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# MARTIN'S AXIOM DOES NOT IMPLY THAT EVERY TWO $\aleph_1$-DENSE SETS OF REALS ARE ISOMORPHIC

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ABSTRACT

Assuming the consistency of ZFC we prove the claim in the title by showing the consistency with ZFC of: There exists a set of reals $A$ such that every function from $A$ to $A$ is order preserving on an uncountable set. We prove related results among which is the consistency with ZFC of: Every function from the reals into the reals is monotonic on an uncountable set.

## §1. Introduction

Martin's Axiom is so powerful and has been used so diversely that one can almost get the impression that there is nothing this axiom cannot settle. There are, however, some consistency constructions which are very similar to the proof used by Solovay and Tennenbaum to get Martin's Axiom, yet the statements whose consistency these constructions give are not known to follow from Martin's Axiom (M.A.). The first example is due to J. Baumgartner who has shown the consistency of ZFC and Martin's Axiom and $2^{\aleph_0} > \aleph_1$ and every two $\aleph_1$-dense sets of reals are order-isomorphic. (A set of reals is $\aleph_1$-dense iff it has no end points and between any two points of the set there are $\aleph_1$ points of the set.) The structure of Baumgartner's proof [2] is like that of Solovay and Tennenbaum [6], but his proof is quite complicated. So, a natural question, asked by Baumgartner, is whether Martin's Axiom and $2^{\aleph_0} > \aleph_1$ already imply that every two $\aleph_1$-dense sets of reals are isomorphic.

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At the intermediate stages of the forcing iteration done in [2] the crucial fact that the continuum hypothesis (C.H.) holds is used as follows: Assuming C.H., Baumgartner proved that for every two $\aleph_1$-dense sets of reals there exists a c.c.c. poset $P$ such that forcing with $P$ makes the two sets isomorphic. So, it is natural to look for a universe where $2^{\aleph_0} > \aleph_1$ holds and in which

(1) There are two $\aleph_1$-dense sets of reals which cannot be made isomorphic by any c.c.c. poset forcing.

Having obtained such a universe, it is possible to extend it further by a c.c.c. poset forcing to get Martin's Axiom and there are two nonisomorphic $\aleph_1$-dense sets of reals, thus showing that Martin's Axiom does not imply isomorphism of $\aleph_1$-dense sets. The strategy to get such a universe is to start with two $\aleph_1$-dense sets which are very far from being isomorphic (in a sense which will be made precise) and iteratively kill one after the other all c.c.c. posets which might make these subsets isomorphic. To kill such a poset we introduce (generically) an uncountable antichain to it thus erasing its c.c.c. property. This is quite a general method of obtaining consistency results with Martin's Axiom and it will be described in §5.

We will get our first example for (1) in another way, using Theorem 2 which is a consistency result on monotonic functions.

Let $R$ *be the set of reals*, a function $f: A \to R$, $A \subset R$ is said to be *order-preserving* iff for $a, b \in A$, $a \leqq b \to f(a) \leqq f(b)$.

2. Theorem. *The following is consistent with* ZFC:

(3) *There exists a set of reals $A$ which is $\aleph_1$-dense such that for every function $f: A \to A$ there is an uncountable subset $A'$ of $A$ such that $f \restriction A'$ is order preserving.*

(3) implies (1) as follows: Suppose $A$ satisfies (3), let $\bar{A} = \{-r: r \in A\}$, then as we shall see, $A$ and $\bar{A}$ are the two $\aleph_1$-dense sets of reals which cannot be made isomorphic by any c.c.c. poset extension. Suppose $P$ is a poset such that in its Boolean universe $A$ and $\bar{A}$ are isomorphic; then it follows that in this universe there is a function $f: A \to A$ which is an anti-isomorphism, i.e., $a < b \to f(a) > f(b)$. For any $a \in A$ pick now some $p_a \in P$ and some $h(a) \in A$ such that $p_a \Vdash f(\boldsymbol{a}) = \boldsymbol{h}(\boldsymbol{a})$ (where $\boldsymbol{x}$ denotes the canonical name of $x \in V$). By our assumption, the function $a \to h(a)$ is order preserving on an uncountable subset $A'$ of $A$. But then for $a, a' \in A'$ if $a \neq a'$ then $p_a$ and $p_{a'}$ are incompatible. Hence $P$ does not satisfy the c.c.c.

Theorem 2 is proved in §2.

C.H. implies that (3) does not hold; in [5], $C_{62}$, where C.H. is assumed, a function $f: A \to A$ is constructed which is not continuous on any uncountable subset, and hence not monotonic on any uncountable subset.

Obviously for a general set $A$ of reals one cannot expect every function $f: A \to R$ to be order preserving on an uncountable subset, so the following theorem answers the natural question.

4. THEOREM. *The following is consistent with* ZFC *and* $2^{\aleph_0} = \aleph_2$:

(5) *For any* $A \subseteq R$ *of cardinality* $\aleph_1$ *and any* $f: A \to R$ *there exists a set* $A' \subset A$ *of cardinality* $\aleph_1$ *such that* $f \restriction A'$ *is monotic.* (*f is said to be* monotonic *iff f is non-decreasing*: $x < y \to f(x) \leqq f(y)$, *or* $f$ *is non-increasing.* $x < y \to f(x) \geqq f(y)$).

The proof of Theorem 4 is given in §3. In §4 we prove the following strengthening of Theorem 4.

6. THEOREM. *It is consistent with* ZFC *and* $2^{\aleph_0} = \aleph_2$ *that for every* $A \subset R$ *of cardinality* $\aleph_1$ *and* $f: A \to R$, *a one-to-one function, f is the union of countably many monotonic functions.*

In §5 we deal with entangled sets of reals (to be defined there) and show that Martin's Axiom and $2^{\aleph_0} > \aleph_1$ is consistent with the existence of an entangled set of reals. These imply that there are uncountable real functions which do not include any uncountable monotonic functions.

The question about the consistence of (5) was asked, independently, by F. Galvin who has noticed the following: $\omega_1 \to [\omega_1]^2_6$ implies (5) which in turn implies cf $2^{\aleph_0} > \aleph_1$. (See [3] lemma 2 and §3 and §4, not for proofs but for some hints.)

The proof that Martin's Axiom does not imply the isomorphism of $\aleph_1$-dense sets is due to Shelah; Avraham extracted Theorem 2 from that proof. Theorem 4 is due to Avraham; it is based on Shelah's method for constructing c.c.c. posets using closed unbounded subsets of $\omega_1$, a method which was invented to investigate consistency results on real orders and to simplify Baumgartner's proof. Theorem 6 and §5 are due to Shelah. Some further applications of these methods will appear in [1].

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## §2. Proof of Theorem 2

We describe first the structure of the proof of Theorem 2. We start with a set $A$ of $\aleph_1$ generic Cohen reals, $A$ satisfies a certain property which enables us, given any function on $A$ into $A$, to force an uncountable subset on which this function is order preserving. We iterate forcing with appropriate c.c.c. posets $\aleph_2$ times, dealing with all possible functions and keeping this property of $A$.

Let $V$ be our universe. First make a generic extension using the poset of all finite functions from $\omega_1 \times \omega$ to $\{0,1\}$, we obtain thus the set $A = \{r_\xi \mid \xi \in \omega_1\}$ where the $r_\xi$'s are $\aleph_1$ generic reals. In $V[A]$, $A$ satisfies the following property, as can be seen:

> Let $\langle \bar{a}_\alpha \mid \alpha < \omega_1 \rangle$ be a sequence of $n$-tuples of increasing countable ordinals,
>
> (7) $$\bar{a}_\alpha = \langle a(\alpha,1), \cdots, a(\alpha,n) \rangle,$$ $$a(\alpha,1) < a(\alpha,2) < \cdots < a(\alpha,n) < \omega_1.$$
>
> Then for some $\alpha < \beta < \omega_1$, $r_{a(\alpha,i)} \leqq r_{a(\beta,i)}$ holds for every $i \leqq n$.

PROOF. Say $P = \{h \mid h$ is a finite function from $\omega_1 \times \omega$ to $\{0,1\}\}$. Let $h \in P$ and let $\tau$ be a name in $P$-forcing such that $h \Vdash \tau$ is a function defined on $\omega_1$, and for $\alpha < \omega_1$ $\tau(\alpha)$ is an increasing $n$-tuple of countable ordinals. Now, for all $\alpha < \omega_1$ find $h_\alpha \in P$, $h_\alpha \geqq h$ and an increasing sequence $\bar{a}_\alpha = \langle a(\alpha,1), \cdots, a(\alpha,n) \rangle$ of countable ordinals such that $h_\alpha \Vdash \tau(\alpha) = \bar{a}_\alpha$. (We should write canonical names for members of $V$, but we won't do it.) Next, using a $\Delta$-system argument we can find an uncountable $I \subseteq \omega_1$ such that: (1) There is an $l < n$ such that for all $\alpha, \beta \in I$, $\alpha < \beta$ we have $a(\alpha,i) = a(\beta,i)$ for $i \leqq l$ and $a(\alpha,i) < a(\beta,j)$ for $l < i,j \leqq n$. (2) For $\alpha,\beta \in I$, $i \leqq n$ and $k < \omega$, $(a(\alpha,i),k) \in \mathrm{Dom}(h_\alpha)$ iff $(a(\beta,i),k) \in \mathrm{Dom}(h_\beta)$, and in this case $h_\alpha((a(\alpha,i),k)) = h_\beta((a(\beta,i),k))$. (3) $\{\mathrm{Dom}(h_\alpha) \mid \alpha \in I\}$ is a $\Delta$-system and the $h_\alpha$'s agree on the intersection of their domains (so that $h_\alpha \cup h_\beta$ is a function for $\alpha, \beta \in I$).

Now, pick any $\alpha, \beta \in I$, $\alpha < \beta$. Let $h' = h_\alpha \cup h_\beta \in P$. Define $h'' \geqq h'$, $h'' \in P$ as follows. For every $l < i \leqq n$ let $m_i$ be the least integer such that $(a(\alpha,i), m_i) \notin \mathrm{Dom}(h_\alpha)$. We set $h''(a(\alpha,i),m_i) = 0$ and $h''(a(\beta,i),m_i) = 1$ (this is possible since for $i > l$ $a(\alpha,i) < a(\beta,i)$). Thinking of reals as sequences of $\{0,1\}$ ordered lexicographically, we obtain $h'' \Vdash \tau(\alpha) = \bar{a}_\alpha$, $\tau(\beta) = \bar{a}_\beta$ and $r_{a(\alpha,i)} \leqq r_{a(\beta,i)}$ holds for every $i \leqq n$.

8. DEFINITION. Let $A = \{r_\xi \mid \xi \in \omega_1\}$ be a set of reals satisfying (7). Any function $f\colon \omega_1 \to \omega_1$ can be naturally viewed as a function from $A$ to $A$; $f(r_\xi) = r_{f(\xi)}$.

Given a function $f\colon \omega_1 \to \omega_1$ we define a poset $Q_f$ with the aid of which we shall obtain an uncountable subset of $A$ where $f$ is order preserving. We can assume $f$ is one to one, as the fixed function is order preserving.

$$Q_f = \{e \subset \omega_1 \mid e \text{ is finite and } f \restriction e \text{ is order preserving}\}.$$

$Q_f$ consists of finite approximations to the desired subset of $A$. We take the partial order of $Q_f$ to be inclusion.

9. LEMMA. *If $A = \{r_\xi \mid \xi \in \omega_1\}$ is a set of reals satisfying* (7), *then for any one-to-one $f\colon \omega_1 \to \omega_1$, $Q_f$ satisfies the* c.c.c.

PROOF. Assume that $e_\xi \subset \omega_1$, $\xi \in \omega_1$, are $\aleph_1$ conditions in $Q_f$; we will find two which are compatible. $e_\xi \cup f[e_\xi] = a_\xi$ is a finite set of ordinals hence we can assume that the $a_\xi$'s form a $\Delta$-system (i.e., by taking an uncountable subset we assume that there is a fixed set $b$ such that $a_\xi \cap a_\eta = b$ for all $\xi < \eta < \omega_1$). Moreover, for every $\xi$ pick a rational number between any two reals in $\{r_\eta \mid \eta \in a_\xi\}$, look at the finite model whose universe $M_\xi$ consists of the ordinals in $a_\xi$, the reals $r_\eta$ for $\eta \in a_\xi$ and the rationals chosen between the reals, and in which we have a (partial) function $f \restriction e_\xi$ and relations $<_{\mathrm{On}}$, $<_{\mathrm{R}}$ which are the orders on the ordinals and on the reals, and the function $\eta \mapsto r_\eta$, $\eta \in a_\xi$. Again we can assume that the set $\{M_\xi \mid \xi < \omega_1\}$ has been appropriately thinned out so that the isomorphism type of the model $M_\xi$ and the rational numbers in $M_\xi$ do not depend on $\xi$ for $\xi < \omega_1$.

Let $\bar{a}_\xi$ be an $n$-tuple enumerating in an increasing ordinal order the elements of $a_\xi$. $\bar{a}_\xi = \langle a(\xi, 1), \cdots, a(\xi, n)\rangle$. Using property (7) of $A$ we get $\alpha < \beta$ such that $r_{a(\alpha,i)} \leqq r_{a(\beta,i)}$ for $i \leqq n$. We claim that $e_\alpha \cup e_\beta$ is in $Q_f$ thus providing two compatible conditions. To prove this, we must show that $f$ is order preserving on $e_\alpha \cup e_\beta$. So let $x, y \in e_\alpha \cup e_\beta$, if both $x$ and $y$ belong to $e_\alpha$ or $e_\beta$ then $f$ is order preserving on $\{r_x, r_y\}$. Assume $x \in e_\alpha$, $y \in e_\beta$, there are two cases: The first is that for some $i \neq j$, $x = a(\alpha, i)$, $y = a(\beta, j)$. Then look at $a(\alpha, j)$, if $r_{a(\alpha,i)} <_{\mathrm{R}} r_{a(\alpha,j)}$ (for example) then there is a rational number $q \in M_\alpha$ such that $r_{a(\alpha,i)} < q < r_{a(\alpha,j)}$, but then, as $M_\alpha$ and $M_\beta$ are isomorphic, we have $q < r_{a(\beta,j)}$ and hence $r_{a(\alpha,i)} < r_{a(\beta,j)}$. Moreover, from $r_{a(\alpha,i)} < r_{a(\alpha,j)}$ follows $r_{f(a(\alpha,i))} < r_{f(a(\alpha,j))}$ and then, using the rational picked between these reals and the isomorphism of $M_\alpha$ and $M_\beta$ we get $r_{f(a(\alpha,i))} < r_{f(a(\beta,j))}$. The second case is that for some $i$, $x = a(\alpha, i)$, $y = a(\beta, i)$. As we wrote above, $r_{a(\alpha,i)} \leqq r_{a(\beta,i)}$. Now if $k$ is such that $f(a(\alpha, i)) = a(\alpha, k)$ then by the isomorphism of $M_\alpha$ and $M_\beta$ we get $f(a(\beta,i))=a(\beta,k)$. But $r_{a(\alpha,k)}\leqq r_{a(\beta,k)}$, so $f$ is order preserving on $\{x,y\}$. We proved that $Q_f$ satisfies the countable chain condition.

To show that the generic filter is uncountable, one uses the following lemma which is well known.

10. Lemma. *If $Q$ is a poset satisfying the* c.c.c. *and* $|Q|>\aleph_0$, *and if $G$ is the name of the generic filter over $Q$ then for some $q\in Q$, $q\Vdash G$ is uncountable.*

Now if we redefine $Q_f$ as all conditions extending some $q$ as in Lemma 10, we get by forcing with $Q_f$ an uncountable subset of $A$ where $f$ is order preserving.

We need some property that will ensure $A$ will continue to satisfy (7) as the iteration goes on. This is done by:

11. Definition. $P$ is called an *appropriate* poset iff for every sequence $\langle (p_\alpha,\bar a_\alpha)\mid \alpha<\omega_1\rangle$ where $p_\alpha\in P$ and $\bar a_\alpha$ is an $n$-tuple of increasing countable ordinals *there are* $\alpha<\beta<\omega_1$ such that $p_\alpha$ and $p_\beta$ are compatible in $P$ and $r_{a(\alpha,i)}\leqq r_{a(\beta,i)}$ for every $i\leqq n$.

12. Lemma. *If $P$ is appropriate, then after forcing with $P$ $A$ satisfies* (7).

Proof. If not, then in $V^P$ there is a counterexample that shows $A$ does not satisfy (7), i.e., there is a name of a sequence $\langle \bar a_\alpha \mid \alpha<\omega_1\rangle$ and a condition forcing that this sequence is a contradiction to (7). Now for each $\alpha<\omega_1$, pick $p_\alpha\in P$ such that $p_\alpha\Vdash \bar a_\alpha=\bar b_\alpha$ for some $\bar b_\alpha$ which is (in $V$) an increasing sequence of ordinals. Applying the appropriateness of $P$ to the sequence $\langle (p_\alpha,\bar b_\alpha)\mid\alpha<\omega_1\rangle$ we get a contradiction.

The proof of Lemma 9 gives with minor changes:

13. Lemma. *If $A$ satisfies* (7) *then $Q_f$ is appropriate.*

Now we iterate $\aleph_2$ times (for example) appropriate c.c.c. posets, like Solovay and Tennenbaum [6] taking the direct limit at the limit stages. The proof that this can be done, i.e., that composition of appropriate posets is appropriate and that direct limits of appropriate posets are appropriate, is like the proof of [6] that the c.c.c. is preserved by iteration. The arguments of [6] show that we can iterate $Q_f$ posets so that after $\omega_2$ steps every function $f$ was dealt with (we assume C.H. in the ground model) and Theorem 2 is proved.

Actually Martin's Axiom holds in the universe we obtained because we get Martin's Axiom for appropriate posets and in the universe we obtain, every non-appropriate poset does not satisfy the c.c.c. Because, if $P$ is non-appropriate, let $\langle (p_\alpha, \bar{a}_\alpha) \mid \alpha < \omega_1 \rangle$ show it. Define a poset $Q = \{e \subset \omega_1 \mid e$ is finite and for $\alpha, \beta \in e$, $\alpha < \beta \Rightarrow r_{a(\alpha,i)} \leqq r_{a(\beta,i)}$ for all $i \leqq n\}$. Then show that $Q$ is an appropriate c.c.c. poset that introduces an uncountable antichain to $P$.

## §3. Proof of Theorem 4

The structure of the proof of Theorem 4 is like that of Baumgartner's proof in [2]. We start from a universe satisfying C.H. and iterate $\aleph_2$ times forcing with c.c.c. posets of cardinality $\aleph_1$. We take direct limits at limit stages. At each stage we get a universe satisfying the C.H. and we are presented with a function $f: A \to R$, where $A$ is an uncountable set of reals. If there is no uncountable subset upon which $f$ is monotonically non-increasing then we will find a poset $P$ such that forcing with $P$ introduces an uncountable subset of $A$ on which $f$ is monotonically increasing. $P$ consists of finite subsets of $A$ on which $f$ is monotonically increasing, but not of all such finite subsets. First we shall construct a certain closed unbounded subset of $\omega_1$ and then we take as conditions in $P$ only finite subsets of $A$ ($\subseteq \omega_1$) whose members are separated by members of the closed unbounded set. We shall describe this proof in detail (see [4] for other results which use the same method).

So we asume $A$ is an uncountable subset of reals, $f: A \to R$ is a function for which there is no uncountable set of reals where $f$ is non-increasing. (Hence we can assume that $f$ is one-to-one.) Assuming C.H., let $\langle r_\alpha \mid \alpha < \omega_1 \rangle$ be an enumeration of all reals.

14. Definition. Define the closed unbounded $C \subseteq \omega_1$ as follows. Pick any increasing and continuous sequence of countable elementary submodules $M_\alpha <$ $H(\aleph_2)$, $\alpha < \omega_1$ (where $H(\aleph_2)$ is the family of all sets of cardinality hereditarily less than $\aleph_2$) such that $A$, $f$, $\langle r_\alpha \mid \alpha < \omega_1 \rangle \in M_0$ and $M_\alpha \cap \omega_1$ is an ordinal and $\langle M_\alpha \cap \omega_1 \mid \alpha < \omega_1 \rangle$ is a closed unbounded set of $\omega_1$. The set of ordinals $\alpha$ such that $M_\alpha \cap \omega_1 = \alpha$ is a closed unbounded set which we call $C$.

Observe that for every real $r$ there is $\alpha \in C$ such that $r \in M_\alpha$. Using the above enumeration of the reals we shall consider $A$ to be a subset of $\omega_1$ and $f$ to be a function on ordinals, let $<_R$ denote the ordering of the countable ordinals induced by the enumeration $\langle r_\alpha \mid \alpha < \omega_1 \rangle$ of the reals.

Now we define $P$ to be the poset of all finite sets of countable ordinals $e = \{a_1, \cdots, a_n\}$ such that $e \subseteq A$ and $f \mid e$ is monotonically increasing and between any two members of $e$ there is a member of $C$. (When writing $e = \{a_1, \cdots, a_n\}$ we imply that $a_1 < a_2 \cdots < a_n$.) The partial order of $P$ is inclusion. Two conditions are incompatible iff their union is not a condition, i.e., two ordinals in it are not separated by $C$ or $f$ is not order preserving on it.

15. Lemma. *$P$ satisfies the* c.c.c.

Proof. Let $e_\xi = \{\xi_1, \cdots, \xi_n\}$, $\xi < \omega_1$ be $\aleph_1$ conditions, we will find two which are compatible. We assume the $e_\xi$'s have the same length and form a $\Delta$-system, i.e., there is a common part $a \subset A$ such that $e_\xi \cap e_{\xi'} = a$ for $\xi \neq \xi'$. By further diluting the sequence $\langle e_\xi \mid \xi < \omega_1 \rangle$ we assume that for $\xi < \xi'$ there is $c \in C$ such that $\bigcup e_\xi < c < \bigcap (e_{\xi'} - a)$. Diluting again we get that the isomorphism type of $(\{\xi_1, \cdots, f(\xi_1), \cdots\}, f, \{\xi_1, \cdots, \xi_n\}, <_{\mathrm{On}}, <_{\mathrm{R}})$ does not depend on $\xi$. Moreover between any two points of $e_\xi \cup f[e_\xi]$ we choose a rational number and we assume that this finite sequence of rational numbers is the same for all choices of $\xi < \omega_1$.

Now because of this fixed sequence of rational numbers, if two conditions $e_\xi$, $e_{\xi'}$ are incompatible, it can only be because for some $i \leqq n$, $\xi_i <_{\mathrm{R}} \xi'_i$ but $f(\xi_i)_{\mathrm{R}}> f(\xi_i)$ (or, $\xi_{i\mathrm{R}}> \xi'_i$ and $f(\xi_i) <_{\mathrm{R}} f(\xi'_i)$). This is because, for $i \neq j$, if for example $\xi_i <_{\mathrm{R}} \xi_j$ then some rational number in between witnesses it and hence $\xi'_j$ is also greater than that rational number so $\xi_i <_{\mathrm{R}} \xi'_j$, but $f(\xi_i) <_{\mathrm{R}} f(\xi_j)$ and some rational witnesses that, hence, again we have $f(\xi_1) <_{\mathrm{R}} f(\xi'_j)$.

As the number of rational intervals is countable, there is $\gamma < \omega_1$ such that $\langle e_\tau \mid \tau < \gamma \rangle$ satisfies the following: For every $\xi < \omega_1$ and every sequence of rational intervals, $I_1, \cdots, I_n, \bar{I}_n, \cdots, \bar{I}_n$ such that $\xi_i \in I_i$ and $f(\xi_i) \in \bar{I}_i$ for $i \leqq n$, there exists $\tau < \gamma$ such that $\tau_i \in I_i$ and $f(\tau_i) \in \bar{I}_i$ for $i \leqq n$. $\langle e_\tau \mid \tau < \gamma \rangle$ can be coded by a real, hence there is $\alpha \in C$ such that $\langle e_\tau \mid \tau < \gamma \rangle \in M_\alpha$. Pick some condition $e_\xi$ such that all ordinals in $e_\xi - a$ are above $\alpha$ (remember $a$ is the common part of the $\Delta$-system). Assume, to simplify the presentation of the argument, that $n = 4$, $e_\xi = \{\xi_1, \cdots, \xi_4\}$ and that $a = \{\xi_1\}$. Using $\xi_1$, $\xi_2$, $\xi_3$, $f$, $\langle e_\tau \mid \tau < \gamma \rangle$ as parameters, $\xi_4$ satisfies the following property $\varphi(x)$.

$\varphi(x)$: For any sequence of rational intervals $I_2$, $I_3$, $I_4$, $\bar{I}_2$, $\bar{I}_3$, $\bar{I}_4$, such that $\xi_i \in I_i$ and $f(\xi_i) \in \bar{I}_i$ for $i = 2, 3$ and $x \in I_4$, $f(x) \in \bar{I}_4$, there is $\tau < \gamma$ such that $\tau_i \in I_i$ and $f(\tau_i) \in \bar{I}_i$ for $2 \leqq i \leqq 4$.

16. Claim. *There are uncountably many $\xi^*_4 \in A$ such that $\varphi(\xi^*_4)$.*

Proof. If there is a countable supremum to the set of ordinals satisfying $\varphi(x)$ then this supremum is in any of the elementary submodels containing the parameters. In particular, if we denote by $c$ the ordinal in $C$ above $\xi_1$, $\xi_2$, $\xi_3$ and below $\xi_4$, we will get this supremum in $M_c$ hence in $M_c \cap \omega_1 = c$, contradicting $c \leqq \xi_4$ and $\varphi(\xi_4)$.

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Remember that we are assuming that there does not exist an uncountable set where $f$ is non-increasing, so there are two ordinals $\xi_4'$, $\xi_4''$ in $A$ above $c$ such that $\varphi(\xi_4')$, $\varphi(\xi_4'')$, $\xi_4' <_R \xi_4''$ and $f(\xi_4') <_R f(\xi_4'')$. Pick now disjoint rational intervals around each of the four ordinals: $\xi_4' \in I_4'$, $f(\xi_4') \in \bar{I}_4'$, $\xi_4'' \in I_4''$, $f(\xi_4'') \in \bar{I}_4''$. Obviously, for any four points $a' \in I_4'$, $\bar{a}' \in \bar{I}_4'$, $a'' \in I_4''$, $\bar{a}'' \in \bar{I}_4''$, the function $a' \to \bar{a}'$, $a'' \to \bar{a}''$ is order preserving (on the real line order).

Look at the following property $\psi(x)$ satisfied by $\xi_3$ and expressed with the parameters $\xi_2$, $I_4'$, $\bar{I}_4'$, $I_4''$, $\bar{I}_4''$, $f$, $\langle e_\tau \mid \tau < \gamma \rangle$.

$\psi(x)$: For every rational interval $I_2$, $I_3$, $\bar{I}_2$, $\bar{I}_3$, such that $\xi_2 \in I_2$, $f(\xi_2) \in \bar{I}_2$, $x \in I_3$, $f(x) \in \bar{I}_3$ there are $\tau'$, $\tau'' < \gamma$ such that $\tau_i', \tau_i'' \in I_i$ and $f(\tau_i')$, $f(\tau_i'') \in \bar{I}_i$ for $i = 2, 3$ and $\tau_4' \in I_4'$, $f(\tau_4') \in \bar{I}_4'$, $\tau_4'' \in \bar{I}_4''$, $f(\tau_4'') \in \bar{I}_4''$.

Repeating the argument of Claim 16, we obtain $\aleph_1$ many members of $A$ satisfying $\psi(x)$ and again, there are two, $\xi_3', \xi_3'' \in A$ such that $\psi(\xi_3')$, $\psi(\xi_3'')$, $\xi_3' <_R \xi_3''$ and $f(\xi_3') < f(\xi_3'')$. As before, pick disjoint rational intervals $\xi_3' \in I_3'$, $f(\xi_3') \in \bar{I}_3'$, $\xi_3'' \in I_3''$, $f(\xi_3'') \in \bar{I}_3''$ and look at the property $\delta(x)$ satisfied by $\xi_2$:

$\delta(x)$: For any rational intervals $I_2$, $\bar{I}_2$ such that $x \in I_2$, $f(x) \in \bar{I}_2$ there are $\tau', \tau'' < \gamma$ such that $\tau_2', \tau_2'' \in I_2$, $f(\tau_2')$, $f(\tau_2'') \in \bar{I}_2$, $\tau_i' \in I_i'$, $f(\tau_i') \in \bar{I}_i'$, $\tau_i'' \in I_i''$, $f(\tau_i'') \in \bar{I}_i''$ for $i = 3, 4$.

Finally, using again the fact that an ordinal $d$ in $C$ separates $\xi_2$ from $\xi_1$, and hence all parameters of $\delta(x)$ appeared in $M_d$, we can find $\xi_1'$, $\xi_2''$ satisfying $\delta(x)$ such that $\xi_2' <_R \xi_2''$ and $f(\xi_2') < f(\xi_2'')$. Pick disjoint rational intervals $\xi_2' \in I_2'$, $f(\xi_2) \in \bar{I}_2'$, $\xi_2'' \in I_2''$, $f(\xi_2'') \in \bar{I}_2''$ and using $\delta(\xi_2')$, $\delta(\xi_2'')$ find $\tau', \tau'' < \gamma$ such that $\tau_i' \in I_i'$, $f(\tau_i') \in \bar{I}_i'$, $\tau_i'' \in I_i''$, $f(\tau_i'') \in \bar{I}_i''$ for $i = 2, 3, 4$. As $\tau_1' = \tau_1''$ we obtain that $e_{\tau'}$ and $e_{\tau''}$ are compatible. So $P$ satisfies the c.c.c.

Using Lemma 10, we make sure that $P$ is redefined so that every generic filter over $P$ is uncountable and the remaining parts of the proof of Theorem 4 are standard.

## §4. Proof of Theorem 6

The following remark, due to M. Rubin and reproduced here with his kind permission, shows why we have to assume in Theorem 6 that the function $f$ is one-to-one.

17. *Construction of a Function which is not a Countable Union of Monotonic Functions*

Let $A$ be any set of reals of cardinality $\aleph_1$. Decompose $A$ as the union of $\aleph_1$

disjoint uncountable subsets – $A = \bigcup_{\alpha<\omega_1} A_\alpha$. Take $\langle r_\alpha \mid \alpha < \omega_1\rangle$ a sequence of different real numbers, and define $f: A \to R$ so that for any $\alpha < \omega_1$ $f \restriction A_\alpha$ has the fixed value $r_\alpha$. We show that $f$ is not the union of countably many monotonic functions. Let $g \subset f$ be any monotonic function. We claim that $\{\alpha < \omega_1 \mid |A_\alpha \cap \mathrm{Dom}(g)| > 1\}$ is countable. This follows from the fact that there cannot be uncountably many disjoint intervals. It follows from the claim that for any countable collection of functions $g_n \subseteq f$, $n < \omega$ there is $\alpha < \omega_1$ such that $|\mathrm{Dom}(g_n) \cap A_\alpha| \leqq 1$ for all $n < \omega$, hence there is a real in $A_\alpha$ which is not in the domain of any $g_n$.

The general outline of the proof of Theorem 6 is like that of Theorem 4, and what we describe in this section is one step of an iteration. The idea is a way to divide the domain of a given real function into two parts so that by forcing, one part will be the domain of countably many increasing functions and the other part of decreasing functions.

As before, we assume C.H. $\langle r_\alpha \mid \alpha < \omega_1\rangle$ is an enumeration of the real numbers $R$, $C \subseteq \omega_1$ is a closed unbounded set such that for $\alpha \in C$, $\alpha = M_\alpha \cap \omega_1$, where $M_\alpha$ is a countable elementary submodel of $H(\aleph_2)$ and $\langle r_i \mid i < \omega_1\rangle$ and $f$ are members of $M_\alpha$. (See Definition 14.) $f: A \to R$, $A \subseteq R$, $f$ is an uncountable function which we assume to be one-to-one. For any $\alpha \in C$, let $\alpha' > \alpha$ be the successor of $\alpha$ in $C$ (i.e., $\alpha'$ is the least member of $C$ which is $> \alpha$), enumerate the interval $[\alpha, \alpha')$ in an $\omega$-sequence $\langle \alpha_n \mid n \in \omega\rangle$ such that $\alpha = \alpha_0$.

18. Definition. Let $\alpha \in C$, a sequence $\bar{t} = \langle t_0, \cdots, t_{n-1}\rangle$ of truth values $(T, F)$ is said to be a *good n-tuple* iff for any formula $\varphi(x_0, \cdots, x_{n-1})$ with parameters in $M_\alpha$ such that $\varphi(\alpha_0, \cdots, \alpha_{n-1})$ holds in $H(\aleph_3)$, there are two $n$-tuples of ordinals $\geqq \alpha$ separated by a member of $C$: $\langle \beta_0, \cdots, \beta_{n-1}\rangle$, $\langle \beta'_0, \cdots, \beta'_{n-1}\rangle$ ($\alpha \leqq \beta_i < c < \beta'_j$ for some $c \in C$) such that $\varphi(\beta_0, \cdots, \beta_{n-1})$, $\varphi(\beta'_0, \cdots, \beta'_{n-1})$ hold in $H(\aleph_3)$ and the truth value of "$f \restriction \{r_{\beta_i}, r_{\beta'_i}\}$ is order preserving" is $t_i$ for $i < n$.

19. Lemma. *Assume $\alpha \in C$ and $\bar{t} = \langle t_0, \cdots, t_{n-1}\rangle$ is a good $n$-tuple, then for some $t_n \in \{T, F\}$, $\langle t_0, \cdots, t_{n-1}, t_n\rangle$ is a good $n+1$-tuple.*

Proof. We have two cases to consider.

*Case $n = 0$.* Assume on the contrary that neither $\langle T\rangle$ nor $\langle F\rangle$ is a good 1-tuple, so we have formulas, $\varphi_T(x_0)$ and $\varphi_F(x_0)$ which are counterexamples to $\langle T\rangle$ and $\langle F\rangle$ respectively. Look at $\varphi = \varphi_T \wedge \varphi_F$, all its parameters are in $M_\alpha$ and $\varphi(\alpha_0)$ holds in $H(\aleph_2)$. By the argument of Claim 16 we know that there are uncountably many ordinals $\xi$ above $\alpha_0$ such that $\varphi(\xi)$ holds. Take any two such

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ordinals $\xi$, $\xi'$ separated by a member of $C$, then "$f \restriction \{r_\xi, r'_\xi\}$ is order preserving" has some truth value $T$ or $F$ and we get a contradiction to the choice of $\varphi_T$ or $\varphi_F$.

*Case* $n > 0$. Assume that $\bar{t} = \langle t_0, \cdots, t_{n-1}\rangle$ is a good $n$-tuple but neither $\bar{t}^\frown\langle T\rangle$ nor $\bar{t}^\frown\langle F\rangle$ is a good $n+1$-tuple. It follows that there are formulas $\varphi_T(x_0, \cdots, x_n)$ and $\varphi_F(x_0, \cdots, x_n)$ which shows that $\bar{t}^\frown\langle T\rangle$ and $\bar{t}^\frown\langle F\rangle$ (respectively) are not good $n+1$-tuples. So, $\psi = \varphi_T \wedge \varphi_F$ is a formula with parameters in $M_\alpha$ such that $\psi(\alpha_0, \cdots, \alpha_n)$ holds in $H(\aleph_2)$. Hence $\exists \xi\,(\bigwedge_{i<n} \xi \neq \alpha_i$ and $\xi > \alpha_0$ and $\psi(\alpha_0, \cdots, \alpha_{n-1}, \xi))$ holds too. Using the assumption that $\bar{t}$ is a good $n$-tuple for this formula, we get two $n$-tuples separated by a member of $C$ as in Definition 18, but as this formula begins with an existential quantifier we get two $n+1$-tuples separated by that same member of $C$ and derive a contradiction.

To conclude, we get an $\omega$-sequence $\bar{t} = \langle t_i \mid i < \omega\rangle$ such that for every $n < \omega$ $\bar{t} \restriction n$ is a good $n$-tuple. Now we define for $\alpha \in C$ a decomposition of $[\alpha, \alpha')$ into two sets: $E_\alpha = \{\alpha_n \mid t_n = T\}$, $D_\alpha = \{\alpha_n \mid t_n = F\}$. Next define $E = \bigcup_{\alpha<\omega_1} E_\alpha$, $D = \bigcup_{\alpha<\omega_1} D_\alpha$. We want to decompose $E \cap \mathrm{Dom}(f)$ into countably many domains on which $f$ is increasing and $f \restriction D$ will be decomposed into countably many decreasing functions. So we define a poset $P$ which consists of all finite functions $g$ such that $\mathrm{Dom}(g) \subseteq \omega$ and for $n \in \mathrm{Dom}(g)$, $g(n)$ is a finite set of $A$ (the domain of $f$ seen as a set of ordinals) with the following properties: (1) $g(n)$ is separated by elements of $C$. (2) If $n$ is even then $g(n) \subseteq E$ and $f \restriction g(n)$ is increasing. (3) If $n$ is odd then $g(n) \subseteq D$ and $f \restriction g(n)$ is decreasing.

Now to prove that $P$ satisfies the c.c.c. one proceeds as in Lemma 15 and uses Lemma 19.

## §5. Entangled sets of reals

20. Definition. Let $E \subseteq R$ be an uncountable set of reals, we say $E$ is *k-entangled* ($k < \omega$) iff for any set $\{e_\xi \mid \xi < \omega_1\}$ of $\aleph_1$ pairwise disjoint $k$-tuples of members of $E$, $e_\xi = \langle \xi_1, \cdots, \xi_k\rangle$, and for any $k$-tuple $\langle t_1, \cdots, t_k\rangle$ of truth values, there are $\xi \neq \xi'$ such that $\xi_i < \xi'_i$ iff $t_i$ is $T$, holds for all $i \leqq k$. Let us denote by $T(e_\xi, e_{\xi'}) = \langle t_1, \cdots, t_k\rangle$ the fact that $\xi_i < \xi'_i \equiv t_i$ for $i \leqq k$.

21. Remarks. (a) Any uncountable set of reals is 1-entangled.

(b) The set of $\aleph_1$ Cohen generic reals is $k$-entangled for every $k < \omega$. Using the C.H. a set of reals of cardinality $\aleph_1$ can be constructed which is $k$-entangled for every $k < \omega$.

(c) Martin's Axiom $+ 2^{\aleph_0} > \aleph_1$ implies that for any set $A \subseteq R$ of cardinality $\aleph_1$, for some $1 < k < \omega$, $A$ is not $k$-entangled.

PROOF. We only sketch the arguments. (a) is the fact that no uncountable set of reals is well ordered or conversely well ordered. For (b), the set of $\aleph_1$ Cohen generic reals is $k$-entangled for every $k < \omega$ by the same arguments used to prove 7 in §2. Assuming C.H. a set of reals of cardinality $\aleph_1$ which is $k$-entangled for every $k < \omega$ is constructed as follows. (Avraham). Let $\langle r_\alpha \mid \alpha < \omega_1 \rangle$ be an enumeration of all reals and $\langle M_\alpha \mid \alpha < \omega_1 \rangle$ an increasing sequence of countable models like in Definition 14. For $\alpha \in C$ let $s_\alpha$ be a Cohen generic real over $M_\alpha$, by diluting $C$ we can assume $s_\alpha \in M_{\alpha'}$, where $\alpha' \in C$ is the first ordinal in $C$ above $\alpha$. Now $E = \{s_\alpha \mid \alpha \in C\}$ is $k$-entangled for every $k < \omega$: Let $\langle e_\xi \mid \xi < \omega_1 \rangle$ be a sequence of disjoint $k$ tuples from $E$, and $\langle t_1, \cdots, t_k \rangle$ is a $k$-tuple of truth values. Pick a countable $M \prec H(\aleph_2)$ such that $\langle e_\xi \mid \xi < \omega_1 \rangle \in M$. Say $\alpha_0 = M \cap \omega_1$, then for some $\alpha_1 < \omega_1$ $\langle e_\xi \mid \xi < \alpha_0 \rangle \in M_{\alpha_1}$. Take now any $\xi$ such that setting $e_\xi = \langle \xi_1, \cdots, \xi_k \rangle$ we have $\alpha_1 < \xi_1$. Then $\langle s_{\xi_1}, \cdots, s_{\xi_k} \rangle$ is a $k$-tuple of reals which is $M_{\alpha_1}$ generic over the $k$-product of the Cohen forcing. For a $k$-tuple of finite functions $\langle f_1, \cdots, f_k \rangle = \bar{f}$, say $\bar{f} \leqq e_i = \langle s_{i_1}, \cdots, s_{i_k} \rangle$ iff each $s_{i_l}$ extends $f_l$. And define $T(e_i, \bar{f})$ when possible, as above. The following set is dense in the $k$-product of Cohen posets: $\{\langle f_1, \cdots, f_k \rangle = \bar{f} \mid$ for no $i < \alpha_0$ does $\bar{f} \leqq e_i$ holds *or* $T(e_i, \bar{f}) = \langle t_1, \cdots, t_k \rangle$ for some $i < \alpha_0\}$. By the genericity of $\langle s_{\xi_1}, \cdots, s_{\xi_k} \rangle$ and the fact that $M$ is an elementary submodel, we get that $T(e_i, e_\xi) = \langle t_1, \cdots, t_k \rangle$ for some $i < \alpha_0$.

Proof of (c). There are two cases: If for some uncountable $A' \subseteq A$ there is an order reversing function $f: A' \to A$, look at the pairs $\langle a, f(a) \rangle$, $a \in A'$ to see that $A$ is not 2-entangled. If there is no such uncountable $A'$, construct a poset $P$ consisting of finite approximation to an anti-isomorphism as follows. Slice $A$ into $\aleph_1$-countable disjoint parts, like Baumgartner [2], and take in $P$ only finite, order reversing functions whose restriction to any slice is a function from that slice to itself. Now if $P$ would satisfy the c.c.c. we would get by Martin's Axiom, an order reversing uncountable function; hence $P$ does not satisfy the c.c.c. We get thus, $\aleph_1$ functions showing that $P$ does not satisfy the c.c.c., after forming a $\Delta$-system we get a counterexample to $k$-entanglement for some $k < \omega$.

Our aim is to show, however, that for any $k < \omega$, Martin's Axiom is consistent with the existence of a set which is $k$-entangled. We are interested in entangled sets because, as we shall see, the existence of an entangled set implies the existence of $2^{\aleph_1}$ non-isomorphic $\aleph_1$-dense sets of reals.

22. DEFINITION. $A, B \subseteq R$ are said to be *far* iff for every uncountable $A' \subseteq A$ there is neither an order preserving nor an order reversing function $f: A' \to B$.

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It can easily be seen that if $A$ is 2-entangled then any two almost disjoint subsets of $A$ are far. It follows that if M.A. $+ 2^{\aleph_0} = \aleph_2$ is consistent with the existence of a 2-entangled set then it is consistent with the existence of $2^{\aleph_1}$ uncountable pairwise far subsets of reals.

23. LEMMA. *Assume* C.H. *Let* $E = \{r_\xi \mid \xi < \omega_1\}$ *be a set of reals which is* $k$*-entangled. Define* $C$ *as in* §3 (Definition 14), *i.e.,* $C$ *is a closed unbounded subset of* $\omega_1$ *and* $\alpha \in C$ *implies that* $\alpha = M \cap \omega_1$ *for some countable elementary submodel* $M$ *of* $H(\aleph_2)$ *such that* $\langle r_\xi \mid \xi < \omega_1 \rangle \in M$ *and a fixed enumeration of the reals is in* $M$. *Let* $\langle e_{\xi,1}{}^\frown e_{\xi,2}{}^\frown \cdots {}^\frown e_{\xi,n} \mid \xi < \omega_1 \rangle$ *be a sequence of* $\aleph_1$ *pairwise disjoint* $k \times n$ *tuples of countable increasing ordinals,* $n < \omega$, *where* $e_{\xi,i} = \langle \xi_{1,i}, \cdots, \xi_{k,i} \rangle$ *and* $\xi_{j,i} < \xi_{m,l}$ *if* $i < l$ *or* $i = l$ *and* $j < m$, *moreover for* $i < n$ *there is a member of* $C > \xi_{k,i}$ *and less than or equal to* $\xi_{1,i+1}$. Then *for any* $k \times n$ *tuple of truth values* $t_{1,1}, \cdots, t_{k,n}$, *there are* $\xi \neq \xi'$ *such that for* $i \leqq n$ $T(e_{\xi,i}, e_{\xi',i}) = \langle t_{1,i}, \cdots, t_{k,i} \rangle$. (*Using this notation we considered tuples of ordinals as tuples of reals using the enumeration* $\{r_\xi \mid \xi < \omega_1\}$.)

PROOF. We give only a sketch of the proof which goes much along the lines of Lemma 15. So let $\langle e_{\xi,1}{}^\frown \cdots {}^\frown e_{\xi,n} \mid \xi < \omega_1 \rangle$ be given, denote $e_\xi = e_{\xi,1}{}^\frown \cdots {}^\frown e_{\xi,n}$. We say that a sequence of rational intervals $\langle I_{j,i} \mid j \leqq k,\ i \leqq n \rangle$ *covers* $e_\xi$ iff for all $j \leqq k$, $i \leqq n$, $r_{\xi_{j,i}} \in I_{j,i}$. (Look again at the lemma to find meaning of $\xi_{j,i}$.) Find, as in Lemma 15, $\gamma < \omega_1$ such that $\langle e_\xi \mid \xi < \gamma \rangle$ satisfies the following: For any sequence $\langle I_{j,i} \mid j \leqq k, i \leqq n \rangle = \bar{I}$ of rational intervals, if for some $\xi < \omega_1$ this sequence $\bar{I}$ covers $e_\xi$ then for some $\xi < \gamma$, $\bar{I}$ covers $e_\xi$. Code $\langle e_\xi \mid \xi < \gamma \rangle$ with a real, say $r$ is that real. Observe next that if for stationary many $\xi < \aleph_1$, $e_\xi$ contains an ordinal below $\xi$ then Fodor's Theorem shows that the $e_\xi$'s are not pairwise disjoint, hence assume that for all $\xi < \aleph_1$, $e_\xi$ does not contain an ordinal below $\xi$. Now take $\xi \in C$, high enough so that $r$ (the code real) is in $M_\alpha$ for $\alpha \in C - \xi$, (see 14), and look at $e_\xi$. We assumed that there exists an ordinal $c \in C$ such that $\xi_{k,n-1} < c \leqq \xi_{1,n}$. Using $e_{\xi,1}, e_{\xi,2}, \cdots, e_{\xi,n-1}$, $\langle e_\zeta \mid \zeta < \gamma \rangle$ as parameters, $e_{\xi,n} = \langle \xi_{1,n}, \cdots, \xi_{k,n} \rangle$ satisfies the following formula $\varphi(x_{1,n}, \cdots, x_{k,n})$.

$\varphi(x_{1,n}, \cdots, x_{k,n})$: For any sequence of rational intervals $\langle I_{j,i} \mid j \leqq k, i \leqq n \rangle$ which covers $e_{\xi,1}{}^\frown \cdots {}^\frown e_{\xi,n-1}{}^\frown \langle x_{1,n}, \cdots, x_{k,n} \rangle$ there exists $\tau < \gamma$ such that $e_\tau$ is covered by that sequence of intervals.

CLAIM. *For any* $\beta < \omega_1$ *there is a sequence* $\langle \xi^*_{1,n}, \cdots, \xi^*_{k,n} \rangle$ above $\beta$ *such that* $\varphi(\xi^*_{1,n}, \cdots, \xi^*_{k,n})$.

PROOF. Like in Claim 16, if there was an ordinal $\beta$ for which the claim does not hold, then we would get such a $\beta$ in $c$. But $e_{\xi,n}$ which is above $c$ contradicts this. Now that the claim is proved we obtain $\aleph_1$ many disjoint $k$-tuples satisfying $\varphi$ and because $E$ is $k$-entangled, we can find two $k$-tuples $e'(n)$ and $e''(n)$ such that $\varphi(e'(n))$, $\varphi(e''(n))$, and $T(e'(n), e''(n)) = \langle t_{1,n}, \cdots, t_{k,n}\rangle$. Then we find disjoint rational intervals $\langle I'_{j,n} \mid j \leqq k\rangle$, $\langle I''_{j,n} \mid j \leqq k\rangle$ which cover $e'(n)$ and $e''(n)$ such that $T(e', e'') = \langle t_{1,n}, \cdots, t_{k,n}\rangle$, for any $k$-tuples $e'$ and $e''$ which are covered by $\langle I'_{j,n} \mid j \leqq k\rangle$ and $\langle I''_{j,n} \mid j \leqq k\rangle$ respectively. Continue this for $n$ steps until two sequences $\langle I'_{j,i} \mid j \leqq k, i \leqq n\rangle$, $\langle I''_{j,i} \mid j \leqq k, i \leqq n\rangle$ of rational intervals are found such that: (1) There is $e_\tau$, $\tau < \gamma$ which is covered by the $I'_{j,i}$ and $e_\tau$, $\tau < \gamma$ which is covered by the $I''_{j,i}$. And (2) if $e_{\tau'}$ and $e_{\tau''}$ are covered by the $I'_{j,i}$ and the $I''_{j,i}$ respectively, then $T(e_{\tau'}, e_{\tau''}) = \langle t_{11}, \cdots, t_{k,n}\rangle$.

This ends the proof.

24. Theorem. *For any $k < \omega$, M.A. $+ 2^{\aleph_0} > \aleph_1 +$ there is a $k$-entangled set of reals of cardinality $\aleph_1$, is consistent.*

Proof. We start from a universe satisfying the C.H. and pick a set $E$ which is $k$-entangled. Fixing some enumeration of $E$ we look at $E$ as a set of ordinals. Now iterate $\aleph_2$ times c.c.c. posets like [6] in order to get Martin's Axiom, but we have one extra concern — to keep $E$ $k$-entangled. At limit stages we take direct limit and standard arguments show $E$ remains $k$-entangled if it was so at every stage. So the problem of keeping $E$ $k$-entangled is at the successor stages. We are at an intermediate stage $V$ where C.H. holds, $E$ is $k$-entangled and $Q$ is a poset that satisfies the c.c.c. There are two cases: If in $V^Q$ $E$ is $k$-entangled, then we continue forcing with $Q$. If, on the other hand, in $V^Q$ $E$ is no longer $k$-entangled, then *our aim is to find a* c.c.c. *poset $P$ such that in $V^P$ $Q$ does not satisfy the* c.c.c. *and $E$ is $k$-entangled.* (And then we continue forcing with $P$, so that finally we obtain Martin's Axiom.)

As $E$ is not $k$-entangled in $V^Q$, we have a $k$-tuple of truth values $\bar{t} = (t_1, \cdots, t_k)$ and an uncountable set $S$ of pairwise disjoint $k$-tuples (in $V^Q$) such that for $e \neq e'$ in $S$, $T(e, e') \neq \bar{t}$ (with Boolean value 1). Now let $C$ be a closed unbounded suset of $\omega_1$ as in Definition 14. Every finite information in $V^Q$ can be described (forced) by a condition in $Q$, so we can find for $\xi \in \omega_1$ a sequence $(q_\xi, e_{\xi,1}, \cdots, e_{\xi,k+1})$ such that

(a) $q_\xi \in Q$. For $1 \leqq i \leqq k+1$, $q_\xi \Vdash^Q e_{\xi,i} \in S$.

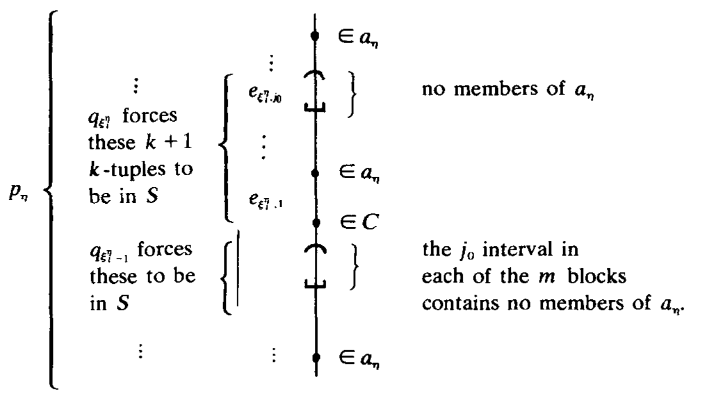
(b) All ordinals of $e_{\xi,i}$ are below the ordinals of $e_{\xi,i+1}$ and moreover there is a member of $C$ in the interval $(\bigcup e_{\xi,i}, \bigcap e_{\xi,i+1})$.

(c) If $\xi < \xi'$ then for some $\alpha \in C$, $\alpha$ is greater than all ordinals in $e_{\xi,k+1}$ and smaller than all ordinals in $e_{\xi',1}$.

The following property holds because of our assumption on $S$: For $\xi \neq \xi'$, if $T(e_{\xi,i}, e_{\xi',j}) = \bar{t}$ then $q_\xi$ and $q_{\xi'}$ are incompatible in $Q$.

Now we define $P$, $p \in P$ iff $p \subseteq \omega_1$ is a finite set such that for $\xi, \xi' \in p$ if $\xi \neq \xi'$ then $T(e_{\xi,n}, e_{\xi',n}) = \bar{t}$ or $T(e_{\xi',n}, e_{\xi,n}) = \bar{t}$ for some $n \leqq k+1$.

It is clear that in $V^P$, $Q$ does not satisfy the c.c.c. (Actually we do something like Lemma 10.) We have to show that $P$ satisfies the c.c.c. and that $E$ remains $k$-entangled in $V^P$. We will prove only that $E$ stays $k$-entangled, the chain condition is easier. Suppose $(p_\eta, a_\eta)$, $\eta < \omega_1$ are such that $p_\eta \in P$ and $\{a_\eta \mid \eta < \omega_1\}$ are disjoint $k$-tuples in $\omega_1$, $\bar{t}^* = (t_1^*, \cdots, t_k^*)$ is a $k$-tuple of truth values, we want $\eta \neq \eta'$ such that $p_\eta$ and $p_{\eta'}$ are compatible and $T(a_\eta, a_{\eta'}) = \bar{t}^*$. (See Lemma 12.) We assume without loss of generality that $(p_\eta, a_\eta)$ are as uniform as possible, so for all $\eta < \omega_1$, $p_\eta$ are of constant cardinality $m$ and pairwise disjoint, say $p_\eta = (\xi_1^\eta, \cdots, \xi_m^\eta)$. Associate with every $\eta$ a $((k+1) \times m) - 1$ tuple of members of $C$ which separate between the $k$-tuples $e_{\xi,j}$ $\xi \in p_\eta$, $j \leqq k+1$. We obtain thus $(k+1) \times m$ successive disjoint intervals such that $e_{\xi_i^\eta,j}$ is contained in the $(k+1) \times (i-1) + j$ interval. By the pigeon hole principle we can find some $j_0 \leqq k+1$ such that no member of $a_\eta$ (which contains $k$ elements) appears in the $(k+1) \times i + j_0$ half open half closed interval for any $i \leqq m$. We assume now that $j_0$ does not depend on $\eta$.




Using the separation by rational numbers as in Lemma 15 we get that for $\eta \neq \eta'$, $1 \leqq i, j \leqq m$ if $i \neq j$ then $\xi_i^\eta$ and $\xi_j^{\eta'}$ are all right, i.e., $T(e_{\xi_i^\eta,n}, e_{\xi_j^{\eta'},n}) = \bar{t}$ or $T(e_{\xi_j^{\eta'},n}, e_{\xi_i^\eta,n}) = \bar{t}$ for some $n \leqq k+1$. So, to get $p_\eta$, $p_{\eta'}$ compatible, we need to take care only for $\xi_i^\eta$ and $\xi_i^{\eta'}$ $i \leqq m$. For any $\eta$ look now at $e_{\xi_i^\eta, j_0}$ $i \leqq m$ and at $a_\eta$ we are in a case of Lemma 23 in the following sense: There are $\alpha_l \in C$, $l = 1, \cdots, 2(m+1)$ such that $e_{\xi_i^\eta, j_0}$ is contained in the interval $(\alpha_{2i}, \alpha_{2i+1})$, $i = 1, \cdots, m$. And the members of the $k$-tuple $a_\eta$ are dispersed in the $[\alpha_{2i-1}, \alpha_{2i})$

intervals, $i = 1, \cdots, (m+1)$. So in the notations of Lemma 2.3 we let $n = 2m+1$, $\langle e_{\eta,1}{}^\frown e_{\eta,2}{}^\frown \cdots {}^\frown e_{\eta,2m+1} \mid \eta < \omega_1 \rangle$ is defined by $e_{\eta,2i} = e_{\xi_i^\eta, j_0}$ and $e_{\eta,2i-1} = [\alpha_{2i-1}, \alpha_{2i}) \cap a_\eta$ (we might have to add some ordinals if we want it to be a full $k$-tuple). The $k \times n$ tuple of truth values is defined according to $\bar{t}$ and $\bar{t}^*$ such that the conclusion of Lemma 23 gives $\eta, \eta'$ such that $T(e_{\xi_i^\eta, j_0}, e_{\xi_i^{\eta'}, j_0}) = \bar{t}$ for $i \leqq m$ and $T(a_\eta, a_{\eta'}) = \bar{t}^*$. Hence $p_\eta$, $p_{\eta'}$ are compatible and as required.

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