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On the number of non conjugate subgroups

SAHARON SHELAH*

Abstract. We shall prove that every group of cardinality $\aleph_1$ has at least $\aleph_1$ non conjugate subgroups, and we shall generalize this theorem to many more uncountable cardinalities. For example under *GCH* for every uncountable cardinal λ and every group G of cardinality λ, G has at least λ non conjugate subgroups.

§0. Introduction

Our work here is motivated by a question of E. Rips who dealt with the problem of finding a lower bound to the number of non conjugate subgroups of a countable group. He constructed a countable group with exactly 3 non conjugate subgroups [Ri]. He asked us what happens in groups of cardinality $\aleph_1$.

DEFINITION 0.1. Let G be a group and H, K subgroups of G (denote by $H, K \leq G$). *H is conjugate to K* if there exists $g \in G$, such that $K = gHg^{-1}$.

Clearly this relation is an an equivalence relation on the subgroups of G.

We shall denote by $nc(G)$ the number of equivalence classes, we call it the number of non conjugate subgroups of G. We shall try to prove:

CONJECTURE. For every uncountable group G, $nc(G) \geq |G|$ (the cardinality of G).

From now on let G be a given uncountable group. We write $\lambda = |G|$ and we shall

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try to prove that $nc(G) \geq \lambda$. We shall prove the conjecture for many families of cardinal numbers. Let us divide the uncountable cardinals to three disjoint families

I. $\aleph_0 < \lambda \leq 2^{\aleph_0}$;
II. $\lambda > 2^{\aleph_0}$ and $\neg(\exists \mu > \aleph_0)[2^\mu = \lambda]$;
III. $\lambda > 2^{\aleph_0}$ and $(\exists \mu > \aleph_0)[2^\mu = \lambda]$.

In §1 we prove the conjecture for $\aleph_0 < \lambda \leq 2^{\aleph_0}$, the theorem we stated in the abstract follows from this one for $\lambda = \aleph_1$. In §2 we prove it for λ of type II. In §3 and §5 we shall prove the conjecture for some of the cardinals of type III, but not for all of them. More exactly we shall prove:

THEOREM. *For every group G of cardinality $\lambda > 2^{\aleph_0}$. If $\mu = \min\{\mu : 2^\mu = \lambda\}$ and $Ded\mu > \lambda$ then then $nc(G) \geq \lambda$ (for definition of $Ded\mu > \lambda$, and explanation see* §3).

OPEN PROBLEM. Prove the conjecture for groups whose cardinality is of type III and not covered by Theorem 3.

However, if $Ded\mu < \lambda, \lambda\mu$ as above, still $\chi < Ded\mu \Rightarrow \chi \leq nc(G)$. Note that under G.C.H. the conjecture is proved. In §4 we shall present various results concerning the problem on finding the number of non conjugate subgroups.

Let us explain the structure of the proof. In each section we shall assume that we are given a fixed group G of uncountable cardinality λ according to the assumption on λ (of type I, II or III). In each section we shall make suitable additional assumptions on G. In the proof in §1 the main ideas are presented and the proofs we have for types II and III are elaborations of the basic idea presented in §1.

NOTATION. $\alpha, \beta, \gamma, \delta, i, j, \xi$ range over ordinals; $\lambda, \mu, \chi, \kappa$ denote infinite cardinals; ω = the first infinite ordinal and also the set of natural numbers; $\mathbf{Z}$ = the integers; $\mathbf{Q}$ = the rationals; $\mathbf{R}$ = the reals; n = ranges over natural numbers; r, s, t range over the reals. Conjugacy will mean inside the group G; subgroups mean subgroups of $G: x \to x^g$ is the map $x \to gxg^{-1}: a, b, \ldots$ denote elements of $G: \langle a, b, \ldots \rangle$ the subgroup of G generated by $\{a, b, \ldots\}$. H, K denote subgroups of G. $\{G_\alpha : \alpha < \kappa\}$ is an increasing continuous chain of subgroups of G if $\alpha < \beta < \kappa \Rightarrow G_\alpha \subseteq G_\beta$ and for $\delta < \kappa$ limit $G_\delta = \bigcup_{\alpha<\delta} G_\alpha$.

Remark. Notice that for groups of cardinality $\aleph_1$ assuming $2^{\aleph_0} = \aleph_1$ the Conjecture is the best possible because there exists a Jónsson group of cardinality $\aleph_1$ (= all proper subgroups are at most countable) and such group has only $\aleph_1$ distinct subgroups $(= \aleph_1^{\aleph_0} = 2^{\aleph_0} = \aleph_1)$.

§1. The conjecture for $\aleph_0<\lambda\le 2^{\aleph_0}$

So we want to prove here:

THEOREM 1.1. *For every G of cardinality λ, if $\aleph_0<\lambda\le 2^{\aleph_0}$ then $nc(G)\ge\lambda$.*

We shall obtain a contradiction by assuming $nc(G)<|G|$.

ASSUMPTION 1.2. Let G be a group of uncountable cardinality $\lambda\le 2^{\aleph_0}$ such that $nc(G)<\lambda$, and choose χ such that $\aleph_0\cdot nc(G)\le\chi<\lambda$.

CLAIM 1.3. Let A be a subset of G. If $\chi^{|A|}<\lambda$, then there exists $a\in G$ which satisfies $(\forall n<\omega)[a^{n+1}\notin\langle A\rangle]$ and $(\forall x\in A)[xa=x]$.

Proof of Claim 1.3. Denote $\kappa=\chi^{|A|+1}$, and define an increasing continuous chain of subgroups $\{G_i: i<\kappa^+\}$ such that:

(i) $A\subseteq G_0$.

(ii) For all $i<\kappa^+$ $|G_i|=\kappa$.

(iii) If $\alpha<\beta\le i<\kappa^+$ and G_α conjugate to G_β (in G), then they are conjugate by an element of G_{i+1}.

The construction of the chain is by induction on $i<\kappa^+$, $G_0=\langle A\rangle$.

If $i=j+1$; assume $\{G_\xi:\xi\le j\}$ already defined, define $\{g_\xi:\xi\le j\}\subseteq G$ by induction on $\xi<j$: if G_ξ is conjugate to G_j let $g_\xi\in G$ an element which witnesses the conjugacy; otherwise let g_ξ be an arbitrary element of G_0. Finally choose $g_j\in G-G_j$ and define $G_i=G_{j+1}=\langle G_j\cup\{g_\xi:\xi\le j\}\rangle$.

If i is limit ordinal then define $G_i=\bigcup_{j<i}G_j$. It is trivial to verify that this construction satisfies our demands (i)–(iii). There must be an ordinal $\xi<\kappa^+$ such that $S=\{i<\kappa^+: G_i$ is conjugate to G_ξ in $\bigcup_{i<\kappa^+}G_i\}$ has cardinality κ^+ [Otherwise in each conjugacy class there are only $\le\kappa$ elements from $\{G_i: i<\kappa^+\}$ so $|\{G_i: i<\kappa^+\}|=\kappa\cdot\kappa=\kappa$ contradiction].

For all $\alpha\in S$ let $g_\alpha\in\bigcup_{i<\kappa^+}G_i$ be an element which conjugates G_α to G_ξ. Without loss of generality we may assume that $\alpha\in S\to G_\xi\subseteq G_\alpha$ [by changing the original ξ to first element of S]. The number of mappings of A to G_ξ is $|G_\xi|^{|A|}$ which is at most κ $[|G_\xi|^{|A|}=\kappa^{|A|}=(\chi^{|A|+1})^{|A|}=(\chi^{|A|})^{|A|}=\chi^{|A|\cdot|A|}=\chi^{|A|}\le\chi^{|A|+1}=\kappa]$. On the other side we have κ^+ maps of A to G_ξ $[\{x\to x^{g_\alpha}:\alpha\in S\}]$; so there are $\alpha,\beta\in S$ $\alpha<\beta$ such that the maps $x\to x^{g_\alpha}$, $x\to x^{g_\beta}$ are the same on A, (there are κ^+ such maps but we shall use only two of them). Define $a=g_\beta^{-1}\cdot g_\alpha$. We shall prove that this is an element as required. Clearly a commutes with every $x\in A$. Now we prove that $n<\omega\Rightarrow a^{n+1}\notin\langle A\rangle$. Clearly $a^{n+1}\notin G_\beta$ is a stronger demand, notice that if $a^{n+1}\in G_\beta$ then $x^{a^{n+1}}$ is an automorphism of G_β so it is enough to prove that $x \to x^{a^{n+1}}$ maps G_β to a proper extension of itself. Prove it by induction on $n<\omega$; it suffices to prove that $x \to x^a$ maps G_β to a proper extension of itself (i.e., for $n=0$). $x \to x^a$ is the composition of the following two maps. First apply $x \to x^{g_\alpha}$ (maps G_α onto G_ξ) and later apply $x \to x^{g_\beta^{-1}}$ (maps G_ξ onto G_β), hence $x \to x^a$ maps G_α onto G_β and if we apply $x \to x^a$ on G_β its image is necessarily a proper extension of G_β (since the map is one to one).

CONCLUSION 1.4. There is $\{a_n : n<\omega\} \subseteq G$ which generates freely an abelian group.

Proof. Define the sequence by induction on $n<\omega$.

For $n=0$; apply Claim 1.3 on $A=\phi$, clearly the assumption of the claim is satisfied and the element a supplied by the Claim will be a_0, it is obvious that $\langle a_0\rangle \cong \mathbf{Z}$. For $n>1$; by induction hypothesis we have $\{a_0, \ldots, a_{n-1}\} \subseteq G$ generators of a free abelian group. Let $A=\{a_0, \ldots, a_{n-1}\}$; clearly $\chi^{|A|}<\lambda$. The element supplied by Claim 1.3 applied on A will be a_n. Again it is easy to verify that $\{a_0, \ldots, a_n\}$ generate free-abelian group, and this implies what we wanted.

Proof of Theorem 1.1. Pick $f:\omega \to \mathbf{Q}$ one to one and onto. For each $r \in R$ let $S_r=\{n<\omega : f(n)<r\}$; by the density of $\mathbf{Q}$ in $\mathbf{R}$ for $r, s \in \mathbf{R}\; r<s \Rightarrow S_r \subsetneq S_s$. By Conclusion 1.4 choose $\{a_n : n<\omega\} \subseteq G$ which generated a free abelian subgroup. For $r \in \mathbf{R}$ let $G_r=\langle\{a_n : n \in S_r\}\rangle$. By Assumption 1.2 there are $r_1>r_2$ such that G_{r_1} is conjugate to G_{r_2}. Let $h \in G$ witness this conjugacy i.e., $x \to x^h$ maps G_{r_1} onto G_{r_2}. Choose $n \in S_{r_1}-S_{r_2}$ (possible since $r_1>r_2$). Define $b_0=a_n$ and for $1 \le k<\omega$, $b_k=b_0^{h^k}$.

The set $\{b_n : n<\omega\}$ freely generates an abelian group [Proof: First prove by induction on $n<\omega$ that $b_n \in G_{r_1}$. So clearly the group generated by $\{b_n : n<\omega\}$ is abelian (it is a subgroup of the abelian group G_{r_1}). Why freely? By our choice $b_0 \notin G_{r_2}$ and no positive power of b_0 is in G_{r_2} (since $\{a_n : n<\omega\}$ generate a free abelian group). It is enough to show that there is no relation among elements from $\{b_n : n<\omega\}$; assume for contradiction that there is such a relation and let b_k be an element with minimal k such that it or a power of it appears non trivially in the relation. Move b_k (or its power) to the left part of the equation and the remaining b_n's to the right. Apply the function $x \to x^{h^{-1}}$ k times on the equation (on both sides) to get a non zero power of b_0 as a combination of a finite number of elements from $\{b_{n+1} : n<\omega\}$. Therefore b_0 or its non trivial power is a member of G_{r_2}, this contradicts what we said earlier on b_0].

Denote $H=\langle\{b_n : n<\omega\}\rangle$, this group has $2^{\aleph_0}$ subgroups. By Assumption 1.2 H has χ^+ distinct subgroups which are conjugate in pairs (by elements of G). For $i<\chi^+$ let H_i be a subgroup of H and $g_i \in G$ such that $i \ne j \Rightarrow H_i \ne H_j$ and

$g_iH_ig_i^{-1}=H_0$. Now for every $i<\chi^+$ define $K_i=\langle b_0, g_i, h\rangle$ a subgroup of G. If we shall find $S\subseteq\chi^+$, $|S|=\chi^+$ such that for $i,j\in S\ i\neq j\Rightarrow K_i\neq K_j$ clearly we obtained a contradiction to the choice of χ (non-isomorphic groups cannot be conjugate).

How shall we do it? View K_i as models of group theory with three additional constants b, g, h (where in K_i b is interpreted by b_0, g by g_i, h by h). Those models cannot be isomorphic since for $i\neq j<\chi^+$, g_i behaves on $\langle b_0, h\rangle$ differently than g_j.

Now we use the following observation: Given a countable group it can be expanded by 3 new constants in only $\aleph_0$ ways ($\aleph_0^3=\aleph_0$). This is a special case of Lemma VIII 1.3 from [Sh 2]. So there is $S\subseteq\chi^+$, $|S|=\chi^+$, such that $i\neq j\Rightarrow\overline{K_i}\neq\overline{K_j}$ when $\overline{K_i}$ is the reduct to the language of group theory (containing on $\mathbf{e}$, $\cdot$) of K_i.

§2. The Conjecture when λ is not a power of 2

THEOREM 2.1. *If $\lambda>2^{\aleph_0}$ and there is no μ such that $\lambda=2^\mu$ then any group G of cardinality λ has at least λ non conjugate subgroups.*

ASSUMPTION 2.2. Let λ be a cardinal as in the hypothesis of Theorem 2.1 and G a group such that $nc(G)<|G|=\lambda$. Choose χ which satisfies $\aleph_0\cdot nc(G)\leq\chi<\lambda$.

As we stated in the introduction we want to use the ideas of the proof of Theorem 1.1, so as our first step in the proof of Theorem 2.1 we want to reprove Claim 1.3 but unfortunately the assumption $\chi^{|A|+1}<\lambda$ is too strong (here it may not hold) so we shall prove here a weaker replacement to Claim 1.1 (weaker in hypothesis and in conclusion), and this will be enough here because of our assumption on λ $(\neg\exists\mu[2^\mu=\lambda])$.

CLAIM 2.3. If A is a subset of G and $2^{|A|}<\lambda$, then there exists $\alpha\in G-\langle A\rangle$ which commutes with every element of $\langle A\rangle$.

Proof of Claim 2.3. For every $a\in G$ define $A_a=\langle A\cup\{a\}\rangle$, and let $\kappa=(2^{|A|})^++\chi^+$. By the definition of χ and the choice of κ as $\geq\chi^+$ there are distinct $a_i\in G$ and $g_i\in G$ for every $i<\kappa$ such that $g_iA_{a_i}g_i^{-1}=A_{a_0}$, and the groups A_{a_i} are distinct.

Moreover we may assume that also $i<j<\kappa$ the maps $x\to x^{g_i}$, $x\to x^{g_j}$ are the same on A (since $\kappa\geq(2^{|A|})^+$, if A is finite then because $\chi\geq\aleph_0$ and $\kappa\geq\chi^+$). Now pick any $i<j<\kappa$ and let $a=g_j^{-1}\cdot g_i$. Then $x\to x^a$ is the identity over A so also over $\langle A\rangle$, so a commutes with the elements of $\langle A\rangle$. If $a\in\langle A\rangle$ then clearly $A_{a_i}=A_{a_j}$ contradiction to our assumption that the groups A_{a_i} where distinct.

CONCLUSION 2.4. Let $\mu = \min\{\mu : 2^\mu \geq \lambda\}$, the group G has an abelian subgroup H of cardinality μ.

Proof. Define H as a union of an increasing continuous chain of abelian subgroups $H_i, i<\mu$ of G such that $i<\mu \Rightarrow |H_i|<\mu$. If we shall be able to construct such a chain clearly $H=\bigcup_{i<\mu} H_i$ will be as required. We define the chain by induction on $i<\mu$:

If $i=0$; choose an arbitrary $a \in G$ and let $H_0=\langle a\rangle$. If i limit ordinal; then let $H_i=\bigcup_{j<i} H_j$. If $i=j+1$; by the induction hypothesis we have H_j an abelian subgroup of G of cardinality $<\mu$, then by the definition of μ, $2^{|H_j|}<\lambda$ so we can apply Claim 2.3 on H_j (in the claim, take $A=H_j$) and we get $a \in G-\langle H_j\rangle$ which commutes with the elements of $\langle H_j\rangle$. Let $H_i=H_{j+1}=\langle H_j \cup \{a\}\rangle$. Clearly H_i abelian and $|H_i|=\aleph_0 \cdot |H_j|<\aleph_0 \cdot \mu=\mu$ (μ must be uncountable by the assumption that $\lambda>2^{\aleph_0}$). The next fact is well known, but since we do not remember a reference to it we shall prove it here.

FACT 2.5. An abelian group H of uncountable cardinality μ contains a subgroup which is a direct sum of μ cyclic groups.

Proof of Theorem 2.1. Let μ be as in Conclusion 2.4 and H an abelian subgroup of G as there. Remember that by our assumption on λ as $>2^{\aleph_0}$ μ must be uncountable, so the hypothesis of Fact 2.5 is satisfied. Apply it and we get that H has 2^μ distinct subgroups hence also G has at least 2^μ subgroups but using our other assumption on λ (that it is not a power of two) necessarily $2^\mu>\lambda$. The size of every conjugacy class of subgroups ($=$the number of subgroups in it) is at most $\lambda(=|G|)$. Since there are λ^+ subgroups, and they are partitioned to classes (the equivalence classes) each of cardinality at most λ there must be at least λ^+ classes so at least λ^+ non conjugate subgroups, contradicting Assumption 2.2.

Now let us go to the proof of Fact 2.5. Recall that if H is an abelian group, a subgroup H' of H is called *pure subgroup* if for all $x \in H'$ and $n \in \mathbf{Z}$ $(\exists y \in H)[y^n=x] \Rightarrow (\exists y \in H')[y^n=x]$.

Proof of Fact 2.5. We shall define an increasing continuous chain of subgroups of H $\{H_i : i<\mu\}$ such that:

(i) $|H_i| \leq |i|+\aleph_0$.

(ii) H_i is a pure subgroup of H.

The construction is by induction on $i<\mu$ and is trivial (it is just a Skolem Löwnheim argument), and we can make the chain increasing since $\mu>\aleph_0$. Notice that for $i<j<\mu$ H_i is pure subgroup of H_j. Define by induction on $i<\mu$ $a_i \in \bigcup_{j<\mu} H_j$ such that $a_i \in H_{i+1}-H_i$ and for every $n<\omega$ $na_i \in H_i \Rightarrow na_i=0$. [Proof: Choose an element $a \in H_{i+1}-H_i$, if for every $n>0$, $na \notin H_i$ we finish. Otherwise choose a minimal n such that $na \in H_i$. By pureness of H_i in H_{i+1} there exists $b \in H_i$ such that $n \cdot a = n \cdot b$; the element $c = a - b$ is a member of $H_{i+1} - H_i$ and $nc = n(a-b) = 0$, now $0 < m < n$ implies $mc \notin H_i$ (otherwise $ma = mc + mb \in H_i$ contradicting m's minimality.)] Therefore $\langle\{a_i : i < \mu\}\rangle = \sum_{i<\mu} \langle a_i\rangle$, and this will be the required subgroup of H.

§3. Proof of the conjecture for some cardinals which are powers of two

The proof which we shall present here will be more similar to what we did in §1, than to what was done in §2. In Theorem 2.1 our life was easy because λ was not a power of 2. So we try to imitate our solution to the case when $\lambda = 2^{\aleph_0}$. We used there the fact that $2^{|\mathbf{Q}|} = |\mathbf{R}| = 2^{\aleph_0} = \lambda$ and the density of $\mathbf{Q}$ in $\mathbf{R}$. So we need a substitute for those facts; to formulate it we need the following.

DEFINITION 3.1. $\lambda < Ded\mu$ iff there exists a linearly ordered set I of cardinality λ and $\{S_t : t \in I\}$ such that $t \in I \Rightarrow S_t \subsetneqq \mu$ and for $r, t \in I$ $r < t \Rightarrow S_r \subsetneqq S_t$ ($r < t$ in the order of I).

Remark. An equivalent formulation and a number of properties of $Ded\mu$ and references can be found in the first section to the appendix of [Sh 2].

THEOREM 3.2. *Let $\lambda > 2^{\aleph_0}$, $\mu = \min\{\mu : 2^\mu = \lambda\}$ and assume that $Ded\mu > \lambda$ and μ is not singular strong limit, then every group of cardinality λ has at least λ non conjugate subgroups.*

Remark. The cardinals which are not covered by this theorem and the earlier theorems when we assume GCH are successors of singulars e.g. $\lambda = \aleph_{\omega+1}$. However we shall prove the Conjecture also for those in §5.

Now we introduce a tool to analyse the distance of a group from countable abelian.

DEFINITION 3.3. Let H be a group. For each ordinal α we define a subset $H^{[\alpha]}$ of H by induction on α:

(i) $H^{[0]} = \{e\}$

(ii) For limit α $H^{[\alpha]} = \bigcup_{\beta<\alpha} H^{[\beta]}$

(iii) $H^{[\alpha+1]} = \langle\{x \in H :$ the normal subgroup of $H/H^{[\alpha]}$ generated by $x/H^{[\alpha]}$ is abelian and at most countable$\}\rangle$.

LEMMA 3.4. *Let H be a given group and $\{H_i : i \in I\}$ a family of groups then the* *following is true for every ordinal α:*

(i) $H^{[\alpha]}$ *is a normal subgroup of* H.

(ii) $\beta<\alpha \Rightarrow H^{[\beta]} \subseteq H^{[\alpha]}$.

(iii) *Let* K *be a subgroup of* H; *then* $H^{[\alpha]} \cap K \subseteq K^{[\alpha]}$.

(iv) $(\prod_{i\in I} H_i)^{[\alpha]} = \prod_{i\in I} H_i^{[\alpha]}$. ($\prod$ *denotes weak direct product*)

(v) *for commutative* K, $K^{[1]}=K$.

Proof of Lemma 3.4.

(i) By induction on α prove that $H^{[\alpha]}$ is a characteristic subgroup of H.

(ii) Follows directly from the definition.

(iii) By induction on α.

(iv) Use induction on α, and the identity $\prod_{i\in I}(H_i/K_i) = \prod_{i\in I} H_i/\prod_{i\in I} K_i$.

(v) trivial

Before Definition 3.3 we promised that the α-th derivative of H helps to measure how abelian H is, the next definition completes our promises.

DEFINITION 3.5. We say that a group H is a *required group of depth* α iff $H/H^{[\alpha]}$ is non trivial free abelian group and for every $\beta<\alpha$ the center of $H/H^{[\beta]}$ is trivial.

ASSUMPTION 3.6. Let λ and μ be as in the statement of Theorem 3.2, and let G be a group of cardinality λ which is a counterexample to the conjecture and choose χ such that $\aleph_0 \cdot nc(G) \leq \chi < \lambda$.

Our replacement of Claims 1.3 and 2.3 is the following.

CLAIM 3.7. For any subset A of G, $|A|<\mu$ and $\alpha<\mu$ there exists a subgroup H such that:

(i) Every element of H commutes with every element of $\langle A\rangle$ and $H\cap\langle A\rangle = \{e\}$.

(ii) H is a required group of depth α.

(iii) $|H| \leq |\alpha| + \aleph_0$.

Let us delay the proof of the Claim and first prove Theorem 3.2 using Claim 3.7 to convince the reader that the notion of "required group of depth" makes sense.

Proof of Theorem 3.2. Choose a family $\{H_\alpha : \alpha<\mu\}$ of subgroups of G such that for every $\alpha<\mu H_\alpha$ is a required group of depth α, $|H_\alpha| \leq |\alpha| + \aleph_0$, and $H_\alpha \cap \langle \bigcup_{\beta<\alpha} H_\beta\rangle = \{e\}$. This is done by induction on α using Claim 3.7. For every $S\subseteq \mu$ let $H_S = \langle \bigcup_{\alpha\in S} H_\alpha\rangle$. It is easy to check using Definition 3.5 and Lemma 3.4 (iv) that $H_S/H_S^{[\alpha]}$ has non trivial center if and only if $\alpha\in S$.

Since the last property is preserved by isomorphism we got $2^\mu(=\lambda)$ non isomorphic subgroups of G, contradiction to Assumption 3.6 ($nc(G)<\lambda$) since non isomorphic subgroups cannot be conjugate.

Now return to prove Claim 3.7. Similarly to Claim 1.3, during the proof we shall need to know that $\mu^{|A|}<\lambda$ (remember $|A|<\mu$). This follows from our choice of μ as the first cardinal which satisfies $2^\mu=\lambda$ and μ is not singular and strong limit cardinal. μ can be only one of the following kinds of cardinals: successor; limit but not strong limit; strongly inaccessible; or singular and strong limit. Now we shall show that $\mu^{|A|}<\lambda$ by checking for each μ according to the classification above:

μ successor: i.e., $\mu=\kappa^+$ so $\mu^{|A|}\leq(2^\kappa)^{|A|}=2^{\kappa\cdot|A|}=2^\kappa<\lambda$ (the last inequality is true because $\mu=\min\{\mu:2^\mu=\lambda\}$).

μ limit but not strong limit. Pick κ such that $|A|<\kappa<\mu$ abd $2^\kappa\geq\mu$; then $\mu^{|A|}\leq(2^\kappa)^{|A|}=2^{\kappa\cdot|A|}=2^\kappa<\lambda$.

μ strongly inaccessible. Since $|A|<\mu\mu^{|A|}=\mu$ which is less than λ by the fact that $2^\mu=\lambda$ i.e., Cantor's Theorem.

μ is singular and strong limit. In this case we do not know and this is the source of the last assumption on μ in the hypothesis of Theorem 3.2.

Before starting the proof of Claim 3.7 we need more terminology:

LEMMA 3.9. *Suppose $G_m (m\in\mathbf{Z})$ are pairwise isomorphic groups, $F^{m(1)}_{m(2)}: G_{m(1)}\to G_{m(2)}$ an isomorphism (onto), $F^{m(2)}_{m(3)}\cdot F^{m(1)}_{m(2)}=F^{m(1)}_{m(3)}$, $G^{[\alpha]}_m=G_m$ but $G^{[\beta]}_m\neq G_m$ for $\beta<\alpha$, and $\bar G=\prod_{m\in\mathbf{Z}} G_n$.*

Suppose $\bar G\subseteq H$, $H=\langle\bar G, a\rangle$, and for $g\in G_m$, $g^\alpha=F^m_{m+1}(g)$. Then H is a required group of order α.

Proof. Let m, n denote integers, and as the case $\alpha=0$ is trivial (as then $\bar G$ is trivial) let $\alpha>0$.

We can identity G_m with its canonical image in $\bar G$. So the members of $G_{m(1)}, G_{m(2)} (m(1)\neq m(2))$ commute, and $\bigcup_{m\in\mathbf{Z}} G_m$ generate $\bar G$. Hence every member of $\bar G$ has a representation $\prod_{-n<m<n} g_m$, $g_m\in G_m$. Let for $g\in G_{m(1)}$, $g^{[m(2)]}=F^{m(1)}_{m(2)}(g)$, so the representation is $\prod_{n<m<n} g^{[m]}_m$ for $g_m\in G_0$. We write $\prod_m g^{[m]}_m$, $g_m=e$ for $m\notin(-n,n)$ for notational simplicity. Similarly, every member of H has a representation $a^n\prod_m g^{[m]}_m$.

FACT A. $H^{[\beta]}=\langle\bigcup_{m\in\mathbf{Z}} G^{[\beta]}_m\rangle$ for $\beta\leq\alpha$.

Proof of Fact A. By induction on β. For $\beta=0$ and β limit, there are no problems. For $\beta=\gamma+1$, note that $G_m/G^{[\gamma]}_m$, $H/H^{[\gamma]}$, $\alpha/H^{[\gamma]}$, $F^m_m/G^{[\gamma]}_m$ satisfies the assumption of the lemma, so it suffices to prove the case $\gamma=0$, i.e., $\beta=1$. Let

$Gr(K)=\{x\in K$: the normal subgroup x generated in K is abelian and at most countable$\}$.

So $K^{[1]}=\langle Gr(K)\rangle$ hence it suffices to prove:

FACT A1. $Gr(G_m)\subseteq Gr(H)$

FACT A2. $Gr(H)\subseteq\langle\bigcup_{m\in\aleph} Gr(G_m)\rangle$

Proof of Fact A1. Let $g\in G_m$, and K_m be the normal subgroup of G_m which g generated, so it is abelian and at most countable. Let $K_n=F_n^m(K_m)$ $K^*=\langle\bigcup_{n\in\mathbf{Z}} K_n\rangle$. As $\{a\}\cup\bigcup_n G_n$ generate H, it is enough to check conjugation by those elements. Using a we see that the normal subgroup which g generates in H includes K^*. On the other hand K^* is closed under $x\to x^a$, (as its set of generators is) and for $h\in G_n$, K_n is closed under $x\to x^h$ (because K is a normal subgroup of G_n and F_n^m maps G_m onto G_n, K_m onto K_n) and the elements in $\bigcup_{\substack{n(1)\in Z\\ n(1)\neq n}} K_{n(1)}$ commutes with h.

So K^* is a normal subgroup of H, hence the normal subgroup which g generates.

Now K^* is $\prod_{n\in\mathbf{Z}} K_n$, each K_n abelian and countable; hence K^* is abelian and countable. So $g\in Gr(H)$ as required, so Fact A1 holds.

Proof of Fact A2. Let $g=a^n\prod g_m^{[m]}\in Gr(H)$, $g_m\in G_0$ and let $g_m=e$ for $m\notin(-m(0),m(0))$.

First assume $n\neq 0$, and we shall get a contradiction. Choose $m(1)>m(0)+|n|+3$, and choose $h\in G_{m(1)}$, $h\neq e$. So g^{-1} and hgh^{-1} belong to the normal subgroup of H which g generates, hence they must commute; let us compute: (let $h^-=F^{m(1)}_{m(1)-n}(h)\in G_{m(1)-n}$, $h^+=F^{m(1)}_{m(1)+n}(h)\in G_{m(1)+n}$, so $a^nh=h^+a^n$, $ha^n=a^nh^-$.

$$\text{a)}\ (hgh^{-1})g^{-1}=\Big(ha^n\prod_m g_m^{[m]}h^{-1}\Big)\Big(\prod_m g_m^{[m]}\Big)^{-1}a^{-n}$$
$$=h\cdot a^n\prod_m g_m^{[m]}a_a^{-n}\cdot a^nh^{-1}a^{-n}\cdot\Big(a^n\prod_m g_m^{[m]}a^{-n}\Big)^{-1}$$
$$=h\cdot\prod_m g_m^{[m+n]}\cdot(h^+)^{-1}\cdot\Big(\prod_m g_m^{[m+n]}\Big)^{-1}=h\cdot(h^+)^{-1}.$$

$$\text{b)}\ g^{-1}(hgh^{-1})=\Big(\prod_m (g_m^{[m]})^{-1}a^{-n}\Big)\Big(ha^n\prod g_m^{[m]}h^{-1}\Big)$$
$$=\Big(a^{-n}\prod_m (g_m^{[m+n]})^{-1}\Big)\Big(a^nh^-\prod g_m^{[m]}h^{-1}\Big)$$
$$=a^{-n}\cdot a^n\cdot\prod(g_m^{[m]})^{-1})\Big(h^-\prod g_m^{[m]}h^{-1}\Big)$$
$$=a^{-n}\cdot a^n\cdot\prod(g_m^{-1}\cdot g_m)^{[m]}\cdot h^-\cdot h^{-1}=h^-\cdot h^{-1}.$$

Note that elements from different G_m's commute, and $g_m = e$ for $m \notin (-m(1), m(1))$. However as $h \in G_{m(1)}$, $n \neq 0$, $h \neq e$, so we get a contradiction.

So $n = 0$, i.e., $g = \prod g_m^{[m]}$. Since $g \in Gr(H)$, it follows at once that each $g_m^{[m]} \in Gr(G_m)$ and hence $g \in \langle \bigcup_m Gr(G_m) \rangle$, so we prove Fact A1. Hence we finish the proof of Fact A.

So clearly $H^{[\alpha]} = \bar{G}$, hence $H/H^{[\alpha]} = H/\bar{G}$ is an infinite cyclic group (hence abelian). Clearly $(H/\bar{G})^{[1]} = H/\bar{G}$, hence $H^{[\alpha+1]} = H$. We have almost proved H is a required group of order α.
We still need:

FACT B. *$H/H^{[\beta]}$ has a trivial center for $\beta < \alpha$.*

Let $g = a^n \prod_{-m(0)<m<m(0)} g_m^{[m]}$ be an element of H which is central mod $H^{[\beta]}$. In the proof of Fact A2 above, choose h to be in $G_{m(1)} - G_{m(1)}^{[\beta]}$. The same calculations as above show that if g^{-1} commutes with hgh^{-1} mod $H^{[\beta]}$, then

$$h \cdot (h^+)^{-1} \cdot h \cdot (h^{-1})^{-1} \in H^{[\beta]} \cap G = \prod_m G_m^{[\beta]}.$$

If $n \neq 0$, we infer that $h^- \in G_{m(1)-n}^{[\beta]}$ and so $h \in G_{m(1)}^{[\beta]}$, contradicting the choice of h. So $n = 0$. Now

$$a^{m(0)} g a^{-m(0)} = \prod_{-m(0)<m<m(0)} g_m^{[m+m(0)]}.$$

So, since g is central mod $H^{[\beta]}$, we have

$$\prod_{-m(0)<m<m(0)} (g_m^{[m]}) \cdot (g_m^{[m+m(0)]})^{-1} \in H^{[\beta]} \cap G = G^{[\beta]}.$$

Hence $g \in \prod_{-m(0)<m<m(0)} G_m^{[\beta]}$, and therefore $g \in H^{[\beta]}$.

LEMMA 3.10. *We may assume without loss of generality that G has trivial center.*

Proof. Define an increasing chain of normal subgroups C_i by induction on i as follows: $C_0 = \{e\}$, for $i \geq 1$ let $C_i = \{x \in G : x/\bigcup_{j<i} C_j$ is in the center of $G/\bigcup_{j<i} C_j\}$.

Since the groups C_i form an increasing chain of normal subgroups the first α such that $C_{\alpha+1} = C_\alpha$ is $<\chi^+$ (otherwise we have χ^+ distinct normal subgroups, which contradict the choice of χ as $\geq nc(G)$, since distinct normal subgroups cannot be conjugate). We shall show that $|C_\alpha| < \lambda$, otherwise let $\beta = \min\{\beta : |C_\beta| \geq \chi^+\}$. Since χ^+ is regular greater than α, β must be a successor i.e., $\beta = \gamma + 1$. By the definition of the sequence C_β/C_γ is an abelian group of cardinality χ^+. Since $\chi \geq \aleph_0$, χ^+ is uncountable, so we can use Fact 2.5 as in the proof of Theorem 2.1 and find 2^{χ^+} distinct subgroups of C_β/C_γ. Since all these subgroups are in the center of G/C_γ they are not conjugate in G/C_γ, and the pre-images of those groups in G (by the canonical epimorphism from G to G/C_γ) are also not conjugate, contradiction to the assumption that $nc(G) < \chi^+$.

Therefore we may assume that $|C_\alpha| < \lambda$. The group G/C_α by our definition of C_α is centerless and since $|C_\alpha| < \lambda$, the power of G/C_α is λ; if we will be able to find χ^+ non conjugate subgroups of G/C_α clearly these groups induce non conjugate subgroups of G. Hence we may replace G by the centerless group G/C_α.

Proof of Claim 3.7. By 3.10 w.l.o.g. the center of G is trivial. Also we may assume that the group generated by A is centerless because if we increase A we get a stronger Claim, and since G has trivial center there exists a subgroup $K \supseteq A$, $|K| < \mu$ which has trivial center. So we may assume that A has trivial center (we use that $\mu > \aleph_0$ which follows from $\lambda > 2^{\aleph_0}$). Now we get a subgroup as required by the Claim. We prove the Claim by induction on $\alpha < \mu$. If $\alpha = 0$, by the choice of μ as the first cardinal to satisfy $2^\mu = \lambda$ since $|A| < \mu$ we have that $2^{|A|} < \lambda$ so we can apply Claim 2.3 to A. Let a be the element supplied by Claim 2.3 and define $H = \langle a \rangle$. It is easy to verify that this group satisfies the demand of Claim 3.7., except $H/H^{[0]}$ free (as an abelian group), but we can use the proof below for $\alpha = 1$ based on this approximation.

So from now on we assume that $\alpha > 0$, and our Claim is true for every $\beta < \alpha$. Using the induction hypothesis for all $\xi < \mu$ we can choose a subgroup of H_ξ of G such that:

(1) The elements of H_ξ commute with the elements of $\langle A \cup \bigcup_{\beta<\xi} H_\beta \rangle$ and $H_\xi \cap \langle A \cup \bigcup_{\beta<\xi} H_\beta \rangle = \{e\}$. Let $i(\xi) < \mu$ and $\beta(\xi) < \alpha$ which satisfy $\xi = \alpha \cdot i(\xi) + \beta(\xi)$ (divide ξ by α and $\beta(\xi)$ is the remainder of the division).

(2) H_ξ is a required group of depth $\beta(\xi)$.

(3) $|H_\xi| \leq |\beta(\xi)| + \aleph_0$.

As we have said the choice of these groups is by using the induction hypothesis of Claim 3.7. Now for every $S \subseteq \mu$ let $H_S = \prod_{i(\xi) \in S} H_\xi$. By the assumption $Ded\mu > \lambda$ let I, $S_t (t \in I)$ be as in Definition 3.1. By the discussion after the proof of Theorem 3.2 we know that $\lambda > \mu^{|A|}$, by choice of χ, $\lambda \geq \chi^+$, together $\lambda \geq \chi^+ + (\mu^{|A|})^+$. So there exists $J \subseteq I$, $|J| = \chi + \mu^{|A|}$ such that $\{\langle H_{S_t}, A \rangle : t \in J\}$ is contained in one conjugacy class of subgroups of G. Let $s \in J$ be such that $|\{t \in J : t > s\}| \geq 2$ and let $g_t \in G$ ($t \in J$ and $t > s$) be such that $x \to x^{g_t}$ conjugates $\langle H_{S_t}, A \rangle$ onto $\langle H_{S_s}, A \rangle$ and every $x \to x^{g_t}$ have the same restriction on A (possible since $|J| > \mu^{|A|}$). Let g_{t_1}, $g_{t_2} t_2 < t_1$ be as above and define $a = g_{t_2}^{-1} g_{t_1}$, it commutes with every element of

A, $a \notin \langle H_{S_{t_2}}, A\rangle$ and $x \to x^a$ maps $\langle H_{S_{t_1}}, A\rangle$ onto $\langle H_{S_{t_2}}, A\rangle$. Since A has trivial center the elements of $\langle H_{S_{t_1}}, A\rangle$ and $\langle H_{S_{t_2}}, A\rangle$ which commute with every element of A are the elements of $H_{S_{t_1}}$ and $H_{S_{t_2}}$ respectively, and since this property is preserved by isomorphism, $x \to x^a$ maps $H_{S_{t_1}}$ onto $H_{S_{t_2}}$. Choose $\xi \in S_{t_1} - S_{t_2}$. Define $G_0 = \langle \bigcup_{\beta<\alpha} H_{\alpha\cdot\xi+\beta}\rangle$. Noting that $|G_0| \leq |\alpha| + \aleph_0$, define $H = \langle G_0 \cup \{a\}\rangle$. We just have to show that H is a required group of depth α. We shall use Lemma 3.9, its hypothesis holds because: Any $g \in G_0$, $h \in H_{S_{t_2}}$ commute and $x \to x^a$ maps $\langle G_0, H_{S_{t_2}}\rangle$ into $H_{S_{t_2}}$ (the argument is exactly like the proof of "$\{b_n : n<\omega\}$ generates freely an abelian group" in the Proof of 1.1). The other requirements of 3.7 are easy.

§4. Various results

DEFINITION 4.1. Let $nc_\kappa(G)$ be the number of non conjugate subgroups of G of cardinality $\leq \kappa$.

CLAIM 4.2. If $\chi = nc_\kappa(G) > 2^\kappa$, then $\chi^\kappa = \chi$.

Proof. Let $H_i (i<\chi)$ be a representative list of non conjugate subgroups of G of cardinality $\leq\kappa$. For every $S \subseteq \chi$, $|S| = \kappa$ let $K_S = \langle \bigcup_{i\in S} H_i\rangle$. If $\chi < \chi^\kappa$ there exists $S_\alpha \subseteq \chi$ $(\alpha < (2^\kappa)^+)$ distinct such that K_{S_α} are conjugate to each other. Hence for every $i \in \bigcup_\alpha S_\alpha$ there exists α such that $i \in S_\alpha$ and there exists an inner automorphism which maps K_{S_α} onto K_{S_0}, therefore K_{S_0} has a subgroup H_i^* which is conjugate to H_i. The groups $\langle H_i^* : i \in \bigcup_a S_a\rangle$ are not conjugate in pairs (so in particular are distinct) and they are subgroups of K_{S_0}, but $|K_{S_0}| \leq \kappa$, hence the group has $\leq 2^\kappa$ subgroups, contradiction.

CONCLUSION 4.3. If $\kappa < \mu$, $|G| = 2^\mu > nc(G)$ and μ is strong limit and singular, then $nc_\kappa(G) < \mu$ or $nc(G) \geq 2^\mu$.

Proof. Since μ strong limit for every $\kappa < \mu$ $2^\kappa < \mu$. By contradiction assume $nc_\kappa(G) \geq \mu$ so for all large enough $\kappa < \mu$, $nc_\kappa(G) > 2^\kappa$. Choose $\kappa < \mu$ and $\kappa \geq cf\mu$, then by Claim 4.2 $nc_\kappa(G)^\kappa = nc_\kappa(G) \leq nc(G) < 2^\mu$, but $nc_\kappa(G)^\kappa \geq \mu^{cf\mu} = 2^\mu$ contradiction.

CONCLUSION 4.4. If $K \leq G$ then the number of non conjugate subgroups of G of cardinality $\leq\kappa$ which include K, non conjugate by inner automorphisms which commute with K is $\leq nc_\kappa(G)^{|K|}$.

§5. The case $\lambda = 2^\mu$, μ strong limit

DEFINITION 5.1. (1) For a group G define $G^{(1)}$ be the subgroup of G generated by the commutatorss.

(2) We define by induction on α, $G^{(\alpha)}$: $G^{(0)} = G$, $G^{(\alpha+1)} = (G^{(\alpha)})^{(1)}$ $G^\delta = \bigcap_{i<\delta} G^{(i)}$ for δ limit.

(3) $G^{(\infty)} = \bigcap_i G^{(i)}$.

This operation is well known, so we know e.g.

CLAIM 5.2. (1) $G^{(\alpha)} \subseteq G^{(\beta)}$ for $\alpha > \beta$, and $(G/G^{(\alpha)})^{(\beta)} = G^{(\beta)}/G^{(\alpha)}$.

(2) $G/G^{(1)}$ is abelian.

(3) For some $i(G) < |G|^+$, $G^{(i(G))} = G^{(i(G)+1)} = G^{(\infty)}$, $G^{(\alpha)} \neq G^{(\beta)}$ for $\alpha < \beta \leq i(G)$

(4) $G^{(\alpha)}$ is a characteristic (hence normal) subgroup of G.

(5) If δ is limit, and $|G/G^{(i)}| \leq \lambda$ for $i < \delta$, then $|G/G_\delta| \leq \lambda^{|\delta|}$.

THEOREM 5.3. *G is a group of power λ, $\lambda = 2^\mu$, μ strong limit and singular, then $nc(G) \geq \lambda$.*

HYPOTHESIS 5.4. G is a counterexample, hence $\mu = \min\{\mu : 2^\mu = \lambda\}$ and $Ded\mu > \lambda$. Choose χ so that $\aleph_0 \cdot nc(G) \leq \chi < \lambda$ and eventually get a contradiction.

CLAIM 5.5. We can assume, w.l.o.g., $G^{(\infty)} = \{e\}$ or $G^{(1)} = G$.

Proof. If $|G/G^{(\infty)}| = \lambda$ then $G/G^{(\infty)}$ is a counterexample too, and $(G/G^{(\infty)})^{(\infty)} = \{e\}$ by 5.2(1). We are left with the case $|G/G^{(\infty)}| = \kappa < \lambda$, so necessarily $|G^{(\infty)}| = \lambda$, and it is enough to prove $nc(G^{(\infty)}) < \lambda$, as then $G^{(\infty)}$ satisfies the conclusion of 5.4. So suppose $nc(G^{(\infty)}) = \lambda$, and we shall derive a contradiction. Let $\{H_i : i < \lambda\}$ be a set of subgroups of $G^{(\infty)}$, pairwise non-conjugate in $G^{(\infty)}$. Let $\{a_\alpha/G^{(\infty)} : \alpha < \kappa\}$ be a list of the members of $G/G^{(\infty)}$.

Let E be the following equivalence relation on λ. Let i, j be E-equivalent iff H_i, H_j are conjugate in G; we know E has $\leq \chi$ equivalence classes, $\chi < \lambda$, hence E has an equivalence class of power $> \chi + \kappa$, so w.l.o.g., $H_i (i < (\chi + \kappa)^+)$ are pairwise conjugate, so let $H_i = g_i H_0 g_i^{-1}$, $g_i \in G$. By the choice of $a_j (j < \kappa)$ there are $\alpha(i) < \kappa$, $h_i \in G^{(\infty)}$ such that $g_i = h_i a_{\alpha(i)}$. Clearly for some $i \neq j$ $\alpha(i) = \alpha(j)$, so $H_i = (h_i \cdot a_{\alpha(i)}) H_0 (h_i a_{\alpha(i)})^{-1} = h_i (a_{\alpha(i)} H_0 a_{\alpha(i)}^{-1}) h_i^{-1}$ hence H_i is conjugate to $a_{\alpha(i)} H a_{\alpha(i)}^{-1}$), in $G^{(\infty)}$, and similarly H_j, hence H_i, H_j are conjugate in $G^{(\infty)}$, contradiction.

Proof of 5.3 in case $G^{(\infty)} = \{e\}$. First note that for each $i < i(G)$ there is

$a_i \in G^{(i)} - G^{(i+1)}$, hence $\langle a_i \rangle \subseteq G^{(i)}, \langle a_i \rangle \subsetneqq G^{(i+1)}$, hence $\{\langle a_i \rangle : i < i(G)\}$ are pairwise non-conjugate. By 4.3 this implies $i(G) < \mu$.

Secondly note that $\sum_{i<i(G)} |G^{(i)}/G^{(i+1)}| \geq \mu$, for suppose it is $\kappa < \mu$, then by 5.2(5) we can prove by induction on $i \leq i(G)$ that $|G/G^{(i)}| \leq \kappa^{|i|}$, and as μ is strong limit, $\kappa^{|i|} < \mu$, contradiction to $|G/G^{i(G)}| = |G| = \lambda$.

Let $\mu = \sum_{\alpha < cf\mu} \mu_\alpha$, such that $\mu_\alpha < \mu$, $\mu_\alpha^{\sum\{\mu_\beta : \beta < \alpha\}} = \mu_\alpha$, and let $i(\alpha)$ be such that $|G/G^{(i(\alpha))}| \leq \mu_\alpha$, $|G/G^{(i(\alpha)+1)}| > \mu_\alpha$ (exists by 5.2.(5) and the above remarks). Clearly $i(\alpha) \geq i(\beta)$ for $\alpha \geq \beta$; and as we can replace $\{\mu_\alpha \mid \alpha < cf\mu\}$ by any cofinal subsequence, we can assume either for every $\alpha i(\alpha) = i(0)$ or for every $\alpha < \beta i(\alpha) < i(\beta)$.

CASE A. For every $\alpha i(\alpha) = i(0)$. So $|G/G^{(i(0))}| < \mu$, $|G/G^{(i(0)+1)}| \geq \mu$.

Let $G/G^{(i(0))} = \{a_\zeta/G^{(i(0))} : \zeta < |G/G^{i(0)}|\}$. It is enough to prove that $nc(K) \geq \lambda$ for $K = G/G^{(i(0)+1)}$. Let H be the subgroup of K generated by $\{a_\zeta/G^{(i(0)+1)} : \zeta < |G/G^{i(0)}|\}$. So $H \subseteq K$, $|H| < \mu$, $|K| \geq \mu$. $K^{(i(0)+1)} = \{e\}$, H contains a set of representatives for $K/K^{(i(0))}$. So any element of K has the form ha, $h \in H$, $a \in K^{(i(0))}$ and so for $x \in K^{(i(0))}$, $(ha)x(ha)^{-1} = h(axa^{-1})h^{-1} = hxh^{-1}$ (as $K^{(i(0))}$ is abelian by 5.2.(2)), so two subgroups of $K^{(i(0))}$ are conjugate in K, iff an inner automorphism $x \to x^h$ $(h \in H)$ witnesses this. So any subgroup of $K^{i(0)}$ has $\leq |H| < \mu$ groups conjugate to it. But $K^{i(0)}$ is abelian of power $\geq \mu$, hence by 2.5, has $\geq 2^\mu = \lambda$ subgroups, so $\geq \lambda$ of them are not conjugate. Hence $nc(K) \geq \lambda$, and we finish.

CASE B. $i(\alpha)(\alpha < cf\mu)$ strictly increasing. Choose for each $i(\alpha)$, $\{a_\xi^\alpha : \xi < \mu_\alpha^+\}$ in $G^{(i(\alpha))} - G^{(i(\alpha)+1)}$. Choose $H_\alpha \subseteq G$ a subgroup of power $\leq \mu_\alpha$, such that $a_\xi^\beta \in H_\alpha$ for $\beta < \alpha$ and H_α contains a complete set of representatives for $G/G^{(i(\alpha))}$. There is, for each α, a set $A_\alpha \in \{a_\xi^\alpha : \xi < \mu_\alpha^+\}$, $|A_\alpha| = \mu_\alpha^+$, such that if $a_\xi^\alpha, a_\zeta^\alpha \in A_\alpha$, $\xi < \zeta$ then $a_\zeta^\alpha \notin \langle G^{(i(\alpha)+1)}, a_\xi^\alpha, H_\alpha \rangle_G$.

Now for any two distinct sequences $\bar{a} = \langle a_\alpha : \alpha < cf\mu \rangle$, $a_\alpha \in A_\alpha$ $(e = 0, 1)$, clearly $\langle \{a_\alpha : \alpha < cf\mu\} > G$ are not conjugate so $nc(G) \geq \prod_\alpha |A_\alpha| \geq \prod_\alpha \mu_\alpha = 2^\mu = \lambda$.

Proof of 5.3 *when* $G^{(1)} = G$. Note that any homomorphic image K of G satisfies $K^{(1)} = K$ hence by Lemma 3.10 we may assume without loss of generality that G has trivial center. Now introduce:

DEFINITION 5.6. A group H is *a good group of depth* α iff H is a required group of depth α and $H^{(\alpha)} = \{e\}$. Notice also that the following trivial lemma is true.

LEMMA 5.7. *Let $\{G_i : i \in I\}$ be a family of groups then for every ordinal α $(\prod_{i\in I} G_i)^{(\alpha)} = \prod_{i\in I} G_i^{(\alpha)}$.*

We repeat everything from §3 using good groups of depth α instead of required groups of depth α. Since in the present case we do not know that $\mu^{|A|} < \lambda$ and we need a replacement; first notice that w.l.o.g. A has trivial center and $A^{(1)} = A$ (we can fulfil the second requirement by taking A to be an elementary submodel of G). Now given $f: \langle H_{S_{t_1}}, A\rangle \to \langle H_{S_{t_2}}, A\rangle$ isomorphism we can prove that $f: A \to A$ and $f: H_{S_{t_1}} \to H_{S_{t_2}}$ by evaluating the ∞-derivative (the angular bracket): everything vanishes except A (use Lemma 5.7 and definition of H_{S_t}). Now there are at most $|A|^{|A|}$ maps from A to A. By the choice of μ as the first such that $2^\mu = \lambda$ and $|A| < \mu$ there are only $2^\mu < \lambda$ maps from A to A so we can choose as before two maps which act on A in the same way.

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Hebrew University of Jerusalem
Jerusalem
Israel