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MORE ON THE WEAK DIAMOND

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We prove that e.g., $2^{\aleph_1} < 2^{\aleph_2}$ does not imply the weak diamond for $\{\delta < \aleph_2 : \mathrm{cf}\,\delta = \aleph_0\}$ (even if CH holds).

Let us define

Definition. For a regular λ and μ let $I^{sm}_{\lambda,\mu} = \{S \subseteq \lambda$: for some function F for every $\eta \in {}^{\lambda}\mu$ for some $g : \lambda \to \lambda$ and closed unbounded subset C of λ, $(\forall \sigma \in S \cap C)$ $[\eta(\delta) = F(g \restriction \delta)]\}$. If $\mu = 2$ we omit it; if $S \subseteq \lambda$, $S \in I^{sm}_{\lambda}$ we call S small.

Essentially by Devlin and Shelah [1], if $\mu = \mu^{<\lambda} < 2^{\lambda}$, then $I^{sm}_{\lambda,\mu}$ is a proper normal ideal, and $I^{sm}_{\lambda} = I^{sm}_{\lambda,(2^{\kappa})}$ (when $2^{\kappa} < 2^{\lambda}$). See also Shelah [5, Ch. XIV].

Notation. $S^{\alpha}_{\beta} = \{\delta < \aleph_\alpha : \mathrm{cf}\,\delta = \mathrm{cf}\,\aleph_\beta\}$.

So if $2^{\aleph_\alpha} < 2^{\aleph_{\alpha+1}}$, then $\aleph_{\alpha+1}$ is not small (as a subset of itself). Naturally the problem arises whether we can say for some β that $S^{\alpha+1}_{\beta}$ is not small. If we assume GCH, and $\mathrm{cf}\,\aleph_\alpha \neq \mathrm{cf}\,\aleph_\beta$ then by Gregory [2] and Shelah [3] even the diamond holds for $S^{\alpha+1}_{\beta}$ (regarding $S^{\alpha+1}_{\alpha}$ in the case that $\aleph_\alpha$ is singular see [4]). What if $\aleph_\alpha$ is regular? The author proves (see Steinhorn and King [6]) that ZFC + GCH + "$S^{\alpha+1}_{\alpha}$ is small" is consistent.

It was still natural to hope that e.g., $2^{\aleph_1} < 2^{\aleph_2}$ implies S^2_0 is not small, and this would have been helpful under some circumstances. Unfortunately we shall prove a consistency result contradicting this (and in fact much more).

Theorem. *Suppose in* V, $\lambda = \lambda^{<\lambda}$, $\mu = 2^{\lambda} = \lambda^+$, $\chi = 2^{\lambda^+}$, $S \subseteq \lambda^+$ *stationary co-stationary*; $(\forall \delta \in S)(\lambda$ *divides* $\delta)$. *Then we can find a forcing notion* P *such that:*

(a) *P is λ-complete.*

(b) *P satisfies the λ^{++}-chain condition.*

(c) *P does not collapse λ^+ (so no cardinalities or cofinalities are changed; use* (a), (b), (c)).

(d) *In V^P: for some F, for every $\eta \in {}^{(\lambda^+)}\mu$, for some $g \in {}^{(\lambda^+)}2$, for every $\delta \in S$,* $\eta(\delta) = F(g \restriction \delta)$.

(e) *In* $V^P : 2^{\lambda} = \lambda^{++}$

Remark. (1) The interesting case is $\chi > \lambda^{++}$.

(2) By Stage D, P has a dense subset of power λ^{++}.

Proof.

Stage A: The forcing. We define iterated forcing

$$\langle P_i, \mathbf{Q}_j : i \leqslant \lambda^{++}, j < \lambda^{++} \rangle$$

such that

(1) Q_α is a forcing in V^{P_α}, $|P_\alpha| \leqslant \chi$.

(2) $P_\alpha = \{f : f$ is a function from a subset of α, for $i \in \mathrm{Dom}\, f$ $f(i)$ is a P_i-name of a member of $\mathbf{Q}_i$, $\{\alpha \in \mathrm{Dom}\, f : \alpha$ even$\}$ has power $<\lambda$, $\{\alpha \in \mathrm{Dom}\, f : \alpha$ odd$\}$ has power $\leqslant \lambda\}$. The order on P_α is the natural one.

(3) For α even, $\mathbf{Q}_\alpha$ is $\{f : f$ is a function from some ordinal $\gamma < \lambda$ to $\{0, 1\}\}$.

(4) For α odd, let $\langle \eta^\alpha_\xi : \xi < \chi \rangle \in V^{P_\alpha}$ be a list of $\eta \in ({}^{(\lambda^+)}\mu)^{V^{P_\alpha}}$. The aim of $\mathbf{Q}_\alpha \in V^{P_\alpha}$ is to force $\langle g^\alpha_\xi : \xi < \chi \rangle$ such that

(4a) $g^\alpha_\xi \in {}^{\lambda^+}2$,

(4b) $g^\alpha_\xi \restriction \delta \notin V^{P_{\alpha-1}}$ for every $\delta \in S$,

(4c) for $\delta \in S$, $\xi < \zeta < \chi$, $g^\alpha_\xi \restriction \delta = g^\alpha_\zeta \restriction \delta \Rightarrow \eta^\alpha_\xi(\delta) = \eta^\alpha_\zeta(\delta)$. So we let

$$\begin{aligned} Q_\alpha = \{p : p = \langle p^\alpha_\xi : \xi \in A_p \rangle, |A_p| \leqslant \lambda \text{ and for some } \delta = \delta_p < \lambda^+,\\ p^\alpha_\xi \text{ is a function from } \delta \text{ to } \{0, 1\} \text{ and for every } \alpha \in S, \alpha \leqslant \delta,\\ p^\alpha_\xi \restriction \delta \notin V^{P_{\alpha-1}} \text{ and } [p^\alpha_\xi \restriction \alpha = p^\alpha_\zeta \restriction \alpha \Rightarrow \eta^\alpha_\xi(\alpha) = \eta^\alpha_\zeta(\alpha)]\}. \end{aligned}$$

The order is: $p \leqslant q$ *iff* $A_p \subseteq A_q$, for $\xi \in A_p$, $p^\alpha_\xi \subseteq q^\alpha_\xi$, and for $\xi \neq \zeta$ in A_p, $q^\alpha_\xi \neq q^\alpha_\zeta$. (The last phrase is in order to ensure λ^+-completeness.) When $\xi \notin A_p$, let $p^\alpha_\xi = \emptyset$. Let $P = P_{\lambda^{++}}$, and we define:

$$p \leqslant^* q \quad \text{iff} \quad p \leqslant q \text{ and for } \alpha \text{ even } p(\alpha) = q(\alpha).$$

Stage B: For α odd $\mathbf{Q}_\alpha$ is as required. We define $\mathbf{Q}_\alpha$-names $\mathbf{g}^\alpha_\xi$ (or $P_{\alpha+1}$-names if you want): $g^\alpha_\xi(i) = n$ *iff* for some p in the generic set $\mathbf{G}_\alpha$ of $\mathbf{Q}_\alpha$, $p^\alpha_\xi(i) = n$. So we have to prove that $Q_\alpha \neq \emptyset$ and that $\mathcal{D}^{\alpha,i}_\xi = \{p \in \mathbf{Q}_\alpha : i \in \mathrm{Dom}\, p^\alpha_\xi\}$ is dense.

Now $\mathbf{Q}_\alpha$ is not empty as in V^{P_α} there is a member of ${}^\lambda 2$ which is not in $V^{P_{\alpha-1}}$ (by the definition of $Q_{\alpha-1}$). The density of $\mathcal{D}^{\alpha,i}_\xi$ is easy too.

Stage C: $Q_{2\alpha}$ is λ-complete, $Q_{2\alpha+1}$ is λ^+-complete, P_α and P are λ-complete.

For α even, $\mathbf{Q}_\alpha$ is trivially λ-complete. For α odd, $\mathbf{Q}_\alpha$ is λ^+-complete as the order (see third demand in its definition) was defined this way. By the definition of the iteration P (and every P_α) are λ-complete too.

Stage D: P does not collapse λ^+. Moreover for any regular κ such that $P_\alpha \in H(\kappa)$, and elementary submodel N of $(H(\kappa), \in)$, to which P_α belongs, if every subset of N of power $<\lambda$ belongs to N and $p \in P_\alpha \cap N$ $(\alpha \leqslant \lambda^{++})$, *then* there is q, $p \leqslant q \in P$, which is (N, P_α)-generic, (i.e., for every predense $I \subseteq P_\alpha$, $I \in N$, the set $I \cap N$ is predense above q). Also, for every $2\alpha + 1 \in \mathrm{Dom}\, q$, $\mathrm{Dom}[q(2\alpha+1)] = N \cap \lambda^+$ and for $2\alpha \in \mathrm{Dom}\, q$, $q(2\alpha) = p(2\alpha)$.

This is by the proof of [5, VIII 1.1]. (There $\lambda = \aleph_1$, but this makes no difference. Condition (4) there holds as $|Q_{2\beta}| = \lambda$, so $h_{2\beta}$ can be chosen one-to-one.)

By the same proof (which, appropriately phrased, could be proven there too).

Stage E: (1) $\mathcal{D}_0 = \{p \in P$: for β even $p(\beta)$ is a real function (not a P_β-name)$\}$ is a dense subset of P_α.

(2) For $\boldsymbol{\xi}$ a P_α-name of an ordinal, and $p_0 \in P_\alpha$ there is $p \in P_\alpha$, $p_0 \leq^* p$, and above p, ξ depends on the generic subsets of $Q_{2\beta}$ only for some λ β's. In fact, there is $B \subseteq \{2\beta : 2\beta < \alpha\}$, $|B| = \lambda$, and a maximal antichain of conditions in $P_B = \{p \in P_\alpha : \mathrm{Dom}\, p \subseteq B,\ p(2\beta) \in V$ for $\beta \in B\}$ (i.e., is not a name but a real function), and a function $F \in V$ with domain P_B into the ordinals such that $p \Vdash$"$\boldsymbol{\xi} = \zeta$ iff $\zeta = F(r)$ for some $r \in \boldsymbol{G}_{P_\alpha} \cap P_B$".

(3) $\mathcal{D}^* = \{p \in P_\alpha \cap \mathcal{D}_0$: for some $\delta \in \lambda^+ - S$, for every odd $\alpha \in \mathrm{Dom}\, p$, $\delta(p(\alpha)) = \delta$, and for some $B_p \subseteq \{2\beta : 2\beta < \alpha\}$, every $p(2\beta + 1)$ depends on the generic subsets of $\boldsymbol{Q}_{2\gamma}$ ($2\gamma \leq 2\beta$, $2\gamma \in B_\beta$) only, and $|B_p| \leq \lambda$ (in fact the dependence is as above), and for β odd, $p(\beta)^\beta_\xi$ ($\xi \in A_{p(\beta)}$) are distinct$\}$ is a dense subset of P.

Stage F: P satisfies the λ^{++}-chain condition. Let $p_i \in P$ for $i < \lambda^{++}$. By E(3), w.l.o.g. $p_i \in \mathcal{D}^*$, and $\delta(p_i(\alpha)) = \delta^*$ for every $i < \lambda^{++}$, $\alpha \in \mathrm{Dom}\, p_i$, α odd, and so $\delta^* \in \lambda - S$. By use of Fodor's Lemma on $\{i < \lambda^{++} : \mathrm{cf}\ i = \lambda^+\}$, w.l.o.g. for some $\alpha(*) < \lambda^{++}$, $\mathrm{Dom}\, p_i \cup B_{p_i} \backslash \alpha(*)$ (for $i < \lambda^{++}$) are pairwise disjoint; hence w.l.o.g. $p_i \in P_{\alpha(*)}$ for every i. Also w.l.o.g. for every odd $\beta < \alpha(*)$, and $\xi < \chi$ either at most for one i, $\Vdash_{P_\beta}$ "$(p_i(\beta))^\beta_\xi \neq 0$" or for all i, j, $\Vdash_{P_\beta}$ "$(p_i(\beta))^\beta_\xi = (p_j(\beta))^\beta_\xi$". Now any two p_i's are compatible: the main problem is (4c) of Stage A, but $\delta^* \notin S$ so it is vacuous.

Stage G: $\Vdash_P$ "$2^\lambda = \lambda^{++}$". Clearly the subsets $\boldsymbol{Q}_{2\beta}$ ($\beta < \lambda^{++}$) add to λ are distinct, hence $\Vdash_p$ "$2^\lambda \geq \lambda^{++}$".

Now for each name $\boldsymbol{g}$ of a function from λ to $\{0, 1\}$, and $p \in P$ we can find p_i ($i \leq \lambda$), $p \leq^* p_i \leq^* p_j$ for $i \leq j \leq \lambda$, such that above p_{i+1}, $\boldsymbol{g}(i)$ depends on the generic subsets of the $Q_{2\beta}$'s only, as in E(2). So above p_λ, $\boldsymbol{g}(i)$ depends on the generic subsets of the $Q_{2\beta}$'s only. As in E(2) this shows that there are $\leq (\lambda^{++}) = \lambda^{++}$ subsets of λ in V^P.

Stage H: We prove (d) from the theorem. For $g \in {}^\delta 2$, $\delta \in S$ we define $F(g)$ as follows: if for some β, ξ, $g = g^\beta_\xi \restriction \delta$, then $F(g) = \eta^\beta_\xi(\delta)$, otherwise it is zero. By the definition of the forcing, F is well defined (see (4b), (4c) of Stage A) and as required (see (4c)).

Further results

(1) We can start with $\mu = 2^\lambda > \lambda^+$, but then, in the iteration, for every α there is a subset E_α of α,

$$|E_\alpha| \leq \lambda^+, \beta \in E_\alpha \to [E_\beta \subseteq E_\alpha (\beta, \beta + \lambda^+) \subseteq E_\alpha],$$

and $P^*_\alpha = \{f : f$ satisfies the demands in A(2) above and $p(i)$ is a name depending on $\langle G_j : j < i,\ j \in E_i \rangle\}$ and for α odd in the definition of Q_α, we work inside $V[G_j : j < i,\ j \in E_i]$ instead $V[G_j : j < i]$ (equivalently, V^{P_i}).

(2) The conclusion on V^P, still holds in $(V^P)^R$ if R is a forcing notion satisfying the λ^+-chain condition of power $\leq \mu$.

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