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On the Nonaxiomatizability of Some Logics by Finitely Many Schemas

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Introduction First a suitable definition of an axiom schema is given. Then it is proved that the class of $(\beth_\omega, \aleph_0)$ two-cardinal models, although recursively axiomatizable [12], cannot be axiomatized by finitely many schemata. Essentially the same proof shows that logic with the quantifier "there exist at least κ many", where κ is a strong limit cardinal, cannot be so axiomatized. Conclusions related to the literature are then drawn.

1 Since the publication by Mostowski of [8] in 1957, researchers have expended much effort in the study of logics extending first-order logic. The completeness problem for such logics, naturally enough, has been perhaps the major concern, and has been settled for various logics with varying degrees of success.

At one end of this spectrum lies the work on logic with the quantifier "there exist uncountably many," $L(Q_{\aleph_1})$. Early research by Vaught [13] and Fuhrken [3] in 1964 revealed, respectively, that $L(Q_{\aleph_1})$ is recursively axiomatizable, and that it is countably compact. The techniques used to prove these results were indirect, however, and consequently Vaught's work gave no clue as to what a complete set of axioms for $L(Q_{\aleph_1})$ might be. This shortcoming was remedied spectacularly by the work of Keisler [6] in 1970, wherein he proved that a simple finite collection of schemata sufficed to axiomatize $L(Q_{\aleph_1})$. Moreover, his direct methods yielded important model-theoretic tools for the study of $L(Q_{\aleph_1})$.

At the other end of this spectrum are various "abstract" completeness theorems. These are results which, as in Vaught's work, [13], on $L(Q_{\aleph_1})$, establish that a logic has a recursive set of axioms by indirect means and do not

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Sh:179

explicitly exhibit such a set of axioms. Notable among such results are those of Helling [4] and Keisler [5]. Helling proved that $L(Q_\kappa)$, for κ a weakly compact cardinal, is recursively axiomatizable and also that for any two weakly compact cardinals κ and λ, the valid sentences for $L(Q_\kappa)$ and $L(Q_\lambda)$ are identical. Keisler proved analogous results in the case that κ and λ are singular strong limit cardinals. In particular, observe that Keisler's work shows that $L(Q_{\beth_\omega})$ is recursively axiomatizable.

Falling somewhere between these two extremes is the work that has been done on the set of valid first-order sentences of the class of $(\beth_\omega, \aleph_0)$ two-cardinal models. In 1964, Vaught [12] proved an "abstract" completeness theorem. The problem of finding an explicit set of axioms remained open until 1975 when Barwise [1] and Schmerl [9], independently, exhibited different complete sets of axioms. However, both solutions, consisting of infinitely many axiom schemata, lacked the perspicuity of Keisler's axioms for $L(Q_{\aleph_1})$.

A fresh approach is taken in this paper to establish that at least to some extent the lack of success for the logics mentioned in the preceding two paragraphs is not accidental. It will be shown that neither the class of $(\beth_\omega, \aleph_0)$-models nor logic with the quantifier "there exist at least κ many", for any strong limit cardinal κ, can have as nice a set of axioms as does $L(Q_{\aleph_1})$ in the sense that none of these can be axiomatized by finitely many schemata.[1]

In Section 2 the formal definition of a schema that is used throughout this paper is given. The principal result, Theorem 3.1, which asserts that the class of $(\beth_\omega, \aleph_0)$ two-cardinal models cannot be axiomatized by finitely many schemata, is proved in Section 3. In the last section, Section 4, modifications are given of the proof of Theorem 3.1 that are needed to yield the further result that logics with the quantifier "there exist at least κ many", for any strong limit cardinal κ, also cannot be axiomatized by finitely many schemata. Finally, some questions motivated by the results are raised.

The notation used in this paper is standard. The only possible exception is $t^{\mathcal{L}}_{(qf)}(\bar{a}, A)$ for a logic $\mathcal{L}$, an $\mathcal{L}$-structure $\mathfrak{M}$, and $\bar{a} \in {}^n\mathfrak{M}$, $A \subseteq \mathfrak{M}$. This simply refers to the (quantifier-free) $\mathcal{L}$-type of $\bar{a}$ with parameters from A.

2 The first and perhaps most important step in carrying out the program of this paper consists in the definition of an axiom schema for a logic $\mathcal{L}$. Let $R_0(v_0^0, \ldots, v_{m_0-1}^0), \ldots, R_{n-1}(v_0^{n-1}, \ldots, v_{m_n-1}^{n-1})$ be m_i-ary relation variables for each $i < n$.

Definition 2.1 A *schema* is an $\mathcal{L}(R_0, \ldots, R_{n-1})$-formula $\Phi(R_0, \ldots, R_{n-1})$ in which each of the variables v_j^i, for $i < n$ and $j < m_i$ is bound to a quantifier of $\mathcal{L}$.

A set $\{\phi_i: i \in I\}$ of $\mathcal{L}$-formulas indexed by I is *schematized* by the $\mathcal{L}(R_0, \ldots, R_{n-1})$-formula $\Phi(R_0, \ldots, R_{n-1})$ if $\{\phi_i: i \in I\}$ is the set of $\mathcal{L}$-formulas obtained from Φ by replacing each R_j by any $\mathcal{L}$-formula $\psi_j(\bar{v}^j; \bar{y}_j)$ where the variables in $\bar{y}_j$ are not among $\{v_j^i: i < n \;\&\; j < m_i\}$.

We remark that the $\bar{y}_j$'s appearing in the preceding definition are to be thought of as parameters. That is, an axiom schema usually involves all universal closures of formulas of a certain type, and the inclusion of the $\bar{y}_j$'s is intended to reflect this.

Sh:179

Via examples, let us see how this definition covers various schemata with which logicians ordinarily deal.

Example 2.2: *The replacement schema in set theory.* Let $\Phi(R(x, y))$ be

$$(\forall x \in u)(\exists! y)R(x, y) \rightarrow (\exists z)(\forall x \in u)(\exists y \in z)R(x, y) \ .$$

Example 2.3: *Keisler's axioms for $L(Q)$.* We indicate how the axiom schema from [6], essentially stating that "the union of a countable collection of countable sets is countable", may be realized within our framework. Just let $\Phi(R(x, y))$ be

$$Qx \exists y R(x, y) \rightarrow \exists y Qx R(x, y) \vee Qy \exists x R(x, y) \ .$$

3 This section is given to the proof of the principal result of this paper, Theorem 3.1. The theorem actually establishes more than that the class of $(\beth_\omega, \aleph_0)$ two-cardinal models cannot be axiomatized by finitely many schemas. That is, no collection of schemas in which there is a uniform finite bound on the number of distinct variables v_j^i, $i < n$ and $j < m_i$, appearing in the new relation symbols $R_0, \ldots, R_{n-1}$ could suffice to axiomatize the class of $(\beth_\omega, \aleph_0)$-models.

Theorem 3.1 *Let $k < \omega$. The class of $(\beth_\omega, \aleph_0)$ two-cardinal models is not axiomatized by any collection of schemas any one of which has the property that the number of distinct variables v_j^i, $i < n$ and $j < m_i$, in the sequence $\langle R_0, \ldots, R_{n-1}\rangle$ of relation variables as in Definition* 2.1 *does not exceed k.*

Let us outline the idea of the argument before embarking upon the series of lemmas that lead to the proof of the theorem. For any given k, we construct a model $\mathfrak{M}$, whose complete first-order theory, by the Erdös-Rado theorem, [2], does not have a $(\beth_\omega, \aleph_0)$-model. On the other hand, we also demonstrate that any schema satisfying the condition for k in the statement of the theorem which also is an axiom for $(\beth_\omega, \aleph_0)$-models must be true in $\mathfrak{M}$. The theorem then follows.

Definition 3.2 A set $\mathcal{U} \subseteq \mathcal{P}(\omega)$ is said to be *strongly independent* if

(a) for any distinct $U_0, \ldots, U_{m-1}, U_m, \ldots, U_{m+n-1} \in \mathcal{U}$,

$$\left| \bigcap_{i<m} U_i \cap \bigcap_{m \le j < n} \omega \setminus U_j \right| = \aleph_0 \ ;$$

(b) for any $d_0, \ldots, d_{p-1}, \ldots, d_{p+q-1} \in \omega$, there are $|\mathcal{U}|$-many $U \in \mathcal{U}$ so that $\{d_0, \ldots, d_{p-1}\} \subseteq U$ and $\{d_p, \ldots, d_{p+q-1}\} \subseteq \omega \setminus U$.

The next result is well-known.

Theorem 3.3 (Hausdorff) *There exists a strongly independent set of power* $2^{\aleph_0}$.

For the remainder of this section let us fix a strongly independent set $\mathcal{U}$ as guaranteed by Theorem 3.3.

Definition 3.4 A *k-degenerate model* $\mathfrak{M} = \langle M \cup \omega, P, R\rangle$ is an L-structure, where the nonlogical symbols of L are a predicate symbol $P(x)$ and a $k+1$ place relation symbol $R(x; y_0, \dots, y_{k-1})$, satisfying:

(i) $P(\mathfrak{M}) = \omega$ and $M \cap \omega = \emptyset$;
(ii) for $a, b_0, \dots, b_{k-1} \in \mathfrak{M}$, if $\mathfrak{M} \models R(a; b_0, \dots, b_{k-1})$ then $\mathfrak{M} \models P(a) \wedge \bigwedge_{j<k} \sim P(b_j) \wedge \bigwedge_{i \neq j} b_i \neq b_j$;
(iii) if σ is a permutation of $\{0, \dots, k-1\}$, $a, b_0, \dots, b_{k-1} \in \mathfrak{M}$ and $\mathfrak{M} \models R(a; b_0, \dots, b_{k-1})$, then $\mathfrak{M} \models R(a; b_{\sigma(0)}, \dots, b_{\sigma(k-1)})$;
(iv) for any distinct $b_0, \dots, b_{k-1} \in M$, $\{a \in \omega: \mathfrak{M} \models R(a; b_0, \dots, b_{k-1})\} \in \mathcal{U}$;
(v) there is no sequence $b_0, \dots, b_{k+1} \in M$ so that for all $b_{i_0}, \dots, b_{i_{k-1}}$, $b_{j_0}, \dots, b_{j_{k-1}} \in \{b_0, \dots, b_{k+1}\}$, $\mathfrak{M} \models (\forall x)(R(x; b_{i_0}, \dots, b_{i_{k-1}}) \leftrightarrow R(x; b_{j_0}, \dots, b_{j_{k-1}}))$.

Intuitively, R partitions k-element subsets of M via elements of $\mathcal{U}$ so that there is no set of power $k+2$ that is homogeneous for the partition. It is apparent that there are many k-degenerate structures. We also note that the Erdös-Rado theorem, [2], implies that there is no k-degenerate structure $\mathfrak{M}$ of power greater than $\beth_k$.

Lemma 3.5
(a) *Let $\omega \subseteq \mathfrak{M}_0 \subseteq \mathfrak{M}_1$ and $\mathfrak{M}_1$ be k-degenerate. Then $\mathfrak{M}_0$ also is k-degenerate.*
(b) *Let $\langle \mathfrak{M}_\beta: \beta < \alpha \rangle$ be an increasing chain of k-degenerate models. Then $\bigcup_{\beta<\alpha} \mathfrak{M}_\beta$ is k-degenerate.*

Proof: Clear.

Lemma 3.6 *Let $\mathfrak{M}_0$, $\mathfrak{M}_1$, and $\mathfrak{M}_2$ be countable k-degenerate structures so that $\mathfrak{M}_0 \subseteq \mathfrak{M}_1, \mathfrak{M}_2$ and $\mathfrak{M}_1 \cap \mathfrak{M}_2 = \mathfrak{M}_0$. Then we can amalgamate $\mathfrak{M}_1$ and $\mathfrak{M}_2$ into a k-degenerate structure with universe $\mathfrak{M}_1 \cup \mathfrak{M}_2$ without introducing any new equalities.*

Proof: Obviously $\mathcal{V} = \{X \subseteq \omega: (\exists y_0 \dots y_{k-1} \in \mathfrak{M}_1) X = R(\mathfrak{M}_1, y_0, \dots, y_{k-1}) \vee (\exists y_0 \dots y_{k-1} \in \mathfrak{M}_2) X = R(\mathfrak{M}_2; y_0, \dots, y_{k-1})\}$ is countable. Thus, for any distinct $b_0, \dots, b_{k-1} \in M_1 \cup M_2$ not all in either M_1 or M_2, we have considerable freedom to choose $Y \in \mathcal{U} \setminus \mathcal{V}$ and set $Y = R(M_1 \cup M_2; b_0, \dots, b_{k-1})$ so that any two new Y's are distinct. This suffices to establish the lemma.

To simplify the presentation, we shall assume CH for the remainder of this section. This does not affect our results, as Theorem 3.1 is quite clearly absolute.

Lemma 3.7 *There exists a universal k-degenerate model of power $\aleph_1$ that is homogeneous over ω. Moreover, such a model can be constructed so that it admits the elimination of quantifiers down to Boolean combinations of atomic formulas and formulas of the form '$(\forall x)(R(x, \bar{y}) \leftrightarrow R(x, \bar{z}))$', and hence actually is homogeneous for complete quantifier-free types.*

Proof: List all countable k-degenerate structures in type ω_1 as $\mathfrak{N}_\xi$, $\xi < \omega_1$. Also fix an ordering ρ of $\omega_1 \times \omega_1$ in type ω_1 so that for each $\alpha < \omega_1$, we have $\rho(\alpha, \beta) > \alpha$.

We shall construct a chain of countable k-degenerate models, $\mathfrak{M}_\xi (\xi < \omega_1)$, whose union, $\mathfrak{M}$, will be the desired model. Furthermore, at each stage $\xi < \omega_1$,

Sh:179

we fix a list p^{ξ}_{α}, $\alpha < \omega_1$, of all quantifier-free types over all countable $A \subseteq \mathfrak{M}_{\xi}$, $\omega \subseteq A$, that must be realized in $\mathfrak{M}$ to ensure the homogeneity of the desired structure. That is, if $f: A_0 \xrightarrow[\text{onto}]{\text{1-1}} A_1$, $\omega \subseteq A_i \subseteq \mathfrak{M}_{\xi}$, $i = 0, 1$, $f \restriction \omega =$ identity, f preserves the truth of quantifier-free formulas and $a \in \mathfrak{M}_{\xi}$, then $p = \{\phi(x, f(a_0), \dots, f(a_{m-1})): \phi(x, a_0, \dots, a_{m-1}) \in t_{qf}(a, A_0)\}$ appears on the list. Also, $\mathfrak{N}_{\xi} \subseteq \mathfrak{M}_{\xi}$ and if $\rho(\alpha, \beta) = \xi$, then p^{α}_{β} will be realized in $\mathfrak{M}_{\xi}$.

Assume the construction has been carried out for all $\nu < \xi$. We indicate how to build $\mathfrak{M}_{\xi}$. First, let $\mathfrak{M}' = \bigcup_{\nu<\xi} \mathfrak{M}_{\nu}$. It is evident that $\mathfrak{M}'$ is k-degenerate. We suppose that $\mathfrak{M}' \cap \mathfrak{N}_{\xi} = \omega$. By Lemma 3.6 we may amalgamate $\mathfrak{M}' \cup \mathfrak{N}_{\xi}$ into a k-degenerate structure, $\mathfrak{M}''$, whose universe is $\mathfrak{M}' \cup \mathfrak{N}_{\xi}$, without introducing any new equalities. Next, we must realize p^{α}_{β}, where $\rho(\alpha, \beta) = \xi$. Let q be the quantifier-free type, which, as in the preceding paragraph, forced the inclusion of p^{α}_{β} in the list. Thus, q is realized by some $a \in \mathfrak{M}_{\alpha}$. By Lemma 3.5, both $M_{\alpha} \restriction dom(q)$ and $\mathfrak{M}_{\alpha} \restriction dom(q) \cup \{a\}$ are k-degenerate models. Also, $\mathfrak{M} \restriction dom(q) \cong \mathfrak{M} \restriction dom(p^{\alpha}_{\beta})$. Let b name a new element, and let us define a k-degenerate model $\mathfrak{M}'''$ with universe $dom(p^{\alpha}_{\beta}) \cup \{b\}$ so that $t_{qf}(b, dom(p^{\alpha}_{\beta})) = p^{\alpha}_{\beta}$. By Lemma 3.6 we may amalgamate $\mathfrak{M}''$and $\mathfrak{M}'''$. We let the resulting structure be $\mathfrak{M}_{\xi}$.

Finally, we must verify that $\mathfrak{M} = \bigcup_{\xi<\omega_1} \mathfrak{M}_{\xi}$ has the asserted properties. That $\mathfrak{M}$ is k-degenerate follows from Lemma 3.5(b). Also, since $\mathfrak{N}_{\xi} \subseteq \mathfrak{M}_{\xi}$, it follows that $\mathfrak{M}$ is universal.

The tedium involved in verifying that $\mathfrak{M}$ admits elimination of quantifiers we leave to the reader. We remark, though, that it is $\mathfrak{U}$ having the property of strong independence that enables the proof to work. That is, the sets $P(a_0, \dots, a_{k-1}) = \{x \in \omega: \mathfrak{M} \models R(x; a_0, \dots, a_{k-1})\} \in \mathfrak{U}$ look very much alike relative to each other and each other's complement in ω by virtue of $\mathfrak{U}$ being strongly independent.

It remains only to verify the homogeneity of $\mathfrak{M}$. Since $\mathfrak{M}$ admits the elimination of quantifiers, we know that for any $\bar{a} \in \mathfrak{M}$ and $A_0 \subseteq \mathfrak{M}$, $t(\bar{a}, A_0)$ is equivalent to $t_{qf}(\bar{a}, A_0)$. Thus we need only concern ourselves with homogeneity relative to quantifier-free types. Consequently, it suffices to prove that if $\omega \subseteq A_i \subseteq \mathfrak{M}$, $|A_i| = \aleph_0$, for $i = 0, 1$, $f: A_0 \xrightarrow[\text{onto}]{\text{1-1}} A_1$ preserves quantifier-free formulas, $f \restriction \omega =$ identity, and $a \in \mathfrak{M}$, then there is some $b \in \mathfrak{M}$ so that

$$t_{qf}(b, A_1) = \{\phi(x, f(a_0), \dots, f(a_{m-1})): a_0, \dots, a_{m-1} \in A_1 \wedge \phi(x, a_0, \dots, a_{m-1}) \in t_{qf}(a, A_0)\} .$$

But A_0, A_1, and a are included in some $\mathfrak{M}_{\xi}$, so it is obvious from the construction (i.e., the realization of the p^{α}_{β}'s) that there exists such a $b \in \mathfrak{M}$. This completes the proof of Lemma 3.7.

Definition 3.8 Let $\mathfrak{N}$ be an L-structure and $\bar{c} \in \mathfrak{N}$. For $k \in \omega$, $L^k_{\bar{c}}$ will denote the language that includes an m-place relation symbol $R_{\phi}(y_0, \dots, y_{m-1})$ for each relation defined in $\langle \mathfrak{N}, \bar{c} \rangle$ by an $L(\bar{c})$-formula $\phi(y_0, \dots, y_{m-1}, \bar{c})$, where $m \leq k$.

Also, for $\mathfrak{N}$ as above, by $\mathfrak{N}^k_{\bar{c}}$ we denote the $L^k_{\bar{c}}$ structure with universe $N \cup \omega$ so that

$$\mathfrak{N}^k_{\bar c} \models R_\phi(a_0, \ldots, a_{m-1}) \quad \text{iff} \quad \mathfrak{N} \models \phi(a_0, \ldots, a_{m-1}, \bar c) \ .$$

Observe that $R(x, y_0, \ldots, y_{m-1})$ does not have an equivalent relation symbol in $L^k_{\bar c}$, precisely because it contains too many places.

Lemma 3.9 *Let $\mathfrak{M}$ be the model constructed in Lemma 3.7 and $\bar c \in \mathfrak{M}$. Then $\mathfrak{M}$ and $\bar c$ have the following property: Let $\omega \subseteq A_i$, for $i = 0, 1$, be countable subsets of $\mathfrak{M}^k_{\bar c}$ and $f\colon A_0 \xrightarrow[\text{onto}]{\text{1-1}} A_1$ be a mapping so that $f \restriction \omega = id$ and for any $a_0, \ldots, a_{k-1} \in A_0$,*

(3.9.1) $$t^L_{qf}(a_0 {}^\frown \ldots {}^\frown a_{k-1}, \bar c) = t^L_{qf}(f(a_0) {}^\frown \ldots {}^\frown f(a_{k-1}), \bar c).$$

Then, for any $a \in \mathfrak{M}$, there is $b \in \mathfrak{M}$ so that we can extend f to $f'\colon A_0 \cup \{b\}$ with $f'(a) = b$ preserving the property expressed in (3.9.1), above.

Corollary 3.10 *For any $\bar c$, $\mathfrak{M}^k_{\bar c}$ is ω_1-homogeneous over ω.*

Proof: Recall all relations in $L^k_{\bar c}$ have no more than k-places. Then it is evident that the back-and-forth property given by Lemma 3.9 enables the construction of an automorphism of $\mathfrak{M}^k_{\bar c}$ starting from any countable A_0, $A_1 \supseteq \omega$ with elementary $f\colon A_0 \xrightarrow[\text{onto}]{\text{1-1}} A_1$, with $f \restriction \omega = id$.

Proof of Lemma 3.9: We go back to $\mathfrak{M}$ itself. First set

$$p(x) = \bigcup_{\substack{B \subseteq A_0 \\ |B| \leq k-1}} t^L_{qf}(a; Bu\{\bar c\}) \ .$$

By the universality and homogeneity of $\mathfrak{M}$ over ω, it suffices to produce a k-degenerate structure $N \supseteq A_1 \cup \{c\}$ with some $d \in \mathfrak{N}$ satisfying $f(p) = \{\phi(x, f(b_0), \ldots, f(b_m), \bar c)\colon \phi(x, b_0, \ldots, b_m, \bar c) \in p\}$. That is, it will be enough to adjoin a new element d to $\mathfrak{M} \restriction A_1 \cup \{\bar c\}$ which satisfies $f(p)$.

The only difficulty in establishing this assertion lies in showing that we may add such an element d in such a way as to preserve k-degeneracy. Let $\mathcal{V} = \{\{y \in \omega\colon R(y; a_0, \ldots, a_{k-1})\}\colon a_0, \ldots, a_{k-1} \in A_1 \cup \{\bar c\}\}$. For $(b_0, \ldots, b_{k-1}) \in (A_1 \cup \{\bar c, d\})^{(k)} \backslash (A_1 \cup \{\bar c\})^{(k)}$, choose $U \in \mathcal{U} \backslash \mathcal{V}$ and set $U = \{y \in \omega\colon R(y; b_0, \ldots, b_{k-1})\}$, except if $\{y \in \omega\colon R(y; b_0, \ldots, b_{k-1})\}$ is already specified by $f(p)$. Also, we insist that distinct sets in $\mathcal{U} \backslash \mathcal{V}$ be chosen for distinct k-tuples from $(A_1 \cup \{\bar c, d\})^{(k)} \backslash (A_1 \cup \{\bar c\})^{(k)}$.

We must show that the structure so obtained is k-degenerate. Suppose it is not. Then, there is $\{b_0, \ldots, b_{k+1}\} \subseteq A_1 \cup \{\bar c, d\}$ so that for any $b_{i_0}, \ldots, b_{i_{k-1}}, b_{j_0}, \ldots, b_{j_{k-1}} \in \{b_0, \ldots, b_{k+1}\}$,

$$\{y \in \omega\colon R(y; b_{i_0}, \ldots, b_{i_{k-1}})\} = \{y \in \omega\colon R(y; b_{j_0}, \ldots, b_{j_{k-1}})\} \ .$$

Clearly, since $\mathfrak{M} \restriction A_1 \cup \{\bar c\}$ is k-degenerate, it cannot be that $\{b_i\colon i < k+2\} \subseteq A_1 \cup \{\bar c\}$. Without loss, we therefore may assume that $b_{k+1} = d$, and $\{b_0, \ldots, b_k\} \subseteq A_1 \cup \{\bar c\}$. Moreover, since $\mathfrak{M} \restriction A_0 \cup \{\bar c, a\}$ is k-degenerate, it must be true that $|\{b_0, \ldots, b_k\} \backslash \{\bar c\}| \geq k - 1$. Suppose, then, that $b_0, \ldots, b_{k-2}$ are distinct elements of $\{b_0, \ldots, b_k\} \backslash \{\bar c\}$. It would follow that

$$\{x \in \omega\colon R(x; b_0, \ldots, b_{k-1})\} = \{x \in \omega\colon R(x; d, b_0, \ldots, b_{k-2})\} \ .$$

This cannot be possible by the construction of $\mathfrak{N}$. Hence, $\mathfrak{N}$ is k-degenerate, as desired.

Sh:179

Lemma 3.11 *Let* $\mathfrak{N}_\xi$, $\xi < \alpha$, *be models for* $L_{\bar{c}}^k$ *satisfying*:

(i) $\mathfrak{N}_\xi \supseteq \omega \cup \{\bar{c}\}$, *for all* $\xi < \alpha$

(ii) *each* $\mathfrak{N}_\xi$ *realizes only those quantifier-free types with parameters from* ω *that are realized in* $\mathfrak{M}_{\bar{c}}^k$

(iii) *for any* $\xi_1, \xi_2 < \alpha$, $\mathfrak{N}_{\xi_1} \upharpoonright \mathfrak{N}_{\xi_1} \cap \mathfrak{N}_{\xi_2} = \mathfrak{N}_{\xi_2} \upharpoonright \mathfrak{N}_{\xi_1} \cap \mathfrak{N}_{\xi_2}$.

Then there is an $\mathfrak{N}_{\bar{c}}^k \supseteq \mathfrak{N}_\xi$, $\forall \xi < \alpha$, *adding no new equalities, so that the quantifier-free types with parameters from* ω *that are realized in* $\mathfrak{N}_{\bar{c}}^k$ *are only those realized in* $\mathfrak{M}_{\bar{c}}^k$, $|\mathfrak{N}_{\bar{c}}^k| < |\alpha| + \sum_{\xi<\alpha} |\mathfrak{N}_\xi| + \aleph_0$, *and* $\mathfrak{N}_{\bar{c}}^k$ *is* $\aleph_0$*-homogeneous for quantifier-free types.*

Proof. We induct on cardinals $\kappa = |\alpha| + \sum_{\xi<\alpha} |\mathfrak{N}_\xi|$.

$\kappa = \aleph_0$. We proceed in a manner similar to that used in the proof of Lemma 3.9. Without loss of generality it may be assumed that $\alpha = \omega$. Consider the $L\left(\bigcup_{m<\omega} N_m\right)$-theory

$$T^* = \bigcup_{m<\omega} \bigcup_{\substack{B \subseteq N_m \\ |B| \leq k}} t_{qf}^L(B, \bar{c}) \cup \Delta\left(\bigcup_{m<\omega} N_m\right)$$

where $\Delta\left(\bigcup_{m<\omega} N_m\right)$ is the appropriate equality diagram. By Lemma 3.7 and Corollary 3.10, it suffices to show that T^* can be completed to the diagram of a k-degenerate structure with universe $N = \bigcup_{m<\omega} N_m$.

As before, the only obstacle to this lies in the preservation of k-degeneracy. Unless specified by T^*, distinct k-tuples $(a_0, \ldots, a_{k-1})$ from N are, as usual, assigned distinct elements from $\mathfrak{U}$ to serve as $\{y \in \omega : R(y; a_0, \ldots, a_{k-1})\}$. With this done, suppose now that there is some $\{a_0, \ldots, a_{k+1}\} \subseteq N$ that violates k-degeneracy. From the definition of T^* and by the construction of $\mathfrak{N}$, there can be at most $k-1$ elements in $\{a_0, \ldots, a_{k+1}\} \backslash \{\bar{c}\}$, which we may assume are $a_0, \ldots, a_{k-2}$. Similarly, it must be the case that for some m, $\{a_0, \ldots, a_{k-2}\} \subseteq N_m$, as otherwise T^* could not force $\{a_0, \ldots, a_{k+1}\}$ to violate k-degeneracy. But then $\mathfrak{N}_m$ would realize a quantifier-free type over ω that is not realized in $\mathfrak{M}_{\bar{c}}^k$, which contradicts hypothesis (ii).

$\kappa > \aleph_0$. Without loss of generality, we may assume that $\alpha = \kappa$. Write $\bigcup_{\xi<\alpha} \mathfrak{N}_\xi \backslash \omega \cup \{\bar{c}\}$ as $\{a_\beta : \beta < \kappa\}$, and let $A_\gamma = \{a_\beta : \beta < \gamma\} \cup \omega \cup \{\bar{c}\}$.

For $\gamma < \kappa$ and $\bar{a} \in A_\gamma^{<\omega}$, let $\xi_{\bar{a}}$ be the least ordinal $\xi < \kappa$ so that $\bar{a} \in \mathfrak{N}_\xi$ if such an ordinal exists, and otherwise let $\xi_{\bar{a}} = 0$. Then, let $\mathcal{O}_\gamma = \{\xi_{\bar{a}} : \bar{a} \in A_\gamma^{<\omega}\}$. Observe that $|\mathcal{O}_\gamma| = |\gamma| < \kappa$.

By induction on $\gamma < \kappa$, we construct $L_{\bar{c}}^k$-structures $\mathfrak{N}^\gamma$ satisfying

(a) $A_\gamma \subseteq \mathfrak{N}^\gamma$, but $\mathfrak{N}^\gamma \cap \mathfrak{N}_\xi \backslash A_\gamma = \phi$ for $\xi < \kappa$,

(b) $\langle \mathfrak{N}^\gamma : \gamma < \kappa \rangle$ is an increasing chain,

(c) $\mathfrak{N}^\gamma$ satisfies the conclusion of the lemma for the collection of $L_{\bar{c}}^k$-structures $\left\{\bigcup_{\beta<\gamma} \mathfrak{N}^\beta\right\} \cup \{\mathfrak{N}_\xi \upharpoonright A_\gamma \cap \mathfrak{N}_\xi : \xi \in \mathcal{O}_\gamma\}$.

Supposing, then, that $\langle \mathfrak{N}^\beta : \beta < \gamma \rangle$ has been constructed, we indicate how to build $\mathfrak{N}^\gamma$.

Since for any $\beta < \gamma$, $|\mathfrak{N}^\beta| < \kappa$, and for any $\xi < \kappa$, $|\mathfrak{N}_\xi \cap A_\gamma| < \kappa$, in order to apply induction hypothesis (on κ) we only have to verify that the collection of structures in (c) satisfies the hypothesis of the lemma for γ. Only the verification that (iii) holds needs any argument. Observe first that for any $\xi \in \mathcal{O}_\gamma$, $(\mathfrak{N}_\xi \restriction \mathfrak{N}_\xi \cap A_\gamma) \cap \left(\bigcup_{\beta<\gamma} \mathfrak{N}^\beta\right)$ has universe $B = A_\gamma \cap \mathfrak{N}_\xi$. Thus, we must show that $\mathfrak{N}_\xi \restriction B = \left(\bigcup_{\beta<\gamma} \mathfrak{N}^\beta\right) \restriction B$ as $L_{\bar{c}}^k$-structures, for $\xi \in \mathcal{O}_\gamma$. Let $\xi \in \mathcal{O}_\gamma$ be fixed and $a_0, \ldots, a_{k-1} \in \mathfrak{N}_\xi \restriction B$. We prove that the same $L_{\bar{c}}^k$-relations hold of $\langle a_0, \ldots, a_{k-1} \rangle$ in both $\mathfrak{N}_\xi \restriction B$ and $\left(\bigcup_{\beta<\gamma} \mathfrak{N}^\beta\right) \restriction B$. For some $\beta < \gamma$, $a_0, \ldots, a_{k-1} \in \mathfrak{N}^\beta$. Consequently, there exists some $\xi' \in \mathcal{O}_\beta$ so that $a_0, \ldots, a_{k-1} \in \mathfrak{N}_{\xi'}$. But by hypothesis (iii) on the set $\{\mathfrak{N}_\xi : \xi < \kappa\}$, $\langle a_0, \ldots, a_{k-1} \rangle$ satisfies the same relations in both $\mathfrak{N}_\xi$ and $\mathfrak{N}_{\xi'}$, and, by induction (on γ), in $\mathfrak{N}_{\xi'}$ and $\mathfrak{N}^\beta$. Therefore, $\langle a_0, \ldots, a_{k-1} \rangle$ satisfies the same relations in $\mathfrak{N}_\xi$ and $\mathfrak{N}^\beta$, completing the proof of the lemma.

We now have all the facts necessary for the proof of Theorem 3.1.

Proof of Theorem 3.1: Let $k < \omega$ be given. It is clear by the Erdös-Rado theorem that the theory of the model $\mathfrak{M}$ built in Lemma 3.7 does not have a $(\beth_\omega, \aleph_0)$ model. We will show that if $\forall \bar{x} \phi$ is an instance of a schema that holds in all $(\beth_\omega, \aleph_0)$ models for which $|\{v_j^i : i < n \ \& \ j < m_i\}| \le k$, then $\mathfrak{M} \models \forall \bar{x} \phi$.

Let us suppose the contrary, and work for a contradiction. So $\mathfrak{M} \models \exists \bar{x} \neg \phi$, that is, for some $\bar{c} \in \mathfrak{M}$, $\mathfrak{M} \models \neg \phi(\bar{c})$. We have $\phi(\bar{c}) \equiv \Phi(\psi_0, \ldots, \psi_{n-1})$ as in Definition 2.1, where the number of free variables in any ψ_i is no greater than k. Consequently, to each $\psi_i(v_0^i, \ldots, v_{m_i}^i, \bar{c})$ there corresponds a relation symbol $R_{\psi_i}(v_0^i, \ldots, v_{m_i}^i)$ in $L_{\bar{c}}^k$. Let ϕ^* denote the $L_{\bar{c}}^k$-sentence $\Phi(R_{\psi_0}, \ldots, R_{\psi_{n-1}})$. We see that $\mathfrak{M}_{\bar{c}}^k \models \neg \phi^*$.

By Lemma 3.11, we may build an $L_{\bar{c}}^k$ model, $\mathfrak{N}_{\bar{c}}^k$, of power $\beth_\omega$ satisfying exactly those quantifier-free types that $\mathfrak{M}_{\bar{c}}^k$ does and ω-homogeneous for quantifier-free types. In particular, $\mathfrak{N}_{\bar{c}}^k$ is a $(\beth_\omega, \aleph_0)$-model, since the interpretation of $P(\cdot)$ remains just ω. Since $\mathfrak{M}_{\bar{c}}^k$ and $\mathfrak{N}_{\bar{c}}^k$ realize exactly the same quantifier-free types, and are ω-homogeneous for the same, the familiar back-and-forth criterion for elementary equivalence is satisfied, whence $\mathfrak{M}_{\bar{c}}^k \equiv \mathfrak{N}_{\bar{c}}^k$.

We then have that $\mathfrak{N}_{\bar{c}}^k \models \neg \phi^*$. But ϕ^* is an instance of a schema that is to hold in all $(\beth_\omega, \aleph_0)$ models, and so it should be the case that $\mathfrak{N}_{\bar{c}}^k \models \phi^*$. Having reached a contradiction, we cannot escape the conclusion that $\mathfrak{M} \models \forall \bar{x} \phi$, completing the proof of Theorem 3.1.

4 We now indicate how the proofs of the results in Section 3 can be modified to decide further questions. The first theorem of this section is:

Theorem 4.1 *No finite collection of schemas suffices to axiomatize the logic $L(Q_{\beth_\omega})$, i.e., logic with the quantifier "there exist at least $\beth_\omega$ many". In fact, the stronger result analogous to the statement of Theorem 3.1 is true also.*

Sh:179

In the paper in which Keisler proved that the set of validities for $L(Q_{\beth_\omega})$ is recursively enumerable [5], he also showed that for any two singular strong limit cardinals κ and λ, the set of validities for $L(Q_\kappa)$ is identical to that for $L(Q_\lambda)$. As an immediate consequence of this and Theorem 4.1 we obtain:

Corollary 4.2 *No finite collection of schemas suffices to axiomatize the logic $L(Q_\kappa)$, for any singular strong limit cardinal κ.*

It only remains to indicate how the proofs of the various lemmas in Section 3 must be adapted to yield a proof of Theorem 4.1.

Sketch of the proof of Theorem 4.1: Let Q be the new quantifier symbol. We deal with exactly the same notion of k-degenerate model as in Section 3. As before, let $\mathfrak{M}$ be the universal, homogeneous k-degenerate model constructed in the proof of Lemma 3.7. We interpret Q for $\mathfrak{M}$ as "there exist uncountably many".

As before, the Erdös-Rado theorem, [2], implies that the $L(Q)$-theory of $\mathfrak{M}$ has no model in which Q is interpreted as "there exist $\beth_\omega$ many". Notice, though, that the $L(Q)$-theory of $\mathfrak{M}$ only can force the cardinality of $P(\cdot)$ to be less than $\beth_\omega$. It also is not difficult to see that $\mathfrak{M}$ admits elimination of quantifiers for formulas of $L(Q)$ also.

We let $L^k_{\bar{c}}(Q)$ be the language that includes m-place relation symbols for each formula $\phi(y_0, \ldots, y_{m-1}, \bar{x})$ of $L(Q)$ where $lh(\bar{x}) = lh(\bar{c})$ and $m \leq k$, and $(\mathfrak{M}^k_{\bar{c}})^*$ be the resulting $L^k_{\bar{c}}(Q)$-structure.

The only care that must be exercised in the final proof of Theorem 4.1 is in the construction of the model $(\mathfrak{N}^k_{\bar{c}})^*$ of power $\beth_\omega$. There we must insist that whenever there are $\aleph_1$ elements in $(\mathfrak{M}^k_{\bar{c}})^*$ of a certain type over ω, that $(\mathfrak{N}^k_{\bar{c}})^*$ contains $\beth_\omega$ many elements of the same type. It is apparent that Lemma 3.11 affords us enough freedom to realize this end. Lastly, as in the original argument, one easily sees that $(\mathfrak{N}^k_{\bar{c}})^* \equiv (\mathfrak{M}^k_{\bar{c}})^*$, completing the proof of Theorem 4.1.

The attentive reader already may have observed that $\beth_\omega$ being a *singular* strong limit cardinal had little bearing on the proof of Theorem 4.1. Consequently, the proof of Theorem 4.1 can be modified with little trouble to yield:

Theorem 4.3 *Let κ be any strong limit cardinal. Then the set of valid sentences of $L(Q_\kappa)$ is not given by finitely many schemata. Moreover, the stronger result analogous to the statement of Theorem 3.1 holds also.*

Before drawing a final corollary, we make one remark. In [10], Schmerl and Shelah extend the result of Helling [4] mentioned in the introduction by proving that if κ is a strongly ω-inaccessible cardinal (strong α-inaccessibility is defined similarly to weak α-Mahloness by starting at the base level with strongly inaccessible cardinals instead of weakly inaccessible cardinals) and λ is a weakly compact cardinal, then the valid sentences of $L(Q_\kappa)$ *and* $L(Q_\lambda)$ are the same. We then have

Corollary 4.4 *Let κ be a strongly ω-inaccessible cardinal (or in particular let κ be weakly compact). Then, although the set of valid sentences of $L(Q_\kappa)$ is recursively enumerable, it cannot be given by finitely many schemas.*

We close by raising a few questions. The strong forms of Theorems 3.1, 4,1, and 4.3 demonstrate not only that the sets of sentences mentioned cannot be given by finitely many schemas, but also that schemas with arbitrarily long quantifier prefixes are necessary. Yet one might ask if the notion of a schema can be liberalized so as to allow arbitrarily long quantifier prefixes and so fall beyond the scope of the theorems of this paper, but also so as to allow any of the logics discussed here to be axiomatized by finitely many liberalized schemas. For example, one might limit the alternation of quantifiers in the prefix but allow arbitrarily long blocks of each quantifier type. We do not have any results in this setting. We note, though, that both Barwise's and Schmerl's axioms for the class of $(\beth_\omega, \aleph_0)$-models have arbitrarily long quantifier alternations, and thus do not settle the question in this case. Lastly, our suggestion for a liberalized notion of schema is not intended to pass as the final word. For example, one also can ask if there are natural and important notions between binding no more than k variables and merely limiting the quantifier prefix, and if any such applies to a logic discussed here.

NOTE

1. More recently, the first author also proved that the same result holds for logic with the two-place Magidor-Malitz quantifier in $\aleph_1$-interpretation (see [7] for definitions). This result will appear in the forthcoming [11].

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Sh:179

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