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STRONG NEGATIVE PARTITION RELATIONS BELOW THE CONTINUUM

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0. Introduction

DEFINITION 1. If λ is a cardinal, $\mathrm{Pr}^+(\lambda)$ means that there is a function $c\colon [\lambda]^2 \to \lambda$ such that if $1 \leqq n < \omega$ and the sets $\{\zeta_\alpha^0, \dots, \zeta_\alpha^{n-1}\}$ are disjoint for $\alpha < \lambda$ and $\zeta_\alpha^0 < \dots < \zeta_\alpha^{n-1}$ then for every $h\colon n \times n \to \lambda$ there are $\alpha < \beta$ such that $c(\zeta_\alpha^i, \zeta_\beta^j) = h(i,j)$ for $i, j < n$.

DEFINITION 2. $\mathrm{Pr}(\lambda)$ is the same but only for every $h\colon n \times n \to \lambda$ with h constant, i.e. $h(i,j) = \gamma$ for $i, j < n$.

LEMMA 1. *If λ is regular, not strong limit, then* $\mathrm{Pr}(\lambda)$ *implies* $\mathrm{Pr}^+(\lambda)$.

PROOF. We use the idea in the proof of the Engelking–Karlowitz theorem. Assume that $\mu < \lambda$ and $2^\mu \geqq \lambda$. Let $\{A_\alpha \colon \alpha < \lambda\}$ be different subsets of μ. Assume that c^- witnesses $\mathrm{Pr}(\lambda)$. Put $G = \{\langle w, g\rangle : w \in [\mu]^{<\omega}, g\colon P(w)^2 \to \lambda\}$. Clearly, $|G| = \lambda$, so we can enumerate it as $\{\langle w_\alpha, g_\alpha\rangle : \alpha < < \lambda\}$. Now put $c(\alpha, \beta) = g_\gamma(A_\alpha \cap w_\gamma, A_\beta \cap w_\gamma)$, where $\gamma = c^-(\alpha, \beta)$.

Assume that $\{\zeta_\alpha^i \colon i < n, \alpha < \lambda\}$ are given as in Definition 1, $h\colon n \times n \to \lambda$. For $\alpha < \lambda$, $i < j < n$, pick $\gamma_\alpha^{i,j} \in A_{\zeta_\alpha^i} \triangle A_{\zeta_\alpha^j}$, and let $w^\alpha = \{\gamma_\alpha^{i,j} \colon i < j < n\}$. As $w^\alpha \subseteqq \mu < \lambda$, we may assume that there exist w, $B_i \subseteqq w$ $(i < n)$, such that $w^\alpha = w$, $A_{\zeta_\alpha^i} \cap w = B_i$ for $\alpha < \lambda$. Let $g\colon P(w)^2 \to \lambda$ be a function satisfying $g(B_i, B_j) = h(i,j)$. There is a $\gamma < \lambda$ with $\langle w, g\rangle = \langle w_\gamma, g_\gamma\rangle$, and by $\mathrm{Pr}(\lambda)$ there are $\alpha < \beta < \lambda$ such that if $i < j < n$, then $c^-(\zeta_\alpha^i, \zeta_\beta^j) = \gamma$. But then $c(\zeta_\alpha^i, \zeta_\beta^j) = g_\gamma(A_{\zeta_\alpha^i} \cap w_\gamma, A_{\zeta_\beta^j} \cap w_\gamma) = g(B_i, B_j) = h(i,j)$, and we are done.

We now state the main result of this paper. We remind the reader that $S \subseteqq \lambda$ is a non-reflecting stationary set if it is stationary and $S \cap \alpha$ is non-stationary in α for every limit $\alpha < \lambda$.

THEOREM. $\mathrm{Pr}(\lambda)$ *holds whenever there exists a nonreflecting stationary set S in λ with $\mathrm{cf}(\alpha) > \omega_1$ for every $\alpha \in S$.*

This work is continued in [10] (see also [11]).

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1. Construction of the coloring

For $\alpha < \lambda$ limit let $C_\alpha \subsetneqq \alpha$ be a closed unbounded set of order type $\mathrm{cf}(\alpha)$ disjoint from S. For $\alpha = \beta + 1$ we let $C_\alpha = \{\beta\}$. For $0 < \alpha < \beta < \lambda$ let $\gamma(\beta,\alpha) = \min(C_\beta - \alpha)$. Obviously, $\alpha \leqq \gamma(\beta,\alpha) < \beta$. We now define $\gamma_\ell(\beta,\alpha)$ for $\ell \leqq k(\beta,\alpha)$ as follows: $\gamma_0(\beta,\alpha) = \beta$, $\gamma_{\ell+1}(\beta,\alpha) = \gamma(\gamma_\ell(\beta,\alpha),\alpha)$. If $\gamma_\ell(\beta,\alpha) = \alpha$ then we terminate the definition and put $k = k(\beta,\alpha) = \ell$. Clearly, $\alpha = \gamma_k(\beta,\alpha) < \ldots < \gamma_0(\beta,\alpha) = \beta$. The string $\varrho(\beta,\alpha) = \langle \gamma_0(\beta,\alpha), \ldots, \gamma_k(\beta,\alpha)\rangle$ is the Todorcevic walk from β to α.

Fix a decomposition $S = \cup\{S^\gamma : \gamma < \lambda\}$ into stationary sets (possible, by Solovay's theorem). Let $H\colon \lambda \to \omega_1$ be a mapping such that for every $i < \omega_1$ the set $S_i = S \cap H^{-1}(\{i\})$ is stationary in λ. Let $\omega_1 = \cup\{R_n : n < \omega\}$ be a partition into stationary sets. For $0 < \alpha < \beta < \lambda$ we let

$$w_1(\beta,\alpha) = \{p > k/2\colon \text{ for every } q < k/2, H(\gamma_p) > H(\gamma_q)\}$$

and $p_1 = \min(w_1)$. Here and in several cases later, we omit (β,α) after w_1, p_1, k etc. if it is obvious what we are speaking of. We now define

$$w_2 = \left\{q < \frac{k}{2}\colon \text{ for every } \frac{k}{2} < p \leqq k, p \notin w_1 \text{ implies } H(\gamma_q) > H(\gamma_p)\right\}.$$

Let p_2 be such that $\min\{H(\gamma_q) : q \in w_2\} \in R_{p_2}$. Now if $0 \leqq p_1 - p_2 \leqq k$ and $\gamma_{p_1-p_2}(\beta,\alpha) \in S^\gamma$ we put $c(\beta,\alpha) = \gamma$ otherwise $c(\beta,\alpha)$ is chosen arbitrarily.

2. Preliminaries

Definition 3. If $s_1 = \langle s_1(0), \ldots, s_1(t_1)\rangle$, $s_2 = \langle s_2(0), \ldots, s_2(t_2)\rangle$ are strings, their *concatenation* $s_1 \wedge s_2$ is $\langle s_1(0), \ldots s_1(t_1 - 1), s_2(0), \ldots, s_2(t_2)\rangle$.

The reason why we are removing the border element is that in our applications $s_1(t_1) = s_2(0)$ holds, so we only remove an immediate repetition.

Lemma 2. *If $\delta \in S$, $\beta > \delta$ then there exists a $\chi(\beta,\delta) < \delta$ such that for every α with $\chi(\beta,\delta) \leqq \alpha < \delta$, $\varrho(\beta,\delta)$ is an initial segment of $\varrho(\beta,\alpha)$. Moreover, $\varrho(\beta,\alpha) = \varrho(\beta,\delta) \wedge \varrho(\delta,\alpha)$.*

Proof. If $\alpha < \delta$ is large enough, $\gamma(\beta,\alpha) = \gamma(\beta,\delta)$. Therefore, if $\alpha \geqq \chi(\gamma(\beta,\delta),\delta)$ also holds, the statement is true. We get, therefore, a proof by induction on β.

Lemma 3. *If $A, B \in [\lambda]^\lambda$, $k < \omega$, then there exist $\alpha \in A$, $\beta \in B$, $\alpha < \beta$ with $k(\beta,\alpha) > k$.*

Proof. We define $C_0 = A'$, and by induction, $C_{i+1} = (S \cap C_i)'$. Pick $\gamma_k \in C_k \cap S$, then $\beta \in B$ with $\beta > \gamma_k$, $\chi_k = \chi(\beta,\gamma_k)$. If γ_{i+1}, χ_{i+1} are found, pick $\gamma_i \in S \cap C_i$ with $\chi_{i+1} < \gamma_i < \gamma_{i+1}$ and χ_i with $\chi_i > \chi(\gamma_{i+1},\gamma_i)$,

$\chi_{i+1} < \chi_i < \gamma_i$. Given γ_0, χ_0 let $\alpha \in A$ satisfy $\chi_0 < \alpha < \gamma_0$, then by Lemma 2, for $\ell \leqq k$ there exists an $m \leqq k(\beta,\alpha)$ such that $\gamma_m(\beta,\alpha) = \gamma_\ell$, so $k(\beta,\alpha) > k$.

DEFINITION 4. $\varrho_H(\beta,\alpha) = \langle H(\gamma_\ell(\beta,\alpha)) \colon \ell \leqq k(\beta,\alpha)\rangle$. If $\sigma \in \omega_1^{<\omega}$, i.e. is a finite string of countable ordinals, then for $i < \omega_1$ σ^i is the following string $|\sigma^i| = |\sigma|$, and

$$\sigma^i(\ell) = \begin{cases} \sigma(\ell) & \text{if } \sigma(\ell) < i, \\ \omega_1 & \text{if } \sigma(\ell) \geqq i. \end{cases}$$

DEFINITION 5. If $T \subseteqq \lambda$, $\delta < \lambda$, $R \subseteqq \omega_1$ stationary, then $U(\delta,T,R)$ denotes the set of those $\varrho \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that for every $i < \omega_1$ there exists a $\beta > \delta$, $\beta \in T$ with $\varrho_H(\beta,\delta)^i = \varrho$ and $\min\{\varrho_H(\ell) : \varrho^i(\ell) = \omega_1\} \in R$. $\varrho \in U(\delta,T,R,\chi)$ denotes that β even satisfies $\chi(\beta,\delta) < \chi$.

LEMMA 4. *If $T \in [\lambda]^\lambda$, then there is a $\delta(T) < \lambda$ such that for $\delta(T) \leqq \delta < \lambda$, $U(\delta,T,R) \neq \emptyset$. If $\mathrm{cf}(\delta) > \omega_1$, then there is a $\chi < \delta$ such that $U(\delta,T,R,\chi) \neq \emptyset$.*

PROOF. For $i < \omega_1$ we let $A_i = \{\delta < \lambda\colon$ if $\beta > \delta, \beta \in T$, then $i \notin \varrho_H(\beta,\delta)\}$.

CLAIM. *$|A_i| < \lambda$ for $i < \omega_1$.*

PROOF OF CLAIM. Suppose that $|A_i| = \lambda$ for some $i < \omega_1$ and select a $\delta \in S_i \cap A_i'$, $\beta \in T$ with $\beta > \delta$. Choose an $\alpha \in A_i$, $\chi(\beta,\delta) < \alpha < \delta$. Then $\delta \in \varrho(\beta,\alpha)$, and $i = H(\delta) \in \varrho_H(\beta,\alpha)$, a contradiction.

Now we define $\delta(T)$ with $\cup\{A_i : i < \omega_1\} \subseteqq \delta(T)$. Assume that $\delta(T) \leqq \delta < \lambda$. For every $i < \omega_1$, there is a $\beta_i > \delta$, $\beta_i \in T$ such that $i \in \varrho_H(\beta_i,\delta)$.

Consider $\{\varrho_H(\beta_i,\delta) : i \in R\}$. There exist a stationary $R_1 \subseteqq R$ and a $k < \omega$ such that for $i \in R_1$, $|\varrho_i| = k$, where $\varrho_i = \varrho_H(\beta_i,\delta)$. We even assume that for every $\ell < k$ either for every $i \in R_1$ $\varrho_i(\ell) < i$ or for every $i \in R_1$ $\varrho_i(\ell) \geqq i$. Applying Fodor's theorem we can find a stationary $R_2 \subseteqq R_1$ and an $\eta \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that $\varrho_H(\beta_i,\delta)^i = \eta$ $(i \in R_2)$. For $i \in R_2$, $\min\{\varrho^i(\ell) : \eta(\ell) = \omega_1\} = i \in R_2 \subseteqq R$, so $\eta \in U(\delta,T,R)$.

If $\mathrm{cf}(\delta) > \omega_1$, $\{\chi(\beta_i,\delta) : i \in R_2\}$ is bounded below δ, so $\eta \in U(\delta,T,R,\lambda)$, if $\lambda > \lambda(\beta_i,\delta)$ $(i \in R_2)$.

DEFINITION 6. If $T \subseteqq \lambda$, $\delta < \lambda$, then $L(\delta,T)$ consists of those $\varrho \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ for which for every $\alpha < \delta$, and large enough $i < \omega_1$ there is a $\beta \in T$, $\alpha < \beta < \delta$ such that $\varrho_H(\delta,\beta)^i = \varrho$. For $T \in [\lambda]^\lambda$ we let $C(T) = \bigcap\{(S_i \cap T')' : i < \omega_1\}$.

Obviously, $C(T)$ is closed unbounded in

LEMMA 5. *If $\delta \in C(T)$, $\mathrm{cf}(\delta) \geqq \omega_1$, then $L(\delta,T) \neq \emptyset$.*

PROOF. *Case 1:* $\mathrm{cf}(\delta) = \omega_1$. Let $\{\delta_i : i < \omega_1\}$ converge to δ. For $i < \omega_1$ pick an $\alpha_i \in S_i \cap T'$, $\delta_i < \alpha_i < \delta$ (possible, as $\delta \in C(T)$). Now choose $\beta_i \in T$, $\delta_i < \beta_i < \alpha_i$ with $\chi(\delta,\alpha_i) < \beta_i$. Then $i = H(\alpha_i) \in \varrho_H(\delta,\beta_i)$.

As in Lemma 4, there is a stationary $X \subseteqq \omega_1$ and a $\varrho \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that $\varrho_H(\delta, \beta_i)^i = \varrho$ $(i \in X)$, so $\varrho \in L(\delta, T)$.

Case 2: $\mathrm{cf}(\delta) > \omega_1$. Let $\{\delta_\alpha : \alpha < \mathrm{cf}(\delta)\}$ converge to δ. For $\alpha < \mathrm{cf}(\delta)$, $i < \omega_1$, pick $\beta_i^\alpha \in T$ with $\delta_\alpha \leqq \beta_i^\alpha < \delta$ as in Case 1. For $\alpha < \mathrm{cf}(\delta)$, there is a $\varrho^\alpha \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that there exist an $X^\alpha \in [\omega_1]^{\omega_1}$ with $\varrho_H(\delta, \beta_i^\alpha)^i = \varrho^\alpha$ for $i \in X^\alpha$. There is a ϱ with $\varrho^\alpha = \varrho$ for $\mathrm{cf}(\delta)$ many α's. Clearly, $\varrho \in L(\delta, T)$.

3. Proof of the theorem

Assume that the sets $\{\zeta_\alpha^0, \dots, \zeta_\alpha^{n-1}\}$ are disjoint $(\alpha < \lambda, n < \omega)$. We may assume that $\alpha < \zeta_\alpha^0 < \zeta_\alpha^1 < \dots < \zeta_\alpha^{n-1}$. There is a closed unbounded set $C \subseteqq \lambda$ such that if $\alpha < \delta$, $\delta \in C$, then $\zeta_\alpha^{n-1} < \delta$.

For $\delta \in S \cap C$, as $\mathrm{cf}(\delta) > \omega_1$, there are $\{\nu_\ell^\delta : \ell < n\}$ such that $\sup\{\alpha < \delta : \varrho_H(\delta, \zeta_\alpha^\ell) = \nu_\ell^\delta\} = \delta$. For a stationary $T_1 \subseteqq S \cap C$, $\nu_\ell^\delta = \nu_\ell$ $(\delta \in T_1)$. By Lemma 5, for $\delta \in S \cap C(T_1)$, $L(\delta, T_1) \neq \emptyset$, so there is a stationary $T_2 \subseteqq S \cap C(T_1)$, and $\tau \in (\omega_1+1)^{<\omega} - \omega_1^{<\omega}$ such that $\tau \in L(\delta, T_1)$ for $\delta \in T_2$. We put $\ell^* = \min\{\ell : \tau(\ell) = \omega_1\}$. Again, by Lemma 5, for $\delta \in S \cap C(T_2)$, $L(\delta, T_2) \neq \emptyset$, so there is a stationary $T_3 \subseteqq S^\gamma \cap C(T_2)$, and ϱ with $\varrho \in L(\delta, T_2)$ $(\delta \in T_3)$.

Since $\lambda > \omega_1$, there is a stationary $T^1 \subseteqq S$ and $\{\nu^\ell : \ell < n\}$ such that $\varrho_H(\zeta_\delta^\ell, \delta) = \nu^\ell$ $(\delta \in T^1)$. By Fodor's theorem, there is a $T^2 \subseteqq T^1$, and $\chi^2 < \lambda$, with $\chi(\zeta_\delta^\ell, \delta) < \chi^2$ for $\delta \in T^2$. By Lemma 4, if $\delta \in S - \delta(T^2)$, then there is a $\chi < \delta$ such that $U(\delta, T^2, R_{\ell^*+|\varrho|}, \chi) \neq \emptyset$, so there are $\eta, \chi^3 > \chi^2$, and $T^3 \subseteqq S - \delta(T^2)$ stationary with $\eta \in U(\delta, T^2, R_{\ell^*+|\varrho|}, \chi^3)$ $(\delta \in T^3)$.

We now apply Lemma 2 with $A = T_3 - (\chi^3+1)$, $B = T^3$ to get a $\beta_3 \in T_3 - (\chi^3+1)$, and $\beta^3 \in T^3$ such that $\beta^3 > \beta_3$ and

$$k(\beta^3, \beta_3) > \max\{|\nu_\ell| : \ell < n\} + |\tau| + |\varrho| + |\eta| + \max\{|\nu^\ell| : \ell < n\}.$$

Choose $i_0 < \omega_1$ which is larger than every countable ordinal in $\varrho_H(\beta^3, \beta_3)$, η, ν^ℓ, ν_ℓ $(\ell < n)$. Since $\varrho \in L(\beta_3, T_2)$, there is a $\beta_2 \in T_2$ with $\chi^3 < \beta_2 < \beta_3$, $\chi(\beta^3, \beta_3) < \beta_2$ such that $\varrho_H(\beta_3, \beta_2)^{i_0} = \varrho$. Pick a χ_2 with $\chi^3 < \chi_2 < \beta_2$, $\chi(\beta^3, \beta_3) < \chi_2$ such that $\chi(\beta_3, \beta_2) < \chi_2$.

Next fix an $i_1 < \omega_1$ which is larger than the ordinals in $\varrho_H(\beta_3, \beta_2)$ and i_0. Then, as $\beta^3 \in T^3$ and $\eta \in U(\beta^3, T^2, R_{\ell^*+|\varrho|}, \chi^3)$, there exists a $\beta \in T^2$, $\beta > \beta^3$ with $\varrho_H(\beta, \beta^3)^{i_1} = \eta$ and $\chi(\beta, \beta^3) < \chi^3$. Since $\beta \in T^2$ we have $\varrho_H(\zeta_\beta^\ell, \beta) = \nu^\ell$ and $\chi(\zeta_b^\ell, \beta) < \chi^3$ $(\ell < n)$.

Finally, choose $i_2 < \omega_1$ which is larger than the countable ordinals in $\varrho_H(\beta, \beta^3)$ and i and use $\tau \in L(\beta_2, T_1)$ to find $\beta_1 \in T_1$ with $\chi_2 < \beta_1 < \beta_2$, $\varrho_H(\beta_2, \beta_1)^{i_2} = \tau$. Also, fix $\chi_1 > \chi(\beta_2, \beta_1)$, $\chi_2 < \chi_1 < \beta_1$. Since $\beta_1 \in T_1$, there is an α, $\chi_1 < \alpha < \beta_1$, such that for $\ell < n$, $\varrho_H(\beta_1, \zeta_\alpha^\ell) = \nu_\ell$.

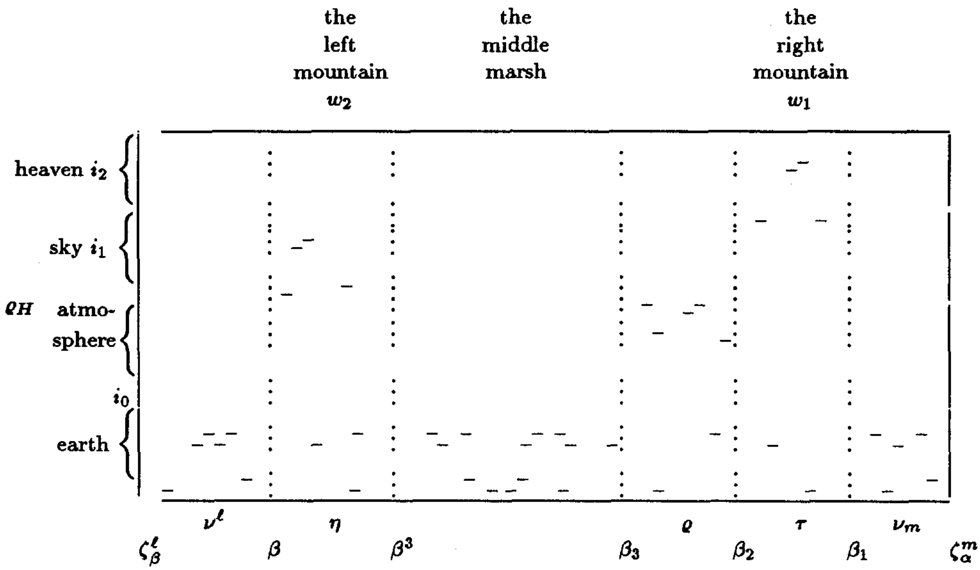

Fig. 1. The sequence $\varrho_H(\zeta^\ell_\beta,\zeta^m_\alpha)$.

Now, by Lemma 2, as $\alpha<\zeta^m_\alpha$,

$$\begin{aligned}
\chi(\zeta^\ell_\beta,\beta)<\chi^3<\chi_2<\chi_1<\alpha \ \text{implies}\ & \varrho(\zeta^\ell_\beta,\zeta^m_\alpha)=\varrho(\zeta^\ell_\beta,\beta)\wedge\varrho(\beta,\zeta^m_\alpha);\\
\chi(\beta,\beta^3)<\chi^3<\alpha \ \text{implies}\ & \varrho(\beta,\zeta^m_\alpha)=\varrho(\beta,\beta^3)\wedge\varrho(\beta^3,\zeta^m_\alpha);\\
\chi(\beta^3,\beta_3)<\chi_2<\alpha \ \text{implies}\ & \varrho(\beta^3,\zeta^m_\alpha)=\varrho(\beta^3,\beta_3)\wedge\varrho(\beta_3,\zeta^m_\alpha);\\
\chi(\beta_3,\beta_2)<\chi_2<\alpha \ \text{implies}\ & \varrho(\beta_3,\zeta^m_\alpha)=\varrho(\beta_3,\beta_2)\wedge\varrho(\beta_2,\zeta^m_\alpha);\\
\chi(\beta_2,\beta_1)<\chi_1<\alpha \ \text{implies}\ & \varrho(\beta_2,\zeta^m_\alpha)=\varrho(\beta_2,\beta_1)\wedge\varrho(\beta_1,\zeta^m_\alpha),
\end{aligned}$$

i.e.

$$\varrho(\zeta^\ell_\beta,\zeta^m_\alpha)=\varrho(\zeta^\ell_0,\beta)\wedge\varrho(\beta,\beta^3)\wedge\varrho(\beta^3,\beta_3)\wedge\varrho(\beta_3,\beta_2)\wedge\varrho(\beta_2,\beta_1)\wedge\varrho(\beta_1,\zeta^m_\alpha).$$

A similar identity holds for ϱ_H.

Now it is obvious that the middle, i.e. the $k(\zeta^\ell_\beta,\zeta^m_\alpha)/2$-th element of the string lies in the $\varrho(\beta^3,\beta_3)$ portion — selected to be so long for this purpose. By the respective selections of i_1,i_2 the largest ϱ_H value of the first half of the string is at least i_1 but less than i_2. It follows that $w_1(\zeta^\ell_\beta,\zeta^m_\alpha)$ consists of those indices p in the $\varrho(\beta_2,\beta_1)$ portion where $\varrho_H(\beta_2,\beta_1)(p)\geqq i_2$, so, in particular, $p_1=s+|\varrho|+\ell^*$ where $s=|\varrho(\zeta^\ell_\beta,\beta_3)|$. $w_2(\zeta^\ell_\beta,\zeta^m_\alpha)$ then consists of those indices q in the $\varrho(\beta,\beta^3)$ portion where $\varrho_H(\beta,\beta^3)(q)\geqq i_1$. By the choices of η and $\varrho(\beta,\beta^3)$ we have that the minimum of $\{H(\gamma_q):q\in w_2\}$ is in $R_{\ell^*+|\varrho|}$, i.e. $p_2=\ell^*+|\varrho|$. From this, $\gamma_{p_1-p_2}=\gamma_s=\beta_3\in S^\gamma$, so $c(\zeta^\ell_\beta,\zeta^m_\alpha)=\gamma$, as required.

4. Corollaries

COROLLARY. *If $\kappa > \omega_1$ is regular, then*
(a) $\mathrm{Pr}^+(\kappa^+)$ *holds;*
(b) *κ^+-c.c.-ness is not a productive property of Boolean algebras;*
(c) *there is a κ^+-separable not κ^+-Lindelöf Hausdorff-space;*
(d) *there is a κ^+-Lindelöf not κ^+-separable Hausdorff-space.*

PROOF. (a) From the Theorem and Lemma 1.
(b) See [6].
(c)–(d) See [1].

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