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S-FORCING IIa: ADDING DIAMONDS AND MORE APPLICATIONS: CODING SETS, ARHANGEL'SKII'S PROBLEM AND $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$

BY

SAHARON SHELAH[a] AND LEE J. STANLEY[b,†]

[a]*Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, Israel; and* [b]*Department of Mathematics, Lehigh University, Bethlehem, PA 18017, USA*

WITH AN APPENDIX BY

JOHN P. BURGESS

Department of Philosophy, Princeton University, Princeton, NJ 08540, USA

ABSTRACT

We continue our earlier paper [20] by proving the equivalence, for regular $\kappa > \omega$, of the existence of $(\kappa, 1)$ morasses with built-in $\diamond$ sequences and a strengthening, $S_\kappa(\diamond)$, of the forcing principle, S_κ of [20]. We obtain various applications of $S_\kappa(\diamond)$, to wit: the existence of a stationary subset of $[\kappa^+]^{<\kappa}$ with sup as coding function, the existence of a counterexample to Arhangel'skii's conjecture ($\kappa = \aleph_1$) and compactness, axiomatizability and transfer properties for the Magidor–Malitz language $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$ ($\kappa = \aleph_1$).

§0. Introduction

(0.1) *Summary*

This paper continues [20], whose notation and terminology are carried over. We formulate a strengthening, $S_\kappa(\diamond)$, of the forcing principle, S_κ, of [20]. Our principal result is:

THEOREM 1. *For regular $\kappa > \omega$, $S_\kappa(\diamond) \Leftrightarrow \exists(\kappa, 1)$-morasses with built-in $\diamond$ sequences.*

The right-to-left implication is proved in §6; the left-to-right implication is proved in §8. In §7, we modify the construction of morasses in L (see [23], §3), to show that

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THEOREM 5. *In L, for all regular $\kappa > \omega$, there are $(\kappa, 1)$-morasses with built-in $\diamondsuit$ sequences.*

In §4, we introduce the principles $S_\kappa(\diamondsuit)$ and the notion of a built-in $\diamondsuit$ sequence for a morass. In §1, we introduce some preliminary notions which serve in the rest of the paper, and notably in the formulation of $S_\kappa(\diamondsuit)$.

In §§2, 5, we obtain applications of $S_\kappa(\diamondsuit)$. In §2, we prove (see [29] for background on this):

THEOREM 2. *For regular $\kappa > \omega$, $S_\kappa(\diamondsuit) \Rightarrow$ there's a stationary subset of $[\kappa^+]^{<\kappa}$ with* sup *as coding function.*

In §5, we prove (see [14], and §9, below, and [10], [13], [16] for background):

THEOREM 3. $S_{\aleph_1}(\diamondsuit) \Rightarrow \mathscr{L}[Q_1^{<\omega}, Q_2^1]$ *has compactness, axiomatizability and transfer properties.*

THEOREM 4. $S_{\aleph_1}(\diamondsuit) \Rightarrow$ *there's a counterexample to the Arhangel'skii conjecture.*

Theorem 3 builds on earlier unpublished work of Burgess which is summarized in §5 and fully developed in an appendix, in §9. §10 is a second appendix, containing various corrections to [20]. §6 also contains corrections to the construction of §6 of [20].

Finally, in §3, we tie up some loose ends from [20]. We formulate a weak form of built-in $\diamondsuit$ sequence for a $(\kappa, 1)$-morass, and prove:

THEOREM 6. $\kappa^{<\kappa} = \kappa \wedge \exists(\kappa, 1)$-*morasses* $\Leftrightarrow$ $\exists(\kappa, 1)$-*morasses with a weak form of a built-in $\diamondsuit$-sequence.*

As a corollary, in (3.2), we give a new proof of an unpublished result of Velleman showing that $\kappa^{<\kappa} = \kappa \wedge \exists(\kappa, 1)$-morasses has a characterization in terms of a forcing principle. We then apply the corresponding forcing principle to show that there are κ^+-closed, κ^{++}-Souslin trees.

(0.2) *Acknowledgements and Historical Remarks*

We are grateful to Baumgartner and Zwicker for suggesting Theorem 2, and to Burgess for providing us with his vintage 1975 manuscript which was transformed into §9, and, in summarized form, much of (5.1). This manuscript, in turn, draws in part upon ideas of Silver first discussed in a different context. We should note that Shelah independently envisaged Theorem 2.

Velleman independently proved analogues of Theorems 1, 4, see [25], and has subsequently improved Theorem 2, by weakening the hypothesis to: There are stationary simplified $(\kappa, 1)$-morasses, see [26]. We are also grateful to Velleman for pointing out, [27], various inaccuracies in [20], which are dealt with in §§6, 10.

We should now set the record straight concerning references to this paper contained in [24], [11], [12], and concerning the evolution of the formulation of the built-in $\diamond$ sequence and the principle $S_\kappa(\diamond)$. Shortly after the completion of [20], we had proved a version of Theorem 1 of this paper which applied to what we called "canonical limit" partial orderings. This is the notion which results if, in the definition of $\mathcal{S}_\kappa$ in [20], we replace κ-directed-closure by a rather convoluted pseudo-closure property which was just what was needed to make the proof of the right-to-left implication work. This pseudo-closure property was also *supposed to* apply to partial orderings like those used in [24], [11], [12] — but didn't, which resulted in a decent burial for the notion of "canonical limit" partial ordering.

The idea behind the "canonical limits" proof was to "saturate" the sufficiently generic set with respect to taking limits other than the canonical one sufficiently often. This involved guessing (via some form of built-in $\diamond$ sequence) arbitrary subsets of different $[\lambda]^{<\kappa}$ for $\kappa \leqq \lambda < \kappa^+$, but did not require arbitrary subsets of $[\kappa^+]^{<\kappa}$ to be guessed.

While the "canonical limit" notion proved to be of doubtful worth, this approach was successfully scaled back to the context of κ-directed-closed partial orderings, yielding earlier and rather different proofs of Theorems 3, 4, where, the reader will see, the sets that need guessing are only subsets of different $[\lambda]^{\aleph_0}$ for $\omega_1 \leqq \lambda < \omega_2$. In the meantime, Velleman and Donder plugged the gaps in the intended applications of the "canonical limit" proof. Velleman, in [28], introduced simplified morasses with linear limits and applied them successfully to problems from [24], [11], [12]; in [6], Donder showed they exist in L. Thus, the references to this paper in [24], [11] and [12] should be replaced by references to [28] and [6].

The developments described above had already resulted in a major delay in the appearance of this paper, when, in early 1983, the authors' attention was drawn to the complex of problems introduced by Zwicker, in [29], and more particularly to the question of the existence of stationary coding sets. This required a reformulation of the built-in $\diamond$ principle and the "saturation" property of the sufficiently generic set. The formulation of §4 resulted. After the fact we realized that the result differed in an inessential way from Velleman's formulations, [25]. Theorem 2 was proved by Spring of 1983. Subsequently, Velleman, [26], improved this result by weakening the hypothesis, as described above.

Since the appearance of this paper had already been so delayed, it was decided that some material we had planned to include would be reserved for *S-Forcing IIb*, to appear. This material includes an application to Manevitz's and Miller's Lindelöf models of the reals, see [15] (a result obtained independently by Velleman), as well as different "saturation" properties of the sufficiently generic set, corresponding to places in the construction where there is "free choice". There are three such places: choice of limits (as in our earlier formulation), choice of extensions (as in the present formulation), and choice of amalgamations. This last saturation property will be dealt with in *S-Forcing IIb*. Finally, material on universal morasses and morasses preserving certain relations will also be included. Among the applications envisioned are extending the material of (5.1) and §9 to the richer language $\mathscr{L}[Q_1^{<\omega}, Q_2^{<\omega}]$, and generalizing the interpretation from $\aleph_1, \aleph_2$ to κ and κ^+. This will end the "gap-one" cycle. Work is now in progress on extending the work of [20] and this paper to higher gap morasses. This will appear as *S-Forcing III*. Further remarks and acknowledgments appear in the body of the paper.

§1. Preliminaries

(1.1) In this paper we shall require some refinements of the notion of S-Forcing, as introduced in §3 of [20]. There, some contorsions were required to make some of the partial orderings satisfy the formal definition of κ-special: in particular, it would have been more natural, for some of these orderings, to take the terms as acting on increasing sequences of κ-cofinal ordinals; as we've come to say, the *set of indiscernibles* should have been taken to be $S_\kappa^{\kappa^+}$, the set of κ-cofinal ordinals below κ^+, rather than κ^+ itself. In this section, we shall show how such situations should be dealt with, rather than dodged. This leads us to the notion of an acceptable set, I, of indiscernibles, of an S-forcing **P** with I as set of indiscernibles, and to the related notion of the support, supp(p), of a condition $p \in P$. Roughly speaking, the support is the set of *all* ordinals mentioned by p, not just the set of indiscernibles. We tie things together by systematizing the dodges of [20], by showing that if **P** has I as acceptable set of indiscernibles we can regard **P** as having κ^+ as set of indiscernibles. In the other direction, we show how we can find a dense subset of **P**, which is an S-forcing with $S_\kappa^{\kappa^+}$ as set of indiscernibles. For the purposes of this paper, when proving results about arbitrary S-forcings, we shall therefore take them to have $S_\kappa^{\kappa^+}$ as set of indiscernibles; when dealing with particular partial orderings we shall take the set of indiscernibles to be the most natural one. As suggested at the outset of this discussion, choosing a set of indiscernibles, I, corresponds closely to restricting the collection of increasing sequences on which terms act (in a less trivial sense than merely restricting to sequences from I). We shall remark on this, at the end of this section.

For the remainder of this subsection, fix regular $\kappa > \omega$.

(1.1.1). Let $I \in [\kappa^+]^{\kappa^+}$. Let $\bar{I}$ = the closure of I in κ^+, so $\bar{I} = I \cup \{\alpha < \kappa^+ : \alpha$ is a limit ordinal $\wedge$ I is cofinal in $\alpha\}$. If $i \in \bar{I}$, let $\hat{i}$ = the immediate successor of i in $\bar{I}$ (so $\hat{i} \in I$).

Definition. I is an *acceptable set of indiscernibles* if:

(1) $S^{\kappa}_{\kappa^+} \subseteq I$,

(2) if $i, j \in \bar{I}$, then o.t. $[i, \hat{i}) =$ o.t. $[j, \hat{j}) = \inf I$,

(3) $I = S^{\kappa}_{\kappa^+}$ or I is $<\kappa$-closed.

(1.1.2) Let I be acceptable. Note that $i_0 \leqq \kappa$, since $\kappa \in S^{\kappa}_{\kappa^+} \subseteq I$, where $i_0 = \inf I$; i_0 is called the width of I.

Definition. If $a \in [I]^{<\kappa}$, we define the *envelope of* a, $\mathrm{ENV}(a)$. We let $b = \bar{a}$, if $I = S^{\kappa}_{\kappa^+}$; otherwise $b = a$. *Then*:

$$\mathrm{ENV}(a) = i_0 \cup \bigcup \{[i, \hat{i}) : i \in b\}.$$

Further, if $X \subseteq I$, we let $\mathrm{ENV}(X) = \bigcup \{\mathrm{ENV}(a) : a \in [X]^{<\kappa}\}$.

(1.1.3) Let I be acceptable, $a_1, a_2 \in [I]^{<\kappa}$, o.t.$(a_1)$ = o.t.(a_2), and let $s : a_1 \to a_2$ be the order isomorphism. We define $\bar{s} \supseteq s$ to be the unique order-isomorphism of $\mathrm{ENV}(a_1)$ and $\mathrm{ENV}(a_2)$. Thus:

(1) $\bar{s} \restriction i_0 = \mathrm{id} \restriction i_0$,

(2) for $i \in \bar{I} \cap \mathrm{ENV}(a_1)$, $\xi < i_0$, $\bar{s}(i) \in \bar{I} \cap \mathrm{ENV}(a_2)$ and $\bar{s}(i + \xi) = \bar{s}(i) + \xi$,

(3) if $i \in (\bar{I} \cap \mathrm{ENV}(a_1)) \setminus I$ (thus, $i \in \overline{(\mathrm{ENV}(a_1) \cap I)}$, and $I = S^{\kappa}_{\kappa^+}$, $b_j = \bar{a}_j$, $j = 1, 2$), then $\bar{s}(i) = \sup s''(a_1 \cap I \cap i)$.

(1.1.4) Definition. If I is acceptable, $\mathbf{P}$ is κ-I-special iff the definition of κ-special (viz. [20], (3.1)) holds for $\mathbf{P}$ *except that all increasing sequences are from* I: i.e. $P = \{\tau(s) : s \in [I]^{\lg \tau}\}$. We should note here that, as was implicit in [20], distinct (τ, s) give rise to distinct conditions.

$\mathbf{P} \in \bar{\mathcal{S}}_\kappa(I)$ iff $\mathbf{P}$ is κ-special, strongly κ-directed-closed, has the indiscernibility and amalgamation properties, and the I-extension property:

$$\text{for } i \in I \ \{p : i \in s^p\} \text{ is dense.}$$

If $\mathbf{P} \in \bar{\mathcal{S}}_\kappa(I)$, we say I is the set of indiscernibles for $\mathbf{P}$.

(1.1.5) Definition. If $\mathbf{P} \in \bar{\mathcal{S}}_\kappa(I)$, $\mathbf{P}$ is *supported* iff there's a function supp: $P \to [\kappa^+]^{<\kappa}$ such that for $p \in P$:

(1) $s^p \subseteq \mathrm{supp}(p) \in [\mathrm{ENV}(p)]^{<\kappa}$,

(2) $p \leqq q \Rightarrow \mathrm{supp}(p) \subseteq \mathrm{supp}(q)$,

(3) if $\tau^p = \tau^q$, $t : s^p \to s^q$ is the order isomorphism, then $\mathrm{supp}(q) = \tilde{t}''\mathrm{supp}(p)$,

(4) if $s = s^p$, $\alpha \in \mathrm{ENV}(s)$, then there's $q \geqq p$ with $s^q = s$ and $\alpha \in \mathrm{supp}(q)$,

(5) if $D \in [P]^{<\kappa}$ is directed then there's $q \in P$, an upper bound for D, with $\mathrm{supp}(q) = \bigcup\{\mathrm{supp}(r) : r \in D\}$.

We say supp is a support function for $\mathbf{P}$. $\mathbf{P} \in \mathcal{S}_\kappa(I)$ iff $\mathbf{P} \in \bar{\mathcal{S}}_\kappa(I)$ and $\mathbf{P}$ is supported. Formally, we should distinguish between different support functions, i.e. $\mathcal{S}_\kappa(I)$ should consist of pairs (partial order, support function), but we slur over this by always considering a $\mathbf{P} \in \mathcal{S}_\kappa(I)$ to come supplied with a support function.

Note that (5) implies strong-directed-κ-closure, and that (1) and (2) imply the last statement in the definition of κ-I-special. (3) is an indiscernibility property for supports.

(1.1.6) We now develop some notation for maps from I to I, and from ordinals to I, related to maps from ordinals to ordinals.

Definition. Suppose f, g are order-preserving maps with domain and range $\subseteq \mathrm{OR}$. We define a partial function $f^* : I \to I$, and a $\hat{g} : \mathrm{dom}\, g \to I$. We first let $(i_\alpha : \alpha < \kappa^+)$, $(\bar{i}_\alpha : \alpha < \kappa^+)$ be the increasing enumerations of I, $\bar{I}$ respectively; we then set:

$$\mathrm{dom}\, f^* = \{i_\alpha : \alpha \in \mathrm{dom}\, f\}, \qquad \text{and if } \alpha \in \mathrm{dom}\, f, \quad f^*(i_\alpha) = i_{f(\alpha)},$$

$$\mathrm{dom}\, \hat{g} = \mathrm{dom}\, g, \qquad \text{for } \alpha \in \mathrm{dom}\, g, \quad \hat{g}(\alpha) = i_{g(\alpha)}.$$

Note that $(f \circ g)^\wedge = f^* \circ \hat{g}$.

In what follows, but in §§4, 6, especially, we shall often abuse notation by using f, g or $f \circ g$ when we formally should be using f^*, $\hat{g}$ or $(f \circ g)^\wedge$. Thus, e.g., if $\tau \in \mathcal{T}$, $s \in [\kappa^+]^{\lg \tau}$, we shall often write $\tau(s)$ when we mean $\tau(\hat{s})$.

(1.1.7) We shall also need notation for maps from ordinals to ordinals which are defined "in chunks".

Definition. Suppose $f : \alpha \to \mathrm{OR}$, $g : \beta \to \mathrm{OR}$ are order preserving and for all $\gamma < \alpha$, $f(\gamma) < g(0)$. We let:

$$f^\frown g : \alpha + \beta \to \mathrm{OR}, \quad (f^\frown g)\big|\alpha = f; \qquad \text{for } \gamma < \beta,$$

$(f^\frown g)(\alpha+\gamma) = g(\gamma)$. In case $g(\gamma) = \xi + \gamma$, for some $\xi \geqq \sup$ range f, we denote $f^\frown g$ by $f^\frown \xi, \beta$.

(1.1.8) We now relate the notions of (1.1.4), (1.1.5) to those of [20]. Note that the $\mathcal{S}_\kappa$ of [20] is now denoted by $\bar{\mathcal{S}}_\kappa(\kappa^+)$.

PROPOSITION. *If I is acceptable and $\mathbf{P} \in \bar{\mathcal{S}}_\kappa(I)$, then $\mathbf{P} \in \bar{\mathcal{S}}_\kappa(\kappa^+)$; further, if $\mathbf{P} \in \bar{\mathcal{S}}_\kappa(\kappa^+)$, then $\mathbf{P}$ is supported with support function* $\mathrm{supp}(p) = s^p$.

PROOF (Sketch). This is the usual dodge, as in [20]. The set of terms for $\mathbf{P}$ as a member of $\bar{\mathcal{S}}_\kappa(\kappa^+)$ will be the same as $\mathcal{T}$, the set of terms for $\mathbf{P}$ as a member of $\bar{\mathcal{S}}_\kappa(I)$. Also, the length function is the same. Now if $\tau \in \mathcal{T}$, $s \in [\kappa^+]^{\lg \tau}$, then $\tau(s)$ is that $p \in P$ such that, regarding $\mathbf{P}$ as a member of $\bar{\mathcal{S}}_\kappa(I)$, $p = \tau(\hat{s})$. The last assertion is clear.

(1.1.9) In (1.1.8), if $\mathbf{P} \in \mathcal{S}_\kappa(I)$, then our way of regarding $\mathbf{P}$ as a member of $\bar{\mathcal{S}}_\kappa(\kappa^+)$, and hence as a member of $\mathcal{S}_\kappa(\kappa^+)$, need not preserve supports. The next Proposition will show that $\mathbf{P} \in \mathcal{S}_\kappa(I)$ always has a dense subset which can be regarded as a member of $\mathcal{S}_\kappa(S^\kappa_{\kappa^+})$, while preserving supports. This will be convenient in §6.

LEMMA. *If $\mathbf{P} \in \mathcal{S}_\kappa(I)$ with support function* supp *then there's dense $\mathbf{P}' \subseteq \mathbf{P}$ with $\mathbf{P}' \in \mathcal{S}_\kappa(S^\kappa_{\kappa^+})$, supported by* supp.

PROOF. $a \in [\kappa^+]^{<\kappa}$ is *nice* if $a \subseteq \mathrm{ENV}'(a \cap S^\kappa_{\kappa^+})$, where ENV$'$ is the envelope in the sense of $S^\kappa_{\kappa^+}$. Let $p \in P'$ iff $\mathrm{supp}(p)$ is nice.

If $p \in P'$, let $x^p = \{\alpha < \lg \tau^p : s^p(\alpha) \in S^\kappa_{\kappa^+}\}$. Let $\theta^p = \text{o.t. } x^p$. Define g^p with domain $\lg \tau^p$ as follows:

(i) if $\alpha \in x^p$, $g^p(\alpha) = \{\xi < \kappa : s^p(\alpha) + \xi \in \text{range } s^p\}$;

(ii) if $\alpha \notin x^p$ but α is a limit point of x^p,

$$g^p(\alpha) = \{\xi < \kappa : \sup(s^{p\,\prime\prime}\alpha) + \xi \in \text{range } s^p\};$$

(iii) if neither (i) nor (ii) applies, $g^p(\alpha) = (\emptyset, \emptyset)$.

Then, let $(\tau')^p = (\tau^p, x^p, g^p)$, and let $\lg'((\tau')^p) = \theta^p$. Define $(s')^p$ as follows. If $\alpha < \theta^p$, let $\beta < \lg p$ be such that the αth member of $x^p = \beta$. Then, we define $(s')^p(\alpha) = s^p(\beta)$. This implicitly defines $\mathcal{T}'$, $\lg'$ and the action of $\mathcal{T}'$ terms on increasing sequences of the appropriate length from $S^\kappa_{\kappa^+}$. Clearly we can reconstruct $p \in P'$ from $(\tau')^p$ and $(s')^p$. As we must, for $p \in P'$, we define $\mathrm{supp}'(p) = \mathrm{supp}(p)$. Since $\mathbf{P}' \subseteq \mathbf{P}$, we clearly have the indiscernibility property, and by construction $\mathbf{P}'$ is κ-$S^{\kappa^+}_{\kappa}$-special. Since $\mathbf{P}'$ is dense in $\mathbf{P}$, $\mathbf{P}'$ has the $S^{\kappa^+}_{\kappa}$-extension property and the amalgamation property (the latter, since $(\tau')^p = (\tau')^q \Rightarrow \tau^p = \tau^q$ and since $(s')^p$, $(s')^q$ have the strong Δ property $\Rightarrow s^p$, s^q do too). Since $\mathbf{P}'$ is dense in $\mathbf{P}$, $\mathbf{P}'$ is κ-directed-closed; for strong κ-directed closure, we merely observe that this is true of $\mathbf{P}$, and that $S^{\kappa^+}_{\kappa} \subseteq I$. In any case, this is superseded by the observation that, by construction, $\mathbf{P}'$ is supported by supp.

(1.2) We now develop some notions which figure in our formulation of morasses with built-in $\diamond$.

(1.2.1) Definition. Let $\mathrm{HF}(X)$ = the hereditarily finite sets over X = the closure of X under the formation of finite subsets; i.e.:

$$\mathrm{HF}(X) = \bigcup_n X_n, \qquad \text{where } X_0 = X \quad \text{and} \quad X_{n+1} = X_n \cup [X_n]^{<\omega}.$$

Note that if $a \subseteq \mathrm{OR}$, then $\mathrm{HF}(a) = \mathrm{HF}(a \setminus \omega)$ and $(\forall z \in (a \setminus \omega))z$ is infinite. This is important in the following context. Suppose $f : X \to Y$. We wish to extend f canonically to $f^{\#} : \mathrm{HF}(X) \to \mathrm{HF}(Y)$. We can easily do this if $(\forall z \in X \cup Y)z$ is infinite, since then we let $f^{\#} = \bigcup_n f^{\#}_n$, where $f^{\#}_n : X_n \to Y_n$ is defined recursively: $f^{\#}_0 = f$; $f^{\#}_{n+1}(x) = (f^{\#}_n)''x$, for $x \in X_{n+1} \setminus X_n$. Thus, we shall write $\mathrm{HF}(a)$, etc., but we really mean $\mathrm{HF}(a \setminus \omega)$.

(1.2.2) Definition. Now suppose $f : X \to Y$, f is 1-1 and $(\forall z \in X \cup Y)$ z is infinite, and suppose $h : \mathrm{HF}(X) \to 2$. We define $h' : \mathrm{HF}(Y) \to 2$ by:

$$h'(y) = 0 \quad \text{if } y \notin \text{range } f^{\#}; \quad \text{otherwise, if } y = f^{\#}(x)$$

$$\text{(our hypotheses guarantee that } f^{\#} \text{ is 1-1)}, \quad h'(y) = h(x).$$

We abuse notation by denoting h' by $f[h]$.

(1.3) We point out, briefly, that we can, by judicious choice of our set of indiscernibles, achieve the effect of taking $P = \{\tau(s) \colon s \in [\kappa^+]^{\lg \tau} \in \mathcal{F}\}$, for certain classes $\mathcal{F}$ of order-preserving maps from ordinals less than κ to κ^+. This will affect §10, below.

Suppose, e.g., that $s \in \mathcal{F}$ iff, in the terminology of [20], s is nice, i.e. $s(0) = 0$, if λ is a limit ordinal, $\lambda \in \operatorname{dom} s$, then $s(\lambda)$ is a limit ordinal and if $\alpha + 1 \in \operatorname{dom} s$, then $s(\alpha + 1) = s(\alpha) + 1$. This can be achieved by taking the set of indiscernibles to be included in the set of non-zero limit ordinals less than κ^+, and taking as the support of P what was s^p when we viewed the set of indiscernibles to be κ^+. Similarly, if in addition we require that membership in $\mathcal{F}$ imply that $\operatorname{cf} s(\alpha) = \kappa$, then we can achieve this by taking $I = S^{\kappa^+}_{\kappa}$.

§2. Stationary subsets of $[\omega_2]^{<\omega_1}$ with sup as coding function

The notions of this section are due to Zwicker. Many people must have noticed that non-thick stationary subsets of $[\omega_2]^{<\omega_1}$ can be added by countably closed, $\aleph_2$-c.c. forcing. Baumgartner mentioned this to Stanley and suggested it as a potential "black-box" application for morasses with built-in $\diamond$. We should note that Velleman [26] has found a direct construction of a stationary subset of $[\omega_2]^{<\omega_1}$ with sup as coding function from a (simplified) stationary $(\omega_1, 1)$-morass. Finally, Shelah [19] has proved theorems which show that, though the material of this section obviously generalizes to $[\kappa^+]^{<\kappa}$ for regular $\kappa > \omega$, in fact $\kappa = \omega_1$ is practically the only interesting case as far as morass applications go: see below, (2.16).

(2.1) DEFINITION. If $X \subseteq [\omega_2]^{<\omega_1}$, $h : X \to \omega_2$ is a *coding function* for X if h is 1-1 and

(*): $$x, y \in X \Rightarrow (x \subsetneqq y \Leftrightarrow h(x) \in y).$$

(2.2) DEFINITION. $X \subseteq [\omega_2]^{<\omega_1}$ is *thick* iff whenever $C \subseteq [\omega_2]^{<\omega_1}$ is club, there are $x, y \in C \cap X$ such that $x \subsetneqq y$ but $x \cap \omega_1 = y \cap \omega_1$.

(2.3) Zwicker, [29], showed that if there is a stationary $X \subseteq [\omega_2]^{<\omega_1}$ with a coding function, then there's $Y \subseteq [\omega_2]^{<\omega_1}$ which is stationary but not thick (clearly thick sets are stationary). The existence of stationary sets with coding functions or of non-thick stationary sets make the theory of club and stationary on $[\omega_2]^{<\omega_1}$ look appealingly like the theory of club and stationary on ω_2. This was what led Zwicker to formulate these notions, see [29].

(2.4) We now introduce an S-forcing, $\mathbf{P}$, which adds a stationary $X \subseteq [\omega_2]^{<\omega_1}$ with sup as coding function.

DEFINITION. $p \in \tilde{P}$ iff $p \in [[\omega_2]^{<\omega_1}]^{<\omega_1}$ and

(*): $$x, y \in [p]^2 \Rightarrow (\sup x \neq \sup y \wedge (x \subsetneqq y \Leftrightarrow \sup x \in y)).$$

For $p, q \in \tilde{P}$, set $p \leqq q \Leftrightarrow p \subseteq q$; $\tilde{\mathbf{P}} = (\tilde{P}, \geqq)$.

(2.5) As usual, we must impose additional, but essentially technical, restrictions on $\tilde{\mathbf{P}}$ to obtain a $P \in \mathcal{S}_{\aleph_1}(I)$.

DEFINITION. Let $I = S^{\aleph_1}_{\aleph_2} = \{\alpha \in \mathrm{Lim} \cap \omega_2 : \mathrm{cf}\, \alpha = \omega_1\}$, let

$$\bar{I} = \{\alpha \in \mathrm{Lim} \cap \omega_2 : \alpha \text{ is an ordinal multiple of } \omega_1\}$$

(so $\bar{I}$ is the closure of $I \cup \{0\}$). For $\alpha < \omega_2$, let $\kappa(\alpha) = \sup(\bar{I} \cap \alpha)$.

Let $x \in [\omega_2]^{<\omega_1}$. x is *nice* iff

(1) $(\forall \alpha < \omega_2)\ (\alpha \in x \Rightarrow \kappa(\alpha) \in x)$,

(2) $(\forall \beta \in \bar{I})\ (\beta = \sup(x \cap \beta) \Rightarrow \beta = \sup(x \cap I \cap \beta))$.

(3) $\forall \beta \in (\bar{I} \setminus I) \cap X)\ (X \cap \beta$ is cofinal in $\beta)$.

(2.6) Definition. Let $p \in P$ iff $p \in \tilde{P}$ and

(**): $$(\forall x \in p) x \text{ is nice},$$

(***): $$x \in p \wedge x \neq \bigcup p \Rightarrow (\exists y \in p) x \subsetneqq y.$$

Let $\mathbf{P} = (P, \geqq)$.

(2.7) Remark. $\{x \in [\omega_2]^{<\omega_1} : x \text{ is nice}\}$ is club.

(2.8) We now start to show that $\mathbf{P} \in \mathcal{S}_{\aleph_1}(I)$.

Lemma. $\mathbf{P}$ *is* $\aleph_1$-I-*special and indiscernible.*

Proof. Let $p \in \mathcal{T} \Leftrightarrow p \in P$ and $(\bigcup p) \cap I$ is an initial segment of I. For $p \in \mathcal{T}$, let $\lg p = \text{o.t.}((\bigcup p) \cap I)$.

If $s : \theta \to \omega_2$ is increasing and $\theta < \omega_1$, let $I(s) = \{\alpha_{s(i)} : i < \theta\}$, where $(\alpha_j : j < \omega_2)$ increasingly enumerates I; thus, $I(s) = \text{range } \hat{s}$, or continuing to abuse notation, $I(s) = \hat{s}$.

Now let $p \in \mathcal{T}$, $s : \lg p \to I$ increasing. Define $p(s)$ by setting $p(s) = \{\bar{s}''x : x \in p\}$. Since $\bar{s}$ is normal, clearly $(*)$ of (2.4) holds, as do (1), (2) of (2.5), so $p(s) \in \tilde{P}$ and all $x \in p(s)$ are nice. Finally (***) of (2.6) is clear, i.e. $p(s) \in P$. Conversely, if $p \in P$, let $\theta = \text{o.t.}(I \cap (\bigcup p))$, and define $s : \theta \to I$ increasing by $s(i) =$ the ith element of $I \cap (\bigcup p)$. Let $\bar{p} = \{\bar{s}^{-1}[x] : x \in p\}$. Then $p = \bar{p}(s)$ and $\bar{p} \in \mathcal{T}$. Clearly $\mathbf{P}$ is indiscernible.

(2.9) Remark. Suppose $p \in P \wedge \bigcup p \notin p$. Then $(\forall x \in p) \sup x < \sup \bigcup p$.

Proof. By (***) of (2.6), if $\bigcup p \notin p$ then no $x \in p$ is maximal for $\subseteq$ among elements of p. But then, by $(*)$ of (2.4), $(\forall x \in p)(\exists y \in p) \sup x \in y$. So, for such y, $\sup x < \sup y \leqq \sup \bigcup p$.

(2.10) The following will be useful in many contexts.

Proposition. (a) *Suppose* $p \in \tilde{P}$ *and* $(\forall x \in p)(x$ *is nice*). *Then there's* $q \in P$, $q \geqq p$ *s.t.* $\bigcup p \cap I = \bigcup q \cap I$.

(b) *Suppose* $p \in P$, $a = \bigcup p$ *and* $a \notin p$. *Then whenever* $a \subseteq b$, $b \in [\omega_2]^{<\omega_1}$, *and* b *is nice*, $p \cup \{b\} \in P$.

PROOF. For (a), note that $p \in P$ unless $\bigcup p \notin p$ and $r \neq \emptyset$, where $r = \{x \in p : x$ is $\subseteq$-maximal among elements of $p\}$. Note that $x \in r \Rightarrow \sup x \notin \bigcup p$ since $\sup x \notin x$ (recall that we're taking sup in the strict sense, i.e. $\sup x =$ the least α s.t. $x \subseteq \alpha$), but if $y \in p \setminus \{x\}$, then $\sup x \in y \Rightarrow x \subsetneq y$, by (2.4)(*), but $x \in r$.

Then, if $p \notin P$ we obtain the desired q as follows.

We let $q = p \cup \{\bigcup p \cup \{\sup x : x \in r\}\}$. Then, q works and $\bigcup q \setminus \bigcup p = \{\sup x : x \in r\}$, which is disjoint from I. Note that $\bigcup p \cup \{\sup x : x \in r\}$ is nice.

(b) is clear by (2.9) and the related fact that $x \in p \Rightarrow \sup x \in \bigcup p$. Note that (b) holds even if $a \in P$, provided we take $\sup a \in b$. Further, note that, if so desired, we can take $b \cap I \subseteq I(s)$, where $\bar{p} = p(s)$, $\bar{p} \in \mathcal{T}$.

(2.11) LEMMA. **P** *has the amalgamation property.*

PROOF. Let $\bar{p} \in \mathcal{T}$, $s^1, s^2 : \lg p \to \omega_2$ order preserving and $\eta < \lg p$ be such that $s^1 \restriction \eta = s^2 \restriction \eta$, range $s^1 \subseteq s^2(\eta)$. Let $p^i = \bar{p}(s^i)$, $i = 1, 2$. Let $p = p^1 \cup p^2$. By (2.10)(a) it will suffice to show that $p \in \tilde{P}$, since clearly $(\forall x \in p)$ x is nice. But this is clear, since to verify (2.4)(*), it suffices to consider the case when $x \in p^1 \setminus p^2$, $y \in p^2 \setminus p^1$. But then $x \not\subseteq I(s^1 \restriction \eta)$, $y \not\subseteq I(s^2 \restriction \eta)$, i.e. $\sup x > s^1(\eta)$, $\sup y > s^2(\eta)$. Then clearly $\sup x \neq \sup y$ and $x \setminus y$, $y \setminus x \neq \emptyset$.

(2.12) LEMMA. **P** *has the I-extension property.*

PROOF. Clear.

(2.13) LEMMA. **P** *is countably closed with union as* lub.

PROOF. Clear.

(2.14) LEMMA. $\mathbf{P} \in \mathcal{S}_{\aleph_1}(I)$, *with support function* $\mathrm{supp}(p) = \bigcup p$.

PROOF. By (2.8), (2.10)–(2.13). Note that the remarks at the end of the proof of (2.10) give (4) of (1.1.5), since given nice a and $\alpha \in \mathrm{ENV}(a \cap I)$, we can easily find nice b with $a \cup \{\alpha, \sup a\} \subseteq b \subseteq \mathrm{ENV}(a \cap I)$.

(2.15) Prior to showing that applying $S_{\aleph_1}(\diamondsuit)$ (a strengthened version of $S_{\aleph_1}$ of [20], introduced in §4, below) yields a stationary subset of $[\omega_2]^{<\omega_1}$ with sup as coding function, and in order to present the main ideas of the proof in a simpler setting, we show:

LEMMA. $\Vdash_{\mathbf{P}}$ "$\bigcup \mathring{G}$ *is a stationary subset of* $[\omega_2]^{<\omega_1}$ *with* sup *as coding function*".

PROOF. In fact, as will be clear from the proof, the Lemma holds with $\bar{\mathbf{P}}$ replacing $\mathbf{P}$.

Of course it suffices to show that $\Vdash_{\mathbf{P}}$ "$\bigcup \mathring{G}$ is stationary in $[\omega_2]^{<\omega_1}$. Note that, in $V^{\mathbf{P}}$, $[\omega_2]^{<\omega_1} = ([\omega_2]^{<\omega_1})^V$.

So, suppose $p \in P$ and $p \Vdash$ "$\mathring{C}$ is club in $[\omega_2]^{<\omega_1}$". Let $q \geqq p$. We construct an increasing ω-sequence $(q_n : n < \omega)$ from P as follows: $q_0 = q$. Having defined q_{2n}, we let $q_{2n+1} \geqq q_{2n}$ be s.t. for some $b_{2n} \supseteq \bigcup q_{2n}$, $q_{2n+1} \Vdash$ "$b_{2n} \in \mathring{C}$", and we let $q_{2n+2} \geqq q_{2n+1}$ be s.t. $\bigcup q_{2n+2} \in q_{2n+2}$, $\bigcup q_{2n+2} \supseteq b_{2n}$ (possible by (2.10)(b)). Finally, we let $q_\omega = \bigcup_{n<\omega} q_n$, and we let $q^* = q_\omega \cup \{\bigcup q_\omega\}$; $q^* \in P$, by (2.10)(b). Also $q^* \Vdash$ "$\bigcup q_\omega \in \mathring{C}$" since for each n, $q^* \geqq q_n$, since $q \Vdash$ "$\mathring{C}$ is club", and since $\bigcup q_\omega = \bigcup_n b_{2n}$. Thus $q^* \Vdash$ "$(\bigcup q_\omega) \in \mathring{G} \cap \mathring{C}$".

Let $D = \{\bar{p} \in \mathcal{T} : \bigcup \bar{p} \in \bar{p}\}$. By (2.10)(b), D is dense in $\mathcal{T}$. Let $D^\#$ be the uniform blow up of D, i.e.

$$D^\# = \{\bar{p}(s) : \bar{p} \in D,\ s : \lg \bar{p} \to \omega_2 \text{ is order-preserving}\}.$$

Now let $C \subseteq [\omega_2]^{<\omega_1}$ be club.

Let $D^\# \mid C = \{\bar{p}(s) \in D^\# : \bigcup \bar{p}(s) \in C\}$. Clearly if G is an ideal in $\mathbf{P}$ and

(1) $$\text{for all club } C \subseteq [\omega_2]^{<\omega_1}, \quad G \cap (D^\# \mid C) \neq \emptyset,$$

then:

(2) $$\bigcup G \text{ is a stationary subset of } [\omega_2]^{<\omega_1} \text{ with sup as coding function.}$$

Therefore, we have:

THEOREM. *$\exists(\omega_1, 1)$-morass with built-in $\diamondsuit \Rightarrow \exists$ stationary $S \subseteq [\omega_2]^{<\omega_1}$ with* sup *as coding function.*

PROOF. We shall prove below that:

(3) $$\exists(\omega_1, 1)\text{-morass with built-in } \diamondsuit \Leftrightarrow S_{\aleph_1}(\diamondsuit),$$

see §§6, 8 below, and:

(4) $$S_{\aleph_1}(\diamondsuit) \Rightarrow \exists \text{ ideal } G \subseteq \mathbf{P} \text{ s.t. (1) holds.}$$

We shall prove (4) in (4.15), below.

(2.16) REMARKS. The material of this section (and of (4.15) below) generalizes from ω_1 to arbitrary regular uncountable κ. However, Shelah has proved, in [19] (among other theorems about stationary sets with coding functions):

THEOREM. *If $\kappa > \aleph_2$ and κ is a successor cardinal or κ is a strongly inaccessible* *and greatly (weakly) Mahlo then there is a stationary subset of* $[\kappa^+]^{<\kappa}$ *with a coding function.*

Actually, much sharper versions of this are proved in [19], including in "higher-gap" situations. It is not clear, for the moment, whether the results of [19] can be improved to have the coding function be sup.

Thus, the morass proofs are of interest for $\kappa = \aleph_1, \aleph_2$ or κ regular (but not strongly inaccessible or not greatly (weakly) Mahlo), or else in order to have sup as the coding function. As we pointed out, above, Velleman has subsequently obtained the same theorems from something weaker than a morass with built-in $\diamond$: a stationary simplified morass, see [26].

§3. Weak $\diamond$ and closed super-Souslin trees

In this section we first prove Theorem 6 of §0, and we then show that $\exists\ (\kappa^+, 1)$ morass $\wedge\ (\kappa^+)^\kappa = \kappa^+ \Rightarrow \exists\ \kappa^+$-closed κ^{++}-super-Souslin tree. Both of these results were obtained independently by Velleman. Velleman first obtained the latter result, "by hand"; Stanley then formulated the weak built-in $\diamond$ principle, proved Theorem 6 and factored Velleman's construction through Theorem 6. Later, Velleman found a more combinatorial proof of Theorem 6. We have chosen to give Stanley's original proof because of the technique involved: thinning a morass by requiring good behavior with respect to a predicate. This theme resurfaces in §8, where the "generic" object, a morass-like structure, is thinned. For the details of the thinning, the reader should consult [22], [23].

We take this opportunity to point out that an error in an early version of [23] has apparently survived in the final version, despite corrections submitted to the editors by the author. This error doesn't affect any of the results of [23], but only the closing remarks, where the claim appears that the thinning can be carried out with respect to any additional predicate $A \subseteq \omega_2$. This is true, and the method of proof outlined works *if* $A \subseteq \omega_1$. However, the claim as it appears in [23] is false. This can be seen as follows. Start from any model of $\mathrm{CH} + 2^{\aleph_1} = \aleph_2 + \exists(\omega_1, 1)$-morass (e.g. L). Force to kill all $\aleph_1$-Souslin trees without adding reals and while preserving cardinals and $2^{\aleph_1} = \aleph_2$ (e.g., Jensen–Johnsbraten forcing, viz. [5], or Shelah's proper forcing approach, viz. [17]). An $(\omega_1, 1)$ morass of the ground model remains one in the extension. Further, by $\mathrm{CH} + 2^{\aleph_1} = \aleph_2$ in the extension let $A \subseteq \omega_2$ be s.t. $L_{\omega_1}[A \cap \omega_1] = H_{\omega_1}$, $L_{\omega_2}[A] = H_{\omega_2}$. If the claim were true as it appears, the morass could be thinned with respect to this A, and the resulting morass would be universal. This, however, would imply that $\diamond$ holds in the extension, contradiction.

(3.1) We now present a result due to Stanley who used it to reprove, in (3.2), a theorem first noticed by Velleman. Stanley's result shows that, in a manner of speaking, CH can be "integrated into a morass" to provide a (very) weak form of built-in $\diamondsuit$-principle. Velleman's result then carries this over to the forcing principle setting, providing a stronger form of $S_{\aleph_1}$, of [20].

In the statement of the lemma and the proof-sketch, S_{ω_1}, S_α refer to morass motions, not to forcing principles. For these, and other morass notions, see [23], §3. The proof of the lemma leans heavily on [23], (5.2); the notation there is slightly different from that of [20]; in [23], $f_{\nu\tau}$ is used in place of $\pi_{\nu\tau}$.

THEOREM 5 (left-to-right). CH + *there is an* $(\aleph_1, 1)$-*morass* $\Rightarrow$ *there is an* $(\aleph_1, 1)$-*morass* $\mathcal{M} = (\mathcal{S}, S^0, S^1, \mapsto\!\!\!\!\prec, \pi_{\nu\tau})_{\nu \mapsto\!\!\prec \tau}$ *and a sequence* $(a_\alpha : \alpha \in S^0)$ *such that setting* $\mu_\alpha = \sup S_\alpha$, a_α *is a bounded subset of* μ_α, *and if* $a \in [\omega_2]^{\leq \aleph_0}$

(**) *there is* $\nu \in S_{\omega_1}$, $\bar{\nu} \mapsto\!\!\!\!\prec \nu$, *such that, setting* $\alpha = \alpha_{\bar{\nu}}$, $a = \pi''_{\bar{\nu}\nu} a_\alpha$.

PROOF. By CH, let $A \subseteq \omega_1$ be such that $\mathcal{P}(\omega) \subseteq L^A$. Let $\bar{\mathcal{M}} = (\bar{\mathcal{S}}, \bar{S}^0, \bar{S}^1, \overline{\mapsto\!\!\!\!\prec}, \bar{\pi}_{\nu\tau})_{\nu \mapsto\!\!\prec \tau}$ be an $(\aleph_1, 1)$-morass. Let $D \subseteq {}^{\wp}\omega_2$ be the predicate coding up the morass, and for $\nu \in \bar{S}^1$, let $D_\nu = D \cap {}^{\wp}\nu$ be the predicate coding up the morass internal to ν, as in [23], (5.2). For $\nu \in \bar{S}^1$, let

$$\mathfrak{A}_\nu = (L_\nu[A \cap \alpha_\nu, D_\nu], \in, A \cap \alpha_\nu, D_\nu).$$

Let $\nu \in S^1$ iff ν is a limit of $(A \cap \alpha_\nu, D_\nu)$-admissibles and $\mathfrak{A}_\nu \models$ "$\alpha_\nu = \aleph_1$". Set $\nu \mapsto\!\!\!\!\prec \tau$ iff $\nu \overline{\mapsto\!\!\!\!\prec} \tau$ and there is $\overset{+}{\pi}_{\nu\tau} \supseteq \pi_{\nu\tau}$, $\overset{+}{\pi}_{\nu\tau} : \mathfrak{A}_\nu \to_Q \mathfrak{A}_\tau$. It is essentially proved in [23], (5.2) that this "thinning" of $\bar{\mathcal{M}}$ results in a new morass

$$\mathcal{M} = (\mathcal{S}, S^0, S^1, \mapsto\!\!\!\!\prec, \bar{\pi}_{\nu\tau}) \qquad \text{where } \mathcal{S} = \{(\alpha_\nu, \nu) : \nu \in S^1\}, \quad S^0 = \{\alpha_\nu : \nu \in S^1\}.$$

In [23], (5.2), the starting structure $\bar{\mathcal{M}}$ wasn't a morass, but a "premorass", and no predicate like A was present, but these differences are totally inessential.

Now we define $(a_\alpha : \alpha \in S^0)$ using the thinnned morass $\mathcal{M}$, the models $\mathfrak{A}_\nu$ and the maps $\overset{+}{\pi}_{\nu\tau}$, by induction on α. Let $\mu_\alpha = \sup S_\alpha$.

$$a_\alpha = \begin{cases} \text{the } <_{L_{\mu_\alpha}[A \cap \alpha, D_{\mu_\alpha}]}\text{-least } a \text{ such that: (!): for all } \nu \in S_\alpha, \\ \text{for all } \bar{\nu} \mapsto\!\!\!\!\prec \nu,\ a \neq \overset{+}{\pi}_{\bar{\nu}\nu}(a_{\alpha_{\bar{\nu}}}), \text{ if there is such an } a; \\ \varnothing, \text{ if not.} \end{cases}$$

Now suppose there is $a \in [\omega_2]^{\leq \aleph_0}$ such that (**) fails. Let a be the $<_{L[A,D]}$-least such. Choose $\nu \in S_{\omega_1}$ such that a is bounded in ν, $a \in L_\nu[A, D_\nu]$ and $L_\nu[A, D_\nu] \models$ "a is the $<_{L[A,D_\nu]}$-least such that (**) fails". Then $L_\nu[A, D_\nu] \models$ "a is the $<_{L[A,D_\nu]}$-least such that (!) is true", letting $\alpha = \omega_1$ in (!).

Let $\bar{\nu}$ be the least $\mapsto$-predecessor of ν such that $a \in \text{range}\, \pi^+_{\bar{\nu}\nu}$, say $\alpha = \alpha_{\bar{\nu}}$ and let $\bar{a}$ be such that $\pi^+_{\bar{\nu}\nu}(\bar{a}) = a$. Then $\bar{\nu} = \mu_{\bar{a}}$ and $L_{\bar{\nu}}[A \cap \alpha, D_{\bar{\nu}}] \models$ "$\bar{a}$ is the $<_{L[A\cap\alpha, D_{\bar{\nu}}]}$-least such that (!) is true", i.e., $\bar{a} = a_\alpha$, contradicting our choice of a.

(3.2) We now give Velleman's theorem characterizing CH + there is an $(\aleph_1, 1)$-morass in terms of forcing principles. Let

$$P^* = \{\tau(s) \in P : \tau \in \mathcal{T},\ s : \lg \tau \to \omega_1 \text{ is order preserving}\}.$$

A collection $(X_a : a \in [\omega_2]^{\leqq \aleph_0})$ of subsets of P is homogeneous if

(a) $a \in [\omega_1]^{\leqq \aleph_0} \Rightarrow X_a$ contains a subset of P^* dense in P^*,

(b) if $\alpha \leqq \beta < \omega_1$, if $s : \beta \to \omega_2$ is order-preserving, if $\bar{p} = \tau(\bar{s}) \in X_\alpha$ where range $\bar{s} \subseteq \beta$, *then* $p = \tau(s \circ \bar{s}) \in X_a$, whre $a = s''\alpha$.

Corollary. CH + *there is an* $(\aleph_1, 1)$-*morass* $\Rightarrow$ *whenever* $\mathbf{P} \in \mathcal{S}_{\aleph_1}$, *whenever* $(E_\alpha : \alpha < \omega_1)$ *is a collection of uniform dense sets and* $(X_a : a \in [\omega_2]^{\aleph_0})$ *is homogeneous, there is an ideal* G *in* $\mathbf{P}$ *meeting all* X_a *and meeting all* E_α *uniformly.*

Proof (sketch). Let $(a_\alpha : \alpha < \omega_1)$ be as in (3.1), and let $\bar{D}_\alpha = \{\tau \in \mathcal{T} : \tau$ belongs to the open dense subset of P^* generated by $X_{a_\alpha} \cap P^*\}$. By the strong directed closure of $\mathbf{P}$, $\bar{D}_\alpha$ is dense in $\mathcal{T}$.

Let G be the ideal in $\mathbf{P}$ constructed as in [20], §6 (see also below, §6) for this choice of $\bar{D}_\alpha$, E_α. By the property of $(a_\alpha : \alpha < \omega_1)$ it is easy to verify that G meets all the X_a.

For the converse, notice that it suffices to show that the principle implies CH, since, being a strengthening of $S_{\aleph_1}$, it clearly implies the existence of an $(\aleph_1, 1)$-morass. For this, let $p \in P \Leftrightarrow p : \text{dom}\, p \to \mathcal{P}(\omega)$, $\text{dom}\, p \in \omega_1$. Set $p \leqq q \Leftrightarrow q \supseteq p$. For $p \in P$, let $\lg p = \text{dom}\, p$. Set $\mathcal{T} = P$, and for $p \in P$, $s : \lg p \to \omega_2$ increasing, let $p(s) = p$. Clearly $\mathbf{P} \in \mathcal{S}_{\aleph_1}$. For $a \in [\omega_2]^{\leqq \aleph_0}$, set

$$X_a = \{p \in P : a \cap \omega \in \text{range}\, p\}.$$

Clearly $(X_a : a \in [\omega_2]^{\leqq \aleph_0})$ is homogeneous, and if $G \subseteq P$ is an ideal meeting all the X_a, then $\mathcal{P}(\omega) \subseteq \bigcup \{\text{range}\, p : p \in G\}$.

(3.3) We now show how the principle of (3.2) yields countably closed super-Souslin trees. Let $\mathbf{P}$ be the super-Souslin tree partial ordering, viz. §2 of [20] and of [25] (see also §10, below).

To do this, we define a homogeneous system $(X_a : a \in [\omega_2]^{\leqq \aleph_0})$ of subsets of $\mathbf{P}$. So, set:

$$p \in X_a \Leftrightarrow a \subseteq t^p \wedge (a \text{ is a} \leqq p\text{-chain} \Rightarrow a \text{ has an upper bound in } (t^p, \leqq^p)),$$

where $p = (x^p, t^p, f^p)$ and $t^p = (t^p, \leqq^p)$.

Then clearly $(X_a : a \in [\omega_2]^{\leqq \aleph_0})$ is homogeneous, and if G is an ideal meeting all the X_a and all of the uniform dense sets as in §2 of [20], then $\bigcup \{t^p : p \in G\}$ is a countably closed $\aleph_2$ super-Souslin tree with $\bigcup \{f^p : p \in G\}$ as witness.

§4. The principles

In this section we present the principle $S_\kappa(\diamond)$, which is a more substantial strengthening of S_κ of [20] than the principle of §3. We also present the notion of $(\kappa, 1)$-morass with built-in $\diamond$. In §§6, 8, we show that the former is equivalent to the existence of the latter. We close this section by completing the proof of Theorem (2.15) in §2. Fix regular $\kappa > \omega$.

(4.1) Definition. If $A \subseteq [\kappa^+]^{<\kappa}$, define:

$$\mathscr{H}_A = \{h : (\exists a \in A) h : \mathrm{HF}(a) \rightarrow 2\}.$$

If $A = [\kappa^+]^{<\kappa}$, we don't mention A; i.e.

$$\mathscr{H} = \{h : (\exists a \in [\kappa^+]^{<\kappa}) h : \mathrm{HF}(a) \rightarrow 2\}.$$

If $h \in \mathscr{H}$, let $a(h) = \mathrm{OR} \cap \mathrm{dom}\, h$; thus, $\mathrm{dom}\, h = \mathrm{HF}(a(h))$.

(4.2) Let $\mathbf{P} \in \mathscr{S}_\kappa(I)$ with support function, supp. Let $b \in [I]^{<\kappa}$, and let $f : b \rightarrow I$ be order-preserving. If $X \subseteq P$ and $(\forall p \in X)(b \subseteq \text{range } s^p)$, define:

$$f[X] = \{\tau^p(f' \circ s^p) : p \in X, f' \supseteq f, f' : \text{range } s^p \rightarrow I \text{ is order preserving}\}.$$

In the spirit of (1.1.6), let $(i_\alpha : \alpha < \kappa^+)$ be the increasing enumeration of I; if $b' = \{\alpha : i_\alpha \in b\}$, and if $f : b' \rightarrow \kappa^+$ is order-preserving, then, by abuse of notation, we let

$$f[X] = f^*[X].$$

(4.3) Definition. Let $O = \{a \in [\kappa^+]^{<\kappa} : a \subseteq \mathrm{ENV}(a \cap I)\}$. Let $A = \{a \in O : a \cap I \text{ is an initial segment of } I\}$. If $a \in O$, define σ_a, $a^\$$ by:

$$\sigma_a = \text{the unique order-isomorphism from an initial segment of } I \text{ to } a \cap I,$$

$$a^\$ = \sigma_a^{-1}[a] = \text{the unique } a' \in A \text{ s.t. } \tilde{\sigma}_a'' a' = a.$$

If $h \in \mathscr{H}$ and $a = a(h) \in O$, let $h^\$ = (\tilde{\sigma}_a)^{-1}[h] = $ the unique $h' \in \mathscr{H}_A$ s.t. $h = \tilde{\sigma}_a[h']$.

Again, as in (1.1.6), if $\theta = \text{o.t.}(a \cap I)$, we let $s : \theta \to \kappa^+$ be order-preserving s.t. for $\xi < \theta$, $s_\xi =$ the ξth member of $a \cap I$; i.e. $\sigma_a = s^*$.

(4.4) DEFINITION. Let $(i_\alpha : \alpha < \kappa^+)$ increasingly enumerate I, and let $I \restriction \kappa = \{i_\alpha : \alpha < \kappa\}$. More generally, if $Y \subseteq \kappa^+$, $I \restriction Y = \{i_\alpha : \alpha \in Y\}$. We say:

$$\mathcal{D} = (D_h : h \in \mathcal{H}_A) \text{ is an } \textit{extension system} \text{ iff for all } h \in \mathcal{H}_A,$$

$$(p \in D_h \Rightarrow a(h) \subseteq \text{supp}(p) \subseteq \text{ENV}(I \restriction \kappa)).$$

(4.5) DEFINITION. If $\mathcal{D} = (D_h : h \in \mathcal{H}_A)$ is an extension system, we define $\tilde{\mathcal{D}} = (D_h : h \in \mathcal{H}_O)$, the blow-up of $\mathcal{D}$, by:

$$D_h = \tilde{\sigma}_{a(h)}[D_{h^s}].$$

If $H : \text{HF}(\kappa^+) \to 2$ and $X \subseteq [\kappa]^{<\kappa}$, then we set:

$$\mathcal{D}(H, X) = \bigcup \{D_h : (\exists a \in X) h = H \restriction \text{HF}(a)\}.$$

H is $\mathcal{D}$-tractable iff:

whenever $f : S \to P$, $S \subseteq [\kappa^+]^{<\kappa}$ is stationary and for all $a \in S$, $\text{supp}(f(a)) = a$, there's $a \in O \cap S$ such that, setting $h = H \restriction \text{HF}(a)$, $(\exists p \in D_h) f(a) \leqq p$.

$\mathcal{D}$ is *reasonable* if $(\exists H : \text{HF}(\kappa^+) \to 2)$ H is $\mathcal{D}$-tractable.

(4.6) DEFINITION. $S_\kappa(\diamondsuit)$ is the principle:

whenever $I \in [\kappa^+]^{\kappa^+}$ is an acceptable set of indiscernibles, $\mathbf{P} \in \mathcal{S}(I)$ and $\mathcal{D} = (D_h : h \in \mathcal{H}_A)$ is reasonable, there's a κ-complete ideal G in $\mathbf{P}$ such that whenever $H : \text{HF}(\kappa^+) \to 2$ is $\mathcal{D}$-tractable and $C \subseteq [\kappa^+]^{<\kappa}$ is club, G meets $\mathcal{D}(H, C)$.

(4.7) We now prove a companion to the Lemma of (1.1.9).

LEMMA. (a) *Whenever I is an acceptable set of indiscernibles, $\mathbf{P} \in \mathcal{S}_\kappa(I)$, $(E_\alpha : \alpha < \kappa)$ is a family of dense subsets of $\mathcal{T}$, $\mathcal{D} = (D_h : h \in \mathcal{H}_A)$ is a reasonable extension system, then, $S_\kappa(\diamondsuit) \Rightarrow$ there's an ideal G in $\mathbf{P}$ meeting all the $E^\#_\alpha$ uniformly and meeting all the $\mathcal{D}(H, C)$ such that $C \subseteq \{\kappa^+\}^{<\kappa}$ is club and $H : \text{HF}(\kappa^+) \to 2$ is $\mathcal{D}$-tractable*;

(b) *in particular, $S_\kappa(\diamondsuit) \Rightarrow S_\kappa$.*

PROOF. For the remainder of this proof, let $S^* = S^\kappa_{\kappa^+}$. We apply (1.1.9) to replace $\mathbf{P}$ by a dense subordering, $\mathbf{P}'$, which is in $\mathcal{S}_\kappa(S^*)$. In what follows, we shall write $\mathcal{T}$, ENV, O, A, supp, etc. for $\mathbf{P}'$'s versions of these notions, as a member of $\mathcal{S}_\kappa(I)$ and $\mathcal{T}'$, ENV$'$, O', A', supp$'$, etc. for $\mathbf{P}'$'s versions of these notions, as a member of $\mathcal{S}_\kappa(S^*)$.

We recall, from (1.1.9), that, for $p \in P$, $p \in P'$ iff $\mathrm{supp}(p) \subseteq \mathrm{ENV}'(\mathrm{supp}(p) \cap S^*)$; an $x \in [\kappa^+]^{<\kappa}$ with this last property was called nice. Let $\mathcal{N}$ be the collection of nice sets; then $\mathcal{N}$ is a club of $[\kappa^+]^{<\kappa}$. Note, further, that $O \cap N \subseteq O'$ and that O (and therefore O') is a club of $[\kappa^+]^{<\kappa}$. Further, if $a \in \mathcal{N} \cap O$, then, letting $b = (a^\$)'$, $b \in A'$ and in fact $b \in \mathcal{N} \cap O$; further $(\sigma_a)^{\sim} = (\sigma'_a)^{\sim} \circ (\sigma_b)^{\sim}$, and so, if $h \in \mathcal{H}_{\mathcal{N} \cap O}$, then, letting $a = a(h)$, $h^\$ = ((\sigma_b)^{\sim})^{-1}[(h^\$)']$. In the same vein, suppose that $a \in N \cap O$, $b = a^\$$, $u = (a^\$)'$ (so $u \in N \cap O$), $X \subseteq P$, and for all $p \in P$, $b \subseteq \mathrm{supp}(p)$. Then $\sigma_a[X] = (\sigma_a)' \circ \sigma_u[X]$.

Further, recall from (1.1.9) that for $p \in P'$, $\mathrm{supp}(p) = \mathrm{supp}'(p)$ and that $(s')^p : \mathrm{o.t.}(\mathrm{supp}(p) \cap S^*) \to \mathrm{supp}(p) \cap S^*$ is the increasing enumeration. Also, $(\tau')^p = (\tau^p, g^p, x^p)$, where $x^p = \langle \alpha < \lg \tau^p : s^p(\alpha) \in S^* \rangle$, and g^p is a function with domain $\bar{x}^p$ such that, setting $\beta(\alpha) = s^p(\alpha)$ for $\alpha \in x^p$, $\beta(\alpha) = \sup s^{p\prime\prime}\alpha$, for $\alpha \in \bar{x}^p \setminus x^p$, $g^p(\alpha) = \{\xi < \kappa : \beta(\alpha) + \xi \in \mathrm{supp}(p)\}$. Thus $\lg(\tau')^p = \mathrm{o.t.}\, x^p = \mathrm{o.t.}\,\mathrm{supp}(p) \cap S^*$ and $(s')^p = s^p \circ \pi$, where π is the increasing enumeration of x^p.

Let $\langle E_\alpha : \alpha < \kappa \rangle$, $\langle D_h : a(h) \in A \rangle$ be as in (a). We shall define an extension system $\mathcal{D}' = \langle D'_h : a(h) \in A' \rangle$ in $\mathbf{P}'$. Our ultimate goal is to prove:

(1) If $H : \mathrm{HF}(\kappa^+) \to 2$, then H is $\mathcal{D}$-tractable $\Rightarrow$ H is $\mathcal{D}'$-tractable; thus, since $\mathcal{D}$ was reasonable, so is $\mathcal{D}'$.

(2) If G' is an ideal in $\mathbf{P}'$ meeting all the $\mathcal{D}'(H, C)$ such that H is $\mathcal{D}'$-tractable and $C \subseteq [\kappa^+]^{<\kappa}$ is club, then letting G be the ideal in $\mathbf{P}$ generated by G', G meets all the $\mathcal{D}(H, C)$ such that $H : \mathrm{HF}(\kappa^+) \to 2$ is $\mathcal{D}$-tractable and $C \subseteq [\kappa^+]^{<\kappa}$ is club *and* G meets all the E^*_α uniformly.

Clearly this will suffice to prove the Lemma. Before defining $\mathcal{D}' = \langle D'_h : a(h) \in A' \rangle$, we replace $\langle D_h : a(h) \in A \rangle$ by a nicer reasonable extension system (in $\mathbf{P}$), $\mathcal{D}^* = \langle D^*_h : a(h) \in A \rangle$. We shall prove (1) for $\mathcal{D}$ and $\mathcal{D}^*$ (in place of $\mathcal{D}$ and $\mathcal{D}'$, respectively) and we shall then prove (1), (2) for $\mathcal{D}^*$ and $\mathcal{D}'$ (in place of $\mathcal{D}$ and $\mathcal{D}'$, respectively). We introduce some unifying notation: if $X \subseteq \mathcal{T}$, we let $\bar{X}$ be the open subset of $\mathcal{T}$ generated by X, we let X^* be the uniform blowup of X, and we let $\hat{X}$ be the uniform blowup of $\bar{X}$. Thus, $\hat{X} = \bar{X}^*$.

Set $p \in D^*_h$ iff $p \in \mathcal{T}$ (i.e. $s^p = \mathrm{id} \restriction \lg \tau^p$), $p \in \bigcap\{\bar{E}_\alpha : \alpha \in \kappa \cap \mathrm{supp}(p)\}$ and $p \in \bar{D}_h$ (i.e. there's $q \in D_h$ with $q \leqq p$). It is straightforward to verify (1) for $\mathcal{D}$ and $\mathcal{D}^*$, using that $\mathbf{P}$ is κ-directed-closed and that $\mathcal{T}$ is dense in $\{q \in P : \mathrm{supp}(1) \subseteq \mathrm{ENV}(I \restriction \kappa)\}$.

We turn to the definition of $\mathcal{D}'$. If $a(h) \in A' \setminus \mathcal{N} \cap O$, we set $D'_h = \emptyset$, so suppose that $a = a(h) \in A' \cap \mathcal{N} \cap O$. We set $D'_h = \{p \in D^*_h \cap P' : \mathrm{supp}(p) \subseteq \mathrm{ENV}(S^* \restriction \kappa)\}$. Several remarks are in order. First, note that, typically, $a(h) \notin A$, so $D_h^* = \sigma_{a(h)}[D_h^{*s}]$. Further, there will typically be $p \in D_h^* \cap P'$ with $\operatorname{supp}(p) \not\subseteq \mathrm{ENV}(S^* \mid \kappa)$, but for every such p, by an argument given below in a slightly different context, there will be an "isomorphic copy" of p which is in D'_h. This will guarantee that in making $\mathscr{D}'$ satisfy the defining property of an extension system, we have not made the D'_h too small.

For (1), let $H : \mathrm{HF}(\kappa^+) \to 2$ be $\mathscr{D}^*$-tractable, let $S \subseteq [\kappa^+]^{<\kappa}$ be stationary, and let $f : S \to P'$ be such that for all $a \in S$, $\operatorname{supp}(f(a)) = a$. Let $S' = S \cap \mathcal{N} \cap O$, and let $f' = f \mid S'$. Then, since H is $\mathscr{D}^*$-tractable, let $a \in S'$, $p \in D_h^*$ be s.t. $f(a) \leqq p$, where $h = H \mid \mathrm{HF}(a)$. Thus, $a = a(h) \in \mathcal{N} \cap O$. Let $\bar{p} \in D_h^{*s}$ and $\sigma_a \subseteq \pi$ be such that $p = \bar{p}(\pi \circ s^\beta)$. Since we may assume, without loss of generality, that $I \neq S^*$, we can also assume that range $\pi \setminus a$ contains only points *below* the least element of S^* which exceeds all elements of a, in other words, that $p \in P'$ (this is the argument alluded to above, following the definition of the D'_h). Clearly, $p \in \bigcap\{\hat{E}_\alpha : \alpha \in a\}$, by our choice of $\bar{p}$ and the definition of D_h^* and since the $\hat{E}_\alpha$ are uniform. Then p is as required. This proves (1).

For (2), suppose G' meets all the $\mathscr{D}^*(H, C)$. Let $H : \mathrm{HF}(\kappa^+) \to 2$ be $\mathscr{D}^*$-tractable (such exists, since $\mathscr{D}^*$ is reasonable). Thus, by (1), H is $\mathscr{D}'$-tractable. Let $C \subseteq [\kappa^+]^{<\kappa}$ be club, let $\alpha < \kappa$, let $s \in [I]^{<\kappa}$. Let $C' = \{a \in C \cap \mathcal{N} \cap O : \text{range } s \cup \{\alpha\} \subseteq a\}$. Let $p' \in G' \cap \mathscr{D}'(H, C)$, say $p' \in D'_h$, where $h = H \mid \mathrm{HF}(a)$, $a \in C'$. But then there is $p \in D_h^*$, s.t. $p \leqq p'$ (so $p \in G$) and, of course, range $s \subseteq \operatorname{supp}(p) \subseteq \operatorname{supp}(p')$ and $p \in \hat{E}_\alpha$. Thus, since G is ideal, G meets E''_α.

(4.8) In anticipation of §6, we need a

Definition. Let $\eta = i_0 =$ the width of I, $\mathbf{P} \in \mathscr{S}_\kappa(I)$. $p \in P$ is *orderly* iff for all $i \in s^p$, all $\zeta < \eta$,

$$i + \zeta \in \operatorname{supp}(p) \Rightarrow \zeta \in \operatorname{supp}(p),$$

i.e.: $\operatorname{supp}(p) \subseteq \{i + \zeta : i \in s^p,\ \zeta \in \operatorname{supp}(p) \cap i_0\}$.

(4.9) Proposition. *If* $\mathbf{P} \in \mathscr{S}_\kappa(I)$ *then*:

(a) *if* $p \in P$ *there's* $q \in P$, $q \geqq p$, q *orderly and* $s^p = s^q$,

(b) *if* D *is directed*, card $D < \kappa$ and for all $p \in D$, p is orderly, *then there's an upper bound* q *for* D *with* $\operatorname{supp}(q) = \bigcup\{\operatorname{supp}(p) : p \in D\}$ *and any such* q *is orderly.*

Proof. (a) is a simple matter of adding enough $\alpha < i_0$ to $\operatorname{supp}(p)$ without changing s^p. This is possible by the fact that $\operatorname{card}(\operatorname{supp}(p)) < \kappa$, and by (4), (5) of (1.1.5). (b) follows immediately from (1.1.5), (5), and the definition of orderly.

(4.10) Definition. If $\mathcal{M} = (\mathcal{S}, S^0, S^1, \mapsto, \pi_{\bar\nu\nu})_{\bar\nu \mapsto \nu}$ is a $(\kappa, 1)$-morass, let:

$\alpha \in (S^0)'$ iff $\alpha \in S^0$, and S_α has a largest element, say ν, which is a limit point in S_α and minimal in $\mapsto$ (this corresponds to case (C2) of §6).

(4.11) Let $I = S^{\kappa^+}_{\kappa}$, and let $(i_\xi : \xi < \kappa^+)$ increasingly enumerate I. If $\tau \in S^1$, let $I(\tau) = \{i_\xi : \xi < \tau\}$. If $\alpha \in S^0$, and $\tau = \max S_\alpha$ then $I(\alpha) = I(\tau)$; otherwise $I(\alpha) = \bigcup\{I(\tau) : \tau \in S_\alpha\}$. If $\bar\tau \mapsto \tau$, let $\sigma_{\bar\tau\tau} = \pi^*_{\bar\tau\tau}$.

If $\alpha \in (S^0)'$, $\bar\nu = \max S_\alpha$, $\xi < \bar\nu$, then ξ is *controlled* if $\xi = 0$ or ξ is a successor ordinal or (ξ is a limit ordinal and there's $\bar\tau \mapsto \tau \in S_\alpha \cap \bar\nu$ s.t. range $\pi_{\bar\tau\tau} \cap \xi$ is cofinal in ξ). We let $\bar I(\alpha) = \{\bar i_\xi : \xi < \bar\nu \wedge \xi$ is controlled$\}$. We let:

$$a(\alpha) = \alpha \cup \{i + \zeta : i \in \bar I(\alpha), \zeta < \alpha\}.$$

Thus $a(\alpha) \cap I = I(\alpha)$, so $a(\alpha) \in \mathrm{ENV}(a(\alpha) \cap I)$; i.e. $a(\alpha) \in O$.

We can now introduce the notion of a built-in-$\diamond$ sequence for a morass.

(4.12) Definition. If $\mathcal{M} = (\mathcal{S}^0, \dots)$ is a $(\kappa, 1)$-morass, then $(h_\alpha : \alpha \in (S^0)')$ is a built-in-$\diamond$ sequence for $\mathcal{M}$ iff

(1) $h_\alpha : \mathrm{HF}(a(\alpha)) \to 2$,

(2) whenever $H : \mathrm{HF}(\kappa^+) \to 2$, S_H is stationary, where:

(*): $a \in S_H \Leftrightarrow$ setting $h = H \restriction \mathrm{HF}(a)$, there's $\alpha \in (S^0)'$, $\nu \in S_\kappa$ such that setting $\bar\nu = \max(S_\alpha)$, $\bar\nu \mapsto \nu \wedge h = \bar\sigma_{\bar\nu\nu}[h_\alpha]$.

(4.13) Remarks. Note that $a \in S_H \Rightarrow a \in O$. Further, the $\alpha \in (S^0)'$ witnessing that $a \in S_H$ is uniquely determined and hence so is the $\bar\nu$; if we require that $a \cap I$ be cofinal in ν, then the ν witnessing $a \in S_H$ is also uniquely determined. We denote these by $\alpha(a)$, $\bar\nu(a)$, $\nu(a)$, respectively. Note that $a(\alpha(a)) = a^{\$}$, and that $\sigma_{\bar\nu\nu} = \sigma_a$, where $\bar\nu = \bar\nu(a)$, $\nu = \nu(a)$. The property (2) of (4.12) is the *oracle* property of the sequence $(h_\alpha : \alpha \in (S^0)')$.

(4.14) It will be useful to develop here some auxiliary notions for built-in-$\diamond$ sequences. These will be useful in §§7, 8. In §7, when we construct a morass with built-in-$\diamond$ in L, it will be natural to seek h_α (and an auxiliary C_α) in L_ρ, where $\rho = \rho(\nu^*)$, $\nu^* = \max S_\alpha$, $\alpha \in (S^0)'$. This will be impossible, however, since the domain of h_α must be a_α, which contains ordinals $\geqq \kappa$. However, this difficulty is easily overcome by "deflating" $a(\alpha)$, since the large ordinals it contains arise only by left-multiplying small ordinals by κ. This motivates the following:

Definition. Let $\alpha \in (S^0)'$, let $\nu^* = \max S_\alpha$, and let $\xi < \nu^*$; say $\xi = \lambda + n$, where $\lambda = \bigcup \lambda$, $n < \omega$. We define $k_\alpha(\xi) = \xi$ if λ is controlled; otherwise

$k_\alpha(\xi)=\xi+1$. We then define $\delta_\alpha : \nu^* \leftrightarrow a(\alpha)$ as follows (recall that, since $\alpha < \nu^*$ and ν^* is p.r. closed, $\nu^* = \alpha \cdot \nu^*$):

$$\delta_\alpha((\alpha \cdot \xi)+\zeta)=(\kappa \cdot k_\alpha(\xi))+\zeta.$$

PROPOSITION. *Suppose $\alpha \in (S^0)'$, $\nu^* = \max S_\alpha$, $\nu^* \mapsto \nu$, $\nu \in S_\kappa$. Suppose further that for $0<\lambda^* = \bigcup \lambda^* < \nu^*$, letting $\lambda = \pi_{\nu^*\nu}(\lambda^*)$:*

(*) λ^* *is controlled iff* $\operatorname{cf}\lambda < \kappa$ *iff* $\lambda = \sup \pi''_{\nu^*\nu}\lambda^*$.

Then: $\pi_{\nu^*\nu} = \sigma_{\nu^*\nu} \circ \delta_\alpha$.

PROOF. Note that:

(1) $\pi_{\nu^*\nu}$ preserves ordinal arithmetic,

(2) $\pi_{\nu^*\nu}$, $\sigma_{\nu^*\nu} \circ \delta_\alpha$ are both order-preserving,

(3) $\pi_{\nu^*\nu} \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi_{\nu^*\nu}(\alpha)=\kappa$, and similarly for $\sigma_{\nu^*\nu}\circ\delta_\alpha$,

(4) $\delta_\alpha : \nu^* \leftrightarrow a(\alpha)$.

Therefore, it will suffice to show that $\bar I \cap \operatorname{range}\pi_{\nu^*\nu} = \bar I \cap \operatorname{range}\sigma_{\nu^*\nu}$.

As noted above, $\bar I \cap a(\alpha) = \bar I(\alpha) = \{\kappa\cdot\xi^* : \xi^* < \nu^* \wedge \xi^*$ is controlled$\}$, so $I(\alpha)=\{\kappa\cdot\xi^* : \xi^*<\nu^* \wedge \xi^*$ is a successor$\}=\{i_{\xi^*} : \xi^* < \nu^*\}$, while $\bar I(\alpha)\setminus I(\alpha) = \{\kappa\cdot\lambda^* : 0<\lambda^* = \bigcup\lambda^* < \nu^* \wedge \lambda^*$ is controlled$\}$. Now $I \cap \operatorname{range}\pi_{\nu^*\nu} = \{\xi \in \operatorname{range}\pi_{\nu^*\nu} : \operatorname{cf}\xi = \kappa\} = \{\kappa\cdot\xi : \xi \in \operatorname{range}\pi_{\nu^*\nu} \wedge \xi$ is a successor$\} \cup \{\kappa\cdot\lambda : 0<\lambda = \bigcup\lambda \in \operatorname{range}\pi_{\nu^*\nu} \wedge \operatorname{cf}\lambda = \kappa\}$. Now by (*), the last term of this union equals $\{\kappa\cdot\pi_{\nu^*\nu}(\lambda^*) : 0<\lambda^* = \bigcup\lambda^* < \nu^* \wedge \lambda^*$ is not controlled$\} = \{i_{\pi_{\nu^*\nu}(\lambda^*)} : 0<\lambda^*<\nu^* \wedge (\lambda^*$ is not controlled$)\}$, while, of course, the first term of the union equals $\{\kappa\cdot\pi_{\nu^*\nu}(\xi^*) : \xi^* \in \nu^* \wedge \xi^*$ is a successor$\} = \{i_{\pi_{\nu^*\nu}(\xi^*)} : 0<\xi^* < \nu^* \wedge (\xi^*$ is a successor $\vee\ \xi^*$ is controlled$)\}$. Finally, $I \cap \operatorname{range}\sigma_{\nu^*\nu} = \{i_{\pi_{\nu^*\nu}(\xi^*)} : \xi^* < \nu^*\}$, so $I \cap \operatorname{range}\pi_{\nu^*\nu} = I \cap \operatorname{range}\sigma_{\nu^*\nu}$.

Now, $(\bar I\setminus I)\cap\operatorname{range}\sigma_{\nu^*\nu} = \{\sigma_{\nu^*\nu}(\kappa\cdot\lambda^*) : 0<\lambda^*=\bigcup\lambda^*<\nu^*\wedge\lambda^*$ is controlled$\} = \{\sup\sigma''_{\nu^*\nu}\kappa\cdot\lambda^* : 0<\lambda^*=\bigcup\lambda^*<\nu^*\wedge\lambda^*$ is controlled$\} = \{\kappa\cdot\sup\pi''_{\nu^*\nu}\lambda^* : 0<\lambda^*=\bigcup\lambda^*<\nu^*\wedge\lambda^*$ is controlled$\}$. Now by (*) this last equals $\{\kappa\cdot\pi_{\nu^*\nu}(\lambda^*) : 0<\lambda^*=\bigcup\lambda^*<\nu^*\wedge\lambda^*$ is controlled$\} = \{\pi_{\nu^*\nu}(\alpha\cdot\lambda^*) : 0<\lambda^*=\bigcup\lambda^*<\nu^*\wedge\lambda^*$ is controlled$\}$, and, again by (*), this equals $(\bar I\setminus I)\cap\operatorname{range}\pi_{\nu^*\nu}$, since $\pi_{\nu^*\nu}(\xi^*)$ is a multiple of κ iff ξ^* is a multiple of α. This completes the proof.

(4.15) COMPLETING THE PROOF OF THEOREM (2.15). We define an extension system $\mathcal{D} = (D_h : h \in \mathcal{H}_A)$. If $h \in \mathcal{H}_O$, let $b(h)=\{\alpha\in a(h) : h(\alpha)=1\}$. Then, for $h \in \mathcal{H}_A$, we define:

(1): $$D_h = \{p \in P : b(h) \in p \wedge \bigcup p \subseteq \mathrm{ENV}(I \restriction \omega_1)\}.$$

Our first task will be to verify:

CLAIM 1. *$\mathcal{D}$ is reasonable.*

PROOF OF CLAIM 1. We first remark:

(2): if $h \in \mathcal{H}_O \setminus \mathcal{H}_A$, $a = a(h)$ and $h' = h^{\$}$, then $b(h) = \sigma''_a(b(h'))$.

This is clear. We also have, equally clearly:

(3): if $h \in \mathcal{H}_O \setminus \mathcal{H}_A$, then $p \in D_h \Leftrightarrow b(h) \in p$ and s^p is an end-extension of $a(h) \cap I$.

Let $H : \mathrm{HF}(\omega_2) \to 2$ be the constant function with value 1. We now show:

(4): H is $\mathcal{D}$-tractable.

So, let $S \subseteq [\omega_2]^{<\omega_1}$ be stationary and let $f : S \to P$ with $\mathrm{supp}(f(a)) = a$, for $a \in S$. Let $S' = S \cap O$. Thus, for $a \in S'$, if $h = H \mid \mathrm{HF}(a)$ (so h = the constant function with value 1 on $\mathrm{HF}(a)$), then $a = b(h)$. Now $a = b(h)$ is nice. Thus, either $b(h) = a = \bigcup f(a) \in f(a)$, in which case $f(a) \in D_h$, by (1) or (3) as appropriate, since here $s^{f(a)} = a(h) \cap I$, or $b(h) = a = \bigcup f(a) \notin f(a)$, in which case by (2.10)(b), $f(a) \cup \{b(h)\} \in P$, and, as in the first case, $f(a) \cup \{b(h)\} \in D_h$. This proves Claim 1.

Now suppose G is an ideal in $\mathbf{P}$ such that whenever $H : \mathrm{HF}(\omega_2) \to 2$ is $\mathcal{D}$-tractable and $C \subseteq [\omega_2]^{<\omega_1}$ is club, then G meets $\mathcal{D}(H, C)$. We show that G meets all the $D^* \mid C$ for $C \subseteq [\omega_2]^{<\omega_1}$ (viz. (2.15)). But this is clear, since if H is as in the proof of Claim 1, C is club, $C' = C \cap O$, then $D^* \mid C' = \mathcal{D}(H, C')$.

§5. Further applications

In this section, we present further applications of $S_\kappa(\diamond)$:

(1) Compactness, Axiomatizability and Transfer for the Magidor–Malitz language $\mathcal{L}[Q_1^{<\omega}, Q_2^{1}](\kappa = \aleph_1)$; this is in (5.1),

(2) Counterexamples to the Arhangel'skii conjecture ($\kappa = \aleph_1$, but generalizes); this is in (5.2).

Velleman has obtained (2), independently, [25]. (1) was suggested by Burgess (this was also envisioned by Shelah), and draws essentially upon his work, presented in the first Appendix, §9. This establishes the general framework for reducing (1) to a statement, (*) of (5.1.1), not obviously involving generalized quantifiers, and shows that $\mathbf{P}$ of (5.1.13) (whose definition is also due to Burgess) is in $\mathcal{S}_{\aleph_1}$ $(S_{\aleph_2}^{\aleph_1})$.

Shelah has suggested extending these methods to $\mathcal{L}[Q_1^{<\omega}, Q_2^{<\omega}]$ as well as to higher cardinals — this will be developed in a sequel. (2) draws on a refinement, due to Hajnal and Juhász, [8], of Shelah's original forcing construction of a counterexample, [18] (see also R. Price's version, [16]). An earlier version of this paper used a further "simplification", [13], but the referee pointed out that there were difficulties in this treatment, certainly for our purposes, and perhaps for actually doing forcing constructions.

(5.1) *Compactness, Axiomatizability and Transfer for* $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$

The quantifiers Q_α^n ($1 \leqq n < \omega$, $\alpha \in \mathrm{OR}$) are defined by:

$\mathfrak{A} \models Q_\alpha^n x_1 \cdots x_n \varphi(x_1, \ldots, x_n) \Leftrightarrow$

$$(\exists X \subseteq |\mathfrak{A}|)(\operatorname{card} X \geqq \aleph_\alpha \wedge (\forall \{x_1, \ldots, x_n\} \in [X]^n) \mathfrak{A} \models \varphi(x_1, \ldots, x_n)).$$

Q_α is just Q_α^1. $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$ is first-order logic (over a countable vocabulary) enriched with the quantifiers Q_1^n ($1 \leqq n < \omega$), and Q_2 ($= Q_2^1$). See [1], [3], and §9 for more on the background and expressive power for such logics. We confine ourselves, here, to recalling that the germinal paper is [14], where, assuming $\diamondsuit$, the compactness of $\mathscr{L}[Q_1^{<\omega}]$ is proved.

In the remainder of this subsection, up to the definition of **P** in (5.1.13), our development will run parallel to that of §9, but in far less detail. Not only will proofs be omitted, but the technical aspects of certain definitions will also be left to §9. Our goal is to give the bare minimum necessary to bring us to the definition of **P**. In (5.1.14), (5.1.15) our treatment becomes somewhat more substantial: we prove that $\mathbf{P} \in \mathscr{S}_\kappa(I)$, where $I = S_2^1$ but this leans heavily on (5.1.11), which isn't proved until §9. Starting in (5.1.16) we give a complete proof (modulo everything up to (5.1.13)) that applying $S_{\aleph_1}(\diamondsuit)$ to **P** yields a statement (*) presented below in (5.1.1); in §9 Burgess shows how (*) implies the desired results for $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$; this is stated as Lemma 2, in (5.1.1). The material of (5.1.1)–(5.1.3), and part of (5.1.4) mirrors (9.1)–(9.6). The remainder of (5.1.4), and (5.1.5)–(5.1.7) mirror (9.7); (5.1.8)–(5.1.11) mirror (9.8).

(5.1.1) *Generalities and Burgess' Reduction*

We assume that $\mathscr{L}$ has just one non-logical symbol, $\underset{\sim}{R}$, which is a binary predicate symbol. This assumption can be eliminated by well-known devices. We also use auxiliary predicate symbols, E (binary), and Γ, Δ (singulary). In §9, Burgess defines a fixed primitive recursive first order theory θ_0 in vocabulary $\{E, \Gamma\}$. The axioms of θ_0 are (a), (b), (c), below, plus (d) which will be introduced in (5.1.3). E is treated as the symbol for the membership relation of set theory.

(a) $\mathrm{ZF}^-(\Gamma)$ (i.e. replacement for formulas containing Γ),

(b) "Γ is a confinal class of ordinals $\wedge$ $V = L[\Gamma]$",

(c) "There is a largest cardinal which is regular and uncountable".

We shall denote by λ the largest cardinal of any model of θ_0, as well as the corresponding θ_0-term. Ordinals smaller than λ are *small*; larger ones are *large*.

Consider the assertion:

(*): Whenever θ is a consistent extension of θ_0, θ has a model $\mathfrak{A}$ such that

(a) the ordinals of $\mathfrak{A}$ form an ω_2-like linear ordering and the small ordinals of $\mathfrak{A}$ form an ω_1-like linear ordering,

(b) for any formula $\varphi(x_1,\dots,x_n)$ in parameters from $|\mathfrak{A}|$ the following are equivalent:

(i) $\mathfrak{A} \models (\sim \exists X)(X$ is a cofinal set of small ordinals and $\forall x_1 < \cdots < x_n$ from X, $\varphi(x_1,\dots,x_n))$,

(ii) $(\sim \exists S)$ S is a cofinal set of small ordinals and $\forall s_1 < \cdots < s_n$ from S, $\mathfrak{A} \models \varphi(s_1,\dots,s_n)$.

In the remainder of (5.1), we present a proof (modulo §9) of:

LEMMA 1. $S_{\aleph_1}(\diamondsuit) \Rightarrow (*)$.

The strategy for doing so is due to Burgess. The proof in (5.1.16)–(5.1.18) is due to Stanley, incorporating ideas of Burgess. As mentioned above, the proof that **P** (of (5.1.13)) is in $\mathcal{S}_{\aleph_1}(S^1_2)$ is essentially due to Burgess and the details are presented in the first Appendix, §9, where it's also shown:

LEMMA 2. (*) $\Rightarrow$ *compactness, axiomatizibility and transfer for* $\mathcal{L}[Q_1^{<\omega}, Q_2^1]$.

(5.1.2) DEFINITION. Let $\mathfrak{A} = (A, E, C) \models \theta_0$. Let $x \in \mathrm{OR}^{\mathfrak{A}}$. $\mathcal{M}_x$ denotes that structure (M_x, E', C') which, in $\mathfrak{A}$, satisfies:

$$\text{“}C' = \Gamma \cap M_x, \quad M_x = L_x[C'], \quad E' = \underset{\sim}{E} \restriction M_x^2\text{”},$$

i.e. $\mathcal{M}_x = (L_x[C], \in \cap L_x[C]^2, C \cap L_x[C])^{\mathfrak{A}}$.

Recall that λ will denote the largest cardinal of any and all models of θ_0. We will write "$\mathfrak{B} \mapsto^* \mathfrak{B}'$" to abbreviate the formula of the language of θ_0 which asserts that $\mathfrak{B}$ is an $\mathcal{L}_{\lambda\lambda}$-elementary substructure of $\mathfrak{B}'$ (for vocabulary $\{\underset{\sim}{E}, \Gamma\}$).

(5.1.3) DEFINITION. For each $n < \omega$, define a formula $\tau_n(x)$ of the language of θ_0 by recursion:

$$\tau_0(x)\text{: “}x \text{ is a large ordinal”},$$

$$\tau_{n+1}(x)\text{: “}\forall y^{\text{ordinal}} \exists z^{\text{ordinal}} (\tau_n(z) \wedge x, y < z \wedge \mathcal{M}_x \mapsto^* \mathcal{M}_z)\text{”}.$$

The list of axioms of θ_0 is completed by: (d): $\{\exists x \tau_n(x): n \in \omega\}$.

(5.1.4) DEFINITION. If $\mathfrak{A} \models \theta_0$ is countable and recursively saturated, $D \subseteq \mathrm{OR}^{\mathfrak{A}}$ is *commendable* iff $(\mathfrak{A}, D) \models \psi$ where ψ is the first-order sentence in vocabulary $\{\underline{E}, \Gamma, \Delta\}$ (Δ is singulary and is interpreted by D), which asserts:

(d'): "Δ is a cofinal class of ordinals $\wedge$
$(\forall x, y)(\Delta(x) \wedge \Delta(y) \wedge x \underline{E} y \Rightarrow \mathcal{M}_x \mapsto\!\!\!\!\!\!\prec^* \mathcal{M}_y)$".

(5.1.5) REMARK. If $\mathfrak{A} \models \theta_0$ is countable and recursively saturated, then there's commendable $D \subseteq \mathrm{OR}^{\mathfrak{A}}$. Also, clearly if $D \subseteq \mathrm{OR}^{\mathfrak{A}}$ is commendable and $a \in D$ then

$$(*): \qquad \text{for all } n < \omega, \quad \mathfrak{A} \models \tau_n(a).$$

An a satisfying (*) is called *praiseworthy*. If a is praiseworthy and $A_a =$ the E-extension of $M_a = \{b \in |\mathfrak{A}| : b E M_a\}$, then $\mathfrak{A} \restriction A_a \mapsto\!\!\!\!\!\prec \mathfrak{A}$. Further, given any praiseworthy $a \in |\mathfrak{A}|$, a commendable $D \subseteq \mathrm{OR}^{\mathfrak{A}}$ can be found with $a =$ the E-least member of D. In fact, we have a converse: if $\mathfrak{A} \models \theta_0$, the integers of $\mathfrak{A}$ are non-standard and there's commendable $D \subseteq \mathrm{OR}^{\mathfrak{A}}$, then $\mathfrak{A}$ is recursively saturated.

(5.1.6) In this and the next paragraph, fix countable, recursively saturated $\mathfrak{A} \models \theta_0$, $\mathfrak{A} = (A, E, C)$, and commendable $D \subseteq \mathrm{OR}^{\mathfrak{A}}$.

DEFINITION. For $a \in A$, adjoin to vocabulary $\{\underline{E}, \Gamma\}$ distinct individual constant symbols, $\underline{a}^+$, $\underline{a}^-$, which are also distinct from the $\underline{b}^+$, $\underline{b}^-$ for $b \in A \setminus \{a\}$. Let Ψ, $\Psi(A)$, $\Psi(\pm A)$ be, respectively, first order logic in vocabulary $\{\underline{E}, \Gamma\}$, first-order logic in vocabulary $\{\underline{E}, \Gamma\} \cup \{\underline{a}^+ : a \in A\}$, first-order logic in vocabulary $\{\underline{E}, \Gamma\} \cup \{\underline{a}^+ : a \in A\} \cup \{\underline{a}^- : a \in A\}$.

(5.1.7) Burgess defines (see (9.7.9), below) a set, $I(\mathfrak{A}, D)$, of $\Psi(\pm A)$ sentences which are *imposed*. $I(\mathfrak{A}, D)$ has the following properties (among others):

(a) it is consistent,

(b) $\varphi(\underline{s}^+, \underline{t}^+, \dots)$, $\varphi(\underline{s}^-, \underline{t}^-, \dots) \in I(\mathfrak{A}, D)$, whenever $\mathfrak{A} \models \varphi(s, t, \dots)$,

(c) for each small ordinal b of $\mathfrak{A}$, the 1-type $\Sigma_b = \{x < \underline{b}\} \cup \{x \neq \underline{c} : \mathfrak{A} \models \underline{c} \underline{E} \underline{b}\}$ is locally omitted (i.e. non-principal) over $I(\mathfrak{A}, D)$.

The rough idea behind $I(\mathfrak{A}, D)$ is that it consists of those sentences which are true in any $\mathfrak{B}$ with the following properties:

(i) $\mathfrak{A} \mapsto\!\!\!\!\!\prec \mathfrak{B}$,

(ii) $\exists j : \mathfrak{A} \to \mathfrak{B}$, elementary embedding, $j \restriction A_a = \mathrm{id} \restriction A_a$, but for all $x \in A$, $x \underline{E}^{\mathfrak{B}} M^{\mathfrak{B}}_{j(a)}$, where a is the least element of D,

(iii) for $a \in A$, $(\underset{\sim}{a}^-)^{\mathfrak{B}} = a$, $(\underset{\sim}{a}^+)^{\mathfrak{B}} = j(a)$,

(iv) the small ordinals of $\mathfrak{A}$ are an $\underset{\sim}{E}^{\mathfrak{B}}$-initial segment of the small ordinals of $\mathfrak{B}$ (we shall abbreviate this by $\mathfrak{A} \subseteq_e \mathfrak{B}$).

In fact, any model of $I(\mathfrak{A}, D)$ which omits the types Σ_b of (c), above, can be isomorphed to yield such a $(\mathfrak{B}, j)$. We shall now turn our attention to other types which can be omitted over $I(\mathfrak{A}, D)$.

(5.1.8) Definition. $S \subseteq A$ is *ominous* for $\mathfrak{A}$ iff it is a cofinal subset of the set of small ordinals of $\mathfrak{A}$. A pair (S, φ) consisting of an ominous set and a $\Psi(A)$ formula $\varphi(x_1, \ldots, x_k)$ is *menacing for* $\mathfrak{A}$ if it constitutes a counterexample to (5.1.1) (*), (b), i.e. (5.1.1)(*) (b) (i) holds but $\forall s_1 < \cdots < s_k$ from S, $\mathfrak{A} \models \varphi(s_1, \ldots, s_k)$.

If (S, φ) is menacing for $\mathfrak{A}$, the *basic dangerous type* for $\mathfrak{A}$, S, φ, $\Sigma_{A,S,\varphi}$ consists of the following formulas:

(a) $\underset{\sim}{s} < x < \lambda$ $(s \in S)$,

(b) $\varphi(\underset{\sim}{s}_1, \ldots, \underset{\sim}{s}_{k-1}, x)$ (whenever $s_1 < \cdots < s_{k-1}$ are from S),

(c) $\sigma(x)$(σ a $\Psi(A)$-formula s.t. for all $s \in S$, $\mathfrak{A} \models \sigma(s)$).

See (9.8.3), below, for a discussion of the intuition involved here.

(5.1.9) The notion of the general dangerous types for $(\mathfrak{A}, S)$ is somewhat technical and will be presented in Appendix 1, (9.8.3), below. Burgess first introduces the notion of a reinforcement of a 1-type. The desired feature is that if $\mathfrak{A}$ locally omits all the reinforcements of Σ, then, under appropriate and mild hypotheses on Σ, a type closely related to Σ is locally omitted by (is non-principal over) $I(\mathfrak{A}, D)$. This is (only slightly) imprecise, but see below (9.8.4). The collection of general dangerous types for $(\mathfrak{A}, S)$ is then defined as the smallest collection of types containing the basic dangerous types and closed for reinforcement.

(5.1.10) Definition. A pair (B, T) is secure for $\mathfrak{A}$ if $B \subseteq A_b$, for some large ordinal b of $\mathfrak{A}$, $\mathfrak{A} \mid B \mapsto\!\!\!\!\rightarrow \mathfrak{A}$, T is ominous for B and $\mathfrak{A}$ omits all (general) dangerous types for $(\mathfrak{A} \mid B, T)$.

(5.1.11) The main Lemma is then

Lemma (Burgess). *Let $\mathfrak{A} \models \theta_0$ be countable, recursively saturated, $\mathfrak{A} = (A, E, C)$. Let $a \in \mathrm{OR}^{\mathfrak{A}}$ be praiseworthy. For $i < \omega$, let S_i be ominous for $\mathfrak{A}$, and let (B_i, T_i) be secure pairs for $\mathfrak{A}$. Then there's $\mathfrak{A}' = (A', E', C')$ and $j : \mathfrak{A} \rightarrow \mathfrak{A}'$ s.t.*

(a) *$\mathfrak{A}'$ is countable, recursively saturated,*

(b) *$\mathfrak{A} \mapsto\!\!\!\!\rightarrow \mathfrak{A}'$, $j : \mathfrak{A} \rightarrow \mathfrak{A}'$ is elementary,*

(c) $j \restriction A_a = \mathrm{id}$, $A \subseteq A'_{j(a)}$,

(d) $\mathfrak{A} \subseteq_e \mathfrak{A}'$,

(e) *all* (A, S_i), (B_i, T_i), $(j''B_i, T_i)$ *are secure for* $\mathfrak{A}'$.

(5.1.12) Let $I = S^{\omega_1}_{\omega_2}$, $\bar{I}$, etc., be as usual. Fix θ, a consistent extension of θ_0.

DEFINITION. $\mathfrak{A} = (A, E, C)$ is *stratified* if it's a countable recursively saturated model of θ, and:

(a) the small ordinals of $\mathfrak{A}$ are just the ordinals less than some countable ordinal, $\mu(\mathfrak{A})$ (though their order in $\mathfrak{A}$ has nothing to do with their natural order),

(b) $A \setminus \mu(A)$ consists of pairs (α, ν) with $\omega_1 \leqq \alpha < \omega_2$, $\nu < \omega_1$; we then define:

$$X(A) = \{\alpha : (\exists \nu)((\omega_1 + \alpha, \nu) \in A)\},$$

and we require:

(c) $X(A) \subseteq \mathrm{ENV}(I \cap X(A))$; $0 \in X(A)$,

(d) if $\alpha \in X(A)$, then $x(\alpha) = (\omega_1 + \alpha, 0) \in A$ and is praiseworthy in $\mathfrak{A}$. Further, if $\alpha \in I$ or $\alpha = 0$ or α is a successor ordinal, then $A_{x(\alpha)} = \mu(A) \cup \{(\omega_1 + \beta, \nu) \in A : \beta < \alpha\}$.

(A, E, C) is *neatly stratified* if in addition:

(e) $X(A)$ is closed.

(5.1.13) DEFINITION. Now fix a neatly stratified $\mathfrak{A}^*$ with $X(\mathfrak{A}^*) = \{\omega_1\}$, and set:

$$P = \{(\mathfrak{A}, \mathcal{S}) : \mathfrak{A} \text{ is stratified, } \mathcal{S} \text{ is a countable set of secure pairs for } \mathfrak{A}\},$$

$$Q = \{(\mathfrak{A}, \mathcal{S}) \in P : \mathfrak{A} \text{ is neatly stratified}\}.$$

For $(\mathfrak{A}, \mathcal{S})$, $(\mathfrak{B}, \mathcal{S}') \in P$; set $(\mathfrak{A}, \mathcal{S}) \leqq (\mathfrak{B}, \mathcal{S}')$ iff:

(a) $\mathfrak{A} \mapsto \mathfrak{B}$,

(b) $\mathfrak{A} \subseteq_e \mathfrak{B}$,

(c) $\mathcal{S} \subseteq \mathcal{S}'$.

(5.1.14) LEMMA. *If* $(\mathfrak{A}, \mathcal{S}) \in P$ *and* S *is ominous for* $\mathfrak{A}$, *there's* $(\mathfrak{B}, \mathcal{S}') \in P$ *such that*:

(a) $(\mathfrak{A}, \mathcal{S}) \leqq (\mathfrak{B}, \mathcal{S}')$, $X(\mathfrak{B}) \cap I$ *is an end-extension of* $X(\mathfrak{A}) \cap I$,

(b) $(A, S) \in \mathcal{S}'$.

Further, such a $(\mathfrak{B}, \mathcal{S}')$ *can be found in* Q, *if desired.*

PROOF. We show how to obtain the desired $(\mathfrak{B}, \mathcal{S}')$ in Q.

Case 1. $(\mathfrak{A}, \mathcal{S}) \in P$. Apply (5.1.11) with $a = x(\omega_1)$, each $S_i = S$,

$\{(B_i, T_i): i \in \omega\} = \mathcal{S}$, to obtain $\mathfrak{A}'$, $\mathcal{S}' = \mathcal{S} \cup \{(A, S)\}$. Now, if necessary, we can rename elements of $A' \setminus A$ to make $(\mathfrak{A}', \mathcal{S}')$ the required $(\mathfrak{B}, \mathcal{S}')$.

Case 2. $(\mathfrak{A}, \mathcal{S}) \notin P$. Let $\beta \in \overline{X(A)} \setminus X(A)$.

Subcase 2a. $\beta = \sup X(A)$. Apply (5.1.11), with $a = x(\omega_1)$, as above, to get $(\mathfrak{A}', \mathcal{S}')$. After renaming, we can assume $(\mathfrak{A}', \mathcal{S}') \in Q$, that (a), (b) hold and that $X(A') = X(A) \cup \{\beta\}$. Details of the renaming are as in [19], (4.11).

Subcase 2b. $\beta < \sup X(A)$. Proceed as in subcase 2a but with $a = x(\inf((X(A) \cap I) \setminus \beta))$.

Then, iterating to exhaust $\overline{X(A)} \setminus X(A)$, taking unions at the end, we obtain $(\mathfrak{B}, \mathcal{S}')$ as required. In all cases, (5.1.11), (d) guarantees that we can take $\mathfrak{B}$ with $X(\mathfrak{B}) \cap I$ an end-extension of $X(\mathfrak{A}) \cap I$.

(5.1.15) Burgess then makes essential use of (5.1.11), (5.1.14) in proving

LEMMA (Burgess). *$\mathbf{P} \in \mathcal{S}_{\aleph_1}(I)$ with support function* $\operatorname{supp}(\mathfrak{A}, \mathcal{S}) = \mu(A) \cup X(A)$.

We shall content ourselves, here, with describing the set of terms and how they act on increasing sequences of the right length to give conditions.

If $p = (\mathfrak{A}, \mathcal{S}) \in P$, set $p \in \mathcal{T}$ iff $X(A) \cap I$ is an initial segment of I. Let $\lg p = \text{o.t.}(X(A) \cap I)$. If $s : \lg p \to I$ is order preserving, define $p(s)$ by letting $p(s) = (\mathfrak{A}(s), \mathcal{S}(s))$, where, for structures $\mathfrak{B}$ with $|\mathfrak{B}| \subseteq \omega_1 \cup ((\text{ENV}(X(A) \cap I) \setminus \omega_1) \times \omega_1)$:

$$\mathfrak{B}(s) \text{ is that structure isomorphic to } \mathfrak{B} \text{ by } \check{s},$$

where

$$\check{s} \restriction \omega_1 = \text{id} \restriction \omega_1 \quad \text{and} \quad \text{for } (\alpha, \nu) \in |\mathfrak{B}| \setminus \omega_1, \quad \check{s}(\alpha, \nu) = (\bar{s}(\alpha), \nu),$$

and

$$\mathcal{S}(s) = \{(\mathfrak{B}(s), T) : (\mathfrak{B}, T) \in \mathcal{S}\}.$$

We now easily obtain that $\mathbf{P}$ is strongly $\aleph_1$-directed-closed and, in fact, that (5) of (1.1.5) holds. This uses (5.1.14) and is analogous to (4.13) of [20], and by transitivity to [1]. Clearly $\mathbf{P}$ is $\aleph_1$-S^1_2-special and indiscernible. The remaining items of (1.1.5) are also clear, again using (5.1.11), (5.1.14). Thus, $\mathbf{P}$ is supported. The extension and amalgamation properties also use (5.1.11), (5.1.14), viz. (4.12), (4.15) of [20] and Lemma 2.7 of [1].

(5.1.16) We now complete (modulo Appendix 1) the proof of Lemma 1 of (5.1.1). We first show that there is a collection of dense subsets of $\mathbf{P}$ which, if met, yield $\mathfrak{A} \models \theta$ as in (*) of (5.1.1); we then show that $S_{\aleph_1}(\diamond)$ allows us to meet these dense sets.

Let $T \in [\omega_1]^{<\omega_2}$, let $C \subseteq [\omega_2]^{\aleph_0}$ be club.

DEFINITION. Let $(\mathfrak{A}, \mathcal{S}) \in D(T, C) \Leftrightarrow$

$$(*) \qquad (\mathfrak{A}, \mathcal{S}) \in P \wedge (\exists \bar{\mathfrak{B}} \mapsto\!\!\!\!\succ \mathfrak{A})$$

$$[X(\bar{\mathfrak{B}}) \in C \wedge T \cap |\bar{\mathfrak{B}}| \text{ is } E^{\bar{\mathfrak{B}}}\text{-cofinal in } \mu(\mathfrak{A}) \cap |\bar{\mathfrak{B}}| \wedge (\bar{\mathfrak{B}}, T \cap |\bar{\mathfrak{B}}|) \in \mathcal{S}].$$

(5.1.17) Now suppose G is an ideal in $\mathbf{P}$ meeting all the $E^{\#}_{\alpha}$ $(\alpha < \omega_1)$ and all the $D(T, C)(T \in [\omega_1]^{<\omega_2}$, C club $\subseteq [\omega_2]^{\aleph_0})$, where $E^{\#}_{\alpha}$ is the uniform blow-up of $E_\alpha = \{(\mathfrak{A}, \mathcal{S}) \in \mathcal{T} : \mu(\mathfrak{A}) \geqq \alpha\}$.

Let $\mathfrak{B} = \bigcup \{\mathfrak{A}^p : p \in G\}$. We claim:

LEMMA. $\mathfrak{B}$ *satisfies* (*) *of* (5.1.1).

PROOF. Clearly $\mathfrak{B}$ satisfies (a). For (b), suppose $\varphi = \varphi(x_1, \dots, x_k)$ is a formula with parameters $b_1, \dots, b_l \in |\mathfrak{B}|$, and suppose

(1) $\mathfrak{B} \models \sim (\exists X)(X$ is a cofinal set of small ordinals $\wedge$ $(\forall x_1 < \cdots < x_k$ from $X)\varphi(x_1, \dots, x_k))$.

Let $T \in [\omega_1]^{<\omega_2}$ and suppose, towards a contradiction, that:

(2) $(\forall s_1 < \cdots < s_\kappa$ from $T)\mathfrak{B} \models \varphi(s_1, \dots, s_k)$.

Let $C = \{a \in [\omega_2]^{\aleph_0} : b_1, \dots, b_l \in a \wedge (\mathfrak{B} \restriction a, T \cap a) \mapsto\!\!\!\!\succ (\mathfrak{B}, T)\}$. Thus $C \subseteq [\omega_2]^{\aleph_0}$ is club.

Let $(\mathfrak{A}, \mathcal{S}) \in D(T, C)$. Then, for some $a \in C$, letting $\bar{\mathfrak{B}} = \mathfrak{B} \restriction a$, $(\bar{\mathfrak{B}}, T \cap |\bar{\mathfrak{B}}|) \in \mathcal{S}$. Since $\bar{\mathfrak{B}} \mapsto\!\!\!\!\succ \mathfrak{A} \mapsto\!\!\!\!\succ \mathfrak{B}$ and $b_1, \dots, b_l \in |\bar{\mathfrak{B}}| = a$, $(T \cap |\mathfrak{B}|, \varphi)$ is menacing for $\bar{\mathfrak{B}}$. Therefore, whenever $(\mathfrak{A}', \mathcal{S}') \in G$ and $(\mathfrak{A}, \mathcal{S}) \leqq (\mathfrak{A}', \mathcal{S}')$, $\mathfrak{A}'$ omits $\Sigma_{\bar{\mathfrak{B}}, T \cap |\bar{\mathfrak{B}}|, \varphi}$.

But now let $t \in T \setminus \mu(\mathfrak{A})$, $t \in |\mathfrak{A}'|$, where $(\mathfrak{A}, \mathcal{S}) \leqq (\mathfrak{A}', \mathcal{S}') \in G$. We claim:

(3) t realizes $\Sigma_{\bar{\mathfrak{B}}, T \cap |\bar{\mathfrak{B}}|, \varphi}$ in $\mathfrak{A}'$,

which yields the sought-after contradiction.

PROOF OF (3). Clearly t realizes (a), (b) of (5.1.7); so let $\sigma(x)$ be a $\Psi(|\bar{\mathfrak{B}}|)$ formula such that for all $s \in T \cap |\bar{\mathfrak{B}}|$, $\bar{\mathfrak{B}} \models \sigma(s)$. But then, letting $\underset{\sim}{U}$ be the new singulary predicate symbol interpreted by $T \cap |\bar{\mathfrak{B}}|$ in $(\bar{\mathfrak{B}}, T \cap |\bar{\mathfrak{B}}|)$ and by T in $(\mathfrak{B}, T)$, we have $(\bar{\mathfrak{B}}, T \cap |\bar{\mathfrak{B}}|) \models (\forall x)(\underset{\sim}{U}(x) \Rightarrow \sigma(x))$, so $(\mathfrak{B}, T) \models (\forall x)(\underset{\sim}{U}(x) \Rightarrow \sigma(x))$, so $\mathfrak{B} \models \sigma(t)$ (since $(\mathfrak{B}, T) \models \underset{\sim}{U}(t)$). But then, since $\mathfrak{A}' \mapsto\!\!\!\!\succ \mathfrak{B}$, $\mathfrak{A}' \models \sigma(t)$; that is, t realizes all the formulas (c) of (5.1.7).

(5.1.18) We now complete the proof, by showing that $S_{\aleph_1}(\diamond) \Rightarrow (\exists G$ ideal in $\mathbf{P})$ G meets all $E^{\#}_{\alpha}$ and all $D(T, C)$.

DEFINITION. Let $h \in \mathcal{H}_A$. Set:

$$p = (\mathfrak{A}, \mathcal{S}) \in D_h \Leftrightarrow (\mathfrak{A}, \mathcal{S}) \in P \wedge a(h) \subseteq \mathrm{supp}(p) \subseteq \mathrm{ENV}(I \mid \kappa)$$

and, setting $b'(h) = \{\alpha \in a(h) \cap \omega_1 : h(\alpha) = 1\}$,

$$(*) \qquad (\exists \mathfrak{B})((\mathfrak{B}, b'(h)) \in \mathcal{S}).$$

Thus, $\mathcal{D} = (D_h : h \in \mathcal{H}_A)$ is an extension system.

Note that if $h \in \mathcal{H}_O \setminus \mathcal{H}_A$, $a = a(h)$, $h' = h^\$$, $a' = a^\$$, then:

(1) $b'(h) = b'(h') \wedge (p = (\mathfrak{A}, \mathcal{S}) \in D_h \Leftrightarrow a \subseteq \mathrm{supp}(p) \wedge X(\mathfrak{A}) \cap I$ is an end-extension of $a \cap I \wedge (\exists \mathfrak{B})((\mathfrak{B}, b'(h)) \in \mathcal{S}))$; this is clear.

Further, if $H : \mathrm{HF}(\omega_2) \to 2$ and $b'(H) = \{\alpha < \omega_1 : H(\alpha) = 1\}$, then:

(2) $b'(H)$ uncountable $\Rightarrow$ H is $\mathcal{D}$-tractable.

PROOF OF (2). Let $E \subseteq [\omega_2]^{<\omega_1}$ be stationary, let $f : E \to P$ be s.t. for all $a \in E$, $a = \mathrm{supp}(f(a))$. Let

$$C = \{a \in O : b'(H) \cap a \text{ is cofinal in } a \cap \omega_1 \text{ for the ordering of } \mathfrak{A}\}.$$

Note that for $a \in O$, letting $h = H \mid \mathrm{HF}(a)$, $b'(h) = b'(H) \cap a$, so $b'(h)$ is cofinal in $a \cap \omega_1$ (for the ordering of $\mathfrak{A}$). Let $a \in E \cap C$, and let $f(a) = (\mathfrak{A}, \mathcal{S})$. But then, applying (5.1.14), with $S = b'(H) \cap a$, there's $(\mathfrak{B}, \mathcal{S}') \geqq (\mathfrak{A}, \mathcal{S})$ with $(A, S) \in \mathcal{S}'$, and $X(\mathfrak{B}) \cap I$ an end-extension of $X(\mathfrak{A}) \cap I$, i.e. $(\mathfrak{B}, \mathcal{S}') \in D_h$.

Thus, $\mathcal{D}$ is reasonable. Now let $T \in [\omega_1]^{<\omega_2}$, and let $H_T : \mathrm{HF}(\omega_2) \to 2$ take value 1 on T and 0 elsewhere. By the above, H is $\mathcal{D}$-tractable. But now, let G be an ideal in $\mathbf{P}$ meeting all the $\mathcal{D}(H, C)$, H $\mathcal{D}$-tractable, $C \subseteq [\omega_2]^{<\omega_1}$ club. Let $C \subseteq [\omega_2]^{<\omega_1}$ be club, let $\mathfrak{B} = \bigcup\{\mathfrak{A}^p : p \in G\}$. Let $C' = \{a \in O : (\mathfrak{B} \mid a, T \cap a) \prec (\mathfrak{B}, T)\}$. But since G meets $\mathcal{D}(H_T, C \cap C')$, clearly G meets $D(T, C)$.

(5.2) *The Arhangel'skiǐ Problem*

(5.2.1) Arhangel'skiǐ proved [10]:

THEOREM. *If (X, T) is a Haudorff, Lindelöf, first countable topological space, then* card $X \leqq 2^{\aleph_0}$, *and then posed what we shall call the Arhangel'skiǐ problem, namely*:

(*): *If (X, T) is Hausdorff, Lindelöf and every point of X is a G_δ (this last property is sometimes stated as: "(X, T) has countable pseudo-character"), must we have* card $X \leqq 2^{\aleph_0}$?

i.e., in the Theorem can we replace "first countable" by "of countable pseudo-character"?

Shelah initially showed, [18], §3, the relative consistency of a negative answer to (*) with ZFC + CH (he also showed (see below, (5.2.7)) that the positive answer to (*) is consistent with ZFC + CH relative to the existence of a weakly compact cardinal). The counterexample (X, T) was, in fact, regular. Later, Hajnal and Juhász, [8], modified Shelah's conditions to find a model in which, letting $L(\mathscr{X})$ be the Lindelöf number of $\mathscr{X}$, i.e. the least κ such that any open cover of $\mathscr{X}$ has a subcover of size $\leqq \kappa$:

(**) There are spaces $\mathscr{X}$, $\mathscr{Y}$ such that $L(\mathscr{X}) = L(\mathscr{Y}) = \aleph_0$ (i.e. $\mathscr{X}$, $\mathscr{Y}$ are Lindelöf), but $L(\mathscr{X} \times \mathscr{Y}) > 2^{\aleph_0}$; further, $\mathscr{X}$, $\mathscr{Y}$ are regular and of countable pseudo-character.

Velleman, [25], has obtained these results independently.

(5.2.2) If $\operatorname{card} X \geqq 2$, let $(X)^2 = \langle (x, y) \in X \times X : x \neq y \rangle$, and suppose $F : (X)^2 \to 2$. We define topologies T_0, T_1 by taking $\mathscr{A}^0$, $\mathscr{A}^1$ as subbases, where, for $i = 0, 1$, and $x \in X$, we define $A^i_x = \{y : F(x, y) = i\}$, and we set:

$$\mathscr{A}^i = \{A^i_x \cup \{x\} : x \in X\} \cup \{A^{1-i}_x : x \in X\}. \tag{1}$$

Hajnal and Juhász called F flexible just in case whenever $y, z \in X$, $y \neq z$, there's $x \in X \setminus \{y, z\}$ such that $F(x, y) \neq F(x, z)$. Thus, it is readily seen that F is flexible iff the (X, T_i) are Hausdorff. Since the $\mathscr{A}^i$ consist of clopen sets in T_i, each (X, T_i) is totally disconnected (0-dimensional) and therefore F is flexible iff the (X, T_i) are regular.

Finally, it is readily seen that if F is flexible and if the (X, T_i) are Lindelöf then:

(2) The diagonal of $X \times X$ is a discrete subspace of $(X, T_0) \times (X, T_1)$,

(3) each (X, T_i) has countable pseudo-character (for each i we use that (X, T_{1-i}) is Lindelöf).

Thus, if we can find flexible $F : (X)^2 \to 2$ such that the (X, T_i) are Lindelöf and such that $\operatorname{card} X > 2^{\aleph_0}$, we'll have proved (**) and given a negative answer to (*). Clearly each (X, T_i) is a counterexample to (*). For (**), take $\mathscr{X} = (X, T_0)$, $\mathscr{Y} = (X, T_1)$. Then $\mathscr{X} \times \mathscr{Y}$ has a discrete subspace of size $\operatorname{card} X$ (the diagonal), and hence $L(\mathscr{X} \times \mathscr{Y}) > \operatorname{card} X > 2^{\aleph_0}$. We assume CH, and we'll take $X = \omega_2$.

(5.2.3) *The Conditions*

If $a \neq \emptyset$, let $\operatorname{Fn}(a) = \{h : h : \operatorname{dom} h \to 2, \operatorname{dom} h \in [a]^{<\omega}\}$. The motivation is as follows: a will be a countable subset of ω_2, $f : (a)^2 \to 2$ an approximation to a flexible function on $(\omega_2)^2$. If $h \in \mathrm{Fn}(a)$, then h codes a set U_h as follows:

$$U_h = \bigcap \{A_y^{h(y)} \cup \{y\} : y \in \mathrm{dom}\, h\}.$$

A slight argument, using flexibility, is required to see that given basic open sets O_0, O_1 of (a, T_0^f), (a, T_1^f), there are finite sets, e_0, e_1 and h such that $O_i = U_h \setminus e_i$ and that this is true not only of (a, T_0^f), (a, T_1^f), but of any (X, T_0^F), (X, T_1^F), where $a \subseteq X$ and $f \subseteq F$; in fact the e_i will both be subsets of dom h.

Since we intend to build F on $(\omega_2)^2$ out of countable pieces, in order to make the (ω_2, T_i) Lindelöf we cannot deal with *all* covers of these spaces at the same time as we build F. We *can*, however, handle, *countably many at a time*, countable covers of an approximation, (a, T_i^f), by basic open sets. We do this by replacing the basic open covers, $C(a)$, of (a, T_0^f), (a, T_1^f) by countable subsets of $\mathrm{Fn}(a)$; each O_i is replaced by an h as after the definition of the U_h, thereby yielding $C^{\&}(a)$. We then require that whenever (a, T_i^f) is a subspace of a larger (countable) $(a', T_i^{f'})$, then the U_h (as interpreted in $(a', T_i^{f'})$) for h in $C^{\&}(a)$ still cover a'. At the end, given a cover, $C = \{O_\xi : \xi < \theta\}$, of (ω, T_i) by basic open sets (and, of course, it suffices to consider such covers), we replace C by $C^{\&} = \{h_\xi : \xi < \theta\}$, where the h_ξ are as above for O_ξ. We then find a countable subspace (a, T_i^f) such that $\{U_{h_\xi} \cap a : h_\xi \in \mathrm{Fn}(a)\}$ is a cover of a. By abuse of notation, we shall also denote $C^{\&} \cap \mathrm{Fn}(a)$ by $C^{\&}(a)$. By density (or as below, by an argument using built-in diamond), $C^{\&}(a)$ was handled at some point, i.e. $\{U_{h_\xi} : h_\xi \in C^{\&}(a)\}$ was required to be a cover of ω_2. Now $\{O_\xi : h_\xi \in C^{\delta}(a)\}$ is not quite a cover of (ω_2, T_i), but it misses at most a subset of $\bigcup\{\mathrm{dom}\, h_\xi : h_\xi \in C^{\&}(a)\}$, which is countable, so adding in countably many more sets from the original cover, as necessary, we obtain a countable subcover. The precise definition follows.

Definition. $p \in P$ iff $p = (a^p, f^p, \mathscr{C}^p)$, where, writing $a = a^p$, $f = f^p$, $\mathscr{C} = \mathscr{C}^p$, we have: $a \in [\omega_2]^{\aleph_0}$, $f : (a)^2 \to 2$ is flexible, $\mathscr{C}$ is a countable collection of infinite subsets, C, of $\mathrm{Fn}(a)$ (thus the $C \in \mathscr{C}$ are codes, not of covers, C, but of the associated collection of slightly larger sets U) and:

(1) $(\forall C \in \mathscr{C})(\forall \alpha \in a)(\exists l \in C)\alpha \in U_l$,

(2) $(\forall C \in \mathscr{C})(\forall \gamma \in a)(\forall g \in \mathrm{Fn}(a \setminus \gamma))(\forall \alpha \in a \setminus \gamma) \quad (\exists k \in C)(\alpha \in U_{k|\gamma} \wedge g \cup k$ is a function),

(3) let $E = E^p$ be the three-place relation defined with respect to p by: $E(\delta, \zeta, \xi)$ iff $\delta \leqq \zeta, \xi$ and $(\forall \tau \in a \cap \delta) f(\tau, \zeta) = f(\tau, \xi)$ (thus, fixing δ, $E(\delta, *, *)$ is an equivalence relation on $a \setminus \delta$, which increases in strength as δ increases). We then require:

$(\forall \delta \in a)$ $(\forall \zeta \in a \setminus \delta)$ $(\forall h \in \mathrm{Fn}(a \setminus \delta))$ $(\exists \xi \in a \setminus (\delta \cup \mathrm{dom}\, h))$ $(E(\delta, \zeta, \xi) \wedge$

$(\forall\beta\in\operatorname{dom}h)f(\beta,\xi)=h(\beta))$ (such a ξ will be said to be a δ-twin of ζ which realizes h in f above δ).

If $p,q\in P$, set $p\leqq q$ iff $a^p\subseteq a^q$, $f^p\subseteq f^q$, $\mathscr{C}^p\subseteq\mathscr{C}^q$ and $E^p\subseteq E^q$. Let $\mathbf{P}=(P,\leqq)$.

Remark. A k as in (2) will be said to γ-cover α. Thus, we have a saturation property of the $C\in\mathscr{C}$: not only are they (codes of) covers by U's, but, given any g, any γ and any α, α can be γ-covered by a $k\in C$ which is compatible with g. Now, γ-covering α is easier than covering α, but it is not entirely obvious that this can be done by a $k\in C$ which is compatible with g. We shall argue, below, at the end of (5.2.6), that we can get away with having in $\mathscr{C}$ only C which have this saturation property. The key to the argument will involve a judicious use of (3).

We clearly have:

Lemma A. *$\mathbf{P}$ is countably closed with least upper bounds.*

Proof. The vector of unions of the coordinates of an increasing sequence of conditions is the lub.

(5.2.4) In proving the Extension and Amalgamation Properties, it will be convenient to divide the work into two steps: we first obtain a pseudo-condition which may fail to be flexible and may fail to satisfy (3) in the definition of P. We then show that such a pseudo-condition, q, can be extended to a real condition, p, with a^p an end-extension of a^q. Clearly, the collection of pseudo-conditions will be countably closed with least upper bounds, and we will have shown that P is dense in the collection of pseudo-conditions (where the collection of pseudo-conditions is ordered by the same ordering as $\mathbf{P}$).

The idea behind the construction of the real p extending the pseudo q is, in fact, simple, but a complete proof, if written out, is notationally complicated, so we content ourselves with a sketch. We shall obtain p as the union of an increasing sequence $(q^n:n\in\omega)$, where $q^0=q$, and $q^n=(a^n,f^n,\mathscr{C}^n)$. We shall have $\mathscr{C}^n=\mathscr{C}^0$, for all n. To obtain q^{n+1} from q^n, we enumerate in type ω all triples (δ,ζ,h) from q^n for which (3) of the definition of P requires us to find a ξ which is a δ-twin of ζ which realizes h above δ. Note that each $h\in\operatorname{Fn}(a^n)$ is handled as the first coordinate and each $\zeta\in a^n$ is handled as the second coordinate of at least one triple, since we can take $\delta=0$. For each triple (δ,ζ,h), we appoint one δ-twin, ξ, of ζ, which will realize h above δ in f^{n+1}. For the ξ's we use the next ω ordinals, α_i, larger than all the members of a^{q^n}; α_i will be used for the ith triple in our enumeration.

For flexibility, we require that if ξ, ξ' are distinct new points added as twins, then $f^{n+1}(\xi,\xi')=0$, and that if $\zeta \in a^n$, then for all i, $f^{n+1}(\alpha_i,\zeta)=1$ iff ζ is the second coordinate of the ith triple in our enumeration. We must do this in such a way as to preserve (1) and (2) in the definition of P, so we fix $i<\omega$, we let (δ,ζ,h) be ith in our enumeration, and we fix an auxiliary enumeration in type ω, this time of triples (C,γ,g), as in (2) of the definition of P, with $C\in\mathscr{C}^0$, $\gamma\in a^n$, $g\in \mathrm{Fn}(a^n\setminus\gamma)$.

We intend to have α_i play the role of α in (1), (2), so we must arrange, for each triple (C,γ,g), that there be $l,k\in C$ such that $\alpha_i\in U_l$ and α_i is γ-covered by k which is compatible with g. Note that, if $\gamma>\delta$, there is a potential conflict, here, between these new requirements and our basic requirement that α_i is to realize h above δ. We shall resolve this conflict by repeatedly using (2) for q^n, in the course of a recursive construction of the function $\pi=f^{n+1}(*,\alpha_i)\restriction(a^n\setminus\delta)$, via finite approximations, using our auxiliary enumeration.

So, we let $\pi\restriction b_0=h$ and at stage j, we suppose that we have defined $\pi\restriction b_j$, where b_j is a finite subset of $a^n\setminus\delta$; we seek to handle the jth triple (C,γ,g) in our auxiliary enumeration, thereby obtaining $\pi\restriction b_{j+1}$. We apply (2) for q^n to C, δ, $\pi\restriction b_j$, ζ to get $k^*\in C$ which δ-covers ζ and is compatible with $\pi\restriction b_j$; let $\pi^*=\pi\restriction b_j\cup k^*\restriction(a\setminus\delta)$. Now, if $\gamma\leqq\delta$, we can let $\pi\restriction b_{j+1}=\pi^*$, $l=k^*$, since ζ is δ-covered by k^*, $k^*\restriction(a^n\setminus\delta)\subseteq\pi$ and we intend to have α_i realize π. By (2), for q^n, and since $\gamma\leqq\delta\leqq\zeta$, there's $k\in C$ which γ-covers ζ and is compatible with g. But then, since we will have α_i being a δ-twin of ζ, k also γ-covers α_i. If $\gamma>\delta$, then we again apply (2), this time to C, δ, ζ and $\hat\pi=\pi^*\restriction\gamma\cup g$ to get $\hat k\in C$ which δ-covers ζ and is compatible with $\hat\pi$. Here, we let $\pi\restriction b_{j+1}=\pi^*\cup(\hat k\restriction(\gamma\setminus\delta))$, and we let $k=l=\hat k$. Then we will have k as desired, since $\hat k$ δ-covers ζ and is compatible with $g\subseteq\hat\pi$, $\hat k\restriction(\gamma\setminus\delta)\subseteq\pi$ which will be realized by α_i, and α_i will be a δ-twin of ζ. Similarly, we will have l as desired.

(5.2.5) We now verify that $\mathbf{P}\in\mathscr{S}_{\aleph_1}(\aleph_2)=\mathscr{S}_{\aleph_1}$.

Lemma B. (Extension Property, Part I). *If $r=(a,f,\mathscr{C})\in P$, $\eta\in\omega_2\setminus a$, there's $p=(a',f',\mathscr{C})\in P$, where $\eta\in a'$ and $p\leqq p'$.*

Proof. By (5.2.4), it will suffice to find a pseudo-condition q with this property. We shall have $a^q=a\cup\{\eta\}$, and for $x\in a$, we shall have $f^q(\eta,x)=0$ and the $f^q(x,\eta)$ will be defined using the method for defining $f^{n+1}(*,\alpha_i)$ in (5.2.4).

Lemma C. (Extension Property, Part II). *If $p=(a,f,\mathscr{C})\in P$, $C\in[\mathrm{Fn}(a)]^{\aleph_0}$ and C satisfies $(\forall\gamma\in a)(\forall g\in\mathrm{Fn}(a\setminus\gamma)\ldots$ (as in (2) of the definition of P), then $(a,f,\mathscr{C}\cup\{C\})\in P$.*

PROOF. Clear.

LEMMA D. **P** *is* $\aleph_1$*-special and indiscernible.*

PROOF. We briefly describe the set of indiscernibles, the supports of conditions, the set of terms, the length of a term, the action of terms on increasing sequences of indiscernibles of the proper length. The set of indiscernibles is ω_2 itself. Thus, the support of p is just a^p which is also its set of indiscernibles. $p \in \mathcal{T} \Leftrightarrow p \in P \wedge a^p \in \omega_1$; for $p \in \mathcal{T}$, $\lg p = a^p$. If $s : \lg p \to \omega_2$ is increasing, $p(s)$ is that $p' \in P$ such that:

(i) $a^{p'} = \text{range } s$,

(ii) $f^{p'}\{s(x), s(y)\} = f^p\{x, y\}$,

(iii) $\mathcal{C}^{p'} = \{\{h \circ s : h \in C\} : C \in \mathcal{C}^p\}$.

More substantial is:

LEMMA E. **P** *has the Amalgamation Property.*

PROOF. Suppose $\bar{q} \in \mathcal{T}$, $s^j : \lg p \to \omega_2$ is increasing, $j = 0, 1$, $\eta < \lg p$, $s^0 \restriction \eta = s^1 \restriction \eta$, range $s^0 \subseteq s^1(\eta)$. Let $q^j = (a^j, f^j, \mathcal{C}^j) = \bar{q}(s^j)$. Let $e = s^{0\prime\prime}\eta = s^{1\prime\prime}\eta$; $b^j = a^j \setminus e$.

We construct a pseudo-condition, $q = (a^0 \cup a^1, f^*, \mathcal{C}^0 \cup \mathcal{C}^1)$, with $f^0 \cup f^1 \subseteq f^*$ and then apply (5.2.4) to obtain a $p \in P$ with $q \leqq p$. We define $f^*(s^0(i), s^1(j)) = f^*(s^1(i), s^0(j))$, for $\eta \leqq i, j < \lg \bar{q}$ as follows. Let R be a set of representatives of $\{[i] : \eta \leqq i < \lg \bar{q}\}$, where $[i]$ is the $E^{\bar{q}}(\eta, *, *)$-equivalence class of i. For $i \in R$, let $\pi_i : (\lg \bar{q} \setminus \eta) \to 2$ be defined from η, i, $\varnothing$ as $f^{n+1}(*, \alpha_i)$ was defined from δ, ζ, h in (5.2.4). Now, for $i, j \in R$ and $i' \in [i]$, $j' \in [j]$, define $f^*(s^0(i'), s^1(j')) = f^*(s^1(i'), s^0(j')) = \pi_j(i')$. It is not difficult, but tedious, to verify we get the same result as we would, had we used j' instead of j in defining π_j.

By modding out to equivalence classes, we insured that $E^{q^1} \subseteq E^{q^*}$. The verification that the $q^i \leqq q^*$ is routine; the verification that q^* is a pseudo-condition essentially follows the arguments of the last paragraph of (5.2.4).

Thus, by Lemmas A, B, D, E, $\mathbf{P} \in \mathcal{S}_{\aleph_1}$.

(5.2.6) In this section, we show:

THEOREM. $S_{\aleph_1}(\diamondsuit) \Rightarrow$ *there's a counterexample to the Arhangel'skii problem.*

PROOF. First, let $F : (\omega_2)^2 \to 2$ be flexible, $T_i = T_i^F$, $i = 0, 1$, let $C = \{O_\xi : \xi < \theta\}$ be a cover of (ω_2, T_i) by basic open sets and h_ξ, C^δ, etc. be as in (5.2.3), preceding the Definition of P.

For $a \in [\omega_2]^{\aleph_0}$, let $\theta(a) = \{\xi < \theta : h_\xi \in \text{Fn}(a)\}$; thus, $C^\delta(a) = \{h_\xi : \xi \in \theta(a)\}$.

Finally, let

$$X_C = \{a \in [\omega_2]^{\aleph_0} : \{U_h : h \in C^\delta(a)\} \text{ is a cover of } a\}.$$

Then, it's readily seen that:

(1) X_C is club.

So, suppose that G is an ideal in $\mathbf{P}$ meeting any collection of $\aleph_1$ uniform dense sets uniformly. Let $F = \bigcup\langle f^p : p \in G\rangle$, $\mathscr{C} = \bigcup\langle \mathscr{C}^p : p \in G\rangle$. Clearly, F is flexible. The difficulty is to see that the (ω_2, T_i) are Lindelöf. Suppose that:

(2) Whenever C is a cover (henceforth understood to mean a cover of ω_2, unless otherwise indicated) in T_i, there's $p \in G$ s.t. $a^p \in X_C$ and $C^\&(a^p) \in \mathscr{C}^p$.

Claim 1. *(2) $\Rightarrow$ (ω_2, T_i) is Lindelöf, $i = 0, 1$.*

Proof. Let C, p be as in (2). We will show that $\omega_2 \setminus \bigcup\{O_\xi : h_\xi \in \mathscr{C}^\&(a^p)\} \subseteq a^p$ (and of course a^p, $C^\&(a^p)$ are countable) so, as promised in (5.2.3), C has a countable subcover. Let $\alpha \in \omega_2 \setminus a^p$. Let $p' \in G$, with $\alpha \in a^{p'}$; WLOG $p \leqq p'$. Then, by (2), and (1) of the definition of P for p', there's $l \in C^\&(a^{p'})$ with $\alpha \in U_l$. Since $\alpha \notin a^p$, in fact, $\alpha \in O_\xi$, where ξ is such that $l = h_\xi$.

It remains to show that:

Claim 2. *$S_{\aleph_1}(\diamondsuit)$ $\Rightarrow$ there's an ideal G in* $\mathbf{P}$ *satisfying* (2).

Proof of Claim 2. We define an extension system $\mathscr{D}$. For $g \in \mathscr{H}_O$, we let $C_g = \{h \in \mathrm{Fn}(a(g)) : g(h) = 1\}$, and, for $g \in \mathscr{H}_A$, we set:

$$p \in D_g \Leftrightarrow p \in P \wedge a(g) = a^p \wedge$$

(*) $\qquad (C_g$ satisfies (2) of the Definition in (5.2.3) for $p \Rightarrow C_g \in \mathscr{C}^p)$.

As usual, if $g \in \mathscr{H}_O \setminus \mathscr{H}_A$, then $p \in D_g \Leftrightarrow a^p$ is an end-extension of $a(g)$ and (*) holds.

We now show that *whenever* $H : \mathrm{HF}(\omega_2) \to 2$, H is $\mathscr{D}$-tractable, so $\mathscr{D}$ is certainly reasonable. So, let $H : \mathrm{HF}(\omega_2) \to 2$, let $S \subseteq [\omega_2]^{<\omega_1}$ be stationary and let $f : S \to P$ be s.t. $\mathrm{supp}(f(a)) = a$, say $f(a) = (a, f^a, \mathscr{C}^a)$. Let $g = H \mid \mathrm{HF}(a)$ and set:

$$\mathscr{C}' = \begin{cases} \mathscr{C}^a, & \text{if } C_g \text{ fails to satisfy (2) of the Definition in (5.2.3) for } f(a), \\ \mathscr{C}^a \cup \{C_g\}, & \text{otherwise.} \end{cases}$$

Thus, by Lemma C, $p' = (a, f^a, \mathscr{C}') \in P$, and clearly $p' \in D_h$, $p' \geqq f(a)$.

So, suppose G is an ideal in $\mathbf{P}$ meeting any collection of $\aleph_1$ uniform dense sets uniformly, and such that for all $H : \mathrm{HF}(\omega_2) \to 2$ and all club $C \subseteq [\omega_2]^{\aleph_0}$, G meets $\mathscr{D}(H, C)$. Let $F = \bigcup\{f^p : p \in G\}$, etc., and let $T_i = T_i^F$. Suppose C is a cover by basic open sets in T_i. Define $H_C : \mathrm{HF}(\omega_2) \to 2$ by: H_C takes value 1 on C, value 0 elsewhere. As above, in (1), X_C is club, so let $p \in G \cap \mathscr{D}(H_C, X_C)$, say $p \in D_g$, where $g = H_C \restriction \mathrm{HF}(a)$ and $a \in X_C$. Clearly, $C_g = C^{\&}(a)$, and so it will suffice to see that $C_g \in \mathscr{C}^p$, i.e., since $p \in D_g$, $a \in X_C$ and $a^p = a$, that C_g satisfies (2) of the Definition in (5.2.3) for p. We argue this using (3) of the Definition for p. So, let $\gamma \in a$, $\sigma \in \mathrm{Fn}(a \setminus \gamma)$ and $\alpha \in a \setminus \gamma$. By (3), taking $\delta = \gamma$, $\zeta = \alpha$, $h = \sigma$, we can find $\xi \in a \setminus (\gamma \cup \operatorname{dom} \sigma)$ which is a γ-twin of α and realizes σ in f^p above γ. By the above, we can find $k \in C_g$ with $\xi \in U_k$, and then, of course, k γ-covers ξ. But then, since ξ is a γ-twin of α, k γ-covers α. Finally, note that ξ realizes both σ and k. The former is by choice of ξ and the latter is since $\xi \in U_k$. But then, clearly σ and k are compatible, as required.

(5.2.7) We should point out that in [18], §5, Shelah proved the consistency of ZFC + CH + "the positive answer to the Arhangel'skii problem", relative to ZF + "there's a weakly compact cardinal".

§6. $(\kappa, 1)$-morasses with built-in $\diamond \Rightarrow S_\kappa(\diamond)$

In this section, we modify the construction of a sufficiently generic set in [20], §6, to show that if the morass in question has a built-in $\diamond$-sequence then, for a given reasonable extension system, $\mathscr{D}$, the sufficiently generic set can be constructed to satisfy the conclusion of $S_\kappa(\diamond)$. At the same time, in (6.2)–(6.6) we correct some inaccuracies in [20], (6.6), which Velleman, [27], pointed out.

We assume the reader is familiar with the structure of the construction of [20], §6, as outlined in (6.1), (6.2), (6.3.1), (6.4) of [20], except that the D_i and all arguments pertaining to them can now be ignored. We shall have two new induction hypotheses, (8), (9), in addition to (1)–(7) of [20], (6.2). The first is: $p(\alpha)$ is orderly (viz. (4.8), above) and satisfies:

(8): (a): $\mathrm{supp}(p(\alpha)) \supseteq \gamma(\alpha) \cup \{\bar{i}_\xi + \zeta : \xi < \rho(\alpha),\ \zeta < \gamma(\alpha)\}$,
$\rho(\alpha)$ is a limit ordinal,
(b): $(\beta \in S^0 \cap \alpha \wedge \zeta \in \mathrm{supp}(p(\beta)) \cap \kappa) \Rightarrow \zeta < \gamma(\alpha)$.

The second new induction hypothesis, (9), will be introduced below in (6.4). The auxiliary condition, $p^*(\alpha)$ of [20], (6.4.2) must now be extended to an orderly condition (viz. (4.8), above), which also satisfies (8). Obtaining $p(\alpha)$ from $p^*(\alpha)$ is routine, using (1.1.5), (1)–(15). Note that since $\mathrm{supp}(p(\alpha)) \subseteq \mathrm{ENV}(\mathrm{supp}(p(\alpha)) \cap I)$,

(!) $$\mathrm{supp}(p(\alpha)) \cap \bar{I} = \{\bar{i}_\xi : \xi < \rho(\alpha)\}.$$

The ideas of [20], (6.3.2) will reappear in (6.2), (6.3) in somewhat different form. The division into cases of [20], (6.6) is essentially unchanged, except that case (C), ν minimal in $\mapsto$, must be further subdivided into:

(C1): ν an immediate successor in S_α,

and

(C2): ν a limit in S_α.

(C1) will be handled exactly as (C) of [20], (6.6). Further, the construction of $p^*(\alpha)$ in cases (A), (B), (E1) is handled exactly as in [20], (6.6).

Some new notions are developed in (6.2), (6.3), leading to the new induction hypothesis, (9) of (6.4). The treatment of (D) is essentially as in [20], but new verifications, arising from the introduction of (9), must be made. This is in (6.5). Finally, the case (E2) is treated, this time correctly, thanks to the introduction of (9), in (6.6). The case (C2) is where we guarantee that the sufficiently generic set will satisfy the conclusion of $S_\kappa(\diamond)$. This case, and the corresponding verifications are done in (6.7). Our treatment will necessitate a further property of the $\gamma(\alpha)$'s which we introduce in (6.7) as induction hypothesis (10).

(6.1) Let I be an acceptable set of indiscernibles, let $\mathbf{P} \in \mathcal{S}_\kappa(I)$ with support function, supp, let $\mathcal{D} = (D_h : h \in \mathcal{H}_A)$ be a reasonable extension system in $\mathbf{P}$, and let

$$\mathcal{M} = (\mathcal{S}, S^0, S^1, \mapsto, \pi_{\bar{\nu}\nu})_{\bar{\nu} \mapsto \nu}$$

be a $(\kappa, 1)$-morass with built-in $\diamond$-sequence $(h_\alpha : \alpha \in (S^0)')$. By (1.1.9) and (4.7), we may assume $I = S^{\kappa}_{\kappa^+}$. Nevertheless, in the interest of readability and consistency with the notation of [20], we shall abuse notation here by denoting conditions by expressions like $\bar{p}(s)$, where $\bar{p} \in \mathcal{T}$ and $s \in [\kappa^+]^{\lg \bar{p}}$ rather than $[I]^{\lg \bar{p}}$. What we really mean is $\bar{p}(\hat{s})$, where $\hat{s}$ is as in (1.1.6). Similarly, ENV(s) means ENV$(\hat{s})$, etc. Where any ambiguity may arise, in (6.7), we will be more explicit.

(6.2) In the spirit of (6.3.2) of [20], if $\bar{\nu} \mapsto \nu$, we define $\bar{\pi}_{\bar{\nu}\nu}$:

(6.2.1) Definition. For $\bar{\eta} < \bar{\nu}$,

$$\bar{\pi}_{\bar{\nu}\nu}(\bar{\eta}) = \begin{cases} \pi_{\bar{\nu}\nu}(\bar{\eta}), & \text{if } \bar{\eta} \text{ is not a limit in } S_{\alpha_{\bar{\nu}}}, \\ \sup \pi''_{\bar{\nu}\nu}\bar{\eta}, & \text{otherwise.} \end{cases}$$

Now suppose $\bar{\eta} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$; let $\tau = \pi_{\bar{\nu}\nu}(\bar{\eta})$ and let $\eta = \bar{\pi}_{\bar{\nu}\nu}(\bar{\eta})$. We then have:

(6.2.2) PROPOSITION. (a) *For all limit* $\theta < \bar{\nu}$, $\sup \pi''_{\bar{\nu}\nu}\theta = \sup \bar{\pi}''_{\bar{\nu}\nu}\theta$,

(b) $\bar{\eta} \mapsto\!\!\!\rightarrow \eta$, $\bar{\pi}_{\bar{\eta}\eta} = \bar{\pi}_{\bar{\nu}\nu} \restriction \bar{\eta}$, range $\pi_{\bar{\eta}\eta}$ *is cofinal in* η,

(c) *if* $\bar{\eta} \mapsto\!\!\!\rightarrow \eta' \mapsto\!\!\!\rightarrow \eta$, *then* $\bar{\pi}_{\bar{\eta}\eta} = \bar{\pi}_{\eta'\eta} \circ \bar{\pi}_{\bar{\eta}\eta'}$.

PROOF. (a) is clear, since $\pi_{\bar{\nu}\nu}$ agree on a cofinal subset of θ. (b) is clear by (M2), if $\eta = \tau$, and by (M2) and (M6), if $\eta < \tau$. (c) follows readily from (b), and the observation that if $\bar{\beta} < \bar{\eta}$ and $\bar{\beta}$ is a limit in $S_{\alpha_{\bar{\eta}}}$, then, setting $\beta = \bar{\pi}_{\bar{\eta}\eta}(\bar{\beta})$, $\beta' = \bar{\pi}_{\bar{\eta}\eta'}(\beta)$ we have:

$$(*): \qquad \sup \pi''_{\eta'\eta}\beta' = \sup \pi''_{\bar{\eta}\eta}\bar{\beta} = \beta.$$

(6.3) In this section we explicitly define $<_{\mathrm{d}}$, a "diagonally below" relation, and order preserving maps $e_{\eta'\nu}$ for $\eta' <_{\mathrm{d}} \nu$. These are implicit in unpublished work of Jensen, [9], and implicit and somewhat inaccurately presented in [20], where, as Velleman observed, [27], they should have been made explicit. Clause (2) in the definition of $e_{\eta'\nu}$ corrects the corresponding erroneous clause in the definition of the h_i in [20]. The result is that (a) of (6.3.2) is now really true. Notions like these have appeared explicitly, in somewhat different form, in [25], [28] and [6]. We have chosen to define $<_{\mathrm{d}}$ in terms of $\bar{\pi}_{\bar{\nu}\nu}$ instead of $\pi_{\bar{\nu}\nu}$ in order to facilitate treatment of the case E2. As a trade off, we have some additional work to do in case (D). The nature of the problem is foreshadowed in (6.3.3).

We shall write $\bar{\nu} \mapsto\!\!\!\rightarrow_* \nu$ *to mean*: "ν *immediately succeeds* $\bar{\nu}$ *in* $\mapsto\!\!\!\rightarrow$".

(6.3.1) DEFINITION. Set $\eta' <_{\mathrm{d}} \nu$ iff:

(i) ν is an immediate successor in $\mapsto\!\!\!\rightarrow$, say of $\bar{\nu}$,

(ii) ν is an immediate successor in S_{α_ν}, *or* range $\pi_{\bar{\nu}\nu}$ is cofinal in ν,

(iii) for some (unique!) $\bar{\eta} \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$, setting

$$\eta = \bar{\pi}_{\bar{\nu}\nu}(\bar{\eta}), \qquad \bar{\eta} \mapsto\!\!\!\rightarrow_* \eta' \mapsto\!\!\!\rightarrow \eta.$$

If $\eta' <_{\mathrm{d}} \nu$, let $\bar{\alpha} = \alpha_{\bar{\nu}}$, $\alpha' = \alpha_{\eta'}$, $\alpha = \alpha_\nu$, $\bar{\gamma} = \gamma(\bar{\alpha})$, $\gamma' = \gamma(\alpha')$, $\gamma = \gamma(\alpha)$, $\bar{\rho} = \rho(\bar{\alpha})$, $\rho' = \rho(\alpha')$. Let $\sigma = \gamma' \cdot \eta'$, let $\bar{\psi} = \bar{\nu} - \bar{\eta}$, let $\sigma' = \sigma + (\bar{\gamma} \cdot \bar{\psi})$, let $\psi = \rho' - \sigma'$; we then define $e_{\eta'\nu} : \rho' \to \gamma \cdot \delta(\nu)$, by:

(1) $e_{\eta'\nu} \restriction \sigma = f_{\eta'\eta} \restriction \sigma$,

(2) $e_{\eta'\nu}(\sigma + ((\bar{\gamma} \cdot \xi) + \zeta)) = (\gamma \cdot \pi_{\bar{\nu}\nu}(\bar{\eta} + \xi)) + \zeta$, for $\xi < \bar{\psi}$, $\zeta < \bar{\gamma}$,

(3) $e_{\eta'\nu}(\sigma' + \xi) = (\gamma \cdot \nu) + \xi$, for $\xi < \psi$.

Thus, in the notation of (1.1.7), $e_{\eta'\nu} = g_1{}^\frown g_2{}^\frown g_3$, where the g_i are as appropriate for (1), (2), (3).

(6.3.2) PROPOSITION. *Suppose* $\eta' <_{\mathrm{d}} \nu$, *and let* $\bar{\nu}$, $\bar{\eta}$, η *witness this.*

(a) $e_{\eta'\nu} \circ f_{\bar{\eta}\eta'} = f_{\bar{\nu}\nu}$,

(b) *if* $\beta' <_{\mathrm{d}} \nu$ *and* $\bar{\nu}$, $\bar{\beta}$, β *witness this, then* $\bar{\beta} < \bar{\eta} \Leftrightarrow \alpha_{\beta'} < \alpha_{\eta'} \Leftrightarrow \beta' <_{\mathrm{d}} \eta'$,

(c) *if* $\beta' <_{\mathrm{d}} \eta'$ *and* $\bar{\eta}$, $\bar{\beta}$, β^* *witness this, then* $\beta' <_{\mathrm{d}} \nu$ *and* $\bar{\nu}$, $\bar{\beta}$, β *witness this, where* $\beta = \bar{\pi}_{\eta'\eta}(\beta^*)$,

(d) further, under the hypotheses of (c), $e_{\beta'\nu} = e_{\eta'\nu} \circ e_{\beta'\eta'}$.

Proof. (a) is now (really) clear by definition. For (b), clearly $\beta' <_{\mathrm{d}} \eta' \Rightarrow (\bar{\beta} < \bar{\eta} \wedge \alpha_{\beta'} < \alpha_{\eta'})$. Further, $\alpha_{\beta'} = \alpha_{\eta'} \Rightarrow \beta' = \eta'$, since $\beta' = \max S_{\alpha_{\beta'}}$, $\eta' = \max S_{\alpha_{\eta'}}$. If $\bar{\beta} = \bar{\eta}$, then $\beta' = \eta'$, since then $\beta = \bar{\pi}_{\bar{\nu}\nu}(\bar{\beta}) = \bar{\pi}_{\bar{\nu}\nu}(\bar{\eta}) = \eta$ and then β', η' are uniquely characterized by $\bar{\beta} \mapsto_* \beta' \mapsto \beta$, $\bar{\eta} \mapsto_* \eta' \mapsto \eta$. So, it suffices to show that

$$(*) \qquad \bar{\beta} < \bar{\eta} \Rightarrow (\alpha_{\beta'} < \alpha_{\eta'} \wedge \beta' <_{\mathrm{d}} \eta').$$

Proof of $(*)$. Let $\bar{\beta} < \bar{\eta}$. Then, by (6.2.2), (b), $\bar{\pi}_{\bar{\eta}\eta} = \bar{\pi}_{\bar{\nu}\nu} \restriction \bar{\eta}$, so $\bar{\pi}_{\bar{\eta}\eta}(\bar{\beta}) = \beta$; thus, by (6.2.2) (b), (c), letting $\beta^* = \bar{\pi}_{\bar{\eta}\eta'}(\bar{\beta})$, we have $\bar{\pi}_{\eta'\eta}(\beta^*) = \beta$ and $\beta^* \mapsto \beta$. But, since $\beta^* < \eta' = \max S_{\alpha_{\eta'}}$, β^* is a limit in $\mapsto$. Thus, $\beta' \mapsto \beta^*$, so $\alpha_{\beta'} < \alpha_{\beta^*} = \alpha_{\eta'}$. Then $\bar{\eta}$, $\bar{\beta}$, β^* witness that $\beta' <_{\mathrm{d}} \eta'$.

For (c), we first note that $\bar{\pi}_{\eta'\eta}(\beta^*) = \bar{\pi}_{\eta'\eta} \circ \bar{\pi}_{\bar{\eta}\eta'}(\bar{\beta}) = \bar{\pi}_{\bar{\eta}\eta}(\bar{\beta}) = \bar{\pi}_{\bar{\nu}\nu}(\bar{\beta})$. The conclusion is then clear.

We now show that $e_{\beta'\nu} = e_{\eta'\nu} \circ e_{\beta'\eta'}$. There are four intervals to consider. Let $\alpha_1 = \alpha_{\beta'}$, $\gamma_1 = \gamma(\alpha_1)$, $\rho_1 = \rho(\alpha_1)$; let $\sigma_1 = \gamma_1 \cdot \beta'$, let $\bar{\psi}_1 = \bar{\eta} - \bar{\beta}$; let $\sigma_1' = \sigma_1 + \bar{\gamma} \cdot \bar{\psi}_1$; let $\bar{\psi}$ be as before, i.e. $\bar{\psi} = \bar{\nu} - \bar{\eta}$; let $\sigma_1'' = \sigma_1' + \bar{\gamma} \cdot \bar{\psi}$; thus, $\bar{\nu} - \bar{\beta} = \bar{\psi}_1 + \bar{\psi}$ and $\sigma_1'' = \sigma_1 + \bar{\gamma} \cdot (\bar{\nu} - \bar{\beta})$. Finally, let $\psi_1 = \rho_1 - \sigma_1''$; we consider the *four* intervals:

$$I_1 = [0, \sigma_1), \quad I_2 = [\sigma_1, \sigma_1'), \quad I_3 = [\sigma_1', \sigma_1''), \quad I_4 = [\sigma_1'', \rho_1).$$

On I_1, $e_{\beta'\nu} = f_{\beta'\beta}$, where $\beta = \bar{\pi}_{\bar{\nu}\nu}(\beta)$, and $f_{\beta'\beta} = f_{\beta^*\beta} \circ f_{\beta'\beta^*}$ (recall that $\beta^* = \bar{\pi}_{\bar{\eta}\eta'}(\bar{\beta})) = f_{\eta'\eta} \circ f_{\beta'\beta^*} = e_{\eta'\nu} \circ e_{\beta'\eta'}$.

On I_2, let $\xi < \bar{\psi}_1$, $\zeta < \bar{\gamma}$; then $e_{\beta'\eta'}(\sigma_1 + ((\bar{\gamma} \cdot \xi) + \zeta)) = (\gamma' \cdot \pi_{\bar{\eta}\eta'}(\bar{\beta} + \xi)) + \zeta < \gamma' \cdot \eta'$, so

$$\begin{aligned} e_{\eta'\nu}(e_{\beta'\eta}(\sigma_1 + ((\bar{\gamma} \cdot \xi) + \zeta))) &= f_{\eta'\eta}((\gamma' \cdot \pi_{\bar{\eta}\eta'}(\bar{\beta} + \xi)) + \zeta) \\ &= (\gamma \cdot \pi_{\bar{\eta}\eta}(\bar{\beta} + \xi)) + \zeta \\ &= (\gamma \cdot \pi_{\bar{\nu}\nu}(\bar{\beta} + \xi)) + \zeta \\ &= e_{\beta'\nu}(\sigma_1 + ((\bar{\gamma} \cdot \xi) + \zeta)). \end{aligned}$$

On I_3, let $\xi < \bar{\psi}$, $\zeta < \bar{\gamma}$; then $e_{\beta'\eta'}(\sigma_1' + ((\bar{\gamma} \cdot \xi) + \zeta)) = \gamma' \cdot \eta' + (\bar{\gamma} \cdot \xi) + \zeta$, so

$$e_{\eta'\nu}(e_{\beta'\eta'}(\sigma_1' + ((\bar{\gamma}\cdot\xi)+\zeta))) = (\gamma\cdot\pi_{\bar{\nu}\nu}(\bar{\eta}+\xi))+\zeta$$
$$= (\gamma\cdot\pi_{\bar{\nu}\nu}((\bar{\beta}+\bar{\psi}_1)+\xi))+\zeta$$
$$= (\gamma\cdot\pi_{\bar{\nu}\nu}(\bar{\beta}+(\bar{\psi}_1+\xi)))+\zeta$$
$$= e_{\beta'\nu}(\sigma_1 + (\bar{\gamma}\cdot(\bar{\psi}_1+\xi))+\zeta)$$
$$= e_{\beta'\nu}(\sigma_1' + (\bar{\gamma}\cdot\xi)+\zeta).$$

Finally, on I_4, let $\xi < \psi_1$; then $e_{\beta'\eta'}(\sigma_1''+\xi) = \gamma'\cdot\eta' + \bar{\gamma}\cdot\bar{\psi}+\xi$, so $e_{\eta'\nu}(e_{\beta'\eta'}(\sigma_1''+\xi)) = e_{\eta'\nu}(\gamma'\cdot\eta'+\bar{\gamma}\cdot\bar{\psi}+\xi) = \gamma\cdot\nu+\xi = e_{\beta'\nu}(\sigma_1''+\xi)$.

(6.3.3) Note that if $\bar{\nu} \mapsto\!\!\!\!\!3_* \nu$, ν a limit in S_{α_ν} and range $\pi_{\bar{\nu}\nu}$ is not cofinal in ν, then ν has no $<_d$-predecessors. A similar conclusion holds in a slightly more complicated situation which we shall encounter in our treatment of case (D).

Suppose $\bar{\nu} \mapsto\!\!\!\!\!3_* \nu$, $\bar{\tau} \in S_{\bar{\alpha}} \cap \bar{\nu}$, where $\bar{\alpha} = \alpha_{\bar{\nu}}$, $\tau = \pi_{\bar{\nu}\nu}(\bar{\tau})$, $\tau^* = \bar{\pi}_{\bar{\nu}\nu}(\bar{\tau}) < \tau$. Let $\bar{\tau} \mapsto\!\!\!\!\!3_* \tau' \mapsto\!\!\!\!\!3\, \tau^*$, but suppose that $\tau' \not\mapsto\!\!\!\!\!3\, \tau$. We have $\tau' <_d \nu$, but we have no obvious relation between τ' and τ. So, we establish a more subtle one. We first let τ'' be defined analogously to τ' but for τ, rather than τ^*, i.e., we let $\bar{\tau} \mapsto\!\!\!\!\!3_* \tau'' \mapsto\!\!\!\!\!3\, \tau$ (note that, as for τ^*, this makes sense since $\tau \in S_\alpha \cap \nu$, so τ is a limit in $\mapsto\!\!\!\!\!3$). We first claim:

(1) range $\pi_{\bar{\tau}\tau''}$ is not cofinal in τ''.

Proof of (1). If it were, then sup range $\pi_{\tau''\tau} =$ sup range $\pi_{\bar{\tau}\tau} = \tau^*$, so $\tau'' \mapsto\!\!\!\!\!3\, \tau^*$, by (M6). But then, since $\bar{\tau} \mapsto\!\!\!\!\!3_* \tau'' \mapsto\!\!\!\!\!3\, \tau^*$, $\tau' = \tau''$ and so $\tau' \mapsto\!\!\!\!\!3\, \tau$, contradicting our hypotheses.

(2) Let $\tau_1 =$ sup range $\pi_{\bar{\tau}\tau''}$. Then, $\tau_1 \mapsto\!\!\!\!\!3\, \tau^*$.

Proof of (2). Note that $\sup \pi''_{\tau''\tau}\tau_1 = \sup \pi''_{\bar{\tau}\tau}\bar{\tau} = \tau^*$. Let $\tau_2 = \pi_{\tau''\tau}(\tau_1)$, so $\tau^* \leqq \tau_2$. But then, if $\tau^* = \tau_2$, $\tau_1 \mapsto\!\!\!\!\!3\, \tau^*$ by (M2), while if $\tau^* < \tau_2$, then $\tau_1 \mapsto\!\!\!\!\!3\, \tau_2$ by (M2), and $\tau_1 \mapsto\!\!\!\!\!3\, \tau^*$, by (M6).

(3) $\tau' \mapsto\!\!\!\!\!3\, \tau_1$; this is clear by (2) and the fact that $\bar{\tau} \mapsto\!\!\!\!\!3_* \tau$.

Let $\alpha'' = \alpha_{\tau''}$, and let $\alpha' = \alpha_{\tau'}$. Recall that $p(\alpha'')$ we constructed according to case (E1) of (6.6) of [19]. For our purposes here, the relevant consequence of this is that there's increasing h' with domain $\gamma''\cdot\tau''$, where $\gamma'' = \gamma(\alpha'')$ such that $p(\alpha'') \geqq \bar{p}(\alpha')(h' \circ f_{\tau'\tau_1})$. h' was defined by:

(a) $h' \mid \gamma''\cdot\tau_1 = \mathrm{id} \mid \gamma''\cdot\tau_1$,

(b) if $\xi < \gamma\cdot\tau'' - \gamma\cdot\tau_1$, $h'((\gamma''\cdot\tau_1)+\xi) = \gamma''\cdot\tau''+\xi$.

We shall also define, for future use in case (D), increasing h with domain $\gamma\cdot(\tau+\alpha)$, where $\alpha = \alpha_\nu = \alpha_\tau$, $\gamma = \gamma(\alpha)$ by:

(c) $h \mid \gamma\cdot\tau = \mathrm{id} \mid \gamma\cdot\tau$,

(d) for $\xi < \bar{\nu} - \bar{\tau}$, $\zeta < \bar{\gamma}$, $h((\gamma \cdot \tau) + (\bar{\gamma} \cdot \xi) + \zeta) = (\gamma \cdot \pi_{\bar{\nu}\nu}(\bar{\tau} + \xi)) + \zeta$,

(e) for $\xi < \gamma \cdot (\tau + \alpha) - [(\gamma \cdot \tau) + \bar{\gamma} \cdot (\bar{\nu} - \bar{\tau})]$, $h((\gamma \cdot \tau) + \bar{\gamma} \cdot (\bar{\nu} - \bar{\tau}) + \xi) = \gamma \cdot \nu + \xi$.

Note that:

(f) $h \restriction \gamma \cdot (\tau + \bar{\alpha}) = \mathrm{id} \restriction \gamma \cdot (\tau + \bar{\alpha})$, $\gamma \cdot (\tau + \alpha) = h(\gamma \cdot (\tau + \bar{\alpha}))$.

More substantial is:

(4) $e_{\tau'\nu} = h \circ f_{\tau''\tau} \circ h' \circ f_{\tau'\tau_1}$.

Proof of (4). First, $h' \restriction f''_{\tau'\tau_1}\gamma' \cdot \tau' = \mathrm{id} \restriction f''_{\tau'\tau_1}\gamma' \cdot \tau'$, so $f_{\tau''\tau} \circ h' \circ f_{\tau'\tau_1} \restriction \gamma' \cdot \tau' = f_{\tau''\tau} \circ f_{\tau'\tau_1} \restriction \gamma' \cdot \tau' = f_{\tau_1\tau_2} \circ f_{\tau'\tau_1} \restriction \gamma' \cdot \tau' = f_{\tau_1\tau^*} \circ f_{\tau'\tau_1} \restriction \gamma' \cdot \tau' = f_{\tau'\tau^*} \restriction \gamma' \cdot \tau' = e_{\tau'\nu} \restriction \gamma' \cdot \tau'$, where, of course, $\gamma' = \gamma(\alpha')$, $\alpha' = \alpha_{\tau'}$.

Second, setting $\sigma = \gamma' \cdot \tau'$, $\bar{\psi} = \bar{\nu} - \bar{\tau}$, for $\xi < \bar{\psi}$, $\zeta < \bar{\gamma}$, $h'(f_{\tau'\tau_1}(\sigma + ((\bar{\gamma} \cdot \xi) + \zeta))) = h'(\gamma'' \cdot \tau_1 + \bar{\gamma} \cdot \xi + \zeta) = \gamma'' \cdot \tau'' + \bar{\gamma} \cdot \xi + \zeta$ and so $f_{\tau''\tau}(h'(f_{\tau'\tau_1}(\cdots))) = \gamma \cdot \tau + \bar{\gamma} \cdot \xi + \zeta$, so $h(f_{\tau''\tau}(h'(f_{\tau'\tau_1}(\cdots)))) = (\gamma \cdot \pi_{\bar{\nu}\nu}(\bar{\tau} + \xi)) + \zeta = e_{\tau'\nu}(\sigma + ((\bar{\gamma} \cdot \xi) + \zeta))$.

Finally, if $\sigma' = \sigma + \bar{\gamma} \cdot \bar{\psi}$, $\psi = \rho' - \sigma'$ (where $\rho' = \rho(\alpha')$), then, for $\xi < \psi$,

$$\begin{aligned} h(f_{\tau''\tau}(h'(f_{\tau'\tau_1}(\sigma' + \xi)))) &= h(f_{\tau''\tau}(h'((\gamma'' \cdot \tau_1) + (\bar{\gamma} \cdot \bar{\psi}) + \xi))) \\ &= h(f_{\tau''\tau}((\gamma'' \cdot \tau'') + (\bar{\gamma} \cdot \bar{\psi}) + \xi)) \\ &= h((\gamma \cdot \tau) + (\bar{\gamma} \cdot \bar{\psi}) + \xi) \\ &= \gamma \cdot \nu + \xi \\ &= e_{\tau'\nu}(\sigma' + \xi). \end{aligned}$$

(6.4) In addition to the induction hypotheses (1)–(7) of [20], (6.2), and (8) above, we shall also require:

(9) Suppose $\nu = \max S_\alpha$ and $\eta' <_d \nu$. Let $\alpha' = \alpha_{\eta'}$; then, $p(\alpha) \geqq \bar{p}(\alpha')(e_{\eta'\nu})$.

Of course, this is vacuous unless ν falls under case (D) or case (E2) of [20], (6.6).

As mentioned above, we still adopt the strategy of constructing an auxiliary condition $p^*(\alpha)$, as in [20], (6.4.2). In addition to the properties (1′)–(3′) of [20], (6.4.2), $p^*(\alpha)$ shall also satisfy:

(4′) if $\nu = \max S_\alpha$ and $\eta' <_d \nu$, then $p^*(\alpha) \geqq \bar{p}(\alpha')(e_{\eta'\nu})$, where $\alpha' = \alpha_{\eta'}$.

Of course, (4′) guarantees that the induction hypothesis, (9), is preserved.

(6.5) In case (D), $\bar{\nu} \mapsto_* \nu$, and ν immediately succeeds τ in S_α. We set $\bar{\alpha} = \alpha_{\bar{\nu}}$. We had $\theta' = \tau + \alpha$, $\bar{\alpha} = \alpha_{\bar{\nu}}$ and we let $\bar{\tau}$ immediately precede $\bar{\nu}$ in $S_{\bar{\alpha}}$, so $\pi_{\bar{\nu}\nu}(\bar{\tau}) = \tau$. We took q to be an upper bound for the directed set $\{p(\bar{\eta}, \eta) : \eta \in$

$S_\alpha \cap \nu$, $\bar\eta \mapsto \eta\}$, with $q = \tilde q(\mathrm{id} \mid \theta')$. Let $\tau^* = \bar\pi_{\bar\nu\nu}(\bar\tau)$. Let $\bar\tau \mapsto_* \tau' \mapsto \tau^*$. Then, by (6.3.2), (b), if $\eta' <_{\mathrm{d}} \nu$ then either:

(1) $\eta' = \tau'$ or

(2) $\eta' <_{\mathrm{d}} \tau'$.

Therefore, setting $\alpha' = \alpha_{\tau'}$, by (8) for α', if $\eta' <_{\mathrm{d}} \tau'$ then $p(\alpha') \geqq \bar p(\alpha_{\eta'})(e_{\eta'\tau'})$. Therefore, in order to show (4′), and hence (9) hold for α, if will suffice to show that $p(\alpha)(= p^*(\alpha)) \geqq \bar p(\alpha')(e_{\tau'\nu})$, since then, by indiscernibility and (6.3.2)(d), if $\eta' <_{\mathrm{d}} \nu$ and $\eta' \neq \nu$, then

$$p(\alpha) \geqq \bar p(\alpha')(e_{\tau'\nu}) \geqq \bar p(\alpha_{\eta'})(e_{\tau'\nu} \circ e_{\eta'\tau'}) = p(\alpha_{\eta'})(e_{\eta'\nu}).$$

Further, by (6.3.2)(a), we'll have $p(\alpha) \geqq p(\bar\nu, \nu)$, since $p(\bar\nu, \nu) = \bar p(\bar\alpha)(f_{\bar\nu\nu})$, but $p(\alpha') \geqq \bar p(\bar\alpha)(f_{\bar\tau\tau'})$ so

$$p(\alpha) \geqq \bar p(\alpha')(e_{\tau'\nu}) \geqq \bar p(\bar\alpha)(e_{\tau'\nu} \circ f_{\bar\tau\tau'}) = \bar p(\bar\alpha)(f_{\bar\nu\nu}).$$

We adopt the notation of (6.3.3). Let $\alpha'' = \alpha_{\tau''}$. Then $p(\alpha'')$ was constructed according to case (E1), so $p(\alpha'') \geqq \bar p(\alpha')(h' \circ f_{\tau'\tau_1})$. Further,

$$q \geqq p(\tau'', \tau) = \bar p(\alpha'')(f_{\tau''\tau}) \geqq \bar p(\alpha')(f_{\tau''\tau} \circ h' \circ f_{\tau'\tau_1}).$$

Recall that $q = \tilde q(\mathrm{id} \mid \gamma \cdot \theta')$. Let $q' = \tilde q(h)$, where h is as defined in (6.3.3). By indiscernibility,

$$q' \geqq \bar p(\alpha'')(h \circ f_{\tau''\tau}) \geqq \bar p(\alpha')(h \circ f_{\tau''\tau} \circ h' \circ f_{\tau'\tau_1}) = \bar p(\alpha')(e_{\tau'\nu}),$$

by (6.3.3), (4). Further, by (6.3.3), (f), q, q' satisfy the hypotheses of the amalgamation property, so we choose a common extension, r, and if ncecessary extend r to $p^*(\alpha) = p(\alpha)$ of the form $\bar p(\alpha)(\mathrm{id} \mid \rho(\alpha))$, for some $\rho(\alpha) < \kappa$. Note, we automatically have $\rho(\alpha) \geqq \gamma \cdot (\nu + \alpha)$, as required.

(6.6) In case (E2), by (6.3.2)(b), $X = \{\eta' : \eta' <_{\mathrm{d}} \nu\}$ is well-ordered by $<_{\mathrm{d}}$ in type o.t. $(S_{\bar\alpha} \cap \bar\nu)$. Further, $\{\alpha_{\eta'} : \eta' \in X\}$ is normal and cofinal in α, by (M7) and the arguments of [20], (6.3.2). Further, if $\eta'_1 <_{\mathrm{d}} \eta'_2$, both in X, then, by (8) for $\alpha_{\eta'_2}$, $\bar p(\alpha_{\eta'_1})(e_{\eta'_1\eta'_2}) \leqq p(\alpha_{\eta'_2})$. So, by indiscernibility, and (6.3.2), (d),

$$\bar p(\alpha_{\eta'_2})(e_{\eta'_2\nu}) \geqq \bar p(\alpha_{\eta'_1})(e_{\eta'_2\nu} \circ e_{\eta'_1\eta'_2}) = \bar p(\alpha_{\eta'_1})(e_{\eta'_1\nu}).$$

So, for $\eta' \in X$, setting $q_{\eta'} = \bar p(\alpha_{\eta'})(e_{\eta'\nu})$, $(q_{\eta'} : \eta' <_{\mathrm{d}} \nu)$ is increasing. Further, for any $\eta' \in X$, by (6.3.2)(a), $p(\bar\nu, \nu) \leqq q_{\eta'}$. We take $p^*(\alpha)$ to be an upper bound for the $q_{\eta'}$, $\eta' \in X$, with $p^*(\alpha) = \bar p^*(\alpha)(\mathrm{id} \mid \gamma \cdot (\nu + \alpha))$, $\mathrm{supp}(p^*(\alpha)) = \bigcup\{\mathrm{supp}(q_{\eta'}) : \eta' \in X\}$. Note that by (4.9), $p^*(\alpha)$ will be orderly, so we can take $p(\alpha) = p^*(\alpha)$. It remains to verify that if $\tau \in S_\alpha \cap \nu$ and $\tau' \mapsto \tau$, then for some

$\eta' \in X$, $p(\tau', \tau) \leqq q_{\eta'}$. This is substantially as in [20], (6 o): we first find $\bar{\tau}_1 \in S_{\bar{\alpha}} \cap \bar{\nu}$ such that $\pi_{\bar{\nu}\nu}(\bar{\tau}_1) > \tau$. We set $\tau_1 = \pi_{\bar{\nu}\nu}(\bar{\tau}_1)$. We then find $\alpha_1 \in S^0 \cap \alpha$ such that $\alpha_1 > \alpha_{\tau'}$ and such that there's $\tau_1' \mapsto\!\!\!\rightarrow \tau_1$ with $\tau \in \text{range}\, \pi_{\tau_1'\tau_1}$, say $\pi_{\tau_1'\tau}(\tau'') = \tau$. Then, $\tau'' \mapsto\!\!\!\rightarrow \tau$ and $\alpha_{\tau''} = \alpha' > \alpha_{\tau'}$, so $\tau' \mapsto\!\!\!\rightarrow \tau''$. Therefore, $p(\tau', \tau'') \leqq p(\alpha')$.

Finally, note that if we set:

$$\eta' \in X' \Leftrightarrow \eta' \in X \text{ and, letting } \bar{\eta} \mapsto\!\!\!\rightarrow_* \eta',\ \bar{\eta} \text{ is an immediate successor in } S_{\bar{\alpha}},$$

then X' is cofinal in X under $<_d$, so $\{\alpha_{\eta'} : \eta' \in X\}$ is cofinal in α. Thus, we can choose $\eta' \in X'$ such that $\alpha_{\eta'} > \alpha'$, and such that, letting $\bar{\eta} \mapsto\!\!\!\rightarrow_* \eta'$, $\bar{\eta} > \bar{\tau}_1$. Since $\eta' \in X'$, letting $\eta = \bar{\pi}_{\bar{\nu}\nu}(\bar{\eta})$, in fact $\eta = \pi_{\bar{\nu}\nu}(\bar{\eta})$. Thus, $\tau_1 \in \text{range}\, \pi_{\bar{\eta}\eta}$, so $\tau_1 \in \text{range}\, \pi_{\eta'\eta}$; say $\pi_{\eta'\eta}(\tau_1'') = \tau_1$. Then $\tau_1'' \mapsto\!\!\!\rightarrow \tau_1$ and $\alpha_{\tau_1'} = \alpha' < \alpha_{\eta'} = \alpha_{\tau_1''}$, so $\tau_1' \mapsto\!\!\!\rightarrow \tau_1''$. Thus, $p(\tau_1', \tau_1'') \leqq p(\alpha_{\eta'})$. Further, $\text{range}\, \pi_{\tau_1''\tau_1} \supseteq \text{range}\, \pi_{\tau_1'\tau_1}$, so $\tau \in \text{range}\, \pi_{\tau_1''\tau_1}$, say $\pi_{\tau_1''\tau_1}(\tau''') = \tau$. Then $\tau'' \mapsto\!\!\!\rightarrow \tau'''$, since $\alpha_{\tau''} = \alpha_{\tau_1'} < \alpha_{\tau_1''} = \alpha_{\tau'''}$.

Thus $p(\tau', \tau''') \leqq p(\alpha^*)$, where $\alpha^* = \alpha_{\eta'}$. But $e_{\eta'\nu} \restriction \gamma(\alpha^*) \cdot (\tau''' + \alpha^*) = f_{\tau'''\tau} \restriction \gamma(\alpha^*) \cdot (\tau''' + \alpha^*)$, and $\text{range}\, f_{\tau'\tau'''} \subseteq \gamma(\alpha^*) \cdot (\tau''' + \alpha^*)$. Thus, by indiscernibility,

$$\bar{p}(\alpha^*)(e_{\eta'\nu}) \geqq \bar{p}(\alpha_{\tau'})(f_{\tau'''\tau} \circ f_{\tau'\tau'''}) = \bar{p}(\alpha_{\tau'})(f_{\tau'\tau}) = p(\tau'\tau);$$

i.e. $q_{\eta'} \geqq p(\tau', \tau)$.

(6.7) In case (C2), we guarantee that G will satisfy the conclusion of $S_\kappa(\diamond)$ for $\mathscr{D} = (D_h : h \in \mathscr{H}_A)$. Let $(\bar{i}_\xi : \xi < \kappa^+)$ increasingly enumerate $\bar{I}$; thus, for limit $\xi < \kappa^+$, if $\text{cf}\, \xi < \kappa$, then $\bar{i}_\xi \notin I$ and for $n < \omega$, $i_{\xi+n} = \bar{i}_{\xi+1+n} = \bar{i}_\xi + ((1+n) \cdot \kappa)$.

Recall that $(h_\alpha : \alpha \in (S^0)')$ is our built-in $\diamond$-sequence, so for $\alpha \in (S^0)'$, letting $\bar{\nu} = \max S_\alpha$, $a(h(\alpha)) = a(\alpha) = \alpha \cup \{\bar{i}_\xi + \zeta : \xi < \bar{\nu},\ \zeta < \alpha,\ \xi \text{ is controlled}\} = \alpha \cup \{\bar{i}_\xi + \zeta : \xi < \alpha \cdot \bar{\nu},\ \zeta < \alpha,\ \xi \text{ is controlled}\}$. Thus, if $\alpha = \gamma(\alpha)$ (which will happen on a club of $(S^0)'$), $a(h(\alpha)) = \gamma(\alpha) \cup \{\bar{i}_\xi + \zeta : \xi < \gamma(\alpha) \cdot \bar{\nu}, \zeta < \gamma(\alpha), \xi$ is controlled$\} = \bigcup\{\text{supp}(p(\bar{\tau}, \tau) : \bar{\tau} \mapsto\!\!\!\rightarrow \tau \in S_\alpha \cap \bar{\nu}\}$, since $\text{supp}\, p(\bar{\tau}, \tau) = \bar{f}_{\bar{\tau}\tau}\, \text{supp}(p(\alpha_{\bar{\tau}}))$.

(6.7.1) So, suppose we're in case (C2), i.e. $\alpha \in (S^0)'$. We take $q =$ an upper bound for $\{p(\bar{\tau}, \tau) : \bar{\tau} \mapsto\!\!\!\rightarrow \tau \in S_\alpha \cap \nu\}$, with

$$\text{supp}(q) = \bigcup\{\text{supp}(p(\bar{\tau}, \tau)) : \bar{\tau} \mapsto\!\!\!\rightarrow \tau \in S_\alpha \cap \bar{\nu}\}.$$

Thus, by (4.8), q is orderly, and $\text{supp}(q) = \bigcup\{\bar{f}''_{\bar{\tau}\tau}\, \text{supp}(p(\alpha_{\bar{\tau}})) : \bar{\tau} \mapsto\!\!\!\rightarrow \tau \in S_\alpha \cap \bar{\nu}\} = \gamma(\alpha) \cup \{\bar{i}_\xi + \zeta : \xi < \gamma(\alpha) \cdot \bar{\nu},\ \zeta < \gamma(\alpha),\ \xi \text{ is controlled}\}$. Thus, if in addition, $\alpha = \gamma(\alpha)$, $\text{supp}(q) = a(h(\alpha)) = a(\alpha)$.

(6.7.2) We now obtain $p^*(\alpha)$ as follows: We first obtain $q^* \geqq q$ and then extend q^* to $p^*(\alpha)$ satisfying (1') of [20], (6.4.2).

If there's no $q' \in D_{h(\alpha)}$ with $q' \geqq q$ we set $q^* = q$. Otherwise, we choose $q^* \geqq q$, $q^* \in D_{h(\alpha)}$.

(6.7.3) We now verify that if G is the ideal generated by $\{p(\bar{\nu}, \nu) : \bar{\nu} \mapsto \nu \in S_\kappa\}$, then, whenever $H : \mathrm{HF}(\kappa^+) \to 2$ is $\mathscr{D}$-tractable and $C \subseteq [\kappa^+]^{<\kappa}$ is club then G meets $\mathscr{D}(H, C)$. So, let H, C be as above. Let $\alpha \in \Gamma$ iff $\alpha \in S^0 \wedge \gamma(\alpha) = \alpha$. Thus, Γ is club in S^0, and so, if we set $a \in \tilde{\Gamma}$ iff $a \cap \kappa \in \Gamma$, then $\tilde{\Gamma}$ is club in $[\kappa^+]^{<\kappa}$.

By the oracle property of the built-in $\diamondsuit$-sequence, S_H is stationary (viz (4.11)). Note further, that if $a \in S_H$, $\alpha = \alpha(a)$, $\nu = \nu(a)$, $\bar{\nu} = \bar{\nu}(a)$, then, as in (4.12), letting $h = H \restriction \mathrm{HF}(a)$,

$$\sigma_{\bar{\nu}\nu} = \sigma_a, \quad a \in O, \quad a = \tilde{\sigma}''_{\bar{\nu}\nu} a(\alpha). \tag{1}$$

Now, let $S' = S_H \cap C \cap \tilde{\Gamma}$, so S' is stationary. Let $a \in S'$, $\alpha = \alpha(a)$, $\nu = \nu(a)$, $\bar{\nu} = \bar{\nu}(a)$, $h = H \restriction \mathrm{HF}(a)$. By (1), $\alpha = a \cap \kappa$. Since $a \in \tilde{\Gamma}$, $\alpha = \gamma(\alpha) \in (S^0)'$. Let $q = \bar{q}((\mathrm{id} \restriction \gamma \cdot \bar{\nu})^\wedge)$ be as in (6.7.1), so, as we observed, q is orderly. Therefore, by induction hypothesis (8) and the fact that α is a limit in S^0, $\alpha = \gamma(\alpha) = \sup\{\gamma(\beta) : \beta \in S^0 \cap \alpha\}$, so we have, by (6.7.1):

$$\mathrm{supp}(q) = a(\alpha). \tag{2}$$

Now, let $q(\bar{\nu}, \nu) = \bar{q}(\hat{\pi}_{\bar{\nu}\nu}) = \bar{q}(\sigma_{\bar{\nu}\nu} \circ (\mathrm{id} \restriction (\gamma \cdot \bar{\nu}))^\wedge)$ (in previous subsections of this section we would have denoted $q(\bar{\nu}, \nu)$ by $\bar{q}(\pi_{\bar{\nu}\nu})$, abusing notation). But from (1), (2), we immediately have:

$$\mathrm{supp}(q(\bar{\nu}, \nu)) = a. \tag{3}$$

Now, define $f : S' \to P$ by setting $f(a) = q(\bar{\nu}, \nu)$, where $\bar{\nu} = \bar{\nu}(a)$, $\nu = \nu(a)$. Thus, for all $a \in S'$ $\mathrm{supp}(f(a)) = a$. But then, since H is $\mathscr{D}$-tractable, there's $a \in S'$ such that, setting $h = H \restriction \mathrm{HF}(a)$, $\bar{\nu} = \bar{\nu}(a)$, $\nu = \nu(a)$, there's $p \in D_h$ with $p \geqq q(\bar{\nu}, \nu)$. Fix such a, $\bar{\nu}$, ν; let $h = H \restriction \mathrm{HF}(a)$, $\alpha = \alpha(a) = \alpha_{\bar{\nu}} = a \cap \kappa$. But then, since $h = \tilde{\sigma}_{\bar{\nu}\nu}[h_\alpha]$, $D_h = \pi^*_{\bar{\nu}\nu}[D_{h_\alpha}] = \sigma_{\bar{\nu}\nu}[D_{h_\alpha}]$ and so there's $p \in D_{h_\alpha}$ with $p \geqq q$. But then, letting q^* be as in (6.7.2), $q^* \in D_{h_\alpha}$ and $p(\alpha) \geqq q^* \geqq q$. We need one last technical observation:

$$f_{\bar{\nu}\nu} \restriction \gamma \cdot \bar{\nu} = \pi_{\bar{\nu}\nu}; \tag{4}$$

this is clear since $\gamma(\kappa) = \kappa$, $\gamma = \alpha$, $\bar{\nu} = \alpha \cdot \bar{\nu} = \gamma \cdot \bar{\nu}$, since $\pi_{\bar{\nu}\nu} \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi_{\bar{\nu}\nu}(\alpha) = \kappa$, and since $\pi_{\bar{\nu}\nu}$ preserves ordinal arithmetic: thus, for $\xi < \bar{\nu}$, $\zeta < \gamma$, $(\gamma \cdot \xi) + \zeta < \bar{\nu}$ and

$$f_{\bar{\nu}\nu}((\gamma \cdot \xi) + \zeta) = \kappa \cdot \pi_{\bar{\nu}\nu}(\xi) + \zeta = \pi_{\bar{\nu}\nu}((\gamma \cdot \xi)) + \zeta = \pi_{\bar{\nu}\nu}((\gamma \cdot \xi) + \zeta).$$

Now, let $q^* = \bar{q}^*(\hat{s})$. Then, by (4), $\bar{q}^*(f^*_{\bar{\nu}\nu} \circ \hat{s}) \in D_h$ and $p(\bar{\nu}, \nu) \geqq \bar{q}^*(f^*_{\bar{\nu}\nu} \circ \hat{s})$, so $\bar{q}^*(f_{\bar{\nu}\nu} \circ \hat{s}) \in G$. Thus, G meets $D_h \subseteq \mathcal{D}(H, C)$, as required.

(6.8) Remarks. If the built-in $\diamond$-sequence satisfies the stronger oracle property, (**), below, then the ideal G constructed above will have a correspondingly stronger property, but which is somewhat technical to state and for which we have, as yet, no applications.

(6.8.1) The following strengthened oracle property is the analogue of the strengthened $\diamond$-principle ($\forall$ stationary $E \subseteq \kappa$) $\diamond_\kappa(E)$:

(**): whenever $H : \mathrm{HF}(\kappa^+) \to 2$, $E \subseteq \{\tilde{\sigma}_{\bar{\nu}\nu} a_\alpha : \alpha \in (S^0)', \ \bar{\nu} = \max S_\alpha, \ \bar{\nu} \mapsto\!\!\!\!\!\!\rightarrow \nu \in S_\kappa\}$ is stationary, then $S_H \cap E$ is stationary, where S_H is as in (4.12).

(6.8.2) It should be clear from the proof in (6.7.3), above, that if the built-in $\diamond$-sequence satisfies (**) of (6.8.1), then in fact the ideal G constructed above satisfies a strengthened version of $S_\kappa(\diamond)$:

(***): whenever $H : \mathrm{HF}(\kappa^+) \to 2$ is $\mathcal{D}$ tractable, $E \subseteq \{\tilde{\sigma}''_{\bar{\nu}\nu} a_\alpha : \alpha \in (S^0)', \ \bar{\nu} = \max S_\alpha, \ \bar{\nu} \mapsto\!\!\!\!\!\!\rightarrow \nu \in S_\kappa\}$ is stationary, G meets $\mathcal{D}(H, E)$.

The defect of (***) as a forcing principle is, of course, that it is not intrinsic — it mentions morass notions in a seemingly essential way.

It should also be clear from the proofs in §§7, 8, below, that morasses whose built-in $\diamond$-sequences satisfy (6.8.1) (**) exist in L, and that if G is an ideal in the partial order $\tilde{\mathbf{P}}$ of §8, which satisfies (***) for the "generic pre-morass", then the built-in $\diamond$-sequence of the thinned morass will satisfy (6.8.1) (**).

§7. Morasses with built-in $\diamond$ in L

In this section, we build upon the construction of natural morasses in L (see [23], §3) to show that there are morasses with built-in $\diamond$ in L, thus proving Theorem 5 of §0. We shall need to modify the morass construction of [23] only slightly; we'll also define a sequence $(h_\alpha : \alpha \in (S^0)')$ for *this morass*, and show that, for this morass, it's a built-in $\diamond$-sequence. Solovay, [21], and Devlin, [4], have proved similar theorems in unpublished work. In fact, our formulation of a morass with built-in $\diamond$ and proof in L owe much to Solovay's [21] (although the formulation and the existence proof in L have a certain air of inevitability).

(7.1) We should immediately point out several differences in notation between this paper and [23]:

(a) the tree relation, $\mapsto\!\!\!\!\!\!\rightarrow$, is denoted by T in [23],

(b) when $\bar{\nu} \mapsto\!\!\!\!\!\!\rightarrow \nu$, the tree-map $\pi_{\bar{\nu}\nu}$ is denoted by $f_{\bar{\nu}\nu}$ in [23].

This said, we recall the salient properties of the morass constructed in [23] (corresponding to (3.14)–(3.23) of [23]).

(1) $\nu \in S^1 \Leftrightarrow \nu < \kappa^+$, ν is not a cardinal, ν a limit of ordinals τ such that $L_\tau \models \mathrm{ZF}^-$, and: $L_\nu \models$ "there's a largest cardinal which is regular"; the largest cardinal of L_ν is denoted α_ν.

(2) $S^0 = \{\alpha_\nu : \nu \in S^1\}$; for $\alpha \in S^0$, $S_\alpha = \{\nu \in S^1 : \alpha = \alpha_\nu\}$; if $\alpha \in S^0 \cap \kappa$, S_α has a largest element.

(3) For $\nu \in S^1$, $\beta(\nu) =$ the least $\beta \geqq \nu$ s.t. $J_{\beta+1} \models$ "ν is not a cardinal"; $n(\nu) =$ the least n s.t. there's a $\Sigma_{n+1}(J_{\beta(\nu)})$ map of a subset of some $\eta < \nu$ onto ν; $\rho(\nu) = \rho^{n(\nu)}_{\beta(\nu)}$, $A(\nu) = A^{n(\nu)}_{\beta(\nu)}$. Then: $\omega \cdot \rho^{n(\nu)+1}_{\beta(\nu)} \leqq \alpha_\nu$, $\nu \leqq \rho(\nu)$.

(4) $p(\nu)$ is the $<_J$-least $p \in J_\rho$ s.t. $J_\rho = h''(\omega \times \alpha_\nu \times \{p\})$, where h is the canonical Σ_1-Skolem function for the amenable structure $(J_{\rho(\nu)}, A(\nu))$.

A new feature we introduce here is that we set $p'(\nu) = p(\nu)$, unless $n(\nu) = 0$ and $\beta(\nu) = \rho(\nu)$ is a successor; when this occurs, we set $p'(\nu) = (p(\nu), \gamma)$, where $\beta(\nu) = \rho(\nu) = \gamma + 1$.

(5) $\mathfrak{A}_\nu = (J_{\rho(\nu)}, A(\nu), p'(\nu))$, and $\bar\nu \mapsto\!\!\!\!\rightarrow \nu$ iff $n(\bar\nu) = n(\nu)$, $\rho(\bar\nu) = \bar\nu$ iff $\rho(\nu) = \nu$ and:

(*) there's $\pi : \mathfrak{A}_{\bar\nu} \to_{\Sigma_1} \mathfrak{A}_\nu$ such that $\pi \restriction \alpha_{\bar\nu} = \mathrm{id} \restriction \alpha_{\bar\nu}$, $\pi(\alpha_{\bar\nu}) = \alpha_\nu$, $\pi \restriction L_{\bar\nu} : L_{\bar\nu} \to_Q L_\nu$, and if $\bar\nu < \rho(\bar\nu)$ then $\pi(\bar\nu) = \nu$.

(6) If $\bar\nu \mapsto\!\!\!\!\rightarrow \nu$, then there's a unique π witnessing (*), and this π is denoted by $\pi_{\bar\nu\nu}$.

(7.2) Our definition of the built-in $\diamondsuit$-sequence $(h_\alpha : \alpha \in (S^0)')$ will involve defining an auxiliary sequence $((\bar h_\alpha, \bar c_\alpha) : \alpha \in (S^0)')$, where, for $\alpha \in (S^0)'$, letting $\nu^* = \max S_\alpha$, $\rho^* = \rho(\nu^*)$, we'll have $(\bar h_\alpha, \bar c_\alpha) \in J_{\rho^*}$ (in cases of interest), $\bar h_\alpha : \mathrm{HF}(\nu^*) \to 2$, $\bar c_\alpha$ a club subset of $\{a \subseteq \nu^* : a \in J_{\rho^*} \wedge J_{\rho^*} \models \text{"card } a < \alpha\text{"}\}$ (henceforth, this set is denoted by: $([\nu^*]^{<\alpha})_{J_{\rho^*}}$). Then, referring back to (4.14), we shall take:

$$h_\alpha = \delta_\alpha[\bar h_\alpha] = \{(\delta^{\#}_\alpha(a), \bar h_\alpha(a)) : a \in \mathrm{HF}(\nu^*)\},$$

and we shall also set $c_\alpha = \{\delta''_\alpha a : a \in \bar c_\alpha\}$. The initial segments $((\bar h_\alpha, \bar c_\alpha) : \alpha \in (S^0)' \cap \beta)$ of the auxiliary sequence will be uniformly $\Sigma_1(L_\beta)$, $\beta \in S^0$. We now turn to the details:

Definition. Let $\beta \in (S^0)'$ and suppose $((\bar h_\alpha, \bar c_\alpha) : \alpha \in (S^0)' \cap \beta)$ has been defined correctly. Let $\nu^* = \max S_\beta$, let $\rho^* = \rho(\nu^*)$. Then, $(\bar h_\beta, \bar c_\beta)$ will be the $<_J$-least pair $(\bar h, \bar c) \in J_{\rho^*}$ satisfying (a), (b), below, if (Case 1) there is such a pair; $(\bar h_\beta, \bar c_\beta) =$ (the constant function with value 1 on $\mathrm{HF}(\nu^*)$, $([\nu^*]^{<\alpha})_{J_{\rho^*}}$) otherwise

(Case 2). Here, (a), (b) are:

(a) in the sense of J_{ρ^*}, $\bar{c}$ is a club subset of $[\nu^*]^{<\alpha}$,

(b) whenever $\tau \in S_\beta \cap \nu^*$, $\alpha \in (S^0)' \cap \beta$, $\bar{\tau} = \max S_\alpha$, if $\bar{\tau} \mapsto\!\!\prec \tau$ and range $\pi_{\bar{\tau}\tau} \in \bar{c}$, then $\pi_{\bar{\tau}\tau}[\bar{h}_\alpha] \neq \bar{h} \mid \mathrm{HF}(\text{range } \pi_{\bar{\tau}\tau})$.

(7.3) REMARKS. (1) Suppose $\beta \in (S^0)'$, $\nu^* = \max S_\beta$, $\nu^* \mapsto\!\!\prec \nu \in S_\kappa$, and suppose that $L_\nu \mapsto\!\!\prec_{\Sigma_1} L_{\kappa^+}$. Then the hypothesis (*) of the Proposition of (4.14) holds. This is because $L_\nu \mapsto\!\!\prec_{\Sigma_1} L_{\kappa^+}$, so if $\mathrm{cf}\,\lambda < \kappa$ then this holds in L_ν. But then, since $L_{\nu^*} \mapsto\!\!\prec_{\Sigma_1} L_\nu$, in L_{ν^*}, $\mathrm{cf}\,\lambda < \alpha_{\nu^*}$, so there's $\beta < \alpha_{\nu^*}$ and $f \in L_{\nu^*}$, $f : \beta \to_{\text{cofinal}} \lambda^*$. Thus $\pi_{\nu^*\nu}(f) = \pi_{\nu^*\nu} \circ f$, so $\pi_{\nu^*\nu}(f) : \beta \to_{\text{cofinal}} \lambda$.

(2) Suppose $\beta \in (S^0)'$, $\nu^* = \max S_\beta$, $\alpha \in (S^0)' \cap \beta$, $\bar{\tau} = \max S_\alpha$, $\tau \in S_\beta \cap \nu^*$, $\bar{\tau} \mapsto\!\!\prec \tau$. Suppose further that $L_\tau \mapsto\!\!\prec_{\Sigma_1} L_{\nu^*}$. Then, we can imitate the proof of the Proposition of (4.14) (the last hypothesis providing the analogue of (*), along the lines of Remark (1)) to show: $\pi_{\bar{\tau}\tau} = \delta_\beta^{-1} \circ \sigma_{\bar{\tau}\tau} \circ \delta_\alpha$.

(3) Building on (2), if the definition of $(\bar{h}_\beta, \bar{c}_\beta)$ falls under Case 1, then (h_β, c_β) satisfies:

(a″) in the sense of J_{ρ^*}, $h_\beta : \mathrm{HF}(a(\beta)) \to 2$ and c_β is a club subset of $\{\delta''_\beta x : x \in [\nu^*]^{<\beta}\}$,

(b′) whenever $\tau \in S_\beta \cap \nu^*$, $\alpha \in (S^0)' \cap \beta$, $\bar{\tau} = \max S_\alpha$, if $\bar{\tau} \mapsto\!\!\prec \tau$, $L_\tau \mapsto\!\!\prec_{\Sigma_1} L_{\nu^*}$, and range $\sigma_{\bar{\tau}\tau} \in c_\beta$ then $\sigma_{\bar{\tau}\tau}[h_\alpha] \neq h_\beta \mid \mathrm{HF}(\text{range } \sigma_{\bar{\tau}\tau})$.

(4) In order to verify the oracle property for the h_α's with respect to the $\sigma_{\bar{\tau}\tau}$, it will suffice to verify an oracle property for the $(\bar{h}_\alpha : \alpha \in (S^0)')$ with respect to the $\pi_{\bar{\tau}\tau}$, to wit:

(!) Whenever $H : \mathrm{HF}(\kappa^+) \to 2$, S'_H is stationary, where $a \in S'_H \Leftrightarrow a \in O$ and for some $\tau \in S_\kappa$, $L_\tau \mapsto\!\!\prec_{\Sigma_1} L_{\kappa^+}$ and for some $\alpha \in (S^0)'$, setting $\bar{\tau} = \max S_\alpha$, $\bar{\tau} \mapsto\!\!\prec \tau$, $a = \text{range } \pi_{\bar{\tau}\tau}$ and, setting $h = H \mid \mathrm{HF}(a)$, $h = \pi_{\bar{\tau}\tau}[h_\alpha]$.

(7.4) LEMMA. *$(h_\alpha : \alpha \in (S^0)')$ is a built-in $\diamondsuit$-sequence.*

PROOF. By (7.3), (4), it will suffice to verify that S'_H is stationary.

If not, let (H_0, C_0) be the $<_J$-least pair (H, C), $H : \mathrm{HF}(\kappa^+) \to 2$, $C \subseteq O$ club such that $S'_H \cap C = \varnothing$.

Now $S'_H \cap C = \varnothing$ is $\Sigma_0(L_{\kappa^{++}})$, in parameter L_{κ^+} and so (H_0, C_0) is the unique witness in $L_{\kappa^{++}}$ to a true Σ_1 condition (in parameter L_{κ^+}). Thus, whenever $L_{\kappa^+} \in X \mapsto\!\!\prec_{\Sigma_1} L_{\kappa^{++}}$, $(H_0, C_0) \in X$.

Let $Y \mapsto\!\!\prec L_{\kappa^{++}}$, card $Y = \kappa$. Then, standard arguments show that L_κ, $L_{\kappa^+} \in Y$, that $Y \cap \kappa^+ \in \kappa^+$ and that, setting $\tau = Y \cap \kappa^+$, $\tau \in S_\kappa$ and $Y \cap L_{\kappa^+} = L_\tau$. Of course, H_0, $C_0 \in Y$. Further, $C_0 \cap Y = C_0 \cap [\tau]^{<\kappa}$ is club in $[\tau]^{<\kappa}$ and $H_0 \mid Y = H_0 \mid \mathrm{HF}(\tau)$. Moreover, if we take $Y =$ the (full elementary) Skolem hull of κ in

$L_{\kappa^{++}}$, then, letting $\psi : L_\gamma \leftrightarrow Y$, we have:

(a) $\gamma = \omega \cdot \gamma$,

(b) $\psi(C_0 \cap Y) = C_0$,

(c) $\psi(H_0 | Y) = H_0$,

(d) $\psi(L_\tau) = L_{\kappa^+}$,

(e) $\beta(\tau) = \gamma + 1$, $n(\tau) = 1$, so $\rho(\tau) = \beta(\tau) = \gamma + 1$, $A(\tau) = \emptyset$; in particular, $\rho(\tau) > \tau$. Further, $J_{\gamma+1} \models$ "γ is a cardinal".

Now, let $\bar{\tau} \mapsto_3 \tau$ be the least $\mapsto_3$-predecessor of τ, and let $\alpha = \alpha_{\bar{\tau}}$. Let $\bar{\beta} = \beta(\bar{\tau})$, $\bar{\rho} = \rho(\bar{\tau})$. Since $n(\bar{\tau}) = n(\tau) = 1$ (by (5) of (7.1)), $\bar{\rho} = \bar{\beta}$ and (again by (5) of (7.1)) $\bar{\rho} = \bar{\beta} > \bar{\tau}$. Further, by (5), (*), of (7.1), $\pi_{\bar{\tau}\tau}(\bar{\tau}) = \tau$, so $\pi_{\bar{\tau}\tau}(L_{\bar{\tau}}) = L_\tau$, so $\psi \circ \pi_{\bar{\tau}\tau}(L_{\bar{\tau}}) = L_{\kappa^+}$. Also, $\psi \circ \pi_{\bar{\tau}\tau} : J_{\bar{\gamma}} \rightarrow_{\Sigma_1} L_{\kappa^{++}}$, so, letting $X = \text{range}\, \psi \circ \pi_{\bar{\tau}\tau}$, $L_{\kappa^+} \in X \mapsto_{3\,\Sigma_1} L_{\kappa^{++}}$, and therefore $(H_0, C_0) \in X$. Further, since $J_{\gamma+1} \models$ "γ is a cardinal", by (1.8)(c) of [7], $\gamma \in \text{range}\, \pi_{\bar{\tau}\tau}$, so $\bar{\rho} = \gamma + 1$, where $\pi_{\bar{\tau}\tau}(\bar{\gamma}) = \gamma$.

If $\psi \circ \pi_{\bar{\tau}\tau}((h_0, c_0)) = (H_0, C_0)$, then $\pi_{\bar{\tau}\tau}((h_0, c_0)) = (H_0 | \text{HF}(\tau),\ C_0 \cap [\tau]^{<\kappa})$. Of course $\bar{\tau} = \max S_\alpha$, since $\bar{\tau}$ is minimal in $\mapsto_3$, and $\bar{\tau}$ is a limit in S_α (since τ is, in S_κ), i.e. $\alpha \in (S^0)'$.

But then, by Σ_1-elementarity, (h_0, c_0) satisfies the defining property of $(\bar{h}_\alpha, \bar{c}_\alpha)$, and setting $a = \pi''_{\bar{\tau}\tau}\bar{\tau}$, $h = H_0 | \text{HF}(a)$, we then have that $h = \pi_{\bar{\tau}\tau}[\bar{h}_\alpha]$, i.e. $H_0 | \text{HF}(a) = \pi_{\bar{\tau}\tau}[h_0]$, since $\pi_{\bar{\tau}\tau}(h_0) = H_0 | \text{HF}(\tau)$.

So, to obtain the sought-after contradiction, it will suffice to show that $a \in C_0 \cap [\tau]^{<\kappa}$.

Towards this end, we need a few preliminary facts.

(1) $\text{cf}\, \tau = \text{cf}\, \bar{\tau} = \text{cf}\, \omega \cdot \rho = \text{cf}\, \omega \cdot \bar{\rho} = \omega$.

Proof. It is known that $\text{cf}\, \tau = \text{cf}\, \omega \cdot \rho$, $\text{cf}\, \bar{\tau} = \text{cf}\, \omega \cdot \bar{\rho}$. But since ρ, $\bar{\rho}$ are successors, $\omega \cdot \bar{\rho}$, $\omega \cdot \rho$ both have cofinality ω.

(2) $\text{cf}\, \alpha = \omega$.

Proof. This is by §2, (4) of [6], since clearly α is not regular.

Now let $(\beta_i : i < \omega)$ be an increasing ω-sequence cofinal in α, let $\eta_i = \bar{\gamma} + i$, let $S_i = S_{\eta_i}$, let h = the canonical Σ_1-Skolem function for $J_{\bar{\rho}}$, and let h be defined by $\exists v(\theta(v, j, \vec{x}, y))$. Then, let:

$$h_i(j, \vec{x}) \simeq \begin{cases} h(j, \vec{x}) \Leftrightarrow \vec{x} \in [\beta_i]^{<\omega},\ p(\bar{\tau}) \in S_i \\ \qquad\qquad \wedge\ (\exists v \in S_i)\theta(v, j, \vec{x}, h(j, \vec{x})) \\ \text{undefined, otherwise.} \end{cases}$$

Then, as usual, $h_i \in J_{\bar{\rho}}$. We now define a $\subseteq$- increasing ω-sequence $(\bar{b}_i : i < \omega)$, each $\bar{b}_i \in J_{\bar{\rho}} \cap \bar{c}_\alpha$ by recursion:

$\bar{b}_0 = \emptyset$; $\bar{b}_{i+1} =$ the $<_J$-least $b \in J_{\bar{\rho}} \cap c_\alpha$ s.t. $b \supseteq \bar{b}_i \cup \bar{\tau} \cap h_i''(\omega \times [\beta_i]^{<\omega})$ (exists, since $J_{\bar{\rho}} \models$ "$\bar{c}_\alpha$ is club in $[\bar{\tau}]^{<\alpha}$", $h_i''(\omega \times [\beta_i]^{<\omega}) \in J_{\bar{\rho}} \wedge J_{\bar{\beta}} \models$ "$h_i''(\omega \times [\beta_i]^{<\omega})$ has cardinality $< \alpha$"). Thus, $\bigcup_{i<\omega} \bar{b}_i = \bar{\tau}$. Now let $b_i = \pi_{\bar{\tau}\tau}(\bar{b}_i)$. Then each $b_i \in C_0$, and $\bigcup_{i<\omega} b_i = a$. Thus $a \in C_0$, since C_0 is closed.

§8. $S_\mu(\diamond) \Rightarrow \exists(\mu, 1)$-morasses with built-in $\diamond$

Let $\mu > \omega$ be regular. In this section, we modify the partial ordering of [20], §5, and show that applying $S_\mu(\diamond)$ to this partial ordering gives a $(\mu, 1)$-morass with built-in $\diamond$, thus proving the left-to-right direction of Theorem 2. For the most part this is routine, following [20], §5. Accordingly, many details will be omitted.

The only substantial new difficulty, which will receive the bulk of our attention, results from the thinning process. The "sufficiently generic set" contributes a "pre-morass" and an oracle sequence $(d_\alpha : \alpha < \kappa)$ for the pre-morass. But the pre-morass must be "thinned", as in §3, to turn it into a morass. We then redefine a sequence, based on $(d_\alpha : \alpha < \kappa)$, and show that it has the oracle property, (4.12), (2), for the resulting morass. As in §3, this requires us to delve somewhat into the details of the thinning, with reference to [22], [23] for still further details.

We will denote by I the set $S^{\mu^+}_\mu$; this was denoted by X in [20], §5. Similarly, what we denote by $\bar{I}$ was denoted by $\bar{X}$ in [20], §5.

(8.1) The following clause was omitted from the definition of acceptable set, [20], (5.5):

(*): if $\eta > 0$ and $\eta \in (S \cap \bar{I}) \setminus I$, then $S \cap I$ is cofinal in η.

Sets S satisfying (*) will be called *strongly acceptable.*

(8.2) For the remainder of this section, $\mathbf{P}'$ denotes the partial ordering of [20], (5.6) and ff.

DEFINITION. $p \in P^* \Leftrightarrow p \in P'$ and

(a) S^p_μ is strongly acceptable, $\max S^p_\mu \cap \mu = \max((\operatorname{dom} t^p) \cap \mu)$,

(b) for all $\alpha \in (\operatorname{dom} t^p) \cap \mu$, γ^p_α is a successor ordinal, say $\gamma^p_\alpha = \bar{\gamma}^p_\alpha + 1$, $\bar{\gamma}^p_\alpha$ is a limit ordinal, and setting $x = (\alpha, \bar{\gamma}^p)$, x is minimal in $\mapsto^p$.

For $p, p' \in P^*$, set $p \leqq^* p' \Leftrightarrow p \leqq p'$ and

(c) $\max((\operatorname{dom} t^p) \cap \mu) < \max((\operatorname{dom} t^{p'}) \cap \mu)$, $\max S^p_\mu < \max S^{p'}_\mu$.

Let $\mathbf{P}^* = (P^*, \geqq^*)$.

Sh:167

(8.3) It is easy to see that (5.9) of [20] goes over with $\mathbf{P}^*$ replacing $\mathbf{P}'$; in fact, the hypotheses (a), (c) of (8.2) somewhat simplify the argument by eliminating certain divisions into cases. Slightly different constructions are required for (5.7), (5.10) of [20]. We give the necessary modifications for (5.7); the treatment of (5.10) is analogous, and equally straightforward. The relevant point in (5.7), [20], is showing that if $t^p \restriction \mu = t^q \restriction \mu$ and S^p_μ, S^q_μ have the strong Δ-property, then p, q are compatible. As in (5.7), [20], one new "level" will be added to $t^p \restriction \mu$. However, we change the definition of S. Adopting the notation of (5.7), [20], and setting $\sigma' = \max S^q_\mu$, we set:

$$S = S^p_\mu \cup \{\sigma + i : 1 \leqq i \leqq \omega\} \cup S^q_\mu \cup \{\sigma' + i : 1 \leqq i \leqq \omega\}.$$

The remainder of the construction is as before.

(8.4) In [20], (5.11) it's argued that $\mathbf{P}'$ is μ-special. In the context of this paper, we need that $\mathbf{P}^*$ is μ-I-special and supported. The argument of (5.11) (and the additional hypothesis, (*) of (8.1)) essentially show that $\mathbf{P}^*$ is μ-I-special. Our support function is:

$$\operatorname{supp}(p) = S^p_\mu \setminus (\{\lambda_\eta : \eta \in S^p_\mu \cap I\} \cup \{\max S^p_\mu\}).$$

It is easy to see that (1.1.5), (1)–(5) hold for this support function.

(8.5) Definition. $(p, \vec{d}) \in \tilde{P}$ iff $p \in P^*$ and setting $\alpha = \max((\operatorname{dom} t^p) \cap \mu)$, $\vec{d} = (d_\beta : \beta < \alpha)$, where $d_\beta : \mathrm{HF}(\bar{\gamma}^p_\alpha) \to 2$.

$$(p, \vec{d}) \leqq (p', \vec{d}') \Leftrightarrow p \leqq^* p' \quad \text{and} \quad \vec{d}' \supseteq \vec{d}.$$

$\tilde{\mathbf{P}} = (\tilde{P}, \geqq)$.

(8.6) Lemma. $\tilde{\mathbf{P}} \in \mathcal{S}_\mu(I)$.

Proof. By (8.3), (8.4), since the addition of the $\vec{d}$ component creates no new difficulties, and since, once again, as in [20], (5.12), the indiscernibility property is clear. By abuse of notation, we let $\operatorname{supp}(p) = \operatorname{supp}((p, \vec{d}))$.

(8.7) We now define our extension system and in (8.8) we show that, in a strong way, it's reasonable.

Definition. If $h \in \mathcal{H}_A$, set $(p, \vec{d}) \in D_h$ iff $\operatorname{supp}(p) \subseteq \operatorname{ENV}(I \restriction \mu)$ and: (*): for some $\beta < \max((\operatorname{dom} t^p) \cap \mu)$, letting

$$x = (\beta, \bar{\gamma}^p_\beta), \qquad \text{for some } \gamma \in S^p_\kappa,$$

setting

$$y = (\mu, \gamma), \quad x \mapsto_3^p y \quad \text{and} \quad h = \pi^p_{xy}[d_\beta].$$

Note that this last implies, in particular, that $a(h) = (\pi^p_{xy})''\bar{\gamma}^p_\beta$.

Thus, if $h \in \mathcal{H}_O \setminus \mathcal{H}_A$, $(p, \vec{d}) \in D_h \Leftrightarrow \text{supp}(p)$ is an end-extension of $a(h)$ and (*), above, holds.

(8.8) Proposition. *$\mathcal{D} = (D_h : h \in \mathcal{H}_A)$ is reasonable; in fact, for all $H : \text{HF}(\mu^+) \to 2$, H is $\mathcal{D}$-tractable.*

Proof. Let $H : \text{HF}(\mu^+) \to 2$, let $S \subseteq [\mu^+]^{<\mu}$ be stationary, let $f : S \to \bar{P}$ with $\text{supp} f(a) = a$. We seek $a \in S \cap O$ s.t. letting $h = H \restriction \text{HF}(a)$, there's $(p', \vec{d}') \in D_h$ with $f(a) \leqq (p', \vec{d}')$. This however is immediate: we let $f(a) = (p, \vec{d})$, let $\alpha = \max((\text{dom}\, t^p) \cap \mu)$, $x = (\alpha, \bar{\gamma}^p_\alpha)$, $\sigma = \max S^p_\mu$. Then $x \mapsto_3^p (\mu, \sigma)$ and

$$\text{range}\, \pi^p_{xy} = S^p_\mu \setminus (\{\lambda_\eta : \eta \in S^p_\mu \cap I\} \cup \{\sigma\}) = \text{supp}(p) = \text{supp}(f(a)) = a.$$

So, we let $\vec{d}' = \vec{d} \cup \{(\alpha, (\pi^p_{xy})^{-1}[h])\}$. We obtain p' by adding one "level" to $t^p \restriction \mu$, and adding the next $\omega + 1$ ordinals greater than σ to S^p_μ. By construction $(p', \vec{d}') \in D_h$.

(8.9) Now let G be an ideal in $\bar{\mathbf{P}}$ meeting all the $\mathcal{D}(H, C)$, $H : \text{HF}(\mu^+) \to 2$, $C \subseteq [\mu^+]^{<\mu}$ club. Let $(T, \overline{\mapsto_3}, \pi_{xy}, d_\alpha)_{x \precneqq y, \alpha < \mu}$ be the triple of unions of co-ordinates of elements of G. Let $H : \text{HF}(\mu^+) \to 2$, and set:

$$a \in S'_H \Rightarrow (\exists \alpha < \kappa)(\exists \gamma < \mu^+)$$

(setting $x = (\alpha, \bar{\gamma}_\alpha)$, $y = (\mu, \gamma)$, $x \overline{\mapsto_3} y$, $a = \text{range}\, \pi_{xy}$ and $H \restriction \text{HF}(a) = \pi_{xy}[d_\alpha]$). Then clearly S'_H is stationary, since if $C \subseteq [\mu^+]^{<\mu}$ is club, $(p', \vec{d}') \in D(H, C)$, say $a \in C$, $p \in D_h$, where $h = H \restriction \text{HF}(a)$, then $a \in S'_H$.

(8.10) The properties of $(d_\alpha : \alpha < \mu)$ developed in (8.9) clearly guarantee that $\mu^{<\mu} = \mu$, so let $A \subseteq \mu$ be s.t. $H_\mu = L_\mu[A]$. Also, as in §3, let $D \subseteq {}^{\omega}\mu^+$ be the predicate coding up the morass. Let $\mathfrak{A} = (L_{\mu^+}[A, D], \in, A, D)$, and let $a \in C^* \Leftrightarrow a \cap \mu$ is transitive $\wedge\, \exists X(a = \text{OR} \cap X \wedge \mathfrak{A} \restriction X \mapsto_3 \mathfrak{A})$. Then C^* is club.

Let $a \in S'_H \cap O \cap C^*$, say $a = \text{range}\, \pi_{xy}$, where there's $\gamma \in (\mu, \mu^+)$, $\alpha < \mu$ s.t. $x = (\alpha, \bar{\gamma}_\alpha)$, $y = (\mu, \gamma)$, $x \overline{\mapsto_3} y$. Let $\lambda = \sup a$, and let $y' = (\mu, \lambda)$. Then $x \overline{\mapsto_3} y'$, $\pi_{xy'} = \pi_{xy}$. Let $D_\lambda = D \cap {}^{\omega}\lambda$, $\mathfrak{A}_\lambda = (L_\lambda[A, D_\lambda], \in, A, D_\lambda)$; then $\mathfrak{A}_\lambda = \mathfrak{A} \restriction L_\lambda[A, D]$, so $\mathfrak{A}_\lambda \prec_{\Sigma_0} \mathfrak{A}$. But then $\mathfrak{A}_\lambda \restriction X \mapsto_{3\,\Sigma_0} \mathfrak{A}_\lambda$ and, since a is cofinal in λ, $\mathfrak{A}_\lambda \restriction X \mapsto_{3\,O} \mathfrak{A}_\lambda$. Further, λ is clearly a limit of limits of (A, D_λ)-admissibles, i.e. $\lambda \in S^1$.

Now, let $\pi : (L_\tau[\bar{A}, \bar{D}], \in, \bar{A}, \bar{D}) \stackrel{\sim}{\leftrightarrow} \mathfrak{A}_\lambda \restriction X$. We then clearly have:

(1) $\tau = \bar{\gamma}_\alpha$,

Sh:167

(2) $\pi \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi(\alpha) = \mu$,

(3) $\pi : (L_\tau[\bar{A}, \bar{D}], \in, \bar{A}, \bar{D}) \to_Q \mathfrak{A}_\lambda$,

(4) $\bar{A} = A \cap \alpha$, $\bar{D} = D \cap {}^{\omega}\bar{\gamma}_\alpha = D_{\bar{\gamma}_\alpha}$, $\pi \restriction \bar{\gamma}_\alpha = \pi_{xy} = \pi_{xy'}$,

(5) $\bar{\gamma}_\alpha \in S^1$, $(L_\tau[\bar{A}, \bar{D}], \in, \bar{A}, \bar{D}) = \mathfrak{A}_{\bar{\gamma}_\alpha}$, π satisfies the definition of $\pi^+_{\bar{\gamma}_\alpha\lambda}$, so $\bar{\gamma}_\alpha \mapsto\!\!\!\!\!\rightarrow \lambda$.

As in §3, and by transitivity, [22], [23], $\mathcal{M} = (\mathcal{S}, S^0, S^1, \mapsto\!\!\!\!\!\rightarrow, \pi_{\bar{\nu}\nu})_{\bar{\nu} \mapsto\!\!\!\rightarrow \nu}$ is a $(\mu, 1)$-morass, $\bar{\gamma}_\alpha$ is minimal in $\mapsto\!\!\!\!\!\rightarrow$ and a limit in S_α, $H \restriction \mathrm{HF}(a) = \pi_{\bar{\gamma}_\alpha\lambda}[d_\alpha]$. Finally, if we let $h_\alpha = \delta_\alpha[d_\alpha]$, then $h_\alpha : \mathrm{HF}(a(\alpha)) \to 2$ and $\sigma_{\bar{\gamma}_\alpha\lambda}[h_\alpha] = H \restriction \mathrm{HF}(a)$.

Summing up, for $\alpha \in (S^0)'$, setting $h_\alpha = \delta_\alpha[d_\alpha]$ if $\max S_\alpha = \bar{\gamma}_\alpha$, $h_\alpha =$ the constant function with value 1 on $\mathrm{HF}(\max S_\alpha)$; otherwise, we have:

LEMMA. *$(h_\alpha : \alpha \in (S^0)')$ is a built-in $\diamond$-sequence for $\mathcal{M}$, and therefore $S_\mu(\diamond) \Rightarrow \exists(\mu, 1)$-morasses with built-in $\diamond$.*

§9. Appendix 1 (by J. P. Burgess): Model theoretic lemmas for (5.1)

The definition of the quantifiers Q^k_α was given in the body of the paper, at the beginning of (5.1). We recall that using Q^1_1 and Q^2_1 (resp. Q^1_1 and Q^1_2) one can write a sentence which has a model iff there's a Souslin (resp. Kurepa) tree. Using Q^1_1, Q^2_1, Q^1_2 together, still more powerful combinatorial principles can be expressed (e.g. those of [2]).

Without entering into a detailed history, we note that compactness for $\mathcal{L}[Q^{<\omega}_1]$ is due to Magidor and Malitz, [13], and for $\mathcal{L}[Q^1_1, Q^1_2]$ (among other languages) to Jensen, [3].

Our goal is to prove (in (9.4)–(9.6)) Lemma 2 of (5.1.1) in the body of the paper and (in the remainder of this section) to prove other model theoretic lemmas which turn the development in (5.1) into a proof of Lemma 1 of (5.1.1).

(9.1) Recall that $\mathcal{L}[Q^{<\omega}_1, Q^1_2]$ is first order logic (over a countable language) enriched with the quantifiers Q^n_1, $n < \omega$, and Q^1_2. Compactness for this logic is as usual:

(**) If T is a countable set of sentences and every finite subset of T has a model, then T has a model.

Implicit in the proof of (**) from $S_{\aleph_1}(\diamond)$, as sketched in (5.1), will be information about axiomatizability and transfer properties of our logic. The model-theoretic part of the proof combines the technique of [14], with an alternative to that of [1] once suggested in a different context by Silver. The set-theoretic part of the proof, as sketched in (5.1), avoids direct use of morasses by appealing to $S_{\aleph_1}(\diamond)$, applied to a partial-ordering (**P** of (5.1.13)) analogous to that of [1] (or, more precisely, a modification of it of the sort suggested by Shelah and independently by Velleman); thus, the approach is analogous to that of [20], §4, and [25].

(9.2) Recall that in (5.1.1)–(5.1.3), the primitive-recursive first-order theory θ_0 in vocabulary $\{E,\Gamma\}$ was introduced, where E, Γ are auxiliary predicate symbols, E binary, Γ singulary. Also, in (5.1.4), another auxiliary singulary predicate symbol, Δ, was introduced. We let θ_0' be the (primitive-recursive) first order theory in vocabulary $\{E,\Gamma,\Delta\}$ whose axioms are (a)–(c) of θ_0 and (d′) of (5.1.4).

(9.3) We recall (*), and Lemmas 1, 2 of (5.1.1).

(*): Whenever θ is a consistent extension of θ_0, θ has a model $\mathfrak{A}$ such that:

(a) the ordinals of $\mathfrak{A}$ form an ω_2-like linear ordering and the small ordinals of $\mathfrak{A}$ form an ω_1-like linear ordering,

(b) for any formula $\varphi(x_1,\dots,x_n)$ in parameters from $|\mathfrak{A}|$, the following are equivalent:

(i) $\mathfrak{A} \models (\sim \exists X)(X$ is a cofinal set of small ordinals and $\forall x_1 < \cdots < x_n$ from $X, \varphi(x_1,\dots,x_n))$,

(ii) $(\sim \exists S)(S$ is a cofinal set of small ordinals and $\forall s_1 < \cdots < s_n$ from S, $\mathfrak{A} \models \varphi(s_1,\dots,s_n))$.

LEMMA 1. $S_{\aleph_1}(\diamondsuit) \Rightarrow (*)$.

LEMMA 2. $(*) \Rightarrow$ *compactness, axiomatizability and transfer for* $\mathcal{L}[Q_1^{<\omega}, Q_2^1]$.

(9.4) We now reveal our strategy for proving Lemma 2. We shall define a primitive recursive map:

$$\varphi \longrightarrow \varphi^{\#}$$

from $\mathcal{L}[Q_1^{<\omega}, Q_2^1]$ sentences φ in vocabulary $\{R\}$ to first order sentences $\varphi^{\#}$ in vocabulary $\{E,\Gamma\}$, and we shall show:

LEMMA 3. *Let $\kappa = \aleph_\alpha$ be a regular uncountable cardinal. Assume $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$. Let T_1 be an $\mathcal{L}[Q_1^{<\omega}, Q_2^1]$ theory in vocabulary $\{R\}$ and T_α the corresponding $\mathcal{L}[Q_\alpha^{<\omega}, Q_{\alpha+1}^1]$ theory. If T_α has a model, then $\theta_0 \cup \{\varphi^{\#} : \varphi \in T_1\}$ is consistent.*

The proof of Lemma 3 will be given below, in (9.6). Note however that, modulo Lemma 3 and its proof, we can prove Lemma 2.

PROOF OF LEMMA 2 (modulo Lemma 3 and its proof). Let T be an $\mathcal{L}[Q_1^{<\omega}, Q_2^1]$ theory such that all finite subsets of T have models. For each finite subset Σ, apply Lemma 3 with $\alpha = 1$, $T_1 = \Sigma = T_\alpha$. Thus, $\theta_0 \cup \{\varphi^{\#} : \varphi \in T\}$ is consistent. Applying (*) to this consistent extension of θ_0, we get an $\{E,\Gamma\}$

structure $\mathfrak{A}^* \models \theta_0 \cup \{\varphi^\# : \varphi \in T\}$, satisfying (a), (b) of (*). It will be clear from the proof of Lemma 3, (9.6), below, how we can recover from $\mathfrak{A}^*$ an $\{R\}$ structure, $\mathfrak{B}^*$, with underlying set the ordinals of $\mathfrak{A}^*$, and satisfying:

(#): $\mathfrak{B}^* \models \varphi$ iff $\mathfrak{A}^* \models \varphi^\#$, for atomic $\mathscr{L}$ sentences φ in vocabulary $\{R\}$ and parameters from $|\mathfrak{B}^*|$.

In fact, it will be clear that (#) can be extended to all $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$ sentences in vocabulary $\{R\}$ and parameters from $|\mathfrak{B}^*|$: by (a) of (*), the interpretations of Q_1^1, Q_2^1 can be handled, and (b) of (*) permits us to handle the Q_1^n for $n > 1$.

We have the immediate corollary of Lemmas 1, 2:

THEOREM. *$V = L \Rightarrow$ compactness (and axiomatizability and transfer) for $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$.*

We should note that Lemma 3 also makes clear how the transfer properties of $\mathscr{L}[Q_1^{<\omega}, Q_2^1]$ will be obtained.

(9.5) We now reveal how $\varphi^\#$ is obtained: as the result of performing the following operations on φ:

replace:	*by*:
xRy	$\Gamma(\pi(0, \pi(x, y)))$
$\exists x$ ——	$\exists^{\text{ordinal}} x$ ——
$Q_1^k x_1, \dots, x_k$ ——	$\exists X(X$ is a set of ordinals $\wedge$ card $X = \lambda \wedge \forall x_1 < \cdots < x_n$ from $X)$ ——
$Q_2^1 x$ ——	$\forall^{\text{ordinal}} y\, \exists^{\text{ordinal}} x > y$ ——

Here π is Godel's ordinal-pairing function, whose definition must be written out in terms of the symbol E for the pseudo-membership relation.

(9.6) PROOF OF LEMMA 3. A routine exercise in coding: By a Lowenheim–Skolem argument, the underlying set of our model $\mathfrak{A}$ of T_α can be assumed to have size $\kappa^+ = \aleph_{\alpha+1}$, and indeed to simply *be* κ^+. Since $2^\kappa = \kappa^+$ there is a $K \subseteq \kappa^+$ such that H_{κ^+} (sets of hereditary cardinality $< \kappa^+$) $= L_{\kappa^+}[K]$. Let

$$C = \{\pi(0, \pi(\alpha, \beta)) : \mathfrak{A} \models \alpha R \beta\} \cup \{\pi(1, \alpha) : \alpha \in K\}.$$

Clearly $\mathfrak{A}^* = (H_{\kappa^+}, \in, C) \models$ (a), (b), (c) of θ_0 and $\models \{\varphi^\# : \varphi \in T_1\}$. Since $2^{<\kappa} = \kappa$, card $\mathscr{L}_{\kappa\kappa} = \kappa$. So by a Lowenheim–Skolem argument we can form a cofinal $D \subseteq \kappa^+$ such that $(L_\alpha[C], \in, C \cap \alpha)$ is an $\mathscr{L}_{\kappa\kappa}$-elementary submodel of $\mathfrak{A}^*$ for each $\alpha \in D$. Then $(\mathfrak{A}^*, D) \models$ (d′) of θ_0' and so $\mathfrak{A}^* \models$ (d) of θ_0, completing the proof.

In order to complete the proof of Lemma 2 in (9.4), above, we need only say that the model $\mathfrak{B}^*$ mentioned there is to be read off from the model $\mathfrak{A}^*$ in the same way the $\mathfrak{B}$ was coded into $\mathfrak{A}$, immediately above.

The remainder of this section is devoted to completing the Proof of Lemma 1, sketched in (5.1).

(9.7) *First Round of Model-Theoretic Lemmas*

Our next goal is a technical lemma, (9.7.10), corresponding to the key model-theoretic step in [14] (Lemma 7, p. 179). Our proof is based on a suggestion of Silver.

We adopt the convention that the underlying sets of structures denoted $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \dots$ will be denoted $A, B, C, \dots$.

The material of this subsection corresponds to (5.1.4)–(5.1.7) in the body of the paper.

(9.7.1) DEFINITION. Let $\mathfrak{A} = (A, E, C) \models \theta_0$. For b a large ordinal of $\mathfrak{A}$, let $A_b = \{c \in A : \mathscr{A} \models cEM_b\}$, $\mathfrak{A}_b = (A_b,\ E$ restricted to A_b, C restricted to $A_b) \subseteq \mathfrak{A}$. Call $b \in A$ *praiseworthy* if $\mathfrak{A} \models \tau_n(b)$ for each $n \in \omega$. Call $D \subseteq A$ *commendable* if $(\mathfrak{A}, D) \models \theta_0'$.

Trivially we have: Any element of a commendable set is praiseworthy. Praiseworthiness is preserved under elementary embedding. If $\mathfrak{A}$ is recursively saturated, then it has a praiseworthy element. Less trivial are the following:

(9.7.2) LEMMA. *If $\mathfrak{A}$ is countable and recursively saturated, then for any praiseworthy $a \in A$ there exists a commendable $D \subseteq A$ having a as its least element.*

PROOF. By recursive saturation, for any ordinal b of $\mathfrak{A}$ there is a praiseworthy a' greater than a and b and having $\mathfrak{A} \models \mathscr{M}_a \mapsto\!\!\!\prec^* \mathscr{M}_{a'}$. Iterating countably many times we obtain a cofinal sequence $a, a', a'', \dots$ constituting a commendable set D as required.

(9.7.3) LEMMA. *If $b \in A$ is praiseworthy, then $\mathfrak{A}_b \mapsto\!\!\!\prec \mathfrak{A}$.*

PROOF. On the one hand, the following are always equivalent:

$$\mathfrak{A} \models (\mathscr{M}_b \models \varphi),$$

$$\mathfrak{A} \models (\varphi \text{ relativized to } \mathscr{M}_b),$$

$$\mathfrak{A}_b \models \varphi.$$

On the other hand, the proof of Levy's Reflection Principle shows that for each $n \in \omega$, $\tau_n(x)$ implies:

$$\forall y_1, \dots, y_k \in M_b (\varphi(y_1, \dots, y_k) \leftrightarrow \mathcal{M}_b \models \varphi(y_1, \dots, y_k))$$

for any Σ_n-formula φ.

(9.7.4) LEMMA. *If the integers of $\mathfrak{A}$ are non-standard and there exists a commendable $D \subseteq A$, then $\mathfrak{A}$ is recursively saturated.*

PROOF. Let Σ be a recursive type involving no more than finitely many parameters from A. If D is commendable, take $b \in D$ large enough that all parameters in Σ come from A_b. If Σ is finitely realized in $\mathfrak{A}$, then it is also finitely realized in $\mathfrak{A}_b$ by (9.7.3). Inside $\mathfrak{A}$ the recursive definition of Σ can be used to define $S = \{n \in \omega : \mathcal{M}_b \models \exists x\ \sigma_n(x)\}$ where σ_n = conjunction of the first n elements of Σ. All standard integers belong to S, so if $\mathfrak{A}$ has non-standard integers, one of them must also belong, call it N. But if $s \in A_b$ is such that $\mathfrak{A} \models (\mathcal{M}_b \models \sigma_N(s))$, then s realizes Σ in $\mathfrak{A}_b$ and hence in $\mathfrak{A}$ by (9.7.3).

(9.7.5) Now until further notice we place ourselves inside some model of θ_0'.

DEFINITION. Let α be the least element of the class Δ (of (d′) of θ_0') and $\Delta' = \Delta \setminus \{\alpha\}$. For $\beta < \gamma$ from Δ let $F(\beta, \gamma)$ be the set of all functions f such that:

(a) $\operatorname{dom} f \subseteq M_\gamma \wedge \operatorname{card} \operatorname{dom} f < \lambda$,

(b) $\operatorname{range} f \subseteq M_\beta \wedge f \restriction \operatorname{dom} f \cap M_\beta = \text{identity}$,

(c) $(M_\beta, f(s))_{s \in \operatorname{dom} f} \mapsto\!\!\!\!\!\rightarrow (M_\gamma, s)_{s \in \operatorname{dom} f}$.

For $f \in F(\beta, \gamma)$ and $s \in \operatorname{dom} f$, write $f^- s$ for $f(s)$, $f^+ s$ for s. (To understand this definition, peek ahead at (9.7.10) in this section, and think of (f, identity) as "approximating" (identity, j).)

(9.7.6) DEFINITION. Further, we introduce for each element s of the universe two constants s^+, s^-. We let Φ be the class of all first-order sentences in vocabulary $\{E, \Gamma, \text{constants } s^+, s^-\}$. For $\beta \in \Delta'$ we write $\Phi(\beta)$ for $\{\varphi \in \Phi$: every constant s^+, s^- in φ comes from some $s \in M_\beta\}$, and $F(\beta)$ for $F(\alpha, \beta)$, and $P(\beta)$ for $\{p \in M_\beta : \operatorname{card}(p) < \lambda\}$.

Let $\varphi(s^+, s^-, t^+, t^-, \dots) \in \Phi(\beta)$. For $f \in F(\beta)$ we write $\beta, f \vdash \varphi$ to indicate that $s, t, \ldots \in \operatorname{dom} f$ and $\mathcal{M}_\beta \models \varphi(f^+s, f^-s, f^+t, f^-t, \dots)$. And for $p \in P(\beta)$ we write $\beta, p \vdash \varphi$ to indicate that $s, t, \ldots \in p$ and $\beta, f \vdash \varphi$ for all $f \in F(\beta)$ with $p = \operatorname{dom} f$ — or equivalently, for all $f \in F(\beta)$ with $p \subseteq \operatorname{dom} f$. We bring this tedious series of definitions to a welcome close with a

(9.7.7) LEMMA. (a) *For $\beta < \gamma$ from Δ, for any $p \in P(\gamma)$ there exists $f \in F(\beta, \gamma)$ with $p = \operatorname{dom} f$.*

(b) *For* $\beta < \gamma$ *from* Δ' *and* $\varphi \in \Phi(\beta)$ *we have* $\exists p \in P(\beta)(\beta, p \vdash \varphi)$ *iff* $\exists p \in P(\gamma)(\gamma, p \vdash \varphi)$. *When* $\exists \beta \in \Delta' \exists p \in P(\beta)(\beta, p \vdash \varphi)$ *we say* φ *is* imposed.

(c) *If* $\varphi_1, \dots, \varphi_m$ *are imposed and* ψ *is a logical consequence of them, then* ψ *is imposed.*

(d) *If* $\mu < \lambda$ *and for* $\nu < \mu$, $\varphi(\nu^-, s^+, s^-, t^+, t^-, \dots)$ *is imposed, then* $\forall x < \mu \varphi(x, s^+, s^-, t^+, t^-, \dots)$ *is imposed.*

PROOF. (a) is immediate from $\mathcal{M}_\beta \mapsto^* \mathcal{M}_\gamma$. (b) is an easy exercise. (Hint: if $\gamma, p \vdash \varphi$ and $g \in F(\beta, \gamma)$ has $p = \operatorname{dom} g$, prove that $\beta, g''p \vdash \varphi$.) (c) follows since for any relevant β, if $\beta, p_i \vdash \varphi_i$, then $\beta, p \vdash \psi$ where $p = \bigcup_i p_i$. (d) is similar to (c).

(9.7.8) LEMMA. *The following are imposed*:

(a) $\varphi(s^*, t^*, \dots)$ *for* $* = +$ *or* $-$, *whenever* $\beta \in \Delta'$ *and* $\mathcal{M}_\beta \models \varphi(s, t, \dots)$,

(b) $s^- = s^+$ *for* $s \in M_\alpha$,

(c) $s^- E M_{\alpha^+}$ *for all* s.

PROOF. An easy exercise. $p = \{\alpha, s, t, \dots\}$ more than suffices as an imposing set in each case.

(9.7.9) Now we step outside to see what we have accomplished. Fix until the end of the next section a countable recursively saturated $\mathfrak{A} = (A, E, C) \models \theta_0$, and a commendable $D \subseteq A$ with least element a. Let Ψ, $\Psi(A)$, $\Psi(\pm A)$ be first-order logic with the respective vocabularies $\{E, \Gamma\}$, $\{E, \Gamma$, constants for $s \in A\}$, $\{E, \Gamma$, constants s^+ and s^- for $s \in A\}$. Let $I(\mathfrak{A}, D)$ be the set of $\Psi(\pm A)$ sentences *imposed* according to the above definitions applied inside the model $(\mathfrak{A}, D)$ of θ_0'.

LEMMA. $I(\mathfrak{A}, D)$ *has the following properties*:

(a) *It is consistent.*

(b) *It contains* $\varphi(s^*, t^*, \dots)$ *for* $* = +$ *or* $-$ *whenever* $\mathfrak{A} \models (s, t, \dots)$.

(c) *It contains* $s^- = s^+$ *for* $s \in A_a$, *and* $s^- E \mathcal{M}_{a^+}$ *for all* $s \in A$.

(d) *It locally omits the type* $\{x < b\} \cup \{x \neq c : \mathfrak{A} \models c < b\}$ *for each small ordinal* b *of* $\mathfrak{A}$.

PROOF. (a) uses (9.7.7) (a), (c). (b) uses (9.7.3), (9.7.8)(a). (c) uses (9.7.8) (b), (c). (d) uses (9.7.7) (d).

(9.7.10) Now let $\mathcal{F}$ be a countable collection of $\Psi(\pm A)$ types locally omitted by $I(\mathfrak{A}, D)$. Let $\mathfrak{A}^*$ be a countable model of $I(\mathfrak{A}, D)$ omitting the types in $\mathcal{F}$ as well as those of (9.7.9)(d), and let $\mathfrak{A}' = (A', E', C')$ be its reduct to vocabulary $\{E, \Gamma\}$. We may suppose the interpretation of s^- is just s itself, and we denote by

$j(s)$ the interpretation of s^+. We may further suppose — $V = L[\Gamma]$ supplying built-in Skolem functions — that every element of A' is definable in $\mathfrak{A}'$ from parameters in $A \cup j''A$.

LEMMA. (a) *$\mathfrak{A}'$ is recursively saturated,*

(b) *$\mathfrak{A} \mapsto\!\!\!\!\ni \mathfrak{A}'$, and $j : \mathfrak{A} \to \mathfrak{A}'$ is an elementary embedding,*

(c) $j \mid A_a =$ identity, *but* $A \subseteq A'_{j(a)}$,

(d) *the small ordinals of $\mathfrak{A}'$ form an end-extension of those of $\mathfrak{A}$ — a relation henceforth abbreviated $\mathfrak{A} \subseteq_e \mathfrak{A}'$.*

PROOF. (b), (c), (d) use the similarly lettered parts of (9.7.9). As for (a), to begin with, we have $\mathfrak{A}'_{j(b)} \mapsto\!\!\!\!\ni \mathfrak{A}'$ for $b \in D$ by (9.7.3). Next, from this and our assumption about the definability of the elements of A', it follows that $j''D$ is cofinal in the ordinals of $\mathfrak{A}'$. So finally (9.7.4) applies.

(9.8) *Second Round of Model-Theoretic Lemmas*

Our next goal is to show that we can arrange for the model $\mathfrak{A}'$ just constructed to omit certain crucial types. We retain the notation of the preceding subsection. The material of this subsection corresponds to (5.1.8)–(5.1.11) in the body of the paper.

(9.8.1) DEFINITION. Let Σ be a $\Psi(A)$ type. For $* = +$ or $-$ we let $\Sigma^* = \{\varphi(s^*, t^*, \ldots) : \varphi(s, t, \ldots) \in \Sigma\}$, which is a $\Psi(\pm A)$ type. For $h : \mathfrak{A} \to \mathfrak{B}$ an embedding we let

$$h\Sigma = \{\varphi(h(s), h(t), \ldots) : \varphi(s, t, \ldots) \in \Sigma\},$$

which is a $\Psi(B)$ type. Clearly, if $I(\mathfrak{A}, D)$ locally omits Σ^- (resp. Σ^+) then by including the latter in $\mathcal{F}$, we can arrange for $\mathfrak{A}'$ to omit Σ (resp. $j\Sigma$).

We say Σ is *bounded* in $\mathfrak{A}$ if for some large ordinal b of $\mathfrak{A}$, all parameters in Σ come from A_b. For each value $* = +$ or $-$ and each Ψ-formula θ in three free variables, we define the *reinforcement* $\Sigma[{*}\theta]$ of Σ to be the $\Psi(A)$ type containing:

(a) x has the form $\pi(y, \pi(z_0, z_1))$,

(b) $\tau_n(z_1)$ [for each $n \in \omega$],

(c) $\mathcal{M}_{z_0} \mapsto\!\!\!\!\ni^* \mathcal{M}_{z_1} \wedge y E M_{z_1}$,

(d) $\forall p \in EP(z_1)$ with yEp $\exists f E F(z_0, z_1)$ with $p \subseteq \operatorname{dom} f$ $\mathcal{M}_{z_1} \models \exists w\, \theta(f^+ y, f^- y, w)$,

(e) $s E M_{z_1}$ [for each parameter s of Σ],

(f) $\exists p E P(z_1)$ with $y, s, t, \ldots$ $Ep\, \forall f E F(z_0, z_1)$ with $p \subseteq \operatorname{dom} f$ $\mathcal{M}_{z_1} \models \forall w [\theta(f^+ y, f^- y, w) \to \sigma(f^* s, f^* t, \ldots, w)]$ [for each $\sigma(s, t, \ldots, x) \in \Sigma$].

(9.8.2) LEMMA. *If Σ is bounded in $\mathfrak{A}$ and $\mathfrak{A}$ omits all its reinforcements $\Sigma[*\,\theta]$, then $I(\mathfrak{A}, D)$ locally omits Σ^*.*

PROOF. Immediate on unpacking the definitions. If Σ^* is not locally omitted, there is a $\Psi(\pm A)$ formula θ "generating" Σ in the sense that $\exists x\,\theta(x)$ is consistent with $I(\mathfrak{A}, D)$, but $\forall x(\theta(x) \rightarrow \sigma(x))$ is a consequence of $I(\mathfrak{A}, D)$ for each $\sigma(x) \in \Sigma$. And θ may be assumed to involve but a single parameter t. Taking $b \in D$ large enough to bound Σ and have tEA_b, then $\pi(t, \pi(a, b))$ as computed in $\mathfrak{A}$ is an element realizing $\Sigma[*\,\theta]$.

(9.8.3) DEFINITION. $S \subseteq A$ is *ominous* for $\mathfrak{A}$ if it is a cofinal subset of the small ordinals of $\mathfrak{A}$. A pair (S, φ) consisting of an ominous set and a $\Psi(A)$ formula $\varphi(x_1, \dots, x_k)$ is *menacing* for $\mathfrak{A}$ if it constitutes a counterexample to (b) of (*) in (9.3) for $\mathfrak{A}$ (i.e. b(i) holds, but S is a counterexample to b(ii)). For each menacing pair we introduce the *basic dangerous* type $\Sigma_{A,S,\varphi}$ for $(\mathfrak{A}, S)$ consisting of:

(a) $s < x < \lambda$ [for each $s \in S$],

(b) $\varphi(s_1, \dots, s_{k-1}, x)$ [for $s_1 < \cdots < s_{k-1}$ from S],

(c) $\psi(x)$ [for each $\Psi(A)$ formula such that $\mathfrak{A} \models \psi(s)$ for all $s \in S$].

The idea here, which has been taken over from Magidor and Malitz, is that in passing from $\mathfrak{A}$ to $\mathfrak{A}'$ we would like to prevent S from "growing", i.e. we would like to omit the type consisting of (a), (b) above. This proves to be too much to hope for, but at least by omitting the type consisting of (a), (b), *and* (c), we can guarantee that any "new members" of S are "disloyal to the original members", in that any of the former can be distinguished from the latter by its failure to satisfy some $\sigma(x)$ *with parameters from* A.

The (*general*) *dangerous* types for $(\mathfrak{A}, S)$ are all basic dangerous types $\Sigma_{A,S,\varphi}$ plus their reinforcements, reinforcements of reinforcements, etc. A pair (B, T) is *secure* for $\mathfrak{A}$ if $B \subseteq A_b$ for some large ordinal of $\mathfrak{A}$, B is the underlying set of an elementary submodel $\mathfrak{B}$ of $\mathfrak{A}$, T is menacing for $\mathfrak{B}$, and $\mathfrak{A}$ omits all dangerous types for $(\mathfrak{B}, T)$.

(9.8.4) LEMMA. *Let S be ominous for $\mathfrak{A}$, Π any dangerous type for $(\mathfrak{A}, S)$. Then $I(\mathfrak{A}, D)$ locally omits Π^-.*

(Why have we not considered Π^+? Well, for $\Pi = \Sigma_{A,S,\varphi}[*\,\theta]$, say, it does us no good to have $\mathfrak{A}'$ omit $j\Pi$, since $j\Sigma_{A,S,\varphi}$ will be unbounded in $\mathfrak{A}'$, and (9.8.2) will be inapplicable. By contrast, $\Sigma_{A,S,\varphi}$ will be bounded by $j(a)$.)

PROOF. We treat only the case of a basic dangerous type $\Sigma_{A,S,\varphi}$, and moreover only the case of $\varphi = \varphi(x_1, x_2)$ with just two free variables. Our excuse is that the added complications of the general case are primarily notational rather than conceptual.

Suppose for contradiction, then, that (S, φ) is menacing, but $I(\mathfrak{A}, D)$ does not locally omit Σ^- for $\Sigma = \Sigma_{A,S,\varphi}$. Then some $\Psi(\pm A)$ formula $\theta(x)$ generates Σ^-. We may suppose (using $V = L[\Gamma]$ and π) that φ contains just one parameter t, an ordinal, and θ just the two constants t^+, t^-. Let a be, as always, the least element of D, and b any element of D greater than a and t, and b' some still greater element of D, and finally let t' be $\pi(t, \pi(a, b))$ as ccomputed in $\mathfrak{A}$.

Since $\exists x\, \theta(x)$ is consistent with $I(\mathfrak{A}, D)$, its negation is not imposed, and the following holds in $\mathfrak{A}$:

(1) $\forall p E P(b)$ with $t E p\, \exists f E F(a, b)$ with $p \subseteq \operatorname{dom} f$ $\mathcal{M}_b \models \exists x\, \theta(f^+t, f^-t, x)$.

Since for each $\sigma(u, v, \ldots, x) \in \Sigma$, $\forall x(\theta(x) \to \sigma(x))$ is a consequence and hence an element of $I(\mathfrak{A}, D)$, each such formula is imposed, and so, provided the parameters $u, v, \ldots$ come from A_b, the following holds in $\mathfrak{A}$:

(2) $\exists p E P(b)$ with $t, u, v, \ldots E p\, \forall f E F(a, b)$ with $p \subseteq \operatorname{dom} f$ $\mathcal{M}_b \models \forall x[\theta(f^+t, f^-t, x) \to \sigma(f^-u, f^-v, \ldots, x)]$.

Moreover the corresponding statements (1′), (2′) with b' in place of b are equally true.

As special cases of (2) applied to (9.8.3) (a), (b) we have:

(3) $\exists p E P(b)$ with $t, s E p\, \forall f E F(a, b)$ with $p \subseteq \operatorname{dom} f$ $\mathcal{M}_b \models \forall x[\theta(f^+t, f^-t, x) \to s < x < \lambda \wedge \varphi(f^-t, s, x)]$

for each $s \in S$. Note $f^-s = f^+s = s$ since $s < \lambda$. Now (3) has the form $\psi(t, a, b, s)$ or let us say $\psi(t', s)$. Hence $\psi(t', x) \in \Sigma_{A,S,\varphi}$ as a case of (9.8.3)(c). Hence applying (2′) we get:

(4) $\exists p E P(b')$ with $t' E p\, \forall f E F(a, b')$ with $p \subseteq \operatorname{dom} f$ $\mathcal{M}_{b'} \models \forall x[\theta(f^+t, f^-t, x) \to \psi(f^-t', x)]$.

We will show that (1′) and (4) together yield the negation of (b), (i) of (*):

(5) $\exists X(X$ is a cofinal set of small ordinals $\wedge$ $\forall x_1 < x_2$ from $X \varphi(t, x_1, x_2))$

thus contradicting the supposed menacing character of (S, φ).

We place ourselves inside $\mathfrak{A}$. We claim:

(6) $\forall X[(X \text{ is a } non\text{-cofinal subset of } \lambda \wedge \forall \xi E X \psi(t', \xi))$
$\rightarrow \exists \eta\, (\psi(t', \eta) \wedge \forall \xi E X(\xi < \eta < \lambda \wedge \varphi(t, \xi, \eta)))]$.

Assuming (6) for the moment, we define inductively:

$$X_0 = \varnothing,$$
$$X_{\nu+1} = X_\nu \cup \{\eta\} \text{ whence } \eta \text{ is obtained by (6)},$$
$$X_\mu = \bigcup_{\nu<\mu} X_\nu \text{ at limits } \mu \leqq \lambda.$$

Then X_λ witnesses the truth of (5).

To prove (6), let X be as in its antecedent. Now (4) above asserts the existence of a $pEP(b')$. For each $\xi E X$, $\psi(t', \xi)$ $(=(3)$ above with ξ for $s)$ implies by (9.7.7)(b) the following $(=(3)$ above with ξ for s and b' for $b)$:

$$\exists p E P(b') \quad \text{with} \quad t, \xi E p \quad \forall f E F(a, b') \quad \text{with} \quad p \subseteq \operatorname{dom} f$$
$$\mathcal{M}_{b'} \models \forall x[\theta(f^+t, f^-t, x) \rightarrow \xi < x < \lambda \wedge \varphi(f^-t, \xi, x)].$$

For each $\xi E X$ fix such a $p_\xi E P(b')$. Let $q E P(b')$ be the union of p and the p_ξ. Now (1′) applied to this q asserts the existence of an $f E F(a, b')$ such that $\mathcal{M}_{b'} \models \exists x\ \theta(f^+t, f^-t, x)$. Let η be such that $\mathcal{M}_{b'} \models \theta(f^+t, f^-t, \eta)$. By the defining property of p, $\mathcal{M}_{b'} \models \psi(f^-t', \eta)$. By the elementarity (9.7.5)(c) of f, $\mathcal{M}_{b'} \models \psi(t', \eta)$. By the absoluteness (cf. (9.7.2)) of ψ for $\mathcal{M}_{b'}$, $\psi(t', \eta)$ is true. By the defining property of p_ξ, elementarity, and absoluteness, for each $\xi E X$ we have $\xi < \eta < \lambda \wedge \varphi(t, \xi, \eta)$. Thus η is as required by the consequence of (6).

We sum up everything proved so far:

(9.8.5) Main Lemma. *Let $\mathfrak{A} = (A, E, C)$ be a countable recursively saturated model of θ_0. Let a be a praiseworthy element of $\mathfrak{A}$. Let S_i for $i \in \omega$ be ominous sets for $\mathfrak{A}$. Let (B_i, T_i) for $i \in \omega$ be secure pairs for $\mathfrak{A}$. Then there exist $\mathfrak{A}' = (A', E', C)$ and $j : \mathfrak{A} \rightarrow \mathfrak{A}'$ such that:*

(a) *$\mathfrak{A}'$ is recursively saturated,*
(b) *$\mathfrak{A} \mapsto\!\!\!\!\!\rightarrow \mathfrak{A}'$, and j is an elementary embedding,*
(c) *$j \mid A_a =$ identity, but $A \subseteq A'_{j(a)}$,*
(d) *$\mathfrak{A} \subseteq_e \mathfrak{A}'$,*
(e) *all (A, S_i), (B_i, T_i), $(j(B_i), j(T_i))$ are secure for $\mathfrak{A}'$.*

(9.9) We established the important properties of $\mathbf{P}$ (of (5.1.13)) in the course of showing $\mathbf{P} \in \mathcal{S}_{\aleph_1}(S^1_2)$, in (5.1.15), above. In (5.1.16)–(5.1.18) it was shown that a "sufficiently generic" subset of $\mathbf{P}$ really is sufficient for (*) of (5.1.1). We complete our development by arguing that in the Cohen extension obtained by adjoining a generic subset, G, of $\mathbf{P}$, the (truly) generic subset is sufficient for (*).

We assume, of course, CH in the ground model, so that the set $\mathscr{T}$ of "terms" of (5.1.15) has power $\aleph_1$. As usual, the amalgamation property then gives us that **P** satisfies the $\aleph_2$-c.c. Thus, since **P** is ω_1-closed ((5.1.15)), cardinals are absolute. The union of the $\mathfrak{A}$ (resp. $\mathscr{S}$) for $(\mathfrak{A}, \mathscr{S}) \in G$ we denote $\mathfrak{B}$ (resp. **S**), and we let $\bar{\mathfrak{B}}$, $\bar{\mathbf{S}}$ be the canonical terms of the forcing language for these items.

(9.9.1) Lemma. *Suppose $p_0 = (\mathfrak{A}_0, \mathscr{S}_0) \in P$, $\bar{T}$ is a term, and $p_0 \Vdash$ "$\bar{T}$ is ominous for $\bar{\mathfrak{B}}$". Then there exist a $p' = (\mathfrak{A}', \mathscr{S}')$ and a pair (C, U) such that:*

(a) $A_0 \subseteq C \subseteq A'$,

(b) $p' \Vdash (\check{C}, \check{U}) \in \bar{\mathbf{S}}$,

(c) $p' \Vdash (\check{\mathfrak{C}}, \check{U}) \mapsto\!\!\!\rightarrow (\bar{\mathfrak{B}}, \bar{T})$

(where $\mathfrak{C}$ is the substructure of $\mathfrak{A}'$ with underlying set C).

Proof. By the Lowenheim–Skolem Theorem p_0 forces:

(*) $\exists$ countable $C, U(\check{A}_0 \subseteq C \wedge U = \bar{T} \cap C \wedge (\mathfrak{C}, U) \mapsto\!\!\!\rightarrow (\bar{B}, \bar{T})$ where $\mathfrak{C}$ is the obvious structure).

So we can find $p_1 \geqq p_0$ such that $p_1 = (\mathfrak{A}_1, \mathscr{S}_1)$ forces certain particular countable $C_1 \subseteq \omega_1 \cup (\omega_2 \times \omega_1)$ and $U_1 \subseteq \omega_1$ to be witnesses to the truth of (*). We may assure that $C_1 \subseteq A_1$. We now iterate the process, obtaining an increasing sequence of $p_i = (\mathfrak{A}_i, \mathscr{S}_i)$ with union $(\mathfrak{A}, \mathscr{S}) \in P$. Note that the underlying set A of $\mathfrak{A}$ is equal to both $\bigcup_i A_i$ and $\bigcup_i C_i$. We set $C = A$ and $U = \bigcup_i U_i$ and apply (5.1.14) to $(\mathfrak{A}, \mathscr{S})$ and U obtaining $(\mathfrak{A}', \mathscr{S}')$ which is readily verified to have all the required properties.

(9.9.2) Proposition. *In the Cohen exclusion of the universe obtained by adjoining a generic subset G of* **P**, *the model $\mathfrak{B}$ satisfies the conditions of (*) of (5.1.1).*

Proof. For (a) of (*) of (5.1.1), on the one hand the extension property (with $V = L[\Gamma]$) guarantees that the set of ordinals of $\mathfrak{B}$ has cardinality $\aleph_2$. On the other hand, if b is a small ordinal of $\mathfrak{B}$, then it is a small ordinal of $\mathfrak{A}$ for some $p = (\mathfrak{A}, \mathscr{S}) \in G$; so since for all $p' = (\mathfrak{A}', \mathscr{S}') \geqq p$ we have $\mathfrak{A} \subseteq_e \mathfrak{A}'$, the predecessors of b in $\mathfrak{B}$ are just its predecessors in $\mathfrak{A}$, and in particular are countable in number.

For (b) of (*) of (5.1.1), we consider the case $k = 2$. Suppose for contradiction that $p_0 \in G$ forces:

(a) $\bar{\mathfrak{B}} \vDash \sim \exists X(X$ is a cofinal set of small ordinals $\wedge\ \forall x_1 < x_2$ from $X\, \varphi(x_1, x_2))$,

(b) $\bar{T}$ is a cofinal subset of the small ordinals of $\bar{\mathfrak{B}} \wedge \forall s_1 < s_2$ from $\bar{T}$, $\bar{\mathfrak{B}} \vDash \varphi(s_1, s_2)$.

Apply (9.9.2) obtaining p' and (C, U). Changing generic set if necessary, we may assume $p' \in G$. Since $p' \Vdash (\check{C}, \check{U}) \in \bar{\mathbb{S}}$, this is true, and as a result $\mathfrak{B}$ must omit the type $\Sigma_{C,U,\varphi}$. But since $p \Vdash (\check{\mathfrak{C}}, \check{U}) \mapsto (\bar{\mathfrak{B}}, \bar{T})$, this is also true, and as a result $\Sigma_{C,U,\varphi}$ will be realized by any element of the denotation of $\bar{T}$ that is greater than all the elements of U. Contradiction!

§10. Corrections to "S Forcing, I..."

In addition to the corrections of imprecisions in [20] given above (§6, §8), we take this opportunity to indicate how, in the framework of this paper, we can overcome the problems, pointed out by Velleman, [27], with the applications in §2 and §4 (the weak $\square$-sequences) of [20]. These are handled, in the spirit of (1.3), by choosing an appropriate set of indiscernibles, which amounts to restricting to certain kinds of functions. This is more or less clear for the application to weak $\square$: the set of indiscernibles is $S^{\kappa}_{\kappa^+}$.

The support of a condition (a, c) is:

$$a \setminus \{\delta_\alpha : \alpha \in a \cap S^{\kappa}_{\kappa^+}\}.$$

For the application to super-Souslin trees, §2 of [20], once again, we take $I = S^{\kappa^+}_{\kappa^{++}}$, and we restrict to conditions $(x, \mathbf{t}, f)$ where x is orderly, and closed for successor and predecessor. Then the support of such a condition is just x.

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