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QUITE COMPLETE REAL CLOSED FIELDS

BY

SAHARON SHELAH*

Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
and
Department of Mathematics, Rutgers University
New Brunswick, NJ 08854, USA
e-mail: shelah@math.huji.ac.il

ABSTRACT

We prove that any ordered field can be extended to one for which every decreasing sequence of bounded closed intervals, of any length, has a nonempty intersection; equivalently, there are no Dedekind cuts with equal cofinality from both sides.

1. Introduction

Laszlo Csirmaz raised the question of the existence of nonarchimedean ordered fields with the following completeness property: any decreasing sequence of closed bounded intervals, of any ordinal length, has nonempty intersection. We will refer to such fields as **symmetrically complete** for reasons indicated below.

THEOREM 1.1: *Let K be an arbitrary ordered field. Then there is a symmetrically complete real closed field containing K.*

The construction shows that there is even a "symmetric-closure" in a natural sense, and that the cardinality may be taken to be at most $2^{|K|^{+}+\aleph_1}$.

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2. Real closed fields

Any ordered field embeds in a real closed field, and in fact has a unique real closure. We will find it convenient to work mainly with real closed fields throughout. Accordingly, we will need various properties of real closed fields. We assume some familiarity with quantifier elimination, real closure, and the like, and we use the following consequence of o-minimality. (Readers unfamiliar with o-minimality in general may simply remain in the context of real closed fields, or, in geometrical language, *semialgebraic geometry*.)

FACT 2.1: Let K be a real closed field, and let f be a parametrically definable function of one variable defined over K. Then f is piecewise monotonic, with each piece either constant or strictly monotonic; this holds uniformly and definably in definable families, with a bound on the number of pieces required, and with each piece an internal whose endpoints are definable from the defining parameters for the function.

2.1. CUTS.

Definition 2.2:

(1) A **cut** in a real closed field K is a pair $C = (C^-, C^+)$ with K the disjoint union of C^- and C^+, and $C^- < C^+$. The cut is a **Dedekind cut** if both sides are nonempty, and C^- has no maximum, while C^+ has no minimum.

(2) The **cofinality** of a cut C is the pair (κ, λ) with κ the cofinality of C^- and λ the coinitiality of C^+ (i.e., the "cofinality to the left"). If the cut is not a Dedekind cut, then one includes 0 and 1 as possible values for these invariants.

(3) A cut of cofinality (κ, λ) is **symmetric** if $\kappa = \lambda$.

(4) A real closed field is **symmetrically complete** if it has no symmetric cuts.

(5) A cut is **positive** if $C^- \cap K_+$ is nonempty.

We will need to consider some more specialized properties of cuts.

Definition 2.3: Let K be a real closed field, C a cut in K.

(1) The cut C is a **Scott cut** if it is a Dedekind cut, and for all $r > 0$ in K, there are elements $a \in C^-$, $b \in C^+$ with $b - a < r$.

(2) The cut C is **additive** if C^- is closed under addition and contains some positive element.

(3) The cut C is **multiplicative** if $C^- \cap K^+$ is closed under multiplication and contains 2.

(4) $C_{\rm add}$ is the cut with left side $\{r \in K : r + C^- \subseteq C^-\}$.

(5) $C_{\rm mlt}$ is the cut with left side $\{r \in K : r \cdot (C^- \cap K_+) \subseteq C^-\}$.

Observe that Scott cuts are symmetric. If C is a positive Dedekind cut which is not a Scott cut, then $C_{\rm add}$ is an additive cut, while if C is an additive cut which is not a multiplicative cut, then $C_{\rm mlt}$ is a multiplicative cut.

2.2. REALIZATION. If $K \subseteq L$ are ordered fields, then a cut C in K is said to be **realized**, or **filled**, by an element a of L if the cut induced by a on K is the cut C.

LEMMA 2.4 ([1]): *Let K be a real closed field. Then there is a real closed field L extending K in which every Scott cut has a unique realization, and no other Dedekind cuts are filled.*

This is called the **Scott completion** of K, and is strictly analogous to the classical Dedekind completion. The statement found in [1] is worded differently, without referring directly to cuts, though the relevant cuts are introduced in the course of the proof. The result is also given in greater generality there.

LEMMA 2.5: *Let K be a real closed field, C a multiplicative cut in K, and L the real closure of $K(x)$, where x realizes the cut C. Then for any $y \in L$ realizing the same cut, we have $x^{1/n} < y < x^n$ for some n.*

Proof: Let $\mathcal{O}_K$ be $\{a \in K : |a| \in C^-\}$, and let $\mathcal{O}_L$ be the convex closure in L of $\mathcal{O}_K$. Then these are valuation rings, corresponding to valuations on K and L which will be called v_K and v_L respectively.

The value group Γ_K of v_K is a divisible ordered abelian group, and the value group of the restriction of v_L to $K(X)$ is $\Gamma_K \oplus \mathbb{Z}\gamma$ with $\gamma = v_L(x)$ negative, and infinitesimal relative to Γ_K. The value group of v_L is the divisible hull of $\Gamma_K \oplus \mathbb{Z}\gamma$.

Now if $y \in L$ induces the same cut C on K, then $v_L(y) = qv_L(x)$ for some positive rational q. Hence $u = y/x^q$ is a unit of $\mathcal{O}_L$, and thus $u, u^{-1} < x^\epsilon$ for all positive rational ϵ. So $x^{q-\epsilon} < y < x^{q+\epsilon}$ and the claim follows. ∎

LEMMA 2.6: *Let $K \subseteq L$ be real closed fields, and C an additive cut in L. Let C' and $C'_{\rm mlt}$ be the cuts induced on K by C and $C_{\rm mlt}$ respectively. Suppose that $C'_{\rm mlt} = (C')_{\rm mlt}$, and that $x, y \in L$ are two realizations of the cut C', with $x \in C^-$ and $y \in C^+$. Then y/x induces the cut $C'_{\rm mlt}$ on K.*

Proof: If $a \in K$ and $ax \geq y$, then $a \in (C_{\rm mlt})^+$, by definition, working in L.

On the other hand, if $a \in K$ and $ax < y$, then $a \in [(C')_{\mathrm{mlt}}]^-$, which by hypothesis is $(C'_{\mathrm{mlt}})^-$. ∎

LEMMA 2.7: *Let $K \subseteq L$ be real closed fields, and C a positive Dedekind cut in L which is not additive. Let C' and C'_{add} be the cuts induced on K by C and C_{add} respectively. Suppose that $C'_{\mathrm{add}} = (C')_{\mathrm{add}}$. Suppose that $x, y \in L$ are two realizations of the cut C', with $x \in C^-$ and $y \in C^+$. Then $y - x$ induces the cut C'_{add} on K.*

Proof: If $a \in K$ and $a + x \geq y$, then $a \in (C_{\mathrm{add}})^+$, by definition, working in L.

On the other hand, if $a \in K$ and $a + x < y$, then $a \in [(C')_{\mathrm{add}}]^-$, which by hypothesis is $(C'_{\mathrm{add}})^-$. ∎

2.3. INDEPENDENT CUTS. We will rely heavily on the following notion of independence.

Definition 2.8: Let K be a real closed field, and $\mathcal{C}$ a set of cuts in K. We say that the cuts in $\mathcal{C}$ are **dependent** if for every real closed field L containing realizations a_C $(C \in \mathcal{C})$ of the cuts over K, the set $\{a_C : C \in \mathcal{C}\}$ is algebraically independent over K.

The following merely rephrases the definition.

LEMMA 2.9: *Let K be a real closed field and $\mathcal{C}$ a set of cuts over K. Then the following are equivalent.*

1. *$\mathcal{C}$ is independent.*
2. *For each set $\mathcal{C}_0 \subseteq \mathcal{C}$, and each ordered field L containing K, if $a_C \in L$ is a realization of the cut C for each $C \in \mathcal{C}_0$, then the real closure of $K(a_C : C \in \mathcal{C}_0)$ does not realize any cuts in $\mathcal{C} \setminus \mathcal{C}_0$.*

Note that this dependence relation satisfies the Steinitz axioms for a dependence relation. We will make use of it to realize certain sets of types in a controlled and canonical way.

LEMMA 2.10: *Let K be a real closed field, and $\mathcal{C}$ a set of cuts over K. Then there is a real closed field L generated over K (as a real closed field) by a set of realizations of some independent family of cuts included in $\mathcal{C}$, in which all of the cuts $\mathcal{C}$ are realized. Furthermore, such an extension is unique up to isomorphism over K. Moreover, L can be embedded into L' over K if L' is a real closed field extending L and realizing every cut in $\mathcal{C}$.*

Proof: Clearly we must take L to be the real closure of $K(a_C : C \in \mathcal{C}_0)$, where $\mathcal{C}_0$ is some maximal independent subset of $\mathcal{C}$; and equally clearly, this works.

It remains to check the uniqueness. This comes down to the following: for any real closed field L extending K, and for any choice of independent cuts $C_1, \dots, C_n$ in K which are realized by elements $a_1, \dots, a_n$ of L, the real closure of the field $K(a_1, \dots, a_n)$ is uniquely determined by the cuts. One proceeds by induction on n. The real closure $\hat{K}$ of $K(a_n)$ is determined by the cut C_n; and as none of the other cuts are realized in it, they extend canonically to cuts $C_1', \dots, C_{n-1}'$ over $\hat{K}$, which are independent over $\hat{K}$. At this point induction applies. ∎

LEMMA 2.11: *Let K be a real closed field, and $\mathcal{C}$ a set of Dedekind cuts in K. Suppose that C is a Dedekind cut of cofinality (κ, λ) which is dependent on $\mathcal{C}$, and let $\mathcal{C}_0$ be the set $\{C' \in \mathcal{C} : \mathrm{cof}(C') = (\kappa, \lambda) \text{ or } (\lambda, \kappa)\}$. Then C is dependent on $\mathcal{C}_0$, and in particular $\mathcal{C}_0$ is nonempty.*

Proof: It is enough to prove this for the case that $\mathcal{C}$ is independent. If this fails, we may replace the base field K by the real closure $\hat{K}$ over K of a set of realizations of $\mathcal{C}_0$. Then since none of the cuts in $\mathcal{C} \setminus \mathcal{C}_0$ are realized, and C is not realized, these cuts extend canonically to cuts over $\hat{K}$, and hence we may suppose $\mathcal{C}_0 = \emptyset$. We may also suppose $\mathcal{C}$ is finite, and after a second extension of K we may even assume that $\mathcal{C}$ consists of a single cut C_0. This is the essential case.

So at this point we have a realization a of C_0 over the real closed field K, and a realization b of C over K, with b algebraic, and hence definable, over a, relative to K. Thus b is the value at a of a K-definable function, not locally constant near a, and by Fact 2.1 it follows that there is an interval about b with endpoints in K which is order isomorphic or anti-isomorphic to an interval about a, with the cuts corresponding. This contradicts the supposition that $\mathcal{C}_0$ has become empty, and proves the claim. ∎

For our purposes, the following case is the main one. We combine our previous lemma with the uniqueness statement.

PROPOSITION 2.12: *Let K be a real closed field, and $\mathcal{C}$ a maximal independent set of symmetric cuts in K. Let L be an ordered field containing K together with realizations a_C of each $C \in \mathcal{C}$. Then the real closure of $K(a_C : C \in \mathcal{C})$ realizes the symmetric cuts of K and no others. Furthermore, the result of this construction is unique up to isomorphism.*

Evidently, this construction deserves a name.

Definition 2.13: Let K be a real closed field. A **symmetric hull** of K is a real closed field generated over K by a set of realizations of a maximal independent set of symmetric cuts.

While this is unique up to isomorphism, there is certainly no reason to expect it to be symmetrically complete, and the construction will need to be iterated. The considerations of the next section will help to bound the length of the iteration.

LEMMA 2.14: *Let K be a real closed field, and L its symmetric hull. Then every Scott cut in K has a unique realization in L.*

Proof: Recall that every Scott cut is symmetric. One can form the symmetric hull of K by first taking its Scott completion K_1, realizing only the Scott cuts (uniquely), and then taking the symmetric hull of K_1. ∎

3. Height and depth

Definition 3.1: Let K be a real closed field.

(1) The **height** of K, $h(K)$, is the least ordinal α for which we can find a continuous increasing sequence K_i $(i \leq \alpha)$ of real closed fields with K_0 countable, $K_\alpha = K$, and K_{i+1} generated over K_i, as a real closed field, by a set of realizations of cuts which are independent.

(2) Let $h^+(K)$ be $\max(|h(K)|^+, \aleph_1)$ ($\aleph_1$ is the first uncountable cardinal strictly greater than $h(K)$).

Observe that the height of K is at most $|K|$ (or is ∞, which by 3.3 does not occur). We need to understand the relationship of the height of K with its order-theoretic structure, which for our purposes is controlled by the following parameter.

Definition 3.2: Let K be a real closed field. The **depth** of K, denoted $d(K)$, is the least regular cardinal κ greater than the length of every strictly increasing sequence in K.

Observe that the depth is uncountable. The following estimate is straightforward, and what we will really need is the estimate in the other direction, which will be given momentarily.

LEMMA 3.3: *Let K be a real closed field. Then $h(K) \leq d(K)$.*

Proof: One builds a continuous tower K_α of real closed fields starting with any countable subfield of K, and realizing maximal sets of independent cuts at each stage. If this continues past $\kappa = d(K)$, then there is a cut over K_κ filled at stage κ by an element $x \in K$. Then the cut determined by x over each K_α for $\alpha < \kappa$ is filled at stage $\alpha + 1$ by an element y_α. Those y_α lying below x form an increasing sequence, by construction, which is therefore of length less than κ; and similarly there are fewer than κ elements $y_\alpha > x$, so we arrive at a contradiction. ∎

PROPOSITION 3.4: *Let K be a real closed field. Then $d(K) \leq h^+(K)$.*

Proof: Let $\kappa > h(K)$ be regular and uncountable, and let K_α ($\alpha \leq h(K)$) be a continuous increasing chain of real closed fields, with K_0 countable, $K_{h(K)} = K$, and K_{i+1} generated over K_i, as a real closed field, by a set of realizations of independent cuts.

For $\alpha \leq h(K)$ and $X \subseteq K$, let $K_{\alpha,X}$ be the real closure of $K_\alpha(X)$ inside K. We recast our claim as follows to allow an inductive argument.

$$\text{For } X \subseteq K \text{ with } |X| < \kappa, \text{ and any } \alpha \leq h(K), \text{ we have } d(K_{\alpha,X}) \leq \kappa.$$

Now this claim is trivial for $\alpha = 0$ as K_0 is countable, and the claim passes smoothly through limit ordinals up to $h(K)$, so we need only consider the passage from α to $\beta = \alpha + 1$. So K_β is $K_{\alpha,S}$ with S a set of realizations of independent cuts over K_α, and similarly $K_{\beta,X}$ is $K_{\alpha,X\cup S}$.

Consider the claim in the following form:

$$d(K_{\alpha,X\cup S_0}) \leq \kappa \quad \text{for } S_0 \subseteq S.$$

In this form, it is clear if $|S_0| < \kappa$, as it is included in the inductive hypothesis for α, and the case $|S_0| \geq \kappa$ reduces at once to the case $|S_0| = \kappa$. So we now assume that $S_0 = (s_i : i < \kappa)$ is a set of realizations of independent types.

We can find a subset S_1 of S_0 of cardinality $\aleph_0 + |X_0|$ such that:

(a) if $s_i \in S_0 \setminus S_1$ then the cut C_i which s_i induces on K_α is not realized in the real closure of K'_α of $K_\alpha(X_0 \cup S_1)$;

(b) the cuts which the $s_i \in S_0 \setminus S_1$ induce on K'_α form an independent family.

Then after moving S_1 into X, we may suppose that S_0 is a set of realizations of cuts which are independent over $K_{\alpha,X}$.

For $\zeta \leq \kappa$, let $L_\zeta = K_{\alpha, X\cup\{s_\epsilon:\epsilon<\zeta\}}$ and let $L = L_\kappa$. We have $d(L_\zeta) \leq \kappa$ for $\zeta < \kappa$, and we claim $d(L) \leq \kappa$.

Let C_i be the cut realized by s_i over L_0. Note that C_i extends canonically to a cut C_i^j on K_j for all $j \leq i$, and for fixed j, the cuts C_i^j are independent for $i \geq j$.

Now suppose, toward a contradiction, that we have $(a_i : i < \kappa)$ increasing in L, and let B_i^ϵ denote the cut induced on L_ϵ by a_i. With ϵ held fixed, and with i varying, as $d(L_\epsilon) \leq \kappa$ we find that the cuts B_i^ϵ stabilize for large i (and furthermore, $a_i \notin L_\epsilon$). Accordingly, for each ϵ we may select $j_\epsilon < \kappa$ such that the cuts B_i^ϵ coincide for all $i \geq j_\epsilon$.

Now fix a limit ordinal $\delta < \kappa$ such that for all $\epsilon < \delta$ we have $j_\epsilon < \delta$. We may also require that $a_i \in L_\delta$ for $i < \delta$. Then $(B_\delta^\delta)^- = \bigcup_{\epsilon<\delta}(B_{j_\epsilon}^\epsilon)^-$, and the cofinality from the left of B_δ^δ is $\mathrm{cof}(\delta)$

Now a_δ is algebraic over $L_\delta(s_i : i \in I_0)$ for some finite subset I_0 of $[\delta, \kappa)$, and hence also over $L_\epsilon(s_i : i \in I_0)$ for some $\epsilon < \delta$. Thus the cut B_δ^ϵ depends on the cuts C_i^ϵ $(i \in I_0)$ over L_ϵ. As $B_\delta^\epsilon = B_{j_\epsilon}^\epsilon$ is realized in L_δ, it follows that this cut is also dependent on the sets $\{C_i^\epsilon : i < \delta\}$ of cuts over L_ϵ. But the cuts C_i^ϵ for $i \geq \epsilon$ are supposed to be independent over L_ϵ, a contradiction. ∎

PROPOSITION 3.5: *Let K be a real closed field. Then $h(K) \leq |K| \leq 2^{|h(K)|}$.*

Proof: The first inequality is clear. For the second, let $\alpha = h(K)$, $\kappa = |\alpha| + \aleph_0$, and let K_i $(i < \alpha)$ be a chain of the sort afforded by the definition of the height. We show by induction on i that $|K_i| \leq 2^\kappa$. Only successor ordinals $i = j + 1$ require consideration, where we suppose $|K_j| \leq 2^\kappa$.

Each generator a of K_i over K_j corresponds to a cut C_a in K_j, and each such cut is determined by the choice of some cofinal sequence S_a in C_a^-. Such a sequence S_a may be taken to have order type a regular cardinal, and will have length less than $d(K)$. Since $d(K) \leq h^+(K)$, we find that the order type of S_a is at most κ. So the number of such sequences is at most $\sum_{\lambda\leq\kappa} |K_j|^\lambda \leq \kappa \times (2^\kappa)^\kappa = 2^\kappa$. ∎

4. Proof of the Theorem

We now consider the following construction. Given a real closed field K, we form a continuous increasing chain K_α by setting $K_0 = K$, taking $K_{\alpha+1}$ to be the symmetric hull of K_α in the sense of Definition 2.13, and taking unions at limit ordinals.

If at some stage K_α is symmetrically complete, that is $K_\alpha = K_{\alpha+1}$, then we have the desired symmetrically complete extension of K, and furthermore our extension is prime in a natural sense. We claim in fact:

PROPOSITION 4.1:

(1) *For K a real closed field, if $\kappa = \max(h^+(K), \aleph_2)$ and K_α ($\alpha \le \kappa$) is the associated continuous chain of symmetric hulls of length $\kappa + 1$, then K_κ is symmetrically complete.*

(2) *Also*
   (i) $|K_\kappa| \le 2^{h^+(K)+\aleph_1}$, *and*
   (ii) *if K' is a symmetrically complete extension of K then K_κ can be embedded into K' over K.*
   (iii) *K is unbounded in K_κ (and no non-Dedekind cut of K is realized in K_κ and no nonsymmetric Dedekind cut of K is realized in K_κ).*

The proof of Proposition 4.1 occupies the remainder of this section.

LEMMA 4.2: *Suppose that K is a real closed field, and that (K_α) is a continuous chain of iterated symmetric hulls of any length. Let $x \in K_\alpha \setminus K$ with $\alpha > 0$ arbitrary. Then the cut induced on K by x is symmetric.*

Proof: Let β be minimal such that the cut in question is filled in $K_{\beta+1}$. Then the cut induced on K_β by x is the canonical extension of the cut induced on K by x, and is symmetric by Proposition 2.12. ∎

We now begin the proof by contradiction of Proposition 4.1(1). We assume therefore that the chain is strictly increasing at every step up to K_κ, and that there is a symmetric cut C over K_κ. Here $\kappa \ge \aleph_2$ is regular and greater than $h(K)$; in particular $\kappa \ge d(K)$ by 3.4. Furthermore, as $\kappa > h(K)$, we can view the chain K_α as a continuation of a chain $\hat{K}_i$ ($i \le h(K)$) of the sort occurring in the definition of $h(K)$, with $\hat{K}_{h(K)} = K_0$; then the concatenated chain gives a construction of K_α of length at most $h(K) + \alpha < \kappa$, and hence $h(K_\alpha) < \kappa$ for all $\alpha < \kappa$, and in particular $d(K_\alpha) \le \kappa$ for all $\alpha < \kappa$ by 3.4.

For $\alpha < \kappa$, let C_α denote the cut induced on K_α by C.

LEMMA 4.3: *For any $\alpha < \kappa$, the cut C_α is symmetric.*

Proof: Suppose C_α is not symmetric. Then the cut C_α is not realized in K_κ, by Lemma 4.2. Hence the cut C is the canonical extension of C_α to K_κ, contradicting its supposed symmetry. ∎

In particular, the cut C_α is realized in $K_{\alpha+1}$, and thus we have the following.

COROLLARY 4.4: *For any limit ordinal $\alpha \leq \kappa$, the two-sided cofinality of C_α is* $\mathrm{cof}(\alpha)$.

After these preliminaries, we divide the analysis of the supposed cut C into a number of cases, each of which leads to a contradiction.

(Case I) C is a Scott cut

In this case, as $d(K_\alpha) \leq \kappa$ for $\alpha < \kappa$, the set E of $\delta < \kappa$ for which C_δ is a Scott cut is a closed unbounded subset of κ.

Fix $\delta \in E$. Then the Scott cut C_δ is filled by a unique element of $K_{\delta+1}$, by Lemma 2.14. So $C_{\delta+1}$ cannot be symmetric, a contradiction.

(Case II) C is a multiplicative cut

Let $\alpha < \kappa$ have uncountable cofinality (recall $\kappa \geq \aleph_2$).

The cut C_α is realized in $C_{\alpha+1}$ by some element a. As C is multiplicative, either all positive rational powers of a lie in C^-, or all positive rational powers of a lie in C^+.

On the other hand, $K_{\alpha+1}$ may be constructed in two stages as follows. First realize all the cuts in a maximal independent set of symmetric cuts in K_α, with the exception of the cut C_α, getting a field K'_α; then take the real closure of $K'_\alpha(a)$, where a fills the canonical extension of the cut C_α to K'_α. As seen in Lemma 2.5, there are only two cuts which may possibly be induced by C on $K_{\alpha+1}$, and each has countable cofinality from one side, and uncountable cofinality from the other.

So $C_{\alpha+1}$ is not symmetric, and this is a contradiction.

(Case III) C is an additive cut

Consider the set E of $\delta < \kappa$ for which $(C_\delta)_{\mathrm{mlt}} = (C_{\mathrm{mlt}})_\delta$, recalling Definition 2.3(5). Taking into account that the two-sided cofinality of C_α is less than κ for all $\alpha < \kappa$, we find that E is a closed and unbounded set in κ.

Fix $\delta \in E$. As the cofinality of C from either side is κ, hence is greater than $\mathrm{cf}(\delta)$, the cofinality of C_δ, we may take $x_\delta \in C^-$, $y_\delta \in C^+$, both of which induce C_δ on K_δ. By Lemma 2.6, the element y_δ/x_δ fills the cut $(C_{\mathrm{mlt}})_\delta$. In particular $(C_{\mathrm{mlt}})_\delta$ is symmetric for $\delta \in E$.

Now consider what happens as δ increases in E. Thinning E, we can extract a decreasing sequence y_δ and an increasing sequence x_δ, so that $z_\delta = (y_\delta/x_\delta)$ is a decreasing sequence with $z_\delta \in (C_{\mathrm{mlt}})^+_\delta$. Accordingly the cofinality of (C_{mlt}) from the right is κ.

Now if the cofinality of $C_{\rm mlt}$ from the left is also κ, then we contradict Case II. On the other hand, if the cofinality of $C_{\rm mlt}$ from the left is less than κ, then from some point onward this cofinality stabilizes; but then, choosing δ large and of some other cofinality (again, since $\kappa \geq \aleph_2$ this is possible), we contradict Lemma 4.3.

(Case IV) C is a positive Dedekind cut, but not a Scott cut

One argues as in the preceding case, considering $C_{\rm add}$ and using Lemma 2.7, which leads to a symmetric additive cut and thus a contradiction to the previous case.

As no cases remain, Proposition 4.1(1) is proved, and thus the construction of a symmetrically complete extension terminates.

As for clause (i) of Proposition 4.1(2), to estimate the cardinality of the resulting symmetrically complete extension, recall that it has height at most $\kappa = \max(h^+(K), \aleph_2) \leq \max(|K|^+, \aleph_2)$ and hence cardinality at most 2^κ. Moreover, similarly for any $\alpha < \kappa, |K_\alpha| \leq 2^{h^+(K)+\aleph_1}$ hence

$$|K| = |\bigcup_{\alpha<\kappa} K_\alpha| \leq \sum_{\alpha<\kappa} |K_\alpha| \leq \sum_{\alpha<\kappa} 2^{h^+(K)+\aleph_1} = \kappa + 2^{h^+(K)+\aleph_1} = 2^{h^+(K)+\aleph_1}.$$

For clause (ii) of Proposition 4.1(2), we define an embedding h_α of K_α into K', increasing continuously with α for $\alpha \leq \kappa$. For $\alpha = 0$, h_0 is the identity; for α limit take the union and for $\alpha = \beta$ use 2.10.

Clause (iii) of Proposition 4.1(2) is easy too. $\blacksquare_{4.1}$

5. Concluding remarks

It should be clear that there are considerably more general types of closure that can be constructed in a similar manner. Let Θ be a class of possible cofinalities of cuts, that is pairs of regular cardinals, and suppose that Θ is symmetric in the sense that $(\theta_1, \theta_2) \in \Theta$ implies $(\theta_2, \theta_1) \in \Theta$. Then we may consider Θ-constructions in which maximal independent sets of cuts, all of whose cofinalities are restricted to lie in Θ, are taken. In order to get such a construction to terminate, all that is needed is the following: (a) for all regular θ_1, there is θ_2 such that the pair (θ_1, θ_2) is not in Θ; (b) for some regular $\kappa \geq h(K) + \aleph_2$, for every θ_1 regular $\theta_2 < \kappa$, there is a $\theta_2 < \kappa$ such that $(\theta_1, \theta_2) \notin \Theta$. The proof is as above; in the symmetric case, $\Theta_{\rm sym}$ consists of all pairs (θ, θ) of equal regular cardinals. Clearly, we may make the closure to be quite as large as we need and κ as in (b) above. Also, in the proof of Proposition 4.1 in the multiplicative case, we choose δ such that $(\aleph_0, \mathrm{cf}(\delta)) \notin \Theta$, but, of course, change the cardinality bound.

Under the preceding mild conditions, such a Θ-construction provides an "atomic" extension of the desired type. So we have Θ-closure, and it is prime (as in clause (ii) of Proposition 4.1(2)). We also can change the cofinality of K.

References

[1] D. Scott, *On complete ordered fields,* in *Applications of Model Theory to Algebra, Analysis, and Probability* (W. A. J. Luxemburg, ed.), Holt, Rinehart, and Winston, New York, 1969, pp. 274–278.