then from $Z_1,Z_2\in\binom{X}{k}$, $|Z_1\cap Z_2|=k^*$, $d\in\mathfrak{C}$ and $d(Z_1)\in Z_2$ it follows $d(Z_1)=d(Z_2)$. Clearly, if $k^*\in N$ then $k^*<k$ (by $Z\neq Y^*$) and $2k-k^*\leqslant|X|$. As in the beginning of the proof of $\boxtimes_1$, we can choose such $k^*>0$. So for the given $d\in\mathfrak{C}$ and $a\in X$, Claim 3.5 below applied to k^*-1, $k-1$, $X\setminus\{a\}$, $(\{Y'\setminus\{a\}\colon a\in Y'$ and $d(Y)=a\}$, $\{Y'\setminus\{a\}\colon\ a\in Y'$ and $d(Y')\neq a\})$. By our assumption, the first family is $\neq\emptyset$. Now clause (α) there gives the desired conclusion (for Y,a as in $\boxtimes_2$). As we know, $k-k^*\leqslant 3$, $k\geqslant 7$, clause (β) is impossible, so we are done.]

Now we get a contradiction: as said above in $\boxtimes_0$, for some $c^*\in\mathfrak{C}$ and $Z\in\mathcal{P}\setminus\{Y^*\}$ we have $c^*(Y^*)\neq c^*(Z)$, choose $Y\in\binom{X}{k}$ such that $\{c^*(Y^*),c^*(Z)\}\subseteq Y$. So by $\boxtimes_2$ we have $d(Y)=d(Y^*)$ and also $d(Y)=d(Z)$, contradiction. $\square$

3.3. Claim. *In Lemma* 3.1 *we can replace* (a) *by*

(a)* (i) $\mathcal{F}$ *is as in Fact* 2.2 (*or just is a clone on* X, $r=3$) *and*
(ii) $g^* \in \mathcal{F}_{[3]}$ *where* (*note* $g^* = g_{3;1,2}$)

$$g^*(x_1,x_2,x_3) = \begin{cases} x_2, & x_2 = x_3, \\ x_1, & \textit{otherwise}, \end{cases}$$

or just
(ii)$^-$ *for any* $\bar a^* \in {}^r X$ *without repetitions, for some* $g = g_{\bar a^*}$, $g(\bar a^*) = a_1^*$ *and if* $\bar a \in {}^r X$ *has repetitions then* $g_{\bar a^*}(\bar a) = g^*(\bar a)$.

Proof. Let $c_1^* \in \mathfrak{C}$, $Y^* \in \binom{X}{k}$, $a_1^* = c_1^*(Y^*)$, $a_2^* \in Y^* \setminus \{a_1^*\}$; we choose c_2^* as in the proof of Lemma 3.1.

Let $\mathcal{P} = \{Y\colon\ Y \in \binom{X}{k},\ Y \neq Y^*,\ c_1^*(Y) \neq c_2^*(Y)\}$; we assume $\mathcal{P} \neq \emptyset$ and shall get a contradiction (this suffices).

$(*)_1$ There are no $Z \in \mathcal{P}$ and $d \in \mathfrak{C}$ such that

$$d(Y^*) = c_2^*(Y^*), \qquad d(Z) \neq c_2^*(Z).$$

[Why? If so, let $c = g(c_1^*, c_2^*, d)$ where g is g^* or just any $g_{\langle c_1^*(Z), c_2^*(Z), d(Z)\rangle}$ (from (a)*(ii)$^-$ of the assumption).

So $c \in \mathfrak{C}$ and

(A) $c(Y^*) = g(c_1^*(Y^*), c_2^*(Y^*), d(Y^*)) = g(c_1^*(Y), c_2^*(Y^*), c_2^*(Y^*)) = c_2^*(Y^*)$;
(B) $c(Z) = g(c_1^*(Z), c_2^*(Z), d(Z)) = c_1^*(Z)$ as $d(Z) \neq c_2^*(Z)$ (just check two cases: if $\langle c_1^*(Z), c_2^*(Z), d(Z)\rangle$ is without repetitions—by the choice of g, otherwise it is equal to $g^*(c_1^*(Z), c_2^*(Z), c_1^*(Z)) = c_1^*(Z)$);
(C) $Y \in \binom{Y}{k}$, $Y \neq Y^*$, $Y \notin \mathcal{P} \Rightarrow c_2^*(Y) = c_1^*(Y) \Rightarrow c(Y) = g(c_1^*(Y), c_2^*(Y), d(Y)) = g^*(c_1^*(Y), c_1^*(Y), d(Y)) = c_1^*(Y)$.

So $(*)_1$ holds by c_2^*'s choice.]

$(*)_2$ if $\pi \in \mathrm{Per}(X)$, $\pi(Y^*) = Y^*$ and $\pi(c_2^*(Y^*)) = c_2^*(Y^*)$ then
(α) $Y \in \mathcal{P}\ \&\ \pi(Y) = Y \Rightarrow \pi(c_2^*(Y)) = c_2^*(Y)$,
(β) $Y \in \mathcal{P} \Rightarrow c_2^*(\pi(Y)) = \pi(c_2^*(Y))$.

[Why? Otherwise may "conjugate" c_2^* by π^{-1} getting $d \in \mathfrak{C}$ which gives a contradiction to $(*)_1$.]

$(*)_3$ let $Z \in \mathcal{P}$ then there are no $d_1, d_2 \in \mathfrak{C}$ such that $d_1(Z) = d_2(Z) \neq c_2^*(Z)$ and $d_1(Y^*) \neq d_2(Y^*)$.

[Why? By $(*)_2$, $d_\ell(Y^*) \neq c_2^*(Y^*)$. Let $g = g_{\langle c_2^*(Y^*), d_1(Y^*), d_2(Y^*)\rangle}$ be as in the proof of $(*)_1$. If the conclusion fails, we let $c = g(c_2^*, d_1, d_2)$ so $c(Y^*) = g(c_2^*(Y^*), d_1(Y^*), d_2(Y^*)) =$

Sh:782

$c_2^*(Y^*)$ as $d_1(Y^*) \neq d_2(Y^*)$ plus choice of g and $c(Z) = g(c_2^*(Z), d_1(Z), d_2(Z)) = d_1(Z) \neq c_2^*(Z)$ as $d_1(Z) = d_2(Z) \neq c_2^*(Z)$. So c contradicts $(*)_1$.]

$(*)_4$ for $Z \in \mathcal{P}$, there are no $d_1, d_2 \in \mathfrak{C}$ such that $d_1(Z) = d_2(Z)$, $d_1(Y^*) \neq d_2(Y^*)$ except possibly when $\{d_\ell(Z)\} = \{c_2^*(Z)\} \in \{Z \cap Y^*, Z \setminus Y^*\}$ for some $\ell = 1, 2$.

[Why? If $d_1(Z) \neq c_2^*(Z)$ use $(*)_3$, so assume $d_1(Z) = c_2^*(Z)$. By the "except possibly" there is $\pi \in \mathrm{Per}(X)$ satisfying $\pi(Y^*) = Y^*$, $\pi(Z) = Z$ and $\pi(c_2^*(Z)) \neq c_2^*(Z)$; now we use it to conjugate d_1, d_2, getting the situation in $(*)_3$; contradiction.]

Let

$$K = \big\{(m)\colon \text{ for some } Z \in \mathcal{P} \text{ we have } |Z \cap Y^*| = m\big\},$$

we are assuming $K \neq \emptyset$. By $(*)_4$ plus symmetry, we know

$(*)_5$ if $(m) \in K$, $1 \neq m < k-1$, and $c_1, c_2 \in \mathfrak{C}$ and $Z_1, Z_2 \in \binom{X}{k}$ satisfies $c_1(Z_1) = c_2(Z_1)$ and $|Z_1 \cap Z_2| = m$, then $c_1(Z_2) = c_2(Z_2)$.

[Why? Let $Z \in \mathcal{P}$, $|Z \cap Y^*| = m$, some $\pi \in \mathrm{Per}(X)$ maps Z_1, Z_2 to Z, Y^*, respectively.]

Case 1. There is $(m) \in K$ such that $1 \neq m < k-1$, let $\mathcal{P}' = \mathcal{P} \cup \{Y^*\}$.

For any $c_1, c_2 \in \mathfrak{C}$ let $\mathcal{P}_{c_1,c_2} = \{Y \in \binom{Y}{k}\colon c_1(Y) = c_2(Y)\}$.

By $(*)_5$ we have $[Y_1, Y_2 \in \binom{X}{k} \wedge |Y_1 \cap Y_2| = m \Rightarrow [Y_1 \in \mathcal{P}_{c_1,c_2} \equiv Y_2 \in \mathcal{P}_{c_1,c_2}]]$.

Let $Y_1 \in \binom{X}{k}$, $c_1 \in \mathfrak{C}$, and let $a = c_1(Y_1)$, $Y_2 \in \binom{X}{k}$ be such that $\{a, b\} = Y_1 \setminus Y_2$ for some $b \neq a$. By conjugation, there is $c_2 \in \mathfrak{C}$ such that $c_2(Y_1) = a = c_1(Y_1)$ and $c_1(Y_2) \neq c_2(Y_2)$. So $Y_1 \in \mathcal{P}_{c_1,c_2}$ and $Y_2 \notin \mathcal{P}_{c_1,c_2}$. To $\mathcal{P}_{c_1,c_2}$ apply Claim 3.5 below; so necessarily $|X| = 2k$, $m = 0$. But as $m = 0$, $(m) \in K$, there is $Y \in \mathcal{P}$ satisfying $|Y \cap Y^*| = m = 0$; hence $Y = X \setminus Y^*$, and by $(*)_2(\alpha)$ we get a contradiction, i.e. we can find π contradicting it.

Case 2. $(m) \in K$, $m = k-1$ and not Case 1 (i.e., for no m').

Let $Z \in \mathcal{P}$ be such that $|Z \cap Y^*| = k-1$, so by $(*)_4$ and $\mathfrak{C}$ being symmetric we have:

$(*)_6$ if $Z_1, Z_2 \in \binom{X}{k}$, $|Z_1 \cap Z_2| = k-1$, $d_1, d_2 \in \mathfrak{C}$, $d_1(Z_1) = d_2(Z_1)$, $d_1(Z_2) \neq d_2(Z_2)$ then $\{d_1(Z_1)\} = Z_1 \setminus Z_2$.

Also,

$(*)_7$ if $Z_1, Z_2 \in \binom{X}{k}$, $|Z_1 \cap Z_2| = k-1$ then for no $d \in \mathfrak{C}$ do we have $d(Z_1) \neq d(Z_2)$ and $\{d(Z_1), d(Z_2)\} \subseteq Z_1 \cap Z_2$.

[Why? Applying appropriate $\pi \in \mathrm{Per}(X)$, we get a contradiction to $(*)_6$.] Case 2 is finished by the following claim (and then we shall continue).

3.4. Claim. *Assume* (a)* *of Claim* 3.3 *and* (b), (c) *of Lemma* 3.1 *and* $(*)_7$ *above* (*on* $\mathfrak{C}$). *Then* $\mathfrak{C}$ *is full.*

Sh:782

Proof of Claim 3.4. Now we state:

$(*)_8$ for every $Z_1, Z_2 \in \binom{X}{k}$, $|Z_1 \cap Z_2| = k-1$ and $a \in Z_1 \cap Z_2$ there is no $d \in \mathfrak{C}$ such that $d(Z_1) = d(Z_2) = a$.

Why? Otherwise we can find Z_1, Z_2 such that $|Z_1 \cap Z_2| = k-1$, $d(Z_1) = d(Z_2) = a$, hence for every $Z_1, Z_2 \in \binom{X}{k}$ such that $|Z_1 \cap Z_2| = k-1$ and $a \in Z_1 \cap Z_2$ there is such d (using appropriate $\pi \in \mathrm{Per}(X)$).

Let $Z_1, Z_2 \in \binom{X}{k}$ such that $|Z_1 \cap Z_2| = k-1$. Let $x \neq y \in Z_1 \cap Z_2$. Choose $d_1 \in \mathfrak{C}$ such that $d_1(Z_1) = d_1(Z_2) = x$. Choose $d_2 \in \mathfrak{C}$ such that $d_2(Z_1) = d_2(Z_2) = y$. Choose $d_3 \in \mathfrak{C}$ such that $d_3(Z_1) = y$, $d_3(Z_2) \in Z_2 \setminus Z_1$.

Why is it possible to choose d_3? Using $\pi \in \mathrm{Per}(X)$, otherwise (using $(*)_7$) we have

$\otimes$ if $Y_1, Y_2 \in \binom{X}{k}$, $|Y_1 \cap Y_2| = k-1$, $d \in \mathfrak{C}$, $d(Y_1) \in Y_1 \cap Y_2$ then $d(Y_2) \in Y_1 \cap Y_2$; hence by $(*)_7$, $d(Y_2) = d(Y_1)$; so for $d \in \mathfrak{C}$ we have (by a chain of Y's):

$$Y_1, Y_2 \in \binom{X}{k}, \quad d(Y_1) \in Y_1 \cap Y_2 \quad \Rightarrow \quad d(Y_2) = d(Y_1).$$

Let $c \in \mathfrak{C}$, $Y_1 \in \binom{X}{k}$, $x_1 = c(Y_1)$. Let $x_2 \in X \setminus Y_1$, $Y_2 = Y_1 \cup \{x_2\} \setminus \{x_1\}$; so if $c(Y_2) \in Y_1 \cap Y_2$, we get a contradiction, therefore $d(Y_2) = x_2$.

Let $x_3 \in Y_1 \cap Y_2$, $Y_3 = Y_1 \cup Y_2 \setminus \{x_3\}$; so $Y_3 \in \binom{X}{k}$, $|Y_3 \cap Y_1| = k-1 = |Y_3 \cap Y_2|$ and clearly $c(Y_1), c(Y_2) \in Y_3$.

If $c(Y_3) \notin Y_1$ then Y_3, Y_1 contradict $\otimes$. If $c(Y_3) \notin Y_2$ then Y_3, Y_2 contradict $\otimes$. But $c(Y_3) \in Y_3 \subseteq Y_1 \cup Y_2$, contradiction. So d_3 exists.

We shall use d_1, d_2, d_3, Z_1, Z_2 to get a contradiction (thus proving $(*)_8$). Let $\{z\} = Z_2 \setminus Z_1$; so $\langle x, y, z\rangle$ is without repetitions. Let $d = g(d_1, d_2, d_3)$; so with $g = g^*$ or $g = g_{\langle x,y,z\rangle}$,

$$d(Z_1) = g\big(d_1(Z_1), d_2(Z_1), d_3(Z_1)\big) = g(x, y, y) = y \quad \text{(see Definition of } g\text{)},$$
$$d(Z_2) = g\big(d_1(Z_2), d_2(Z_2), d_3(Z_2)\big) = g(x, y, z) = x$$

by Definition of g as $y \neq z$ because $y \in Z_1$, $z \notin Z_1$.

So Z_1, Z_2, d contradicts $(*)_7$ and we have proved $(*)_8$.

$(*)_9$ if $|Z_1 \cap Z_2| = k-1$, $Z_1, Z_2 \in \binom{X}{k}$, $d \in \mathfrak{C}$, $d(Z_1) \in Z_1 \cap Z_2$, then $d(Z_2) \in Z_2 \setminus Z_1$.

[Why? By $(*)_7$, $d(Z_2) \notin Z_1 \cap Z_2 \setminus \{d(Z_1)\}$ and by $(*)_8$, $d(Z_2) \notin \{d(Z_1)\}$.]

Let $c \in \mathfrak{C}$ and $x_1, x_2 \in X$ be distinct and $Y \subseteq X \setminus \{x_1, x_2\}$, $|Y| = k$. Let $x_3 = c(Y)$, $x_4 \in Y \setminus \{x_3\}$ and $x_5 \in Y \setminus \{x_3, x_4\}$.

So $Y_1 = Y \cup \{x_1\} \setminus \{x_4\}$ belongs to $\binom{X}{k}$, satisfies $|Y_1 \cap Y| = k-1$ and $c(Y) = x_3 \in Y_1 \cap Y$; hence by $(*)_9$ we have $c(Y_1) = x_1$.

Let $Y_2 = Y \cup \{x_2\} \setminus \{x_4\}$, so similarly $c(Y_2) = x_2$. Let $Y_3 = Y \cup \{x_1, x_2\} \setminus \{x_4, x_5\}$, so $Y_3 \in \binom{X}{k}$, $Y_3 \setminus Y_1 = \{x_2\}$ and $Y_3 \setminus Y_2 = \{x_1\}$. The proof now splits into three cases:

- If $c(Y_3) \in Y$, then $c(Y_3) \in Y_3 \cap Y = Y \setminus \{x_4, x_5\} \subseteq Y_1$, hence $c(Y_3) \in Y_3 \cap Y_1$. Recall that $c(Y_1) = x_1 \in Y_3 \cap Y_1$ and $c(Y_3) \neq x_1$ as $x_1 \notin Y$, so (Y_3, Y_1, c) contradicts $(*)_7$.
- If $c(Y_3) = x_1$, then recalling $c(Y_1) = x_1$ clearly c, Y_3, Y_1 contradicts $(*)_8$.
- If $c(Y_3) = x_2$, then recalling $c(Y_2) = x_2$ clearly c, Y_3, Y_2 contradicts $(*)_8$.

Together contradiction, so we have finished proving Claim 3.4 hence Case 2 in the proof of Claim 3.3. □

Continuation of the proof of Claim 3.3.

Case 3. Neither Case 1 nor Case 2. As $\mathcal{P} \neq \emptyset$ (otherwise we are done), clearly $K = \{(1)\}$. So easily follows (clearly $2k - 1 \leqslant |X|$ as $(1) \in K$):

$\boxtimes_1$ if $|Y_1 \cap Y_2| = 1$, $Y_1 \in \binom{X}{k}$, $Y_2 \in \binom{X}{k}$ and $d \in \mathfrak{C}$ then $d(Y_1) \in Y_1 \cap Y_2$ or $d(Y_2) \in Y_1 \cap Y_2$.

[Why? Otherwise by conjugation we can get a contradiction to $(*)_4$ above.]

$\boxtimes_2$ $Y_1, Y_2 \in \binom{X}{k}$, $|Y_1 \cap Y_2| = k - 1$, $d \in \mathfrak{C}$, $d(Y_1), d(Y_2) \in Y_1 \cap Y_2$ is impossible.

[Why? Assume this fails. Let $x \in Y_1 \setminus Y_2$ and $y \in Y_2 \setminus Y_1$; we can find $Y_3 \in \binom{X}{k}$ such that $Y_3 \cap (Y_1 \cup Y_2) = \{x, y\}$, so $Y_3 \cap Y_1 = \{x\}$, $Y_3 \cap Y_2 = \{y\}$; this is possible as $|X| \geqslant 2k - 1$. Apply $\boxtimes_1$ to Y_3, Y_1, d and as $d(Y_1) \neq x$ (as $d(Y_1) \in Y_2$), we have $c(Y_3) = x$.

Apply $\boxtimes_1$ to Y_3, Y_2, d and as $d(Y_2) \neq y$ (as $d(Y_2) \in Y_1$), we get $d(Y_3) = y$. But $x \neq y$, contradiction.]

By $\boxtimes_2$ we can use the proof of Case 2 from $(*)_7$, i.e. Claim 3.4 to get contradiction. □

3.5. Claim. *Assume*:

(a) $k^* < k < |X| < \aleph_0$;
(b) $\mathcal{P} \subseteq \binom{X}{k}$;
(c) *if* $Z, Y \in \binom{X}{k}$, $|Z \cap Y| = k^*$ *then* $Z \in \mathcal{P} \Leftrightarrow Y \in \mathcal{P}$;
(d) $2k - k^* \leqslant |X|$ (*this is equivalent to clause* (c) *being non-empty*).

Then

(α) $\mathcal{P} = \emptyset \vee \mathcal{P} = \binom{X}{k}$ *or*
(β) $|X| = 2k$, $k^* = 0$ *and so* $E = E_{X,k} := \{(Y_1, Y_2)\colon Y_1 \in \binom{X}{k}, Y_2 \in \binom{X}{k}, (Y_1 \cup Y_2 = X)\}$ *is an equivalence relation on* X, *with each equivalence class a doubleton and* $\mathcal{P}$ *a union of a set of* E*-equivalence classes.*

Proof. If not clause (α), then for some $Z_1 \in \mathcal{P}$, $Z_2 \in \binom{X}{k} \setminus \mathcal{P}$ we have $|Z_1 \setminus Z_2| = 1$. Let $Z_1 \setminus Z_2 = \{a^*\}$, $Z_2 \setminus Z_1 = \{b^*\}$.

Case 1. $2k - k^* < |X|$.

We can find a set $Y^+ \subseteq X \setminus (Z_1 \cup Z_1)$ with $k - k^*$ members (use $|Z_1 \cup Z_2| = k + 1$, $|X \setminus (Z_1 \cup Z_2)| = |X| - (k+1) \geqslant (2k - k^* + 1) - (k+1) = k - k^*$).

Let $Y^- \subseteq Z_1 \cap Z_2$ be such that $|Y^-| = k^*$. Let $Z = Y^- \cup Y^+$; so $Z \in \binom{X}{k}$, $|Z \cap Z_1| = |Y^-| = k^*$, $|Z \cap Z_2| = |Y^-| = k^*$; hence $Z_1 \in \mathcal{P} \Leftrightarrow Z \in \mathcal{P} \Leftrightarrow Z_2 \in \mathcal{P}$, contradiction.

Case 2. $2k - k^* = |X|$ and $k^* > 0$.

Let $Y^+ = X \setminus (Z_1 \cup Z_2)$, so

$$\left|Y^+\right| = (2k - k^*) - (k+1) = k - k^* - 1.$$

Let $Y^- \subseteq Z_1 \cap Z_2$ be such that $|Y^-| = k^* - 1$ (OK, as $|Z_1 \cap Z_2| = k - 1 \geqslant k^*$).

Let $Z = Y^+ \cup Y^- \cup \{a^*, b^*\}$. So $|Z| = (k - k^* - 1) + (k^* - 1) + 2 = k$, $|Z_1 \cap Z| = |Y^- \cup \{a^*\}| = k^*$, $|Z_2 \cap Z| = |Y^- \cup \{b^*\}| = k^*$ and as in Case 1 we are done. $\square$

3.6. Claim. *Assume* $k \geqslant 7$, $|X| - k \geqslant 5$. *If* $r(\mathcal{F}) < \infty$ *then Lemma* 3.1 *or Claim* 3.3 *apply, so* $\mathfrak{C}$ *is full.*

Remark. Recall $r(\mathcal{F}) = \inf\{r\colon$ some $f \in \mathcal{F}_{[r]}$ is not a monarchy$\}$, see Definition 2.4.

Proof. Case 1. $r(\mathcal{F}) \geqslant 4$. Let $f \in \mathcal{F}_{[r]}$ exemplify it, so by Claim 2.5 we have $k \geqslant r$ and for some $\ell(*)$:

$$\bar{a} \in {}^r X \text{ with repetitions} \quad \Rightarrow \quad f(\bar{a}) = a_{\ell(*)}.$$

As f is not a monarchy for some $k(*) \in \{1, \dots, r\}$ and $\bar{a}^* \in {}^r X$, we have $f(\bar{a}^*) = a_{k(*)} \neq a_{\ell(*)}$. Without loss of generality, $\ell(*) = 1$, $k(*) = 2$ and Lemma 3.1 applies.

Case 2. $r(\mathcal{F}) = 3$.

Let $f^* \in \mathcal{F}_{[r]}$ exemplify it. Now apply Lemma 2.11; if (a) there holds, apply Lemma 3.1, if (b) there holds, apply Claim 3.3.

Case 3. $r(\mathcal{F}) = 2$.

By Claim 4.7 below, clause (a) of Lemma 3.1 holds, so we are done. $\square$

4. The case $r = 2$

This is revisited in Section 6 (non-simple case), and we can make presentation simpler (e.g. Fact 6.4).

4.1. Hypothesis. As in Hypothesis 2.1 and

(a) $r(\mathcal{F}) = 2$,
(b) $|X| \geqslant 5$ (have not looked at 4).

4.2. Claim. *Choose* $\bar{a}^* = \langle a_1^*, a_2^* \rangle$, $a_1^* \neq a_2^* \in X$.

4.3. Claim. *For some* $f \in \mathcal{F}_{[2]}$ *and* $\bar{b} \in {}^2X$, *we have*

(a) $f(\bar{a}^*) = a_2^*$;
(b) $\bar{a}^* {}^\wedge \bar{b}$ *has no repetition*;
(c) $f(\bar{b}) = b_1 \neq b_2$.

Proof. There is $f \in \mathcal{F}_{[2]}$ non-monarchical, so for some $\bar{b}, \bar{c} \in {}^2X$,

$$f(\bar{b}) = b_1 \neq b_2, \qquad f(\bar{c}) = c_2 \neq c_1.$$

If $\mathrm{Rang}(\bar{b}) \cap \mathrm{Rang}(\bar{c}) = \emptyset$, we can conjugate $\bar{c}$ to $\bar{a}^*$, f to f', which is as required. If not, find $\bar{d} \in {}^2X$, $d_1 \neq d_2$ satisfying $\mathrm{Rang}(\bar{d}) \cap (\mathrm{Rang}(\bar{a}) \cup \mathrm{Rang}(\bar{b})) = \emptyset$, so $\bar{d}, \bar{b}$ or $\bar{d}, \bar{c}$ are like $\bar{c}, \bar{b}$ or $\bar{b}, \bar{c}$, respectively. $\square$

4.4. Claim. *There is* $f^* \in \mathcal{F}_{[2]}$ *such that*

(a) $f^*(\bar{a}^*) = \bar{a}_2^*$;
(b) $b_1 \neq b_2 \in X, \{b_1, b_2\} \subseteq \{a_1^*, a_2^*\} \Rightarrow f(b_1, b_2) = b_2$;
(c) $b_1 \neq b_2, \{b_1, b_2\} \nsubseteq \{a_1^*, a_2^*\} \Rightarrow f(b_1, b_2) = b_1$.

Proof. Choose f such that

(i) $f \in \mathcal{F}_{[2]}$;
(ii) $f(\bar{a}^*) = a_2^*$;
(iii) $n(f) = |\{\bar{b} \in {}^2X\colon\ f(\bar{b}) = b_1\}|$ is maximal under (i) + (ii).

Let $\mathcal{P} = \{\bar{b} \in {}^2X\colon\ f(\bar{b}) = b_1\}$. In each case we can assume that the previous cases do not hold for any f satisfying (i)–(iii).

Case 1. There is $\bar{b} \in {}^2(X \setminus \{a_1^*, a_2^*\})$ such that $f(\bar{b}) = b_2 \neq b_1$.

There is $g \in \mathcal{F}_{[2]}$, $g(\bar{a}^*) = a_2^*$, $g(\bar{b}) = b_1$ (by Claim 4.3 plus conjugation). Let $f^+(x, y) = f(x, g(x, y))$. So

(A) $f^+(\bar{a}^*) = f(a_1^*, g(\bar{a}^*)) = f(a_1^*, a_2^*) = a_2^*$;
(B) $f^+(\bar{b}) = f(b_1, g(\bar{b})) = f(b_1, b_1) = b_1$;
(C) if $\bar{c} \in \mathcal{P}$ then $f(\bar{c}) = c_1$.

[Why does (C) hold? If $g(\bar{c}) = c_1$ then $f^+(\bar{c}) = f(c_1, g(\bar{c})) = f(c_1, c_1) = c_1$. If $g(\bar{c}) = c_2$ then $f^+(\bar{c}) = f(c_1, g(\bar{c})) = f(c_1, c_2) = f(\bar{c}) = c_1$ (the last equality as $\bar{c} \in \mathcal{P}$).]

By the choice of f, the existence of f^+ is impossible, so

$(*)$ $\bar{b} \in {}^2(X \setminus \{a_1^*, a_2^*\}) \Rightarrow f(\bar{b}) = b_1 \Rightarrow \bar{b} \in \mathcal{P}$ (if $b_1 = b_2$—trivial).

Case 2. There are $b_1 \neq b_2$ such that $\{b_1, b_2\} \nsubseteq \{a_1^*, a_2^*\}$, $f(b_1, b_2) = b_2$ and $b_1 \neq a_1^* \wedge b_2 \neq a_2^*$.

There is $g \in \mathcal{F}_{[2]}$ such that $g(a_1^*, a_2^*) = a_2^*$, $g(b_1, b_2) = b_1$.

[Why? There is $\pi \in \mathrm{Per}(X)$, $\pi(b_1) = a_1^*$, $\pi(b_2) = a_2^*$, $\pi^{-1}(\{b_1, b_2\})$ is disjoint to $\{a_1^*, a_2^*\}$. Conjugate f by π^{-1}, getting g, so $g(a_1^*, a_2^*) = g(\pi b_1, \pi b_2) = \pi(f(b_1, b_2)) = \pi(b_2) = a_2^*$; let c_1, c_2 be such that $\pi(c_1) = b_1$, $\pi(c_2) = b_2$, so

$$g(b_1, b_2) = g(\pi c_1, \pi c_2) = \pi\big(f(c_1, c_2)\big) = \pi(c_1) = b_1$$

(third equality as $c_1, c_2 \notin \{a_2^*, a_2^*\}$ by not Case 1). So there is such $g \in \mathcal{F}$.]

Let $f^+(x, y) = f(x, g(x, y))$; as before, f^+ contradicts the choice of f.

Case 3. For some $b' \neq b'' \in X \setminus \{a_1^*, a_2^*\}$ we have $f(a_1^*, b') = b' \wedge f(a_1^*, b'') = a_1^*$.

As in Case 2, using $\pi \in \mathrm{Per}(X)$ such that $\pi(a_1^*) = a_1^*, \pi(a_2^*) = a_2^*, \pi(b') = b''$.

Case 4. For some $b' \neq b'' \in X \setminus \{a_1^*, a_2^*\}$ we have $f(b', a_2^*) = a_2^* \wedge f(b'', a_2^*) = b''$.

As in Case 3, recall that without loss of generality, Cases 1–4 fail.

Case 5. For some $b', b'' \in X \setminus \{a_1^*, a_2^*\}$, we have $f(a_1^*, b') = b' \wedge f(b'', a_2^*) = a_2^*$.

As Cases 1–4 fail, this holds for every such b', b''; so without loss of generality, $b' \neq b''$ and prove as in Case 2 conjugating by $\pi \in \mathrm{Per}(X)$ such that $\pi(b') = a_2^*$, $\pi(a_1^*) = a_1^*$ and $\pi(b'') = b''$, getting g which satisfies $g(a_1^*, a_2^*) = g(\pi a_1^*, \pi b') = \pi(f(a_1^*, b')) = \pi(b') = a_2^*$ and $g(b'', a_2^*) = g(\pi b'', \pi b') = \pi(f(b'', b')) = \pi(b'') = b''$, whereas $f(b', a_2^*) = a_2^*$; so $f^+(x, y) = f(x, g(x, y))$ contradicts the choice of f.

Without loss of generality, Cases 1–5 fail.

Case 6. For some $b \in X \setminus \{a_1^*, a_2^*\}$ we have $f(a_1^*, b) = b$ and $f(a_2^*, b) = a_2^*$ follows.

Subcase 6A. $f(a_2^*, a_1^*) = a_1^*$. Let $\pi \in \mathrm{Per}(X)$, $\pi(a_1^*) = a_2^*$, $\pi(a_2^*) = a_1^*$ (and $\pi(a) = a$ for $a \in X \setminus \{a_1^*, a_2^*\}$); then $g = \pi f \pi^{-1}$ satisfies $g(a_1^*, a_2^*) = a_2^*$, $g(a_2^*, a_1^*) = a_1^*$ but for $b \in X \setminus \{a_1^*, a_2^*\}$, $g(a_1^*, b) = g(\pi a_2^*, \pi b) = \pi(f(a_2^*, b)) = \pi a_2^* = a_1^*$, easy contradiction (or as below)).

Subcase 6B. So as Cases 1–5 and 6A fail, we have

$$\circledast \quad (\forall b_1, b_2 \in X)[f(b_1, b_2) \neq b_1 \Leftrightarrow (b_1 = a_1^* \;\&\; b_2 \neq a_1^*)].$$

Hence for every $c \in X$ there is $f_c \in \mathcal{F}_{[2]}$ such that

$$\circledast_{f_c} \quad (\forall b_1, b_2 \in X)[f_c(b_1, b_2) \neq b_1 \Leftrightarrow (b_1 = c \;\&\; b_2 \neq c)].$$

Let $a \neq c$ be from X and define $f_{a,c} \in \mathcal{F}_{[2]}$ by $f_{a,c}(x, y) = f_a(x, f_c(y, x))$. Assume $b_1 \neq b_2$, so $f_{a,c}^*(b_1, b_2) = b_2 \neq b_1$ implies $f_c(b_2, b_1) \in \{b_1, b_2\}$, $f_{a,c}(b_1, b_2) = f_a(b_1, f_c(b_2, b_1))$ and so (by the choice of f_a) $b_1 = a$ and $f_c(b_2, b_1) = b_2$, which (by the choice of f_c) implies ($b_1 = a$ and) $b_2 \neq c$. But $b_1 = a$, $b_2 \neq c$ and $b_1 \neq b_2$ imply

$f_c(b_2, b_1) = b_2$, $f_{a,c}(b_1, b_2) = f_a(b_1, b_2) = b_2$. So $f_{a,c}(b_1, b_2) = b_2 \neq b_1$ iff $b_1 = a$, $b_2 \neq c$ and $b_2 \neq b_1$.

Let $a = a_1^*$. Let $\langle c_i \colon i < i^* = |X| - 2\rangle$ list $X \setminus \{a_1^*, a_2^*\}$. We define by induction on $i \leqslant i^*$ a function $f_i \in \mathcal{F}_{[2]}$ by

$$f_0(x, y) = y, \qquad f_{i+1}(x, y) = f_i\big(x, f_{a,c_i}(x, y)\big)$$

and let $f' = f_{i^*}$. Now by induction on i, we can show that $f_i(a_1^*, a_2^*) = a_2^*$ and $f'(b_1, b_2) = b_2 \neq b_1$ imply $(\forall i < i^*)(f_{a,c_i}(b_1, b_2) = b_2 \neq b_1)$.

So $f' \in \mathcal{F}_{[2]}$, $f'(a_1^*, a_2^*) = a_2^*$ and $b_1 \neq b_2 \wedge (b_1, b_2) \neq (a_1^*, a_2^*)$ imply $f'(b_1, b_2) = b_1$. By the choice of f (minimal $n(f)$), we get a contradiction.

Case 7. For some $b \in X \setminus \{a_1^*, a_2^*\}$, we have $f(b, a_2^*) = a_2^*$ and $f(a_1^*, b) = a_2^*$ follows.

Similar to Case 6.

Subcase 7A. $f(a_2^*, a_1^*) = a_1^*$. Similar to 6A.

Subcase 7B. That is, as there, without loss of generality, for every $a \in X$ and for some $f_a \in \mathcal{F}_{[2]}$, we have

$$\circledast \quad (\forall b_1, b_2 \in X)[(f_a(b_1, b_2) = b_2 \neq b_1 \Leftrightarrow b_2 = a \neq b_1)].$$

Let $a \neq c \in X$ and $f_{a,c}(x, y) = f_a(f_c(y, x), x)$. So for $b_1 \neq b_2 \in X$,

(i) $f_{a,c}(b_1, b_2) = b_2$ $(\neq b_1)$ implies $f_a(f_c(b_2, b_1), b_1) = b_2$, which implies $b_2 = c$ and $f_c(b_2, b_1) = b_2$, which implies $b_2 = c$ and $b_1 \neq a$.

We continue as there.

Case 8. Not Cases 1–7; not the conclusion.

So for $\bar{a} = (a_1, a_2) = {}^2X$, $a_1 \neq a_2$ there is $f_{\bar{a}} \in \mathcal{F}$ such that

$$\{b_1, b_2\} \nsubseteq \{a_1, a_2\} \quad \Rightarrow \quad f_{\bar{a}}(b_1, b_2) = b_1,$$
$$f_{\bar{a}}(a_1, a_2) = a_2$$

and (as "not the conclusion")

$$f_{\bar{a}}(a_2, a_1) = a_2.$$

Let $\langle \bar{b}^i \colon i < i^* = |X|^2 - |X| - 2\rangle$ list the pairs $\bar{b} = (b_1, b_2) \in {}^2X$ such that $b_1 \neq b_2$, $\{b_1, b_2\} \neq \{a_1^*, a_2^*\}$.

Define $g_i \in \mathcal{F}_{[2]}$ by induction on i: let $g_0(x, y) = x$ and $g_{i+1}(x, y) = f_{\bar{b}^i}(g_i(x, y), y)$. We can prove by induction on $i \leqslant i^*$ that $g_i(a_1^*, a_2^*) = a_1^*$, $g_i(a_2^*, a_1^*) = a_2^*$, and for $j < i$, $g_i(\bar{b}^j) = b_2^j$. So g_{i^*} is as required interchanging 1 and 2, that is $g(x, y) := g_{i^*}(y, x)$ is as required. $\square$

4.5. Definition/choice. For $b \neq c \in X$, let $f_{b,c}$ be like f in Claim 4.4 with (b, c) instead of (a_1^*, a_2^*), so $f_{c,b}(c, b)$ is b, $f(b, c) = c$ and $f(x_1, x_2) = x_1$ if $\{x_1, x_2\} \nsubseteq \{b, c\}$.

4.6. Claim. *Let $a_1, a_2, a_3 \in X$ be pairwise distinct. Then for some $g \in \mathcal{F}_{[3]}$:*

(i) $\bar{b} \in {}^3X$ *with repetitions* $\Rightarrow g(\bar{b}) = b_1$,
(ii) $g(a_1, a_2, a_3) = a_2$.

Proof. Without loss of generality, we replace a_2 by a_3 in (ii). Let h_ℓ for $\ell = 1, 2, 3, 4$ be the three-place functions

$$h_1(\bar{x}) = f_{a_1,a_2}(x_1, x_2), \qquad h_2(\bar{x}) = f_{a_1,a_3}(x_1, x_3),$$
$$h_3(\bar{x}) = f_{a_2,a_3}\big(h_1(\bar{x}), h_2, (\bar{x})\big), \qquad h_4(\bar{x}) = f_{a_1,a_3}\big(x, h_3(\bar{x})\big).$$

Clearly $h_1, h_2, h_3, h_4 \in \mathcal{F}_{[3]}$. We shall show that h_4 is as required.

To prove clause (ii), note that for $\bar{a} = (a_1, a_2, a_3)$ we have $h_1(\bar{a}) = a_2$, $h_2(\bar{a}) = a_3$, $h_3(\bar{a}) = f_{a_2,a_3}(a_2, a_3) = a_3$ and $h_4(\bar{a}) = f_{a_1,a_3}(a_1, a_3) = a_3$, as agreed above. To prove clause (i), let $\bar{b} \in {}^3X$ be such that $\bar{b} \neq \bar{a}$ and we show that by $(\bar{b}) = b_1$.

Case 1. $b_1 \neq a_1, a_3$, so

$$h_4\big(\bar{b}\big) = f_{a_1,a_3}\big(b_1, h_3\big(\bar{b}\big)\big) = b_1 \quad \text{as } b_1 \neq a_1, a_3.$$

Case 2. $b_1 = a_1$, $b_2 \neq a_2$, hence $b_1 \neq a_2, a_3$, so

$$h_1\big(\bar{b}\big) = f_{a_1,a_2}(b_1, b_2) = f_{a_1,a_2}(a_1, b_2) = a_1 = b_1, \quad \text{as } b_2 \neq a_2 \text{ (if } b_2 = a_1 \text{ also OK)},$$
$$h_3\big(\bar{b}\big) = f_{a_2,a_3}\big(h_1\big(\bar{b}\big), h_2\big(\bar{b}\big)\big) = f_{a_2,a_3}\big(b_1, h_2\big(\bar{b}\big)\big) = b_1 \quad \text{as } b_1 \neq a_2, a_3,$$
$$h_4\big(\bar{b}\big) = f_{a_1,a_3}\big(b_1, h_3\big(\bar{b}\big)\big) = h_{a_1,a_3}(b_1, b_1) = b_1.$$

Case 3. $b_1 = a_1$, $b_2 = a_2$, $b_3 \neq a_3$, so

$$h_1\big(\bar{b}\big) = f_{a_1,a_2}(b_1, b_2) = f_{a_1,a_2}(a_1, a_2) = a_2 = b_2,$$
$$h_2(\bar{b}) = f_{a_1,a_3}(b_1, b_3) = f_{a_1,a_3}(a_1, b_3) = a_1 = b_1 \quad \text{as } b_3 \neq a_3 \text{ (if } b_3 = a_1,\ \text{fine)},$$
$$h_3\big(\bar{b}\big) = f_{a_2,a_3}\big(h_1\big(\bar{b}\big), h_2\big(\bar{b}\big)\big) = h_{a_2,a_3}(b_2, b_1) = b_2 \quad \text{as } b_1 = a_1 \neq a_2, a_3,$$
$$h_4\big(\bar{b}\big) = f_{a_1,a_3}\big(b_1, h_3\big(\bar{b}\big)\big) = f_{a_1,a_3}(b_1, b_2) = b_1 \quad \text{as } b_2 = a_2 \neq a_1, a_3.$$

Case 4. $b_1 = a_3$, $b_3 \neq a_1$. So

$$h_1\big(\bar{b}\big) = f_{a_1,a_2}(b_1, b_2) = b_1 \quad \text{as } b_1 = a_3 \neq a_1, a_2,$$

$$h_2(\bar b) = f_{a_1,a_3}(b_1, b_3) = f_{a_1,a_3}(a_3, b_3) = a_3 = b_1$$
$$\text{as } b_3 \neq a_1 \text{ (if } b_3 = a_3 \text{ then } b_3 = b_1, \text{ so OK too)},$$
$$h_3(\bar b) = f_{a_2,a_3}\big(h_1(\bar b), h_2(\bar b)\big) = f_{a_2,a_3}(b_1, b_1) = b_1,$$
$$h_4(\bar b) = f_{a_1,a_3}\big(b_1, f_3(\bar b)\big) = f_{a_1,a_3}(b_1, b_1) = b_1.$$

Case 5. $b_1 = a_3$, $b_3 = a_1$.

$$h_1(\bar b) = f_{a_1,a_2}(b_1, b_2) = b_1 \quad \text{as } b_1 = a_3 \neq a_1, a_2,$$
$$h_2(\bar b) = f_{a_1,a_3}(b_1, b_3) = b_3 \quad \text{as } \{b_1, b_3\} = \{a_1, a_3\},$$
$$h_3(\bar b) = f_{a_2,a_3}\big(h_1(\bar b), h_2(\bar b)\big) = f_{a_2,a_3}(b_1, b_3) \equiv b_1 \quad \text{as } b_3 = a_1 \neq a_2, a_3,$$
$$h_4(\bar b) = f_{a_1,a_3}\big(b_1, f_3(\bar b)\big) = f_{a_1,a_3}(b_1, b_1) = b_1,$$

as required. □

4.7. Claim. *Let $\bar a^* = (a_1^*, a_2^*, a_3^*, a_4^*) \in {}^4X$ be with no repetitions. Then for some $g \in \mathcal{F}_{[4]}$ we have*:

(i) *if $\bar b \in {}^4X$ is with repetitions then $f(\bar b) = b_1$,*
(ii) $g(\bar a^*) = a_2^*$.

Proof. For any $\bar a \in {}^3X$ without repetitions, let $f_{\bar a}$ be as in Claim 4.6 for the sequence $\bar a$. Let us define (with $\bar x = (x_1, x_2, x_3, x_4)$) $g(\bar x) = g_0(x_1, g_2(x_1, x_2, x_4), g_3(x_1, x_3, x_4))$ with $g_0 = f_{\langle a_1^*, a_2^*, a_3^*\rangle}$, $g_2 = f_{\langle a_1^*, a_2^*, a_4^*\rangle}$, $g_3 = f_{\langle a_1^*, a_3^*, a_4^*\rangle}$. So

(A) $g(\bar a^*) = g_0(a_1^*, g_2(a_1^*, a_2^*, a_3^*), g_3(a_1^*, a_3^*, a_4^*)) = g_0(a_1^*, a_2^*, a_3^*) = a_2^*$;
(B) if $\bar b \in {}^4X$ and $\langle b_1, b_2, b_4\rangle$ has repetitions then $g_2(b_1, b_2, b_4) = b_1$, hence $g(\bar b) = g_0(b_1, b_1, g_3(b_1, b_3, b_4)) = b_1$;
(C) if $\bar b \in {}^4X$ and $\langle b_1, b_3, b_4\rangle$ has repetitions then $g_3(b_1, b_3, b_4) = b_1$, hence $g(\bar b) = g_0(b_1, g_2(b_1, b_2, b_4), b_1) = b_1$;
(D) $\bar b \in {}^4X$ has repetitions, but neither (B) nor (C), then necessarily $b_2 = b_3$, so $\langle b_1, b_2, b_3\rangle$ has repetitions, so $g(\bar b) = g_0(b_1, b_2, b_3) = b_1$. □

Part B: Non-simple case

5. Fullness for the non-simple case

5.1. Context. As in Section 1: $\mathfrak{C}$ is a (X, k)-FCF, $\mathcal{F} = \bigcup\{\mathcal{F}_{[r]}\colon r < \infty\}$ and $\mathcal{F} = \{f\colon f \in \mathrm{AV}(\mathfrak{C})\}$, so

$$\mathcal{F}_{[r]} = \left\{ f\colon\ f \text{ is (not necessarily simple) function written } f_Y(x_1, \dots, x_r),\ \text{for } Y \in \binom{X}{k}, \right.$$

$$\left.\begin{array}{l} x_1, \dots, x_r \in Y \text{ such that } f_Y(x_1, \dots, x_r) \in \{x_1, \dots, x_r\} \text{ and } \mathfrak{C} \text{ is closed under } f, \\ \text{i.e., if } c_1, \dots, c_r \in \mathfrak{C} \text{ and } c = f(c_1, \dots, c_r), \text{ i.e. } c(Y) = f_Y\big(c_1(Y), \dots, c_r(Y)\big), \\ \text{then } c \in \mathfrak{C} \end{array}\right\}$$

and we add (otherwise use Part A; alternatively combine the proofs):

5.2. Hypothesis. If $f \in \mathcal{F}$ is simple then it is a monarchy.

5.3. Definition. (1) $\mathcal{F}[Y] = \{f_Y\colon f \in \mathcal{F}\}$.
(2) $\mathcal{F}_{[r]}(Y) = \{f_Y\colon f \in \mathcal{F}_{[r]}\}$.

5.4. Observation. If $f \in \mathcal{F}_{[r]}$, $Y \in \binom{X}{k}$, then f_Y is an r-place function from Y to Y and

$(*)$ $\mathcal{F}[Y]$ is as in Fact 2.2 on Y.

5.5. Definition. (1) $r(\mathcal{F}) = \min\{r\colon r \geqslant 2$, some $f \in \mathcal{F}_{[r]}$ is not a monarchy$\}$ where
(2) f is a monarchy if for some t we have $(\forall Y)(\forall x_1, \dots, x_r \in Y)[f_Y(x_1, \dots, x_r) = x_t]$.

5.6. Claim. (1) *For proving that $\mathfrak{C}$ is full, it is enough to prove, for some $r \in \{3, \dots, k\}$:*

$(*)$ *for every $Y \in \binom{X}{k}$ and $\bar a \in {}^rY$ which is one-to-one, there is $f = f^{\bar a, Y} \in \mathcal{F}$ such that*
(i) $f_Y(\bar a) = a_2$,
(ii) *if $Z \in \binom{X}{k}$, $Z \neq Y$, $\bar b \in {}^rZ$ then $f_Z(\bar b) = b_1$.*

(2) *If $r \geqslant 4$, we can weaken $f_Z(\bar b) = b_1$ in clause* (ii) *to* $[b_3 = b_4 \vee b_1 = b_2 \vee b_1 = b_3 \vee b_2 = b_3] \Rightarrow f_Y(\bar b) = b_1$.

Proof. The proof is as in the proof of Claim 5.8 below, only we choose $c_3, c_4, \dots, c_r$ such that $\bar a = \langle c_\ell(Y)\colon \ell = 1, 2, \dots, r\rangle$ is without repetitions and $f = f^{\bar a, Y}$ from $(*)$. $\square$

5.7. Claim. *In Claim* 5.6 *we can replace* $(*)$ *by*: $r = 3$ *and*

$(*)$ *if $Y \in \binom{X}{k}$ and $\bar a \in {}^3Y$ is one-to-one* (*or just $a_2 \neq a_3$*), *then for some $g \in \mathcal{F}_{[r]}$,*
(i) $g_Y(\bar a) = a_1$,
(ii) *if $Z \in \binom{Y}{k}$, $Z \neq Y$, $\bar b \in {}^3Z$ is not one-to-one then $g_Z(\bar b) = b_2$ for $b_2 = b_3$, and is b_1 otherwise* (*i.e.* $g_{3;1,2}(\bar b)$).

Proof. Like for Claim 3.3. Let $c_1^* \in \mathfrak{C}$, $Y^* \in \binom{X}{k}$, $a_1^* = c_1(Y^*)$, $a_2^* \in Y^* \setminus \{a_1^*\}$; we choose c_2^* as in the proof of Claim 5.6, i.e. Lemma 3.1, that is $c_2^*(Y^*) = a_2^*$ and with $|\mathcal{P}|$ minimal where $\mathcal{P} = \{Y\colon Y \in \binom{X}{k}, Y \neq Y^*, c_1^*(Y) \neq c_2^*(Y)\}$. As there suffices to prove that $\mathcal{P} = \emptyset$. Now otherwise

$\boxtimes$ there are no $Z \in \mathcal{P}$ and $d \in \mathfrak{C}$ such that

$$d(Y^*) = c_2^*(Y^*), \qquad d(Z) \neq c_2^*(Z).$$

[Why? If so, let $c = g^*(c_1^*, c_2^*, d)$ where g is from $(*)$ for Z, $a_1 = c_1^*(Z)$, $a_2 = c_2^*(Z)$, $a_3 = d(Z)$.] Continue as there: the $g_{\bar{a}}$ depends also on Y, and we write $c(Y) = f_Y(c_1(Y), \ldots, c_r(Y))$. $\square$

5.8. Claim. *Assume* $r(\mathcal{F}) = 2$ ($\mathfrak{C}, \mathcal{F}$ *as usual*) *and*

$(*)$ *for every* $a_1 \neq a_2 \in Y \in \binom{X}{k}$, *for some* $f = f^Y_{\langle a_1, a_2\rangle} \in \mathcal{F}$, *we have*:
 (i) $f_Y(\bar{a}) = a_2$,
 (ii) $Z \in \binom{Y}{k}$, $Z \neq Y$, $\bar{b} \in {}^2Z \Rightarrow f_Z(\bar{b}) = b_1$.

Then $\mathfrak{C}$ *is full.*

Remark. $\mathfrak{C}$ is full iff every choice function of $\binom{X}{k}$ belongs to it.

Proof. If $\mathfrak{C}$ is not full, as $\mathfrak{C} \neq \emptyset$, there are $c_1 \in \mathfrak{C}$, $c_0 \notin \mathfrak{C}$, c_0 a choice function for $\binom{X}{k}$. Choose such a pair (c_1, c_0) with $|\mathcal{P}|$ minimal where $\mathcal{P} = \{Y \in \binom{X}{k}: c_1(Y) \neq c_0(Y)\}$. So clearly $\mathcal{P}$ is a singleton, say $\{Y\}$. By symmetry, for some $c_2 \in \mathfrak{C}$ we have $c_2(Y) = c_0(Y)$. Let f be $f^Y_{c_1(Y), c_0(Y)} = f^Y_{c_1(Y), c_2(Y)}$ from the assumption, so $f \in \mathcal{F}$ and let $c = f(c_1, c_2)$; so clearly $c \in \mathfrak{C}$ (as $\mathfrak{C}$ is closed under every member of $\mathcal{F}$).

Now

(A) $c(Y) = f_Y(c_1(Y), c_2(Y)) = c_2(Y) = c_0(Y)$;
(B) if $Z \in \binom{X}{k} \setminus \{Y\}$ then $c(Z) = f_Z(c_1(Z), c_2(Z)) = c_1(Z) = c_0(Y)$.

So $c = c_0$, hence $c_0 \in \mathfrak{C}$, contradiction. $\square$

5.9. Claim. *Assume* $r(\mathcal{F}) = 2$ *and* $\boxtimes(f^*)$ *of Claim* 6.9 (*see Definitions* 6.3, 6.6) *below holds. Then* $\mathfrak{C}$ *is full.*

Proof. We use conventions from Definition 6.6 and Claims 6.7, 6.9 below. In $\boxtimes(f^*)$ there are two possibilities:

Possibility (i). This holds by Claim 5.8.

Possibility (ii). Similar to the proof of Claim 5.8. Again $\mathcal{P} = \{Y\}$ where $\mathcal{P} = \{Y \in \binom{X}{k}: c_1(Y) \neq c_0(Y)\}$. We choose $c_2 \in \mathfrak{C}$ such that $c_2(Y) = c_0(Y)$ and $c_2(X \setminus Y) = c_1(X \setminus Y)$, continue as before. Why is this possible? Let $\pi \in \mathrm{Per}(X)$ be such that $\pi(Y) = Y$, $\pi(c_1(Y)) = c_0(Y)$, $\pi(c_1(X \setminus Y)) = c_1(X \setminus Y)$ (and of course, $\pi(X \setminus Y) = X \setminus Y$). Now conjugating c_1 by π gives c_2 as required. $\square$

5.10. Claim. *If* $r(\mathcal{F}) < \infty$ *then* $\mathfrak{C}$ *is full.*

Proof. Let $r = r(\mathcal{F})$.

Case 1. $r = 2$.

So Hypothesis 6.1 holds.

If $\boxtimes(f)$ of Claim 6.9 holds for some $f \in \mathcal{F}_{[r]}$, by Claim 5.9 we know that $\mathfrak{C}$ is full. If $\boxtimes(f)$ of Claim 6.9 fails for every $f \in \mathcal{F}_{[r]}$ then Hypothesis 6.11 holds hence 6.12–6.18 holds. So by Claim 6.18 we know that $(*)$ of Claim 5.6 holds (and $\mathcal{P}_{\pm}$ is a singleton, see Conclusion 6.17(c) plus Claim 6.18(2)). So by Claim 5.6, $\mathfrak{C}$ is full.

Case 2. $r \geqslant 4$.

So Hypothesis 7.1 holds. By Claim 7.5 clearly $(*)$ of Claim 5.6 holds hence by Claim 5.6(2) we know that $\mathfrak{C}$ is full.

Case 3. $r = 3$.

Let $f^* \in \mathcal{F}_{[3]}$ be not a monarchy. So for $\bar{b} \in {}^3Y$ not one-to-one, $Y \in \binom{X}{k}$, clearly $f^*_Y(\bar{b})$ does not depend on Y, so we write $f^-(\bar{b})$. If for some $\ell(*)$, $f^-(\bar{b}) = b_{\ell(*)}$ for every such $\bar{b}$ then easily Claim 5.6(1) apply. If $f^-(\bar{b}) = g_{r;1,2}(\bar{b})$, let $\bar{a} \in {}^3Y$, $Y \in \binom{X}{k}$, $\bar{a}$ is one-to-one, so $f_Y(\bar{b}) = a_k$ for some k; by permuting the variables, f^- does not change while we have $k = 1$, so Claim 5.7 applies. If both fail, then by repeating the proof of Claim 2.8, for some $f' \in \mathcal{F}_{[3]}$, for $\bar{b} \in {}^3X$ not one-to-one, we have $\bar{b} \in {}^3Y \Rightarrow f'_Y(\bar{b}) = f_{\langle 1,2,1\rangle}(\bar{b})$ or for $\bar{b}$ not one-to-one $\bar{b} \in {}^3Y \Rightarrow f'_Y(\bar{b}) = f_{\langle 1,2,2\rangle}(\bar{b})$. By the last paragraph of the proof of Claim 2.8 we can assume that Case 2 holds. In this case, repeat the proof of the case $\eta = \langle 1, 2, 2\rangle$ in the end of the proof of Claim 2.8. □

6. The case $r(\mathcal{F}) = 2$

For this section

6.1. Hypothesis. $r = 2$.

6.2. Discussion. So (α) or (β) holds where

(α) there are $Y \in \binom{X}{k}$ and $f \in \mathcal{F}_{[r]}(Y)$ which is not monarchy. Hence by Section 4, i.e. Claim 4.4 for $a \neq b \in Y$ there is $f = f^Y_{a,b} \in \mathcal{F}_2[Y]$,

$$f_Y(x, y) = \begin{cases} y, & \text{if } \{x, y\} = \{a, b\}, \\ x, & \text{otherwise;} \end{cases}$$

(β) every f_Y is a monarchy but some $f \in \mathcal{F}_{[r]}$ is not.

6.3. Definition/choice. Choose $f^* \in \mathcal{F}_2$ such that

(a) $\neg(\forall Y)(\forall x, y \in Y)(f_Y(x, y) = x)$;

Sh:782

(b) under (a), $n(f) = |\mathrm{dom}_1(f)|$ is maximal where $\mathrm{dom}_1(f) = \{(Z, a, b)\colon f_Z(a, b) = a \neq b,\ Z \in \binom{X}{k}$ and $\{a, b\} \subseteq Z$ of course$\}$.

6.4. Fact. If $f_1, f_2 \in \mathcal{F}_{[2]}$ and f is $f(x, y) = f_1(x, f_2(x, y))$ (formally $f(Y, x, y) = f_1(Y, x, f_2(Y, x, y))$ but we shall be careless) then $\mathrm{dom}_1(f) = \mathrm{dom}_1(f_1) \cup \mathrm{dom}_1(f_2)$.

Proof is easy.

6.5. Claim. *If $Z \in \binom{X}{k}$, $f^*_Z(a^*, b^*) = b^* \neq a$ then*

(a) $(\forall x, y \in Z)[f^*_Z(x, y) = y]$ *or*
(b) $x, y \in Z\ \&\ \{x, y\} \nsubseteq \{a^*, b^*\} \Rightarrow f^*_Z(x, y) = x$.

Proof. As in Claim 4.4 (plus Definition/choice 6.3 and Fact 6.4), recalling 5.4, i.e., that $\mathcal{F}[Z]$ is a clone. □

6.6. Definition. Let

(1) $\mathcal{P}_1 = \mathcal{P}_1(f^*) = \{Z \in \binom{X}{k}\colon (\forall a, b \in Z)(f^*_Z(a, b) = a\}$;
(2) $\mathcal{P}_2 = \mathcal{P}_2(f^*) = \{Z \in \binom{X}{k}\colon (\forall a, b \in Z)(f^*_Z(a, b) = b\}$;
(3) $\mathcal{P}_\pm = \mathcal{P}_\pm(f^*) = \binom{X}{k} \setminus \mathcal{P}_1(f^*) \setminus \mathcal{P}^*_2(f^*)$.

6.7. Claim. *For $Y \in \binom{X}{k}$ we have*:

(1) *$Y \in \mathcal{P}_\pm(f^*)$ iff $Y \in \binom{X}{k}$ and $(\exists a, b \in Y)$ $(f^*_Y(a, b) = a \neq b)$ and also $(\exists a, b \in Y)$ $(f^*_Y(a, b) = b \neq a)$.*
(2) *If $Y \in \mathcal{P}_\pm$, then there are $a_Y \neq b_Y \in Y$ such that $f^*_Y(a_Y, b_Y) = b_Y$ and*

$$\{a, b\} \subseteq Y, \quad \{a, b\} \nsubseteq \{a_Y, b_Y\} \quad \Rightarrow \quad f^*_Y(a, b) = a.$$

Proof. By Claim 6.5. □

6.8. Claim. (1) *$\langle \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_\pm \rangle$ is a partition of $\binom{X}{k}$.*

(2) *For $Y \in \mathcal{P}_\pm$ the pair (a_Y, b_Y) is well defined (but maybe (b_Y, a_Y) can serve as well).*

Proof. (1) By Definition 6.6. (2) By Claim 6.7. □

6.9. Claim. *If $\mathcal{P}_2(f^*) \neq \emptyset$ then*

$\boxtimes(f^*)$ (i) *$\mathcal{P}_2 = \mathcal{P}_2(f^*)$ is a singleton, $\mathcal{P}_\pm = \emptyset$ or*
(ii) *$2k = |X|$, $\mathcal{P}_2$ is $\{Y^*, Y^{**}\} \subseteq \binom{X}{k}$ where $Y^* \cup Y^{**} = X$ and $\mathcal{P}_\pm = \emptyset$.*

Proof. Assume $\mathcal{P}_2 \neq \emptyset$, let $Y^* \in \mathcal{P}_2$. As f^* is not a monarchy

$(*)_1$ $\mathcal{P}_1 \cup \mathcal{P}_\pm \neq \emptyset$.

By Definition 6.6 and Fact 6.4, $f^* \in \mathcal{F}_{[r]}$ satisfies

$(*)_2$ (i) $f^*_{Y^*}(a,b) = b$ for $a, b \in Y^*$;
(ii) if $g \in \mathcal{F}_{[r]}$, $g_{Y^*}(a,b) = b$ for $a, b \in Y^*$ then $\mathrm{dom}_1(f^*) \supseteq \mathrm{dom}_1(g)$.

Hence

$(*)_3$ if $Y_1 \in \mathcal{P}_2$, $Y_2 \notin \mathcal{P}_2$, $k^* = |Y_1 \cap Y_2|$ and $Y \in \binom{Y}{k}$, $|Y \cap Y^*| = k^*$, then $Y \notin \mathcal{P}_2$ (even $Y \in \mathcal{P}_1 \Leftrightarrow Y_2 \in \mathcal{P}_1$).

[Why? By $(*)_2$ as we can conjugate f^* by $\pi \in \mathrm{Per}(X)$ which maps Y^* onto Y_1 and Y onto Y_2.]

So by Claim 3.5 (applied to k^*) and $(*)_1$

$(*)_4$ (i) $\mathcal{P}_2$ is the singleton $\{Y^*\}$ or
(ii) $\mathcal{P}_2$ is a $\{Y^*, Y^{**}\}$, $2k = |X|$ and $Y^{**} = X \setminus Y^*$;

$(*)_5$ if $Z \in \mathcal{P}_\pm$, then (α) or (β):
(α) $\{a_Z, b_Z\} = Z \cap Y^*$, $f^*_Z(b_Z, a_Z) = a_Z$,
(β) $\{a_Z, b_Z\} = Z \setminus Y^*$, $f^*_Z(b_Z, a_Z) = a_Z$.

[Why? If $\{a_Z, b_Z\} \notin \{Z \cap Y^*, Z \setminus Y^*\}$ then, as $k \geqslant 3$, we can choose $\pi \in \mathrm{Per}(X)$, $\pi(Y^*) = Y^*$, $\pi(Z) = Z$ such that $\pi''\{a_Z, b_Z\} \nsubseteq \{a_Z, b_Z\}$ and use Definition 6.3 and Fact 6.4 on a conjugate of f^*. So $\{a_Z, b_Z\} \in \{Z \cap Y^*, Z \setminus Y^*\}$ and if $f^*_Z(b_Z, a_Z) \neq a_Z$, we use $\pi \in \mathrm{Per}(X)$ such that $\pi(Y^*) = Y^*$, $\pi(Z) = Z$ and $\pi(a_Z) = b_Z$, $\pi(b_Z) = a_Z$ and 6.4.]

It is enough by $(*)_4$ to prove $\mathcal{P}_\pm = \emptyset$. So assume toward contradiction $\mathcal{P}_\pm \neq \emptyset$. By $(*)_5$ one of the following two cases occurs.

Case 1. $Z^* \in \mathcal{P}_\pm$, $|Z^* \cap Y^*| = k - 2$.

As we are allowed to assume $k + 4 < |X|$, there is Y$\in \binom{X}{k}$ such that $|Y \cap Y^*| = k - 1$ and $Y \cap Z^* = Y^* \cap Z^*$. Now (by $(*)_5$) we have $Y \notin \mathcal{P}_\pm$ and (by $(*)_4$) we have $Y \notin \mathcal{P}_2$ so $Y \in \mathcal{P}_1$. So there is $\pi \in \mathrm{Per}(X)$ such that $\pi(Y^*) = Y$, $\pi \restriction Z^* =$ identity, let $f = (f^*)^\pi$ so by Fact 6.4 we get a contradiction to the choice of f^*.

Case 2. $Z^* \in \mathcal{P}_\pm$, $|Z^* \cap Y^*| = 2$.

A proof similar to Case 1 works if $Z^* \cup Y^* \neq X$. Otherwise let $\pi \in \mathrm{Per}(X)$ be the identity on $Z^* \cap Y^*$ and interchange Z^*, Y^*. Apply Fact 6.4 on f^*, $(f^*)^\pi$, so $(a_{Z^*}, b_{Z^*}) \notin \mathrm{dom}_1(f^*) \cup \mathrm{dom}_1((f^*)^\pi)$, etc., easy contradiction. □

6.10. Remark. If $\boxtimes(f^*)$ of Claim 6.9 holds for some f^* then (in the context of Section 5) $\mathfrak{C}$ is full by Claim 5.9.

6.11. Hypothesis. For no $f \in \mathcal{F}_{[r]}$ is $\boxtimes(f)$.

6.12. Conclusion. (1) $\mathcal{P}_2(f^*) = \emptyset$.

(2) $\mathcal{P}_\pm \neq \emptyset$.

(3) $\mathcal{P}_1 \neq \emptyset$.

(4) If $Y \in \mathcal{P}_\pm$ and $|Y \cap Z_1| = |Y \cap Z_2|$ and $a_Y \in Z_1 \Leftrightarrow a_Y \in Z_2$ and $b_Y \in Z_1 \Leftrightarrow b_Y \in Z_2$ where, of course, $Y, Z_1, Z_1 \in \binom{X}{k}$, then $Z_1 \in \mathcal{P}_\pm \Leftrightarrow Z_2 \in \mathcal{P}_\pm$.

Proof. (1) By Hypothesis 6.11 and Claim 6.9.

(2) Otherwise f^* is a monarchy.

(3) Assume not, so $\mathcal{P}_\pm = \binom{X}{k}$. Let $Y \in \mathcal{P}_\pm$, $Z \in \binom{X}{k}$, $Z \cap \{a_Y, b_Y\} = \emptyset$ and[2] $|Z \cap Y| > 2$ and $|Z \setminus Y| > 2$, we can get a contradiction to $n(f^*)$'s minimality.

(4) By Definition/choice 6.3 and Fact 6.4 as we can find $\pi \in \mathrm{Per}(X)$ such that $\pi(Y) = Y$, $\pi(Z_1) = Z_2$, $\pi(a_Y) = a_Y$, $\pi(b_Y) = b_Y$. $\square$

6.13. Claim. *If $Y, Z \in \mathcal{P}_\pm$ and $Y \neq Z$, then there is no $\pi \in \mathrm{Per}(X)$ such that*

$$\begin{aligned} &\pi(Y) = Y, && \pi(Z) = Z, \\ &\pi(a_Y) = a_Y, && \pi(b_Y) = b_Y, && \bigl\{\pi(a_Z), \pi(b_Z)\bigr\} \not\subseteq \{a_Z, b_Z\}. \end{aligned}$$

Proof. By Definition/choice 6.3 and Fact 6.4. $\square$

6.14. Claim. *If $Y \in \mathcal{P}_\pm$, $Z \in \mathcal{P}_\pm$, $2 < |Y \cap Z| < k - 2$ then $\{a_Z, b_Z\} = \{a_Y, b_Y\}$.*

Proof. By Claim 6.13. Except when $Y \cap Z = \{a_Y, b_Y, a_Z, b_Z\}$. Then choose $Z_1 = Z$ and $Z_2 \in \binom{X}{k}$ $Z_2 \cap (Y \cap Z) = \{a_Y, b_Y\}$, $|Y \cap Z_1| = |Y \cap Z|$, $Z_1 \backslash Y \cap Z = Y' \setminus Y'_* \setminus Y \cap Z$ where $Y_* \subseteq Y \setminus Z$ has $|Y \cap Z|_{-2}$ members.

By 6.12(2), $Z_2 \in \mathcal{P}^\pm$, so as in the original case $Y \cap Z_2 = \{a_Y, b_Y, a_{Z_2}, b_{Z_2}\}$ and for Z_1, Z_2 the original case suffices. (Alternatively as a lemma $4 < |Y \cap Z| < k - 4$, and in 6.12 replace 4 by 6.) $\square$

6.15. Claim. *If $Z_0, Z_1 \in \mathcal{P}_\pm$ and $|Z_1 \setminus Z_0| = 1$ then $\{a_{Z_0}, b_{Z_0}\} = \{a_{Z_1}, b_{Z_1}\}$.*

Proof. We shall choose by induction $i = 0, 1, 2, 3, 4$ a set $Z_i \in \mathcal{P}_\pm$ such that $j < i \Rightarrow |Z_i \setminus Z_j| = i - j$. By Claim 6.14 we have $i - j = 3, 4 \leqslant 4 \Rightarrow \{a_{Z_i}, b_{Z_i}\} = \{a_{Z_j}, b_{Z_j}\}$, as this applies to $(j, i) = (0, 4)$ and $(j, i) = (1, 4)$, we get the desired conclusion by transitivity of equality.

To choose Z_i, let $x_i \in X \setminus (Z_0 \cup \cdots \cup Z_{i-1})$; possible as we exclude $k + i - 1$ elements and choose $y_i \in Z_0 \cap \cdots \cap Z_{i-1} \setminus \{a_{Z_{i-1}}, b_{Z_{i-1}}\}$. Now let $Z_i = Z_{i-1} \cup \{y_i\} \setminus \{x_i\}$ easily $j < i \Rightarrow |Z_i \setminus Z_j| = i - j$ and $Z_i \in \mathcal{P}_\pm$ by Conclusion 6.12(4) with Y, Z_1, Z_2 there standing for Z_{i-1}, Z_{i-2}, Z_i here. $\square$

6.16. Choice. $Y^* \in \mathcal{P}_\pm$.

[2] I am sure that after careful checking we can improve the bound.

6.17. Conclusion.

(a) $Y^* \in \mathcal{P}_\pm$.
(b) If $Y \in \mathcal{P}_\pm$ then $(\{a_Y, b_Y\} = \{a_{Y^*}, b_{Y^*}\}$.
(c) One of the following possibilities holds.
  (α) $\mathcal{P}_\pm = \{Y^*\}$;
  (β) $\mathcal{P}_\pm = \{Y \in \binom{X}{k}$: $\{a_{Y^*}, b_{Y^*}\} \subseteq Y\}$;
  (γ) $\mathcal{P}_\pm = \{Y^*, Y^{**}\}$ where $Y^{**} = (X \setminus Y^*) \cup \{a_{Y^*}, b_{Y^*}\}$ and $|X| = 2k - 2$ (hence $\{a_{Y^{**}}, b_{Y^*}\} = \{a_{Y^*}, b_{Y^*}\}$).

Proof. Note that

($*$) if $Y_1, Y_2 \in \mathcal{P}_\pm$, $|Y_1 \setminus Y_2| = 1$ and $Y_3 \in \mathcal{P}_\pm$, $Y_4 \in \binom{X}{k}$, $|Y_3 \setminus Y_4| = 1$ and $\{a_{Y_3}, b_{Y_3}\} = \{a_{Y_1}, b_{Y_1}\} \subseteq Y_4$ then $Y_4 \in \mathcal{P}_\pm$ (hence $\{a_{Y_4}, b_{Y_4}\} = \{a_{Y_3}, b_{Y_3}\} = \{a_{Y_1}, b_{Y_1}\}$).

[Why? As there is a permutation π of X such that $\pi(a_{Y_1}) = a_{Y_1}$, $\pi(b_{Y_1}) = b_{Y_1}$, $\pi(Y_3) = Y_1$, $\pi(Y_4) = Y_2$. By Fact 6.4 we get a contradiction to the choice of f^*.] The hence of (c)(γ) is by 6.13.

By the choice of $Y^* \in \mathcal{P}_\pm$, we have (a), now (b) follows from (c) so it is enough to prove (c). Assume (α), (γ) fail and we shall prove (β). So there is $Z_1 \in \mathcal{P}_\pm$ such that $Z_1 \notin \{Y^*, (X \setminus Y^*) \cup \{a_{Y^*}, b_{Y^*}\}\}$. We can find $c_1, c_2 \in X \setminus \{a_{Y^*}, b_{Y^*}\}$ such that $c_1 \in Y^* \Leftrightarrow c_2 \in Y^*$ and $c_1 \in Z_1 \Leftrightarrow c_2 \notin Z_1$.

[Why? if $Y^* \cup Z_1 \neq X$ any $c_1 \in X \setminus Y^* \setminus Z_1$, $c_2 \in Z_1 \setminus Y^*$ will do; so assume $Y^* \cup Z_1 = X$; so as $k + 2 < |X|$, clearly $|Y^* \cap Z| < k - 2$; hence by Claim 6.14, $|Z_1 \cap Y^*| \leqslant 2$. As not case ($\gamma$) of (c), that is by the choice of Z_1, necessarily $\{a_{Y^*}, b_{Y^*}\} \nsubseteq Y^* \cap Z_1$ and using $\pi \in \mathrm{Per}(X)$, $\pi \restriction Z_1 = \mathrm{id}$, $\pi(Y^*) = Y^*$, π the identity on Z_1 and $\{\pi(a_{Y^*}), \pi(b_{Y^*})\} = \{a_{Y^*}, b_{Y^*}\}$; now by Claim 6.13 we contradict Definition/choice 6.3 and Fact 6.4.]

Let $Z_2 = Z_1 \cup \{c_1, c_2\} \setminus (Z_1 \cap \{c_1, c_2\})$, so $Z_1, Z_2 \in \binom{X}{k}$, $|Z_2 \cap Y^*| = |Z_1 \cap Y^*|$ and $Z_1 \cap \{a_{Y^*}, b_{Y^*}\} = Z_2 \cap \{a_{Y^*}, b_{Y^*}\}$; hence by Conclusion 6.12(4) we have $Z_2 \in \mathcal{P}_\pm$ and clearly $|Z_1 \setminus Z_2| = 1$.

By Claim 6.15 we have $\{a_{Z_1}, b_{Z_1}\} = \{a_{Z_2}, b_{Z_2}\}$. Similarly by ($*$) we can prove by induction on $m = |Z \setminus Z_1|$ that $\{a_{Z_1}, b_{Z_1}\} \subseteq Z \in \binom{X}{k} \Rightarrow Z \in \mathcal{P}_\pm$ & $\{a_Z, b_Z\} = \{a_{Z_1}, b_{Z_1}\}$. If ($\beta$) of (c) fails, then there is $Z_3 \in \mathcal{P}_\pm$ satisfying $\{a_{Z_1}, b_{Z_1}\} \nsubseteq Z$. Easily $\{a_{Z_3}, b_{Z_3}\} \subseteq Z \in \binom{X}{k} \Rightarrow Z \in \mathcal{P}_\pm$ & $\{a_Z, b_Z\} = \{a_{Z_3}, b_{Z_3}\}$. As we are assuming $k \geqslant 4$, we can find $Y \in \binom{X}{k}$ such that $\{a_{Z_1}, b_{Z_1}, a_{Z_3}, b_{Z_3}\} \subseteq Y$; contradiction. □

6.18. Claim. (1) *The* ($*$) *of Claim* 5.8 *holds.*
(2) *In Conclusion* 6.17 *clause* (c), *clause* (α) *holds.*

Proof. (1) Obvious by part (2) from (α).

(2) First assume (β), so by Conclusion 6.17(b), Definition/choice 6.3 and Fact 6.4, we have without loss of generality either $\{a, b\} = \{a_{Y^*}, b_{Y^*}\} \subseteq Y \in \binom{X}{k} \Rightarrow f^*_Y(a, b) = b$ or $\{a_{Y^*}, b_{Y^*}\} \subseteq Y \in \binom{X}{K} \Rightarrow f^*_Y(a_{Y^*}, b_{Y^*}) = b_{Y^*} = f(b_{Y^*}, a_{Y^*})$. In both cases, f^* is simple and not a monarchy contradiction, to Hypothesis 5.2.

Second, assume (γ). Let $\langle \pi_i \colon i < i^* \rangle$ be a list of the permutations π of X such that $\pi(a_{Y^*}, b_{Y^*}) = (a_{Y^*}, b_{Y^*})$.

Let f_i^* be f^* conjugated by π_i. Now define g^i for $i \leqslant i^*$ by induction on i: $g_Y^0(x_1, x_2) = x_1$, $g_Y^{i+1}(x_1, x_2) = f_i^*(g_Y^i(x_1, x_2), x_2)$. So $g^{i^*} \in \mathcal{F}_{[2]}$ and $\mathrm{dom}_2(g^{i^*}) = \bigcap_{i<i^*} \mathrm{dom}_2(f_i^*)$ where $\mathrm{dom}_2(g) = \{(Z, a, b) \colon a, b \in Z \in \binom{X}{k}$ and $g_Z(a, b) = b \neq a\}$, so $\mathrm{dom}_1(g^{i^*}) = \bigcup_{i<i^*} \mathrm{dom}_1(f_i^*)$ hence

$(*)_1$ $g_Y^{i^*}(a_1, a_2) = a_2$ if $\{a_1, a_2\} = \{a_{Y^*}, b_{Y^*}\}$,
$(*)_2$ $g_Y^{i^*}(a_1, a_2) = a_1$ if $\{a_1, a_2\} \neq \{a_{Y^*}, b_{Y^*}\}$.

Now g is simple but non-monarchical contradiction to Hypothesis 5.2. $\square$

7. The case $r \geqslant 4$

7.1. Hypothesis. $r = r(\mathcal{F}) \geqslant 4$.

7.2. Claim. (1) *For every $f \in \mathcal{F}_r$ there is $\ell(f) \in \{1, \ldots, r\}$ such that*

$\circledast$ *if $Y \in \binom{X}{k}$, $\bar{a} \in {}^r Y$ and $|\mathrm{Rang}(\bar{a})| < r$ (i.e. $\bar{a}$ is not one-to-one) then $f_Y(\bar{a}) = a_{\ell(f)}$.*

(2) $r \leqslant k$.

Proof. (1) Clearly there is a two-place function h from $\{1, \ldots, r\}$ to $\{1, \ldots, r\}$ such that if $y_1, \ldots, y_r \in Y \in \binom{X}{k}$, $y_\ell = y_k$ and $\ell \neq k$ then $f_Y(y_1, \ldots, y_r) = y_{h(\ell,k)}$; we have some freedom, so let without loss of generality

$\boxtimes$ $\ell \neq k \Rightarrow h(\ell, k) \neq k$.

Assume toward contradiction that the conclusion fails, i.e., there is no $\ell(f)$ as required; i.e.

$\circledast'$ $h \restriction \{(m, n) \colon 1 \leqslant m < n \leqslant r\}$ is not constant.

Case 1. For some $\bar{x} \in {}^r Y$, $Y \in \binom{X}{k}$ and $\ell_1 \neq k_1 \in \{1, \ldots, r\}$, we have

$$|\mathrm{Rang}(\bar{x})| = r - 1, \qquad x_{\ell_1} = x_{k_1}, \qquad f_Y(\bar{x}) \neq x_{\ell_1},$$

equivalently: $h(\ell_1, k_1) \notin \{\ell, k\}$. Without loss of generality, $\ell_1 = r - 1$, $k_1 = r$, $f_Y(\bar{x}) = x_1$ (as by a permutation σ of $\{1, \ldots, r\}$, we can replace f by f^σ: $f_Y^\sigma(x_1, \ldots, x_2) = f_Y(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$).

We can choose $Y \in \binom{X}{k}$ and $x \neq y$ in Y, so $h(x, y, \ldots, y) = x$; hence $\ell \neq k \in \{2, \ldots, r\} \Rightarrow h(\ell, k) = 1$.

Now for $\ell \in \{2, \ldots, r\}$ we have agreed $h(1, \ell) \neq \ell$ (see $\boxtimes$), as we can assume $h \restriction \{(m, n) \colon 1 \leqslant m < n \leqslant r\}$ is not constantly 1, by $\circledast'$ for some such ℓ, $h(1, \ell) \neq 1$ so without loss of generality, $\ell = 2$; so $h(1,2) \neq 1, 2$, so without loss of generality, $h(1,2) = 3$, but as $r \geqslant 4$, we have that if $x \neq y \in Y \in \binom{X}{k}$ then $f_Y(x, x, y, y, \ldots, y)$ is y for $h(1,2) = 3$ and is x for $h(3,4) = 1$, contradiction. So

$\circledast$ $h \restriction \{(\ell, k)\colon 1 \leqslant \ell < k \leqslant r\}$ is constantly 1.

hence

$$\bar{x} \in {}^r X \text{ has repetitions} \quad \Rightarrow \quad h(\bar{x}) = x_1,$$

as required.

Case 2. Not Case 1.

So $h(\ell, k) \in \{\ell, k\}$ for $\ell \neq k \in \{1, \ldots, r\}$. Now let $Y \in \binom{X}{k}$, $x \neq y \in Y$ and look at $f_Y(x, x, y, y, \ldots)$ it is both x as $h(1,2) \in \{1,2\}$ and y as $h(3,4) \in \{3,4\}$, contradiction.

(2) This follows as if $f \in \mathcal{F}_{[r]}$, $k < r(\mathcal{F})$ and $\ell(*)$ is as in part (1) then $f_Y(\bar{x}) = x_{\ell(*)}$ always, as $x_{\ell(*)}$ has repetitions by pigeon-hole. $\square$

Recall

7.3. Definition. $f = f_{r;\ell,k} = f^{r;\ell,k}$ is the r-place function

$$f_Y(\bar{x}) = \begin{cases} x_\ell, & \bar{x} \text{ has repetitions,} \\ x_k, & \text{otherwise.} \end{cases}$$

7.4. Claim. (1) *If $f_{r;1,2} \in \mathcal{F}$ then $f_{r;\ell,k} \in \mathcal{F}$ for $\ell \neq k$.*

(2) *If $f_{r;1,2} \in \mathcal{F}$, $r \geqslant 3$ then $f_{r+1;1,2} \in \mathcal{F}$.*

Proof. (1) Trivial.

(2) For $r \geqslant 5$ let $g(x_1, \ldots, x_{r+1}) = f_{r,1,2}(x_1, x_2, \tau_3(x_1, \ldots), \ldots, \tau_r(x_1, \ldots))$ where $\tau_m \equiv f_{r,1,m}(x_1, \ldots, x_m, x_{m+2}, \ldots, x_{r+1})$, that is x_{m+1} is omitted. Continue as in the proof of Claim 2.7. $\square$

7.5. Claim. *Assume $Y \in \binom{X}{k}$, $\bar{a} \in {}^r Y$ is one-to-one. There is $f = f^{Y,\bar{a}} \in \mathcal{F}_r$ such that $f_Y^{Y,\bar{a}}(\bar{a}) = a_2$ and $f_Z^{Y,\bar{a}}(\bar{b}) = b_1$ if $Z \in \binom{X}{k}$ and $\bar{b} \in {}^r X$ is not one-to-one* (*so* $(*)$ *of Claim* 5.6(2) *holds*).

Proof. Let $f \in \mathcal{F}_r$ be non-monarchical, and without loss of generality, $\ell(*) = 1$ in Claim 7.2. By being not a monarchy, for some Y, $\bar{a}$ and some $k \in \{2, \ldots, r\}$, we have $f_Y(\bar{a}) = a_k \neq a_1$; necessarily $\bar{a}$ is one-to-one. Conjugating by $\pi \in \mathrm{Per}(X)$ and permuting $[2, r]$, we get $f^{Y,\bar{a}}$ as required, in particular $f^{Y,\bar{a}}(\bar{a}) = a_2$. $\square$

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