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Covering a function on the plane by two continuous functions on an uncountable square – the consistency

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Abstract

It is consistent that for every function $f : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ there is an uncountable set $A\subseteq\mathbb{R}$ and two continuous functions $f_0, f_1 : D(A)\to\mathbb{R}$ such that $f(\alpha,\beta)\in\{f_0(\alpha,\beta),\ f_1(\alpha,\beta)\}$ for every $(\alpha,\beta)\in A^2$, $\alpha\neq\beta$. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Suppose that X is a topological space and $f : X\to\mathbb{R}$ is a real-valued function on X. Is there a "large" subset of X such that the restriction $f\restriction X$ is continuous? Obviously, if $A\subseteq X$ is a discrete subspace, then $f\restriction A$ is continuous. Hence in the case when $\mathrm{dom}(f)=\mathbb{R}$, we can always find an infinite subset on which f is continuous. The problem whether there is such "large" set has been investigated by Abraham et al. [1]. They proved that it is consistent that every function from $\mathbb{R}$ to $\mathbb{R}$ is continuous on some uncountable set. Later Shelah [4] showed that every function may be continuous on a non-meager set.

In this paper we consider functions on the plane, $\mathbb{R}\times\mathbb{R}$. The reasonable question to ask in this case is: is there a "large" set $A\subseteq\mathbb{R}$ such that on $A\times A$ the function f can be covered by two continuous functions? Note that we could not hope for f to be just continuous on $A\times A$, e.g., if g is a Sierpinski partition, then for every uncountable

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set A, g is not continuous on $A\times A$. The main result of this paper is the following theorem. For technical reasons we consider squares without the diagonal, i.e. for a set A we consider $D(A)=\{(x,y)\colon x,y\in A,\ x\neq y\}$.

Theorem. *Assume* $2^{\aleph_l}=\aleph_{l+1}$ *for* $l<4$, *and* $\diamondsuit_s(\aleph_4,\aleph_1,\aleph_0)$, *see below. Then there is a forcing notion* P *which preserves cardinals and cofinalities and such that in* V^P, $2^{\aleph_0}=\aleph_4$ *and for every function* $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *there is an uncountable set* $A\subseteq\mathbb{R}$ *and two continuous functions* $f_0, f_1:D(A)\to\mathbb{R}$ *such that* $f(\alpha,\beta)\in\{f_0(\alpha,\beta), f_1(\alpha,\beta)\}$ *for every* $(\alpha,\beta)\in D(A)$.

The proof is separated into two parts. In Section 2, we prove the consistency of a guessing principle, diamond for systems. Then, in Section 3, we give the proof of the theorem.

Remark. (1) We can replace $\aleph_0$ by any $\mu=\mu^{<\mu}$.

(2) Our main goal was to prove the consistency of the statement in the theorem with $2^{\aleph_0}<\aleph_\omega$. We get $2^{\aleph_0}=\aleph_4$ naturally from the proof, but the values $\aleph_3$ or $\aleph_2$ may be possible.

1.1 Notation. We use standard set-theoretic notation. Below we list some frequently used symbols.

- For A,B subsets of ordinals of the same order type, $OP_{B,A}$ is the order preserving isomorphism from A to B.
- If C is a set of ordinals, then $(C)'$ denotes the set of accumulation points.
- Let λ,χ be cardinals, χ regular. $S^\lambda_\chi=\{\alpha\in\lambda\colon \mathrm{cf}(\alpha)=\chi\}$.
- For a statement ϕ we define $TV(\phi)=0$ if ϕ is true, otherwise $TV(\phi)=1$.
- $\mathbb{R}={}^\omega 2$.
- If M is a model, $X\subseteq M$, then $\mathrm{Sk}(X)$ is the Skolem hull of X in M.
- $\mathscr{L}[\kappa,\theta)$ is a "universal" vocabulary of cardinality $\kappa^{<\theta}$, arity $<\theta$.

2. Diamond for systems

In this section we prove the consistency of a guessing principle, diamond for systems $\diamondsuit_s$.

Definition 2.1. A sequence $\bar{M}=\langle M_u\colon u\in[B]^{\leqslant 2}\rangle$ is a system of models (of some fixed language) if:

(1) $M_u\subseteq\mathrm{Ord}$, $B\subseteq\mathrm{Ord}$,

(2) $B\cap M_u=u$ for every $u\in[B]^{\leqslant 2}$,

(3) for every $u,v\in[B]^{\leqslant 2}$, $|u|=|v|$, the models M_u and M_v are isomorphic and OP_{M_u,M_v} is the isomorphism from M_v onto M_u, $OP_{M_u,M_v}(v)=u$,

(4) for every $u,v\in[B]^{\leqslant 2}$, $M_u\cap M_v\subseteq M_{u\cap v}$,

(5) if $|u|=|v|$, $u'\subseteq u$, $v'=\{\alpha\in v\colon (\exists\beta\in u')(|\beta\cap u|=|\alpha\cap v|)\}$, then $OP_{M_{u'},M_{v'}}\subseteq OP_{M_u,M_v}$, and $OP_{M_u,M_u}=id_{M_u}$, and if $|w|=|u|$, then $OP_{M_u,M_v}\circ OP_{M_v,M_w}=OP_{M_u,M_w}$.

Remark. See [3] on the existence of "nice" systems of models for λ a sufficiently large cardinal, e.g., measurable. Here we do not use large cardinals, and try to get a model in which the continuum is small, i.e., less than $\aleph_\omega$. For this we need a suitable guessing principle.

Definition 2.2 (*Diamond for systems* $\diamondsuit_s(\lambda, \sigma, \kappa, \theta)$). Let $\{C_\alpha: \alpha \in \lambda\}$ be a square sequence on λ. $\langle \bar{M}^\alpha: \alpha \in W\rangle$ is a $\diamondsuit_s(\lambda, \sigma, \kappa, \theta)$ sequence (or $\diamondsuit_s(\lambda, \sigma, \kappa, \theta)$-diamond for systems) if:

(A) $W \subseteq \lambda$ and for every $\alpha \in W$, $\bar{M}^\alpha = \langle M^\alpha_u: u \in [B_\alpha]^{\leqslant 2}\rangle$ is a system of models, M^α_u is a model of cardinality κ, universe $\subseteq \alpha$, vocabulary of cardinality $\leqslant \kappa$, arity $< \theta$, a subset of $\mathscr{L}[\kappa, \theta)$.

(B) $B_\alpha \subseteq \alpha = \sup(B_\alpha)$, $\mathrm{otp}(B_\alpha) = \sigma$, so $\sigma = \mathrm{cf}(\alpha)$.

(C) if M is a model with universe λ, vocabulary of cardinality $\leqslant \kappa$, arity $< \theta$, a subset of $\mathscr{L}[\kappa, \theta)$, then for stationarily many $\alpha \in W$ for all $u \in [B_\alpha]^{\leqslant 2}$, $M^\alpha_u \prec M$,

(D) if $\alpha, \beta \in W$ and $\mathrm{otp}(C_\alpha) < \mathrm{otp}(C_\beta)$, then
  (i) for some $\zeta \in B_\beta$, $\bigcup\{M^\beta_u: u \in [B_\beta]^{\leqslant 2}\} - \bigcup\{M^\beta_u: u \in [B_\beta \cap \zeta]^{\leqslant 2}\}$ is disjoint from $\bigcup\{M^\alpha_u: u \in [B_\alpha]^{\leqslant 2}\}$,

(E) if $\alpha \neq \beta$ in W, $\mathrm{otp}(C_\alpha) = \mathrm{otp}(C_\beta)$, then there is a one-to-one map h from $\bigcup_{u \in [B_\alpha]^{\leqslant 2}} M^\alpha_u$ onto $\bigcup_{u \in [B_\beta]^{\leqslant 2}} M^\beta_u$, order preserving, mapping B_α onto B_β, M^α_u onto $M^\beta_{h(u)}$ which is the identity on the intersection of these sets and the intersection is an initial segment of $\bigcup_{u \in [B_\alpha]^{\leqslant 2}} M^\alpha_u$ and $\bigcup_{u \in [B_\beta]^{\leqslant 2}} M^\beta_u$.

(F) if $\sigma = \kappa$ we may omit σ.

Lemma 2.3. *Assume*: $\kappa < \mu < \lambda$ *are uncountable cardinals*, $\lambda = \chi^+$, $2^\mu = \chi$, $\square_\lambda$, $\diamondsuit_{S^\chi_\sigma}$, $\kappa = \kappa^{<\theta}$, $\mu^\kappa = \mu$, σ, χ, κ *regular cardinals.*

Then there exists a diamond for systems on λ, $\diamondsuit_s(\lambda, \sigma, \kappa, \theta)$.

Proof. Let $\bar{C} = \langle C_\gamma: \gamma \in \lambda\rangle$ be a square sequence on λ. We assume that each C_γ is closed unbounded in γ, if γ is a limit. Let $C_\gamma = \{\alpha^\gamma_\zeta: \zeta < \mathrm{otp}(C_\gamma)\}$. First choose a sequence $\langle b^\alpha_i: i < \chi\rangle$ for every $\alpha < \lambda$ such that $b^\alpha_i \subseteq \alpha$, $|b^\alpha_i| < \chi$, b^α_i increasing, continuous in i, $\alpha = \bigcup\{b^\alpha_i: i < \chi\}$. Next, choose a_α for $\alpha < \lambda$ such that

(1) $a_\alpha \subseteq \alpha$,
(2) if $\mathrm{cf}(\alpha) < \chi$, then $|a_\alpha| < \chi$,
(3) if $\beta \in (C_\alpha)'$, then $a_\beta \subseteq a_\alpha$,
(4) if $\beta \in C_\alpha$ and $i = \mathrm{otp}(C_\alpha)$, then $b^\beta_i \subseteq a_\alpha$,
(5) if $\mathrm{otp}(C_\alpha)$ is a limit of limit ordinals, then $a_\alpha = \bigcup_{\beta \in (C_\alpha)'} a_\beta$.

Note that if $\alpha \in S^\lambda_\chi$, then there is a club $C'_\alpha \subseteq C_\alpha$ such that $\langle \alpha_\beta: \beta \in C'_\alpha\rangle$ is an increasing, continuous sequence of subsets of α of cardinality $< \chi$ with union α. Let H_0, H_1 be functions which witness that $\lambda = \chi^+$, i.e., H_0, H_1 are two place functions, for every $\alpha \in [\chi, \lambda)$, $H_0(\alpha, -)$ is a one-to-one function from α onto χ and $H_1(\alpha, H_0(\alpha, i)) = i$ for every $\alpha \in [\chi, \lambda)$ and $i < \alpha$.

Now by induction on $\alpha < \lambda$ we define the truth value of "$\alpha \in W$", and if we declare it to be true, then we also define $\bar{M}^\alpha$. Suppose we have defined $W \cap \alpha$ and $\bar{M}^\beta$ for

$\beta \in W \cap \alpha$. Now consider the following properties of an ordinal $\alpha \in \lambda$:

(a) $a_\alpha \cap \chi = \mathrm{otp}(C_\alpha)$,

(b) a_α is closed under H_0 and H_1,

(c) for every $\gamma \in a_\alpha$ we have

  (i) if $\mathrm{cf}(\gamma) < \chi$, then $a_\alpha \cap \gamma = b^\gamma_{\mathrm{otp}(C_\alpha)}$ and $C_\gamma \subseteq a_\alpha$ and $\mathrm{otp}(C_\gamma) \leqslant \mathrm{otp}(C_\alpha)$,

  (ii) if $\mathrm{cf}(\gamma) = \chi$, then $\sup(a_\alpha \cap \gamma) = \alpha^\gamma_{\mathrm{otp}(C_\alpha)}$ and $C_{\alpha^\gamma_{\mathrm{otp}(C_\alpha)}} \subseteq a_\alpha$,

(d) $\mathrm{cf}(\alpha) = \sigma$.

If α does not satisfy one of the conditions (a)–(d), then we declare that $\alpha \notin W$. So suppose that α satisfies (a)–(d). Let $\langle M_\zeta\colon \zeta \in \chi \rangle$ be the diamond sequence for S^χ_σ, i.e., each M_ζ is a model on ζ, vocabulary as above, and for every model M on χ, there are stationarily many $\zeta \in S^\chi_\sigma$, such that $M \cap \zeta = M_\zeta$. We say that M_ζ is suitable if it is of the form $(\zeta, <^*_\zeta, M^*_\zeta)$, where $<^*_\zeta$ is a well-ordering of ζ. For each ζ such that M_ζ is suitable, let $\xi_\zeta = \mathrm{otp}(\zeta, <^*_\zeta)$. Let $h_\zeta : \zeta \to \xi_\zeta$ be the isomorphism between $(\zeta, <^*_\zeta)$ and $(\xi_\zeta, <)$. Let $M^\oplus_\zeta$ be the model with universe ξ_ζ, such that h_ζ is the isomorphism between M^*_ζ and $M^\oplus_\zeta$. For $\alpha \in \lambda$ let $\zeta(\alpha) = \mathrm{otp}(C_\alpha)$. Consider the following properties of $\alpha \in \lambda$:

(e) there is a system $\bar{N}^{\zeta(\alpha)} = \langle N^{\zeta(\alpha)}_s\colon s \in [\bar{B}_{\zeta(\alpha)}]^{\leqslant 2} \rangle$, $N^{\zeta(\alpha)}_s \prec M^\oplus_{\zeta(\alpha)}$, $\|N^{\zeta(\alpha)}_s\| = \kappa$, $\bar{B}_{\zeta(\alpha)}$ cofinal in $\xi_{\zeta(\alpha)}$, $\mathrm{otp}(\bar{B}_{\zeta(\alpha)}) = \sigma$,

(f) $\mathrm{otp}(a_\alpha) = \xi_{\zeta(\alpha)}$.

If α does not satisfy (e), and (f), then declare $\alpha \notin W$. So assume that α satisfies (e) and (f). Let $g_\alpha : \xi_{\zeta(\alpha)} \to a_\alpha$ be the order preserving isomorphism. Let $\bar{M}^\alpha = \langle M^\alpha_u\colon u \in [B_\alpha]^{\leqslant 2} \rangle$ be the system of models on a_α, which is isomorphic to $\bar{N}^{\zeta(\alpha)}$ and the isomorphism is g_α. If this system satisfies:

(g) for every $\beta \in (C_\alpha)'$ there is $v \in B_\alpha$ such that $a_\beta \cap \bigcup\{M^\alpha_u\colon u \in [B_\alpha]^{\leqslant 2}\} \subseteq \bigcup\{M^\alpha_u\colon u \in [B_\alpha \cap v]^{\leqslant 2}\}$,

then we declare $\alpha \in W$. This finishes the definition of the diamond for systems sequence, $\langle \bar{M}^\alpha\colon \alpha \in W \rangle$.

We have to prove that it is as required. Clauses (A) and (B) are clear.

Proof of clause (C). We need the following fact, it is proved essentially in [5], but for completeness we give the proof at the end of the section.

Lemma 2.4. *Assume*:

(1) $\lambda = (2^\mu)^+$, $\mu = \mu^\kappa$, $\kappa = \mathrm{cf}(\kappa) > \aleph_0$, $\kappa^{<\theta} = \kappa$,

(2) *M is a model with universe λ, at most κ functions each with $< \theta$ places and $\leqslant \kappa$ relations including the well-ordering of λ.*

Then for some club E of λ for every $\delta \in E$ of cofinality $\geqslant \mu^+$ we can find $I \subseteq \delta = \sup(I)$ and $\langle N_t\colon t \in [I]^{\leqslant 2}, s \in I \rangle$ such that

(α) *$\langle N_t\colon t \in [I]^{\leqslant 2} \rangle$ is a system of elementary submodels of M, $\|N_t\| = \kappa$.*

Suppose the $\mathscr{A}$ is a model on λ, C a club on λ. We have to find $\alpha \in C \cap W$ such that $M^\alpha_u \prec \mathscr{A}$ for every $u \in [B_\alpha]^{\leqslant 2}$. Let $E \subseteq \lambda$ be the club given by Lemma 2.4. W.l.o.g. we can assume that $E \subseteq C'$, where C' is the set of limit points of C, (so if $\delta \in E$, then

$C\cap\delta$ is a club in δ). Fix $\delta\in S^{\lambda}_{\chi}\cap E$. Let $f_\delta:\delta\to\chi$ be a bijection and let

$$D_1=\{\zeta<\chi\colon \zeta \text{ is a limit, } f_\delta \text{ maps } a_{\alpha^\delta_\zeta} \text{ onto } \zeta\}.$$

D_1 is a σ-club, i.e., unbounded, closed under σ-sequences. Let $\mathscr{A}^{[\delta]}$ be $(\chi, f''_\delta(<\restriction\delta), f''_\delta(\mathscr{A}\restriction\delta))$. Note that by Lemma 2.4 we have a system of submodels on $\mathscr{A}\restriction\delta$, we transfer this system on $\mathscr{A}^{[\delta]}$ by the bijection f_δ and, choosing a subsystem if necessary, we can assume that we have an end-extension system on $\mathscr{A}^{[\delta]}$ which is cofinal in χ, i.e., we have $\bar{N}^*=\langle N^*_u\colon u\in I\rangle$, $I\subseteq\chi$, $\sup(I)=\chi$, $N^*_u\prec\mathscr{A}^{[\delta]}$ and if $\xi<\zeta$ in I, then $\min(N^*_{\{\zeta\}}\backslash N^*_{\emptyset})>\sup(N^*_{\{\xi\}})$, and if u is an initial segment of v, then N^*_u is an initial segment of N^*_v. Hence the set

$$D_2=\left\{\zeta<\chi\colon \bigcup_{u\in[\zeta\cap I]^{\leqslant 2}} N^*_u\subseteq\zeta\right\}$$

is a club of χ and such that for every $\zeta\in D_2$ there is a system of models on ζ, $(\langle N^*_u\colon u\in[\zeta\cap I]^{\leqslant 2}\rangle)$. Note that the set

$$D_3=\{\zeta<\chi\colon \alpha^\delta_\zeta\in C \text{ and } \alpha^\delta_\zeta \text{ satisfies conditions (a)–(d)}\}$$

is a σ-club of χ. Note that $\mathscr{A}^{[\delta]}$ is a model on χ. Hence by $\diamondsuit_{S^\chi_\sigma}$, for stationary many $\zeta\in S^\chi_\sigma$ we have guessed it, i.e., the set

$$S=\{\zeta\in S^\chi_\sigma\colon M_\zeta=\mathscr{A}^{[\delta]}\restriction\zeta\}$$

is a stationary. Now if $\zeta\in S\cap(D_1)'\cap D_2\cap D_3$ then $\alpha^\delta_\zeta\in C$, and α^δ_ζ satisfies conditions (a)–(d). Note that $\zeta(\alpha^\delta_\zeta)=\mathrm{otp}(C_{\alpha^\delta_\zeta})=\zeta$. Moreover, as $\zeta\in D_1\cap S$ we have $\xi_\zeta=\mathrm{otp}(a_{\alpha^\delta_\zeta})$, i.e., condition (f) holds. By the construction it follows that condition (e) holds, (the system of submodels on ξ_ζ is isomorphic to the system on $a_{\alpha^\delta_\zeta}$ given by Lemma 2.4). Finally (g) holds, as $\zeta\in(D_1)'$ and the system of models of $\mathscr{A}^{[\delta]}$ is end-extending.

Hence $\alpha^\delta_\zeta\in W\cap C$, and $\bar{M}^{\alpha^\delta_\zeta}$ is a system of models as required.

Proof of clause (E). Suppose $\alpha,\beta\in W$, $\xi=\mathrm{otp}(C_\alpha)=\mathrm{otp}(C_\beta)$. By the construction, both a_α and a_β are isomorphic to $M^\oplus_\xi$ and the isomorphisms are order preserving functions. Hence a_α is order isomorphic to a_β. Note that $a_\alpha\cap\chi=a_\beta\cap\chi=\xi$. Since both a_α and a_β are closed under H_0 and H_1 it follows that $a_\alpha\cap a_\beta$ is an initial segment of both a_α and a_β.

Proof of clause (D). Suppose that $\alpha,\beta\in W$ and $\mathrm{otp}(C_\alpha)<\mathrm{otp}(C_\beta)$. As above, since a_α and a_β are closed under H_0 and H_1, it follows that $a_\alpha\cap a_\beta$ is an initial segment of a_α. Let $\gamma=\sup(a_\alpha\cap a_\beta)$. We have four cases, we will show that the first three never occur.

Case 1: $\gamma\in a_\alpha\cap a_\beta$. We can assume that each a_α is closed under successor, so this case can never happen.

Case 2: $\gamma\in a_\alpha-a_\beta$. Note that $C_\gamma\subseteq a_\alpha$. Let $\gamma_1=\min(a_\beta-\gamma)$. By (c)(i) for a_β it follows that we must have $\mathrm{cf}(\gamma_1)=\chi$. Now by (c)(ii), $\gamma=\sup(a_\beta\cap\gamma_1)=\alpha^{\gamma_1}_{\mathrm{opt}(C_\beta)}$. So

$\gamma \in C_{\gamma_1}$ and $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\beta)$. Note that $\mathrm{cf}(\gamma) < \chi$. Hence by (c)(i) for a_α we have $\mathrm{otp}(C_\gamma) \leqslant \mathrm{otp}(C_\alpha)$, a contradiction.

Case 3: $\gamma \notin (a_\alpha \cup a_\beta)$. Let $\gamma_0 = \min(a_\alpha - \gamma)$ and, $\gamma_1 = \min(a_\beta - \gamma)$. As above we have $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\alpha)$ and $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\beta)$, a contradiction.

Case 4: $\gamma \in a_\beta - a_\alpha$. Let $\gamma_0 = \min(a_\alpha - \alpha)$. We have $\mathrm{cf}(\gamma_0) = \chi$ and $\mathrm{otp}(C_\gamma) = \mathrm{otp}(C_\alpha)$, so $C_\gamma \subseteq a_\alpha$. Note that $a_\alpha \cap \gamma = \bigcup_{\zeta \in C_\gamma} (a_\alpha \cap \zeta)$. But for $\zeta \in a_\alpha$ with $\mathrm{cf}(\zeta) < \chi$ we have $a_\alpha \cap \zeta = b^\zeta_{\mathrm{otp}(C_\alpha)}$. Hence $a_\alpha \cap \gamma = \bigcup_{\zeta \in (C_\gamma)'} b^\zeta_{\mathrm{otp}(C_\alpha)} \subseteq a_{\beta_1}$, for some $\beta_1 \in (C_\beta)'$ large enough. Hence by (g) in the definition of the diamond for systems sequence, the conclusion follows. $\square$

Proof of Lemma 2.4. We prove slightly more. In addition to the sequence $\langle N_t \colon t \in [I]^{\leqslant 2} \rangle$ there is a sequence $\langle N'_{\{\alpha\}} \colon \alpha \in I \rangle$ such that:

(β) $N_{\{\alpha\}}, N'_{\{\alpha\}}$ realize the same $L_{\theta,\theta}$-type over M, for $\alpha \in I$,

(γ) we have $N'_{\{\alpha\}} \prec N_{\{\alpha\}}$ for $\alpha \in I$ and for $\alpha < \beta$ in I we have $N_{\{\alpha,\beta\}} = Sk(N_{\{\alpha\}} \cup N'_{\{\beta\}})$.

Remark. (1) Note that for $\alpha < \beta$, $N_{\{\beta\}}$ is not necessarily a subset of $N_{\{\alpha,\beta\}}$. (2) The idea of the proof is to define $N^*_{\{0\}}$, $N^{*\prime}_{\{1\}}$ and $N^*_{\{0,1\}}$ (and more, see definition of a witness below). Then we use it as a blueprint and "copy" it many times using elementarity, to obtain a suitable system.

We can assume that M has Skolem functions, even for $L_{\theta,\theta}$. Let χ^* be large enough. Let for $i < \lambda$, $\mathscr{B}_i \prec (H(\chi^*), \in, <^*_{\chi^*})$ such that $\|\mathscr{B}_i\| = 2^\mu < \lambda$, and $M \in \mathscr{B}_i$, $\mathscr{B}_i$ increasing continuous with i, and if $\mathrm{cf}(i) \geqslant \mu^+$ or i non-limit, then $\mathscr{B}_i \prec_{L_{\mu^+,\mu^+}} (H(\chi^*), \in, <^*_{\chi^*})$. Let $E = \{\delta < \lambda \colon \delta$ is a limit and $\mathscr{B}_\delta \cap \lambda = \delta\}$, it is a club of λ. Fix $\delta \in E \cap S^\lambda_{\geqslant \mu^+}$. Note that $\mathscr{B}_\delta \prec_{L_{\mu^+,\mu^+}} (H(\chi^*), \in, <^*_{\chi^*})$.

We say that $(N^*_\emptyset, N^*_{\{0\}}, N^{*\prime}_{\{1\}}, N^*_{\{0,1\}}, \alpha_0, \alpha_1)$ is a witness if:

(1) $N^*_u \prec M$, $|N^*_u| = \kappa$, $N^*_{\{0\}} \cap N^{*\prime}_{\{1\}} = N^*_\emptyset$, $N^*_\emptyset, N^*_{\{0\}} \prec M \restriction \mathscr{B}_\delta$, $N^*_{\{0,1\}} = Sk(N^{*\prime}_{\{1\}} \cup N^*_{\{0\}})$,

(2) $N^*_{\{1\}} \cap \mathscr{B}_\delta = N^*_\emptyset$, $\alpha_0 \in N^*_{\{0\}} - N^*_\emptyset$, $\alpha_1 \in N^{*\prime}_{\{1\}} - N^*_\emptyset$,

(3) if $\alpha \in N^*_{\{0,1\}} \backslash N^{*\prime}_{\{1\}}$, $\beta = \min(N^{*\prime}_{\{1\}} \backslash \alpha)$, then $\mathrm{cf}(\beta) \geqslant \mu^+$,

(4) for every $A \subseteq \mathscr{B}_\delta$, $|A| \leqslant \mu$ there are $N'_{\{1\}} \prec N_{\{1\}}$ and $N_{\{0,1\}}$ such that
   - (a) $N'_{\{1\}}$, $N_{\{0,1\}} \prec M \cap \mathscr{B}_\delta$,
   - (b) $N'_{\{1\}}$ is order isomorphic to $N^{*\prime}_{\{1\}}$,
   - (c) $N_{\{1\}}$ is order isomorphic to $N^*_{\{0\}}$,
   - (d) $OP_{N_{\{0,1\}}, N^*_{\{0,1\}}}$ is an isomorphism from $N^*_{\{0,1\}}$ onto $N_{\{0,1\}}$ which is the identity on $N^{*\prime}_{\{1\}}$, maps $N^*_{\{0\}}$ onto $N_{\{0\}}$,
   - (e) for $\alpha \in N^*_{\{0,1\}} \backslash N^{*\prime}_{\{1\}}$, if $\beta = \min(N'_{\{1\}} - \alpha)$, then $OP_{N_{\{0,1\}}, N^*_{\{0,1\}}}(\alpha) \in \sup(A \cap \beta, \beta)$.

Claim 2.5. *There is a witness.*

We can find $\mathscr{C} \prec_{L_\mu, L_\mu} (H(\chi^*), \in, <^*_{\chi^*})$ such that $\|\mathscr{C}\| = \mu$, ${}^\kappa\mathscr{C} \subseteq \mathscr{C}$, $\mu + 1 \subseteq \mathscr{C}$ and $(M, \mathscr{B}_\delta, \delta) \in \mathscr{C}$. As $\mathscr{B}_\delta \prec_{L_{\mu^+,\mu^+}} (H(\chi^*), \in, <^*_{\chi^*})$ it follows that there is a function f, $\mathrm{dom}(f) = \mathscr{C}$, $\mathrm{rang}(f) \subseteq \mathscr{B}_\delta$, $f \restriction \mathscr{C} \cap \mathscr{B}_\delta$ is the identity, f preserves satisfaction of L_{μ^+,μ^+} formulas, i.e. f is an isomorphism.

Let $\mathscr{N} \prec (H(\chi^*), \in, <^*_{\chi^*})$ be such that $\{\mathscr{B}_\delta, \mathscr{C}, f, \delta\} \in \mathscr{N}$, $\|\mathscr{N}\| = \kappa$. Let $\mathscr{N}_1 = \mathscr{N} \cap \mathscr{C}$, $\mathscr{N}_0 = \mathscr{N} \cap \mathscr{B}_\delta$. Let $\mathscr{N}_0' = f(\mathscr{N}_1)$, note that $\mathscr{N}_0' \subseteq \mathscr{N}_0$. Let $\delta_0 = f(\delta_1)$. W.l.o.g. we can assume that $\mathscr{N} = Sk(\mathscr{N}_0, \mathscr{N}_1)$. Let $\mathscr{N}_\emptyset = \mathscr{B}_\delta \cap \mathscr{C} \cap \mathscr{N}$. We claim that $(\mathscr{N}_\emptyset, \mathscr{N}_0, \mathscr{N}_1', \mathscr{N}, \delta_0, \delta_1)$ is a witness. Note that

(∗) if $\alpha \in \mathscr{N} \cap (\delta + 1)$, then $\min(\mathscr{C} - \alpha) \in \mathscr{N}_1$.

Let us check condition (3). Suppose that $\alpha \in \mathscr{N} - \mathscr{N}_1$ and let $\beta = \min(\mathscr{N}_1 - \alpha)$. Note that by (∗) we have $\beta = \min(\mathscr{C} - \alpha)$. But as $\mu + 1 \subseteq \mathscr{C}$ and $\mathscr{C} \prec (H(\chi^*), \in, <^*_{\chi^*})$ we must have $\mathrm{cf}(\beta) \geqslant \mu^+$.

Now to verify (4), suppose that there is a set A such that the conclusion of (4) fails. Then A is definable from: $\mathscr{N}_1$, the isomorphism type of $\mathscr{N}$ over $\mathscr{N}_1$ and the isomorphism type of $\mathscr{N}_0$ over $\mathscr{N}_0'$. As $\mathscr{N}_1$, $\mathscr{N}_\emptyset$ are in $\mathscr{C}$ and $\mathscr{C} \prec_{L_\mu, L_\mu} (H(\chi^*), \in, <^*_{\chi^*})$ and $\kappa < \mu$ it follows that such set A is in $\mathscr{C}$. But now the witness itself is a counterexample. Note that clause (e) follows from (∗).

Claim 2.6. *If there is a witness, then there is a system as required, (for our $\delta \in E \cap S^\lambda_{\geqslant \mu^+}$).*

By induction on $\alpha < \mu^+$ we define $\delta_\alpha < \delta$ and a system $\langle N'_{\{\alpha\}}, N_{\{\alpha\}}, N_{\{\alpha,\beta\}}\rangle$, for $\beta < \alpha$.

Suppose that we have defined the system for all $\beta < \alpha$. Let $A = \bigcup \{N_u\colon u \in [\{\delta_\beta\colon \beta < \alpha\}]^{\leqslant 2}\}$. Let $N'_{\{\alpha\}}$ and $N_{\{\alpha\}}$, $N_{\{0,\alpha\}}$ be as in the definition of a witness, for the above A. For $\beta < \alpha$ let $N_{\{\beta,\alpha\}} = Sk(N_{\{\beta\}}, N'_{\{\alpha\}})$. It follows that N_α is isomorphic to $\mathscr{N}_0$ and $N_{\{\beta,\alpha\}}$ is isomorphic to $\mathscr{N}$. Let $\delta_\alpha = OP_{N_{\{0,\alpha\}}, N^*_{\{0,1\}}}(\alpha_0)$. Note that $I = \{\delta_\alpha\colon \alpha < \mu^+\}$ is such that $\sup(I) = \delta$ and $N_u \cap I = u$ for every $u \in [I]^{\leqslant 2}$. This finishes the proof.

3. Proof of the theorem

Start with a model satisfying the assumptions of the theorem, i.e. we have $2^{\aleph_l} = \aleph_{l+1}$ for $l < 4$, $\{C_\alpha\colon \alpha \in \omega_4\}$ is a square sequence and $\langle \bar{M}^i\colon i \in W\rangle$ is a diamond for systems, $\diamondsuit_s(\aleph_4, \aleph_1, \aleph_1, \aleph_0)$. Let $\bar{M}^i = \langle M^i_u\colon u \in [\bar{B}_i]^{\leqslant 2}\rangle$ and let $\bar{B}_i = \{\alpha^i_\varepsilon\colon \varepsilon < \omega_1\}$ be the increasing enumeration.

Definition 3.1. (1) A set $b \subseteq \alpha$ is $\bar{Q} \restriction \alpha$-closed, i.e. $\alpha \in b \Rightarrow a_\alpha \subseteq b$.
(2) $\mathscr{K} = \mathscr{K}_\mu$ is the family of FS-iterations $\bar{Q} = \langle P_\alpha, Q_\alpha, a_\alpha,\colon \alpha < \alpha^*\rangle$ such that:
 (a) $a_\alpha \subseteq \alpha$,
 (b) $|a_\alpha| \leqslant \mu$,
 (c) $\beta \in a_\alpha \Rightarrow a_\beta \subseteq a_\alpha$,
 (d) for $b \subseteq \alpha$, $P^*_b = \{p \in P_\alpha\colon \mathrm{dom}(p) \subseteq b$ and $(\forall \beta \in \mathrm{dom}(p))\, p(\beta)$ is a $P^*_{b\cap\alpha}$ name$\}$,
 (e) Q_α is a $P^*_{a_\alpha}$-name (see 3.2 below),
 (f) $P^*_{\alpha^*}$ has the property K (= *Knaster*).

Remark. The above definition proceeds by induction on α^*, so part (d) is not circular.

Lemma 3.2. *Suppose* $\bar{Q} = \langle P_\alpha, Q_\alpha, a_\alpha,: \alpha < \alpha^* \rangle \in \mathscr{K}$. *If* $b \subseteq \alpha^*$ *is* $\bar{Q}$*-closed, then* $P_b^* \lessdot P_{\alpha^*}^*$.

Proof. Straightforward, see [2, 3].

Let $f : {}^{\omega_1 >}2 \to \aleph_1$ be one to one, such that if $\eta \triangleleft \nu$, then $f(\eta) \triangleleft f(\nu)$. For $\rho \in {}^{\omega_1}2$ let $w_\rho = \{f(\rho \restriction i): i < \aleph_1\} \in [\aleph_1]^{\aleph_1}$. Note that if $\rho_1 \neq \rho_2$ in ${}^{\omega_1}2$, then $|w_{\rho 1} \cap w_{\rho 2}| < \aleph_1$. Let R be the countable support forcing adding $\aleph_4$ many Cohen subsets of ω_1, ρ_i $(i < \omega_4)$. Note that in V^R, $\{w_{\rho_i}: i \in \omega_4\}$ is a family of almost disjoint, uncountable subsets of ω_1. Let $B_i = \{\alpha^i_\varepsilon: \varepsilon \in w_{\rho_i}\}$. Note that $\{M^i_u: u \in [B_i]^{\leqslant 2}\}$ is still a system of models on i, hence without loss of generality we can assume that $w_{\rho_i} = \omega_1$. For $\zeta \in \omega_1$ define $B_i(\zeta) = \{\alpha^i_\varepsilon: \varepsilon < \zeta\}$. In V^R we shall define an iteration $\langle P_i, Q_i, a_i: i < \chi \rangle \in \mathscr{K}_{\aleph_4}$. Working in V^R, we define $\bar{Q} \restriction i$, by induction on $i < \omega_4$, and we prove that it is as in 3.1 (in V^R).

We call i good if it satisfies: $i \in W$, each M^i_u has a predetermined predicate describing $\bar{Q} \restriction M^i_u$ (as an R-name, with the limit $\underset{\sim}{P}^i_u$ and an $R \restriction M^i_u * \underset{\sim}{P}^i_u$-name $\underset{\sim}{f}$ for a function from ${}^\omega 2 \times {}^\omega 2$ into ${}^\omega 2$ and each M^i_u is $\bar{Q}$-closed. (Recall that we do not distinguish between the model M^i_u and its universe.) In this case we put $a_i = \bigcup \{M^i_u: u \in [B_i]^{\leqslant 2}\}$ and define Q_i below.

If i is not good we put $a_i = \emptyset$ and define Q_i to be the Cohen forcing, i.e., $Q_i = ({}^{\omega >}2, \triangleleft)$. We can assume that if $\alpha \in B_i$, then Q_α is Cohen, (or just replace B_i by $\{\alpha + 1: \alpha \in B_i\}$). For $\alpha \in B_i$, let r_α be the Cohen real forced by Q_α.

Remark. The reason we add $\aleph_4$ almost disjoint subsets of ω_1 is that, in V^R, if $i \neq j$ are good and $\mathrm{otp}(C_i) = \mathrm{otp}(C_j)$, then the systems associated with i and j are almost disjoint, i.e., there is $\zeta \in \omega_1$ such that

$$\left(\bigcup \{M^i_u: u \in [B_i]^{\leqslant 2}\}\right) \cap \left(\bigcup \{M^j_u: u \in [B_j]^{\leqslant 2}\}\right)$$
$$\subseteq \left(\bigcup \{M^i_u: u \in [B_i(\zeta)]^{\leqslant 2}\}\right) \cap \left(\bigcup \{M^j_u: u \in [B_j(\zeta)]^{\leqslant 2}\}\right)$$

Note that if $\mathrm{otp}(C_i) \neq \mathrm{otp}(C_j)$ then we have almost disjointness by Definition 2.2(D)(i).

Notation. For $\xi, \zeta \in \omega_1$ let $Z^i_{\xi,\zeta} = M^i_{\{\alpha^i_\xi, \alpha^i_\zeta\}} \cup M^i_{\{\alpha^i_\xi\}} \cup M^i_{\{\alpha^i_\zeta\}}$, $Z^i_\xi = M^i_{\{\alpha^i_\xi\}}$.

Now we fix a good i. Our goal is to define Q_i.

Definition 3.3. For $p, q \in R$ (or in $P^*_{\omega_4}$), $\mathrm{dom}(p), \mathrm{dom}(q) \subseteq Z^i_{0,1}$ we say that p and q are dual if $OP_{Z^i_1, Z^i_0}(p \restriction Z^i_0) = q \restriction Z^i_1$ and $OP_{Z^i_1, Z^i_0}(q \restriction Z^i_0) = p \restriction Z^i_1$.

Using $G_{R \restriction M^i_\emptyset}$ we choose, by induction on $k < \omega$, conditions r^i_η, $r^{i,l}_\eta \in R$ for $\eta \in {}^k 2$, $l < 2$, such that:

(a) $r^i_\eta \in (R \restriction Z^i_0)/G_{R \restriction M^i_\emptyset}$.

(b) $\nu \triangleleft \eta \Rightarrow r^i_\nu \leqslant r^i_\eta$.

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(c) if $l=m+1$, if $\eta\in{}^{m}2$, $l<2$, then $r_\eta^{i,l}\in(R\restriction Z_{0,1}^i)/G_{R\restriction M_\emptyset^i}$ and $r_\eta^i\leqslant r_\eta^{i,l}\restriction Z_0^i\leqslant r_{\eta\frown\langle l\rangle}^i$ and $OP_{Z_1^i,Z_0^i}(r_\eta^i)\leqslant r_\eta^{i,l}\restriction Z_1^i\leqslant OP_{Z_1^i,Z_0^i}(r_{\eta\frown\langle 1-l\rangle}^i)$, and $r_\eta^{i,0}$ and $r_\eta^{i,1}$ are dual.

(d) $r_\eta^{i,l}$ forces that $A_k^{\eta,l}=\{\underset{\sim}{p}_{k,n}^{\eta,l}\colon n\in\omega\}$ is a predense subset of $P_{Z_{0,1}^i}^*$, such that each $\underset{\sim}{p}_{k,n}^{\eta,1}$ forces the value $f_{k,n}^{\eta,l}$ of $\underset{\sim}{f}(r_{\alpha_0^i},r_{\alpha_1^i})\restriction k$.

(e) $A_k^{\eta,0}$ and $A_k^{\eta,1}$ are dual, i.e. for every $m\in\omega$, $\underset{\sim}{p}_{k,m}^{\eta,0}$ and $\underset{\sim}{p}_{k,m}^{\eta,1}$ are dual. Moreover if $k_1<k_2$, then $A_{k_2}^{\eta,l}$ refines $A_{k_1}^{\eta,l}$.

Suppose we have r_η^i. We define $r_\eta^{i,0},r_\eta^{i,1}$ and $A_k^{\eta,0},A_k^{\eta,0}$ as follows.

1. Let $r_1=r_\eta^i\cap OP_{Z_1^i,Z_0^i}(r_\eta^i)$.
2. Let $r_{1,0}\geqslant r_1$, $r_{1,0}\in R\restriction Z_{0,1}$, forces a maximal antichain $A_{1,0}$ of $P_{Z_{0,1}}^*$, such that each element of $A_{1,0}$ forces a value of $\underset{\sim}{f}(r_{\alpha_0^i},r_{\alpha_1^i})\restriction k$.
3. Let $r_2=OP_{Z_1^i,Z_0^i}(r_{1,0}\restriction Z_0^i)\cup OP_{Z_0^i,Z_1^i}(r_{1,0}\restriction Z_1^i)$. Let $r_{2,1}\geqslant r_2$, $r_{2,1}\in R\restriction Z_{0,1}$ forces $A_{2,1}$ to be a predense subset of $P_{Z_{0,1}}^*$ such that each element of $A_{2,1}$ forces a value of $\underset{\sim}{f}(r_{\alpha_0^i},r_{\alpha_1^i})\restriction k$. Moreover, $A_{2,1}=\bigcup\{A_p\colon p\in A_{1,0}\}$, which for every $q\in A_p$ we have $q\geqslant OP_{Z_1^i,Z_0^i}(p\restriction Z_0^i)\cup OP_{Z_0^i,Z_1^i}(p\restriction Z_1^i)$.
4. Let $r_3=OP_{Z_1^i,Z_0^i}(r_{2,1}\restriction Z_0^i)\cup OP_{Z_0^i,Z_1^i}(r_{2,1}\restriction Z_1^i)$.
5. Let $r_{3,0}=r_3\cup r_{1,0}$ (note: $r_{3,0}$ is dual to $r_{2,1}$). Let $A_{3,0}=\{p\cup OP_{Z_1^i,Z_0^i}(q\restriction Z_0^i)\cup OP_{Z_0^i,Z_1^i}(q\restriction Z_1^i)\colon q\in A_p\}$.
6. Let $r_\eta^{i,0}=r_{3,0}$, $r_\eta^{i,1}=r_{2,1}$, $A_k^{\eta,0}=A_{3,0}$ and $A_k^{\eta,1}=A_{2,1}$.

Let for $\eta\in{}^{\omega}2$, $r_\eta^i=\bigcup_{k<\omega}r_{\eta\restriction k}^i$. In V choose $\langle\eta_\varepsilon^*\colon\varepsilon<\omega_1\rangle$, distinct members of ${}^{\omega}2$. Recall that ρ_j $(j<\aleph_4)$ are the Cohen subsets of ω_1 forced by R. In $V[\langle\rho_j\colon j\in\{i\}\cup a_i\rangle]$ we can find $w^i\in[\omega_1]^{\omega_1}$ such that

(α) if $\varepsilon\in w^i$ then $OP_{Z_\varepsilon^i,Z_0^i}(r_{\eta_\varepsilon^*})\in G_{R\restriction Z_\varepsilon^i}$,

(β) if $\varepsilon_0<\varepsilon_1$ are in w^i, $l=TV(\eta_{\varepsilon_0}^*<_{lx}\eta_{\varepsilon_1}^*)$, then

$$OP_{Z_{\varepsilon_0,\varepsilon_1}^*,Z_{0,1}^i}(r_{\eta_{\varepsilon_0}^*\cap\eta_{\varepsilon_1}^*}^{i,l})\in G_{R\restriction Z_{\varepsilon_0,\varepsilon_1}^i}.$$

We choose the members of w^i inductively using the fact that R has $(<\aleph_1)$-support.

Notation. For $\xi\in w^i$ denote $r_\xi^i=r_{\alpha_\xi^i}$.

Let H be R-generic and G be $P_{a_i}^*$-generic. In $V[H][G]$ we define Q_i. A condition in Q_i is $(u,v,\bar v,\bar m,F_0,F_1)$, where:

(1) u is a finite subset of w^i.

(2) v is a finite set of elements of the form (η,ρ), where

   (a) $\eta,\rho\in{}^{\omega>}2$, $\mathrm{lh}(\eta)=\mathrm{lh}(\rho)$, $\rho\neq\eta$,

   (b) $\eta\lhd r_\alpha^i$, $\rho\lhd r_\beta^i$ for some $\alpha,\beta\in u$ and if $v=\eta_\alpha^*\cap\eta_\beta^*$ then for every $\gamma\in u$ we have: if $\eta\lhd r_\gamma^i$, then $\eta_\gamma^*\restriction(\mathrm{lh}(v)+1)=\eta_\alpha^*\restriction(\mathrm{lh}(v)+1)$, and if $\rho\lhd r_\gamma^i$, then $\eta_\gamma^*\restriction(\mathrm{lh}(v)+1)=\eta_\beta^*\restriction(\mathrm{lh}(v)+1)$.

(3) $\bar v$ is a function from v into ${}^{\omega>}2$ such that for $(\eta,\rho)\in v$ we have: $\bar v(\eta,\rho)$ is such that there is $\alpha,\beta\in u$ such that $\eta\lhd r_\alpha^i$, $\rho\lhd r_\beta^i$ and $\bar v(\eta,\rho)=\eta_\alpha^*\cap\eta_\beta^*$, ($\bar v$ is well defined by (2)).

(4) $\bar{m}$ is a function from v to ω. For $(\eta,\rho)\in v$, $\bar{m}(\eta,\rho)$ is such that for every $\alpha,\beta\in u$ such that $\eta\lhd r^i_\alpha$, $\rho\lhd r^i_\beta$, we have $OP_{Z^i_{\alpha,\beta},Z^i_{0,1}}(p^{v,l}_{lh(\eta),\bar{m}(\eta,\rho)})\in G$, where $l=TV(\eta^*_\alpha<_{lx}\eta^*_\beta)$ and $v=\eta^*_\alpha\cap\eta^*_\beta$.
(5) For $l=0$, 1, F_l is a function from v into ${}^{\omega>}2$, defined by: for $(\eta,\rho)\in v$, $F_l(\eta,\rho)$ is the value of $\underset{\sim}{f}(r_0,r_1)\restriction lh(\eta)$ forced by $p^{\bar{v}(\eta,\rho),l}_{lh(\eta),\bar{m}(\eta,\rho)}$.
(6) For (η,ρ), $(\eta_1,\rho_1)\in v$, if $\eta\lhd\eta_1$ and $\rho\lhd\rho_1$, then $F_l(\eta,\rho)\lhd F_l(\eta_1,\rho_1)$, for $l=0,1$.
Order: $(u,\ v,\ \bar{v},\ \bar{m},\ F_0,\ F_1)\leqslant(u^1,v^1,\bar{v}^1,\bar{m}^1,F^1_0,F^1_1)$ if
(7) $u\subseteq u^1$,
(8) $v\subseteq v^1$,
(9) $F_l=F^1_l\restriction v$, $\bar{v}=\bar{v}^1\restriction v$, $\bar{m}=\bar{m}^1\restriction v$, $l=0,1$.

Lemma 3.4. *Suppose (q_α,p_α), (for $\alpha\in\omega_1$), are in $P^*_{a_i}*Q_i$, q_α forces p_α to be a real 6-tuple in Q_i, not just a $P^*_{a_i}$-name of such a tuple,* $\mathrm{dom}(q_\alpha)$ $(\alpha\in\omega_1)$ *form a delta system with the root Δ, $\zeta\in\omega_1$. Let $b=\bigcup\{M^i_u\colon u\in[B_i(\zeta)]^{\leqslant 2}\}$. Suppose $\Delta-\{i\}\subseteq b$ and* $\mathrm{dom}(q_\alpha)\cap b=\Delta$ *for $\alpha\in\omega_1$.*

Then there is an uncountable set $E\subseteq\omega_1$ such that for every $\alpha,\beta\in E$, (q_α,p_α) and (q_β,p_β) are compatible, moreover if $q\in P^*_b$, $q\geqslant q_\alpha\restriction b$, $q_\beta\restriction b$, then $q,(q_\alpha,p_\alpha)$ and (q_β,p_β) are compatible.

Proof. By thinning out we can find an uncountable set $E\subseteq\omega_1$ such that:
(a) For $\alpha\in E$ let $w_\alpha=\bigcup\{u\in[B_i]^{\leqslant 2}\colon \mathrm{dom}(q_\alpha)\cap M^i_u\neq\emptyset\}$, (each w_α is finite). The sets w_α, $(\alpha\in E)$ form a delta system with the root w and if $\alpha<\beta$, $\xi\in w_\alpha$, $\zeta\in w_\beta$, then $\xi\leqslant\zeta$.
(b) u^{p_α} $(\alpha\in E)$ form a delta system with the root u and $\alpha<\beta$, $\xi\in u^{p_\alpha}$, $\zeta\in u^{p_\beta}$, then $\xi\leqslant\zeta$, $|u^{p_\alpha}|=n^*$.
(c) $v^{p_\alpha}=v^*$ for $\alpha\in E$ and the structures $(u^{p_\alpha},\{q_\alpha(\xi)\colon \xi\in u^{p_\alpha}\},\ v^*,\ \{\eta^*_\xi\restriction m^*\colon \xi\in u^{p_\alpha}\})$ are isomorphic, (isomorphism given by the order preserving bijection between respective u^{p_α}'s), where m^* is such that $\mathrm{lh}(\eta^*_\xi\cap\eta^*_\zeta)<m^*$ for every $\xi\neq\zeta$ in u^{p_α}.

Lemma 3.5. *P_{i+1} has the property K.*

Proof. Let $\{p_\alpha\colon \alpha\in\omega_1\}$ be an uncountable subset of P_{i+1}. W.l.o.g. we can assume that $\mathrm{dom}(p_\alpha)$, $(\alpha\in\omega_1)$ form a delta system with the root Δ. We have to find an uncountable subset $E\subseteq\omega_1$ such that for any $\alpha,\beta\in E$, p_α and p_β are compatible. We prove it by induction on $k=|\Delta|$.

For $k=0$, trivial. For the induction step assume that $\Delta=\{i_0,\ldots,i_k\}$ ordered by $\lhd$, where for $\alpha,\beta<\omega_4$, we define $\alpha\lhd\beta$ iff $\mathrm{otp}(C_\alpha)<\mathrm{otp}(C_\beta)$ or $\mathrm{otp}(C_\alpha)=\mathrm{otp}(C_\beta)$ and $\alpha<\beta$.

By the induction hypothesis there is an uncountable set $E'\subseteq\omega_1$ such that for $\alpha,\beta\in E'$, $p_\alpha\restriction\bigcup_{l<k}a_{i_l}$ and $p_\beta\restriction\bigcup_{l<k}a_{i_l}$ are compatible. Note that there is $\zeta\in\omega_1$ such that $a_{i_k}\cap(\bigcup_{l<k}a_{i_l})\subseteq\bigcup\{M^{i_k}_u\colon u\in[B_{i_k}(\zeta)]^{\leqslant 2}\}$, (see Definition 2.2(D)). Now use the previous lemma.

Now suppose that $G(i)$ is Q_i-generic. Let

$$A' = \bigcup \{u\colon \exists(v, \bar{v}, \bar{m}, F_0, F_1), (u, v, \bar{v}, \bar{m}, F_0, F_1) \in G(i)\}.$$

In $V[G]$ let $A = \{r^i_\alpha\colon \alpha \in A'\}$ and let $f_l : [A]^2 \to {}^\omega 2$ be defined by

$$f_l(r^i_\alpha, r^i_\beta) = \bigcup \{F_l(\eta, \rho)\colon \exists(u, v, \bar{v}, \bar{m}, F_0, F_1) \in G(i), \alpha, \beta \in u, (\eta, \rho) \in v, \eta \lhd r^i_\alpha, \rho \lhd r^i_\beta\}.$$

Let $\mathscr{V} = \bigcup \{v\colon \exists(u, \bar{v}, \bar{m}, F_0, F_1)\colon (u, v, \bar{v}, \bar{m}, F_0, F_1) \in G(i)\}$.

Lemma 3.6. (1) *For every* $\alpha, \beta \in A'$ *and* $n \in \omega$, *there is* $(\eta, \rho) \in \mathscr{V}$ *such that* $\mathrm{lh}(\eta) = \mathrm{lh}(\rho) \geqslant n$ *and* $\eta \lhd r_\alpha$ *and* $\rho \lhd r_\beta$,

(2) *A is uncountable,*

(3) f_0, f_1 *are continuous,*

(4) *for every* $(\alpha, \beta) \in [A]^2$, *if* $l = TV(\eta^*_\alpha <_{lx} \eta^*_\beta)$, *then* $f(r^i_\alpha, r^i_\beta) = f_l(r^i_\alpha, r^i_\beta)$.

Proof. (1) and (2) follow by a density argument. To prove (1) suppose that $(p, q) \in P_i * Q_i$, p forces that $\alpha, \beta \in u^q$. W.l.o.g. $\alpha, \beta \in \mathrm{dom}(p)$. Let $p_1 \in P_i$ be such that $\mathrm{dom}(p) = \mathrm{dom}(p_1)$, $p(\zeta) = p_1(\zeta)$ for $\zeta \in \mathrm{dom}(p) \setminus \{\alpha, \beta\}$, $p(\alpha) \lhd p_1(\alpha)$, $p(\beta) \lhd p_1(\beta)$, $\mathrm{lh}(p_1(\alpha)) = \mathrm{lh}(p_1(\beta)) \geqslant n$, (remember that Q_α, Q_β are Cohen). Let $\eta = p_1(\alpha)$, $\rho = p_1(\beta)$, $v = \eta^*_\alpha \cap \eta^*_\beta$, $l = TV(\eta^*_\alpha <_{lx} \eta^*_\beta)$. Let $m \in \omega$ be such that $OP_{Z_{\alpha,\beta}, Z_{0,1}}(p^{v,l}_{\mathrm{lh}(\eta), m})$ is compatible with p_1, and let p_2 be the common upper bound. Now define $q_1 \geqslant q$ as follows. $u^{q_1} = u^q$, $v^{q_1} = v^q \cup \{(\eta, \rho)\}$, $\bar{v}^{q_1}(\eta, \rho) = v$, $\bar{m}^{q_1}(\eta, \rho) = m$, $F^{q_1}_l(\eta, \rho)$ is the value forced by $p^{v,l}_{\mathrm{lh}(\eta), m}$. Hence $(p_2, q_1) \geqslant (p, q)$ and it forces what is required.

To prove (2) it is enough to show, in V^R, that for every $\alpha \in \omega_1$ and $(p, q) \in P_i * Q_i$ there is $\beta > \alpha$ and $(p_1, q_1) \geqslant (p, q)$, such that $\beta \in u^{q_1}$. Let $\beta > \alpha$ be such that $\mathrm{dom}(p) \cap Z^i_{\gamma,\beta} \subseteq M^i_\emptyset$ and $\beta > \gamma$ for every $\gamma \in u^q$. Let $\gamma \in u^q$ be such that $(\eta^*_{\gamma_1} \cap \eta^*_\beta) \lhd (\eta^*_\gamma \cap \eta^*_\beta)$ for every $\gamma_1 \in u^q$. Define condition $q_1(\beta) = q(\gamma)$ and let p_1 be a condition extending p and each of conditions $OP_{Z^i_{\gamma_1,\beta}, Z^i_{0,1}}(p^{\bar{v}(\eta,\rho), l}_{\mathrm{lh}(\eta), \bar{m}(\eta,\rho)})$ such that $(\eta, \rho) \in v$, $\eta \lhd q(\gamma_1)$, $\rho \lhd q(\gamma)$ and $l = TV(\eta^*_{\gamma_1} < \eta^*_\beta)$. Finally extend q to q_1 such that $u^{q_1} = u^q \cup \{\beta\}$.

Condition (3) follows from (1), (5) and (6) in the definition of Q_i.

To prove (4) it is enough to show that for every $n \in \omega$, $f(r^i_\alpha, r^i_\beta) \restriction n = f_l(r^i_\alpha, r^i_\beta) \restriction n$. By condition (1) there is $(\eta, \rho) \in V$ such that $k = \mathrm{lh}(\eta) \geqslant n$ and $\eta \lhd r^i_\alpha$ and $\rho \lhd r^i_\beta$. Recall that $p = p^{\bar{v}(\eta,\rho), l}_{\mathrm{lh}(\eta), \bar{m}(\eta,\rho)}$ forces that $f(r^i_0, r^i_1) \restriction k = h$ for some fixed h. Now working in V consider $(r^{i,l}_{\eta^*_\alpha \cap \eta^*_\beta}, p) \in R * P_i \restriction Z^i_{0,1}$. By the construction the condition $(r', p) = OP_{Z^i_{\alpha,\beta}, Z^i_{0,1}}(r^{i,l}_{\eta^*_\alpha \cap \eta^*_\beta}, p) \in H * G$, and forces that $f(r^i_\alpha, r^i_\beta) = h$. On the other hand, by definition $F_l(\eta, \rho) = h$ and $F_l(\eta, \rho) \lhd f_l(r^i_\alpha, r^i_\beta)$ This finishes the proof.

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