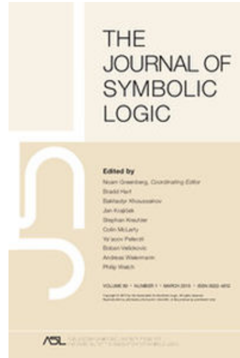

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CANONIZATION THEOREMS AND APPLICATIONS

SAHARON SHELAH[1]

Abstract. We improve the canonization theorems generalizing the Erdös-Rado theorem, and as a result complete the answer to "When does a Hausdorff space of cardinality χ necessarily have a discrete subspace of cardinality κ?" We also improve the results on existence of free subsets.

Introduction. Ramsey's theorem is well known and widely used. Generalizing it, Erdös and Rado defined $\lambda \to (\kappa)^r_\mu$ by: if $|A| = \lambda$, $<$ wellorders A, f an r-place function from A into some C, $|C| \le \mu$, *then* for some $B \subseteq A$, $|B| = \kappa$, f is constant on all increasing sequences from B. That is, if $b_0 < \cdots < b_{r-1} \in B, b'_0 < \cdots < b'_{r-1} \in B$ then $f(b_0, \ldots, b_{r-1}) = f(b'_0, \ldots, b'_{r-1})$. (Alternatively Dom f is the family of subsets of A of power r; there is no real difference.) First they prove: $(2^\lambda)^+ \to (\lambda^+)^2_\lambda$ (and trivially $\lambda^+ \to (\lambda^+)^1_\lambda$). For getting similar results for higher r they need a lemma which will be the induction step.

For this they introduce an end-homogeneous set. For an r-place function f, we say B (wellordered by $<$) is end-homogeneous for f, if for every $b_0 < \cdots < b_{r-2} \in B$, and $b_{r-2} < b'_{r-1} \in B$, $b_{r-2} < b''_{r-1} \in B$

$$f(b_0, \ldots, b_{r-2}, b'_{r-1}) = f(b_0, \ldots, b_{r-2}, b''_{r-1}).$$

Then they define: $\lambda \to \langle\kappa\rangle_\mu$ if for any r, for every $(A, <)$ of order-type λ, f r-place function with range C, $|C| \le \mu$, there is $B \subseteq A$ of power κ end-homogeneous for f. Now they prove $(2^\lambda)^+ \to \langle\lambda^+\rangle_{(2^\lambda)}$ and that $\lambda \to \langle\lambda'\rangle_\mu$, $\lambda' \to (\kappa)^r_\mu$ implies $\lambda \to (\kappa)^{r+1}_\mu$. So they were able to prove the Erdös-Rado theorem: $\beth_{r-1}(\lambda)^+ \to (\lambda^+)^r_\lambda$. See Erdös-Hajnal-Rado [EHR]; they also prove that the theorem above is best possible.

However, already in [EHR] they raise more complicated relations, when we consider several A_ξ ($\xi < \theta$) and an r-place function f on $\bigcup_\xi A_\xi$ and want to get $B_\xi \subseteq A_\xi$ (as large as possible) so that, e.g., if $\xi(0) < \cdots < \xi(r-1)$, $b_l \in B_{\xi(l)}$, $b'_l \in B_{\xi(l)}$ then $f(b_0, \ldots, b_{r-1}) = f(b'_0, \ldots, b'_{r-1})$. More complicatedly, e.g., if $\xi(0) < \xi(1)$, $b_0 < b_1 \in B_{\xi(0)}$, $b'_0 < b'_1 \in B_{\xi(0)}$, $b_2 < b_3 \in B_{\xi(1)}$, $b'_2 < b'_3 \in B_{\xi(1)}$ then $f(b_0, b_1, b_2, b_3) = f(b'_0, b'_1, b'_2, b'_3)$.

In a manuscript of a book [EHMR], Erdös, Hajnal, Mate and Rado formulate when $\langle\lambda_\xi : \xi < \theta\rangle$ has $\langle\kappa_\xi : \xi < \theta\rangle$—canonization for r, μ: *if* $|A_\xi| = \lambda_\xi$, $<$ wellorders $\bigcup_{\xi<\theta} A_\xi$, $\xi < \zeta \Rightarrow A_\xi < A_\zeta$, f an r-place function from $\bigcup_{\xi<\theta} A_\xi$ into C, $|C| = \mu$, *then* there are $B_\xi \subseteq A_\xi$, $|B_\xi| = \kappa_\xi$ such that:

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(*) if $b_0, \ldots, b_{r-1} \in \bigcup_{\xi<\theta} B_\xi$ and $b'_0, \ldots, b'_{r-1} \in \bigcup_{\xi<\theta} B_\xi$, and $(\forall \xi < \theta)(\forall l < r)$ $[b_l \in B_\xi \equiv b'_l \in B_\xi]$ and $l < m < r \Rightarrow [b_l < b_m \equiv b'_l < b'_m]$ then $f(b_0, \ldots) = f(b'_0, \ldots)$.

They prove that $\langle \beth_n(\kappa_\xi)^+ : \xi < \theta \rangle$ has $\langle \kappa_\xi : \xi < \theta \rangle$—canonization for r, μ, when $\mu \leq \kappa_0$, and n is quite bigger than r (of the order of magnitude of r^2).

In that manuscript they also give some application to the discrete subspace problem of a Hausdorff space, and the free subset problem.

We suggest here a definition which is a mixture of canonization and end-homogeneity (Definition 1). It includes all the cases mentioned above. The question is whether this is the right definition, i.e. still comprehensible and reasonably easy to handle, but already enough to cover all reasonable application. The reader can return to this after reading.

We then formulate and prove a lemma (Lemma 2) which will be the induction step in proving the satisfaction of canonization theorems. The improvement in it is that by lowering the power from $\beth_n(\lambda)^+$ to $\beth_{n-1}(\lambda)^+$, we get, many times, that the function does not depend on the two last elements (instead of one), however this can be done only if they come from different A's. This lemma is very similar to [Sh 1, 1.1]. The major change is that possibly $\lambda_i = \lambda_{i+1}$, a case ignored there.

However Lemma 2 is quite general, so in Lemma 6 we derive from it the simple canonization theorems, which we shall need as an induction step (the parallels of $\lambda \to \langle \kappa \rangle_\mu$).

The Composition Claim 5 tells us how to use Lemma 6 as an induction step; more exactly how subsequent applications of two canonization theorems (with the "output" of the first a suitable "input" of the second) give a third canonization theorem.

In 7, 8 we get actual canonization theorems. E.g., if $\beth_{r-1}(\kappa(\xi))$ is strictly increasing for $\xi < \theta$, and $|\theta| \leq \kappa(0)$, then (1) $\langle \beth_{r-1}\kappa(\xi)^+ : \xi < \theta \rangle$ has $\langle \kappa(\xi)^+ : \xi < \theta \rangle$ canonization for $\kappa_{(0)}$, r and (2) if $|A_\xi| = \beth_{r-1}\kappa(\xi)^+$ for $\xi < \theta$, f a $2r$-place function from $\bigcup_{\xi<\theta} A_\xi$ to C, $|C|^{|\theta|} \leq \kappa_0$, we can find $B_\xi \subseteq A_\xi$, $|B_\xi| = \kappa(\xi)^+$, such that $\xi(0) < \cdots < \xi(2r-1) < \theta$, $b_0, b'_0 \in B_{\xi(0)}, \ldots$ implies $f(b_0, \ldots) = f(b'_0, \ldots)$.

Though the second result is a bigger improvement, it is not clear whether it is best possible (whereas (1) is). Notice also that by Shelah [Sh 3] it is consistent (assuming the consistency of some large cardinals) that much better canonization theorems hold.

In 3, 9, 10 we deal with the case "θ is finite". In the end we deal with the applications.

The first question is "suppose X is a Hausdorff space of cardinality λ, does it have a discrete subspace of cardinality κ?"

Juhasz and Hajnal prove that if $\kappa = \mu^+$, $\lambda > 2^{2^\mu}$ the answer is yes, but if $\lambda \leq 2^{2^\mu}$, it is consistent with ZFC that the answer is no (see [J]). If κ is inaccessible, or singular but $\langle 2^{2^\mu} : \mu < \kappa \rangle$ is eventually constant, the situation is similar. In the remaining case, in [EHMR] Erdös, Hajnal, Mate and Rado prove the answer is yes if $\lambda > \sum_{\mu<\kappa} 2^{2^\mu}$; if $\lambda \leq 2^{2^\mu}$, $\mu < \kappa$ this reduces to the previous case. There remains open the case κ singular, $\lambda = \sum_{\mu<\lambda} 2^{2^\mu}$, but $\langle 2^{2^\mu} : \mu < \kappa \rangle$ is not eventually constant. We prove that the answer is yes.

The free subset problem is: suppose f is an r-place function from A to A, $|A| = \lambda$, we want to find $B \subseteq A$, $|B| = \kappa$, which is free, i.e., $b_0, \ldots, b_{r-1} \in B \Rightarrow f(b_0, \ldots, b_{r-1}) \notin B - \{b_0, \ldots, b_{r-1}\}$. (Alternatively, $f(b_0, \ldots, b_{r-1})$ is a small subset of A, and we want $f(b_0, \ldots, b_{r-1}) \cap B \subseteq \{b_0, \ldots, b_{r-1}\}$.)

There are canonization theorems which implies it. In [EHMR] they use the Erdös-Rado theorem, hence get a positive answer when λ is, approximately, $\beth_r(\kappa)$ i.e., the "distance" from κ to λ is r exponentiation. We succeed in reducing it to about $r/2$ exponentiation (but with some application of the successor operation) by using a canonization theorem for $\langle \lambda : \xi < \kappa \rangle$ like (2) above.

A first version of this was [Sh 2].

Notation. Natural numbers are denoted by k, l, m, n, r, ordinals by $i, j, \alpha, \beta, \gamma, \xi, \zeta, \eta, \nu, l$, cardinals by $\lambda, \kappa, \mu, \chi$. We define $\beth_\alpha(\lambda)$ by induction on α: $\beth_0(\lambda) = \lambda$, and $\beth_\alpha(\lambda) = \sum_{\beta<\alpha} 2^{\beth_\beta(\lambda)}$ for $\alpha > 0$. Let $\lambda^{<\mu} = \sum_{\kappa<\mu} \lambda^\kappa$.

If $<$ orders A, $B \subseteq A$, $C \subseteq A$, $a \in A$ *then* $B < a$ means $(\forall x \in B)\ x < a$, $B < C$ means $(\forall x \in B)(\forall y \in C)(x < y)$, etc.

Let

$$[A]^\kappa = \{B : B \subseteq A,\ |B| = \kappa\}.$$
$$[A]^{<\kappa} = \{B : B \subseteq A, |B| < \kappa\}.$$

DEFINITION 1. $\langle \lambda_\xi : \xi < \theta \rangle$ has a $\langle \kappa(\xi) : \xi < \theta \rangle$—canonical form for $\Gamma = \{\bar{r}(i)^{l(i)}_{\chi(i)} : i < \alpha\}$ [where $\chi(i)$ is a nonzero cardinal, and $\bar{r}(i) = \langle n_1(i); \ldots; n_k(i) \rangle$, $n_m(i) \geq 0$ and $l(i)$ are natural numbers, and for each $\bar{r} = \langle n_1; \ldots; n_k \rangle$ we denote $n(\bar{r}) = \sum_{i=1}^{k} n_i$, $k(\bar{r}) = k$, $n_m(\bar{r}) = n_m$] *if* for every set A_ξ $(\xi < \theta)$, $|A_\xi| = \lambda_\xi$ (and $<$ wellorders $\bigcup_{\xi<\theta} A_\xi$, $A_\xi < A_\eta$ for $\xi < \eta$) and functions f_i $(i < \alpha)$, f_i an $n(\bar{r}(i))$-place function from $\bigcup_\xi A_\xi$ to $\chi(i)$ *there are* $B_\xi \subseteq A_\xi$, $|B_\xi| = \kappa(\xi)$ such that for every i, f_i is $\bar{r}(i)^{l(i)}$-canonical on $\langle B_\xi : \xi < \theta \rangle$. This means that when $\xi_1 < \cdots < \xi_{k(\bar{r}(i))} < \theta$, $a_1 < \cdots < a_{n_1(\bar{r}(i))} \in B_{\xi_1}$, $a_{n_1(\bar{r}(i))+1} < \cdots < a_{n_1(\bar{r}(i))+n_2(\bar{r}(i))} \in B_{\xi_2}$, etc. then $f_i(a_1, \ldots, a_{n(\bar{r}(i))})$, depends on $\xi_1, \ldots, \xi_k, a_1, \ldots, a_{n(\bar{r}(i))-l(i)}$ only (and not on $a_{n(\bar{r}(i))-l(i)+1}, \ldots, a_{n(r(i))}$).

THE MAIN LEMMA 2. *Suppose θ is an ordinal $S \subseteq \theta$, and for each $\xi < \theta$ λ_ξ is a regular cardinal, and λ^*_ξ are such that $\xi \in S \Rightarrow \lambda_\xi = (\lambda^*_\xi)^+$, $\xi \notin S \Rightarrow \lambda_\xi = \lambda^*_\xi$. Suppose also $\kappa(\xi) \leq \lambda^*_\xi$, and $\prod_{\eta<\xi} (\lambda^*_\eta)^{\kappa(\eta)} < \lambda_\xi$ for each $\xi < \theta$ (so λ_ξ is nondecreasing).*

Suppose $|A_\xi| = \lambda_\xi$, $<$ wellorders $\bigcup_\xi A_\xi$, $A_\xi < A_\eta$ for $\xi < \eta$, and $F_i(i < \alpha \leq \omega)$ is an $m(i)$-place symmetric function from $\bigcup_{\xi<\theta} A_\xi$ into $\chi(i)$. Let $\chi = \sum_{i<\alpha} \chi(i)$; suppose further that for every $\xi < \theta$, $B_\eta \subseteq A_\eta$ $(\eta < \xi)$, $a_\eta \in A_\eta$ $(\eta < \theta)$, $|B_\eta| \leq \kappa(\eta)$ and $C \subseteq A_\xi$, $|C| = \lambda^*_\xi$ there is $B_\xi \subseteq C$, $|B_\xi| \leq \kappa(\xi)$ such that $P_\xi(\langle B_\eta : \eta \leq \xi \rangle, \langle a_\eta : \eta < \theta \rangle)$ [P_ξ—a specific property].

Then there are $a^*_\xi \in A_\xi$, $B^*_\xi \subseteq A_\xi$ such that

(1) *for every $\xi < \theta$, $P_\xi(\langle B^*_\eta : \eta \leq \xi \rangle, \langle a^*_\eta : \eta < \theta \rangle)$,*

(2) *$|B^*_\eta| \leq \kappa(\eta)$ and for $\eta \in S$, $B^*_\eta < a^*_\eta$,*

(3) *if $\chi^{\sum_{\eta<\xi} \kappa(\eta) + |\alpha|} < \lambda_\xi$ then for all $a_1, \ldots \in \bigcup_{\eta<\xi} B^*_\eta$ and $b \in B^*_\xi$, and $i < \alpha$, $F_i(a_1, \ldots, a_{m(i)-1}, b) = F_i(a_1, \ldots, a_{m(i)-1}, a^*_\xi)$,*

(4) *If $\chi^{|\theta|} + \chi^{\sum_{\eta<\xi} \kappa(\eta)} + \chi^{|\alpha|} < \operatorname{cf} \lambda^*_\xi$ then for each η such that $\chi^{\sum_{\nu<\eta} \kappa(\nu) + |\alpha|} < \lambda_\eta$, $\xi < \eta < \theta$ and for all $a_1, \ldots \in \bigcup_{\rho<\xi} B^*_\rho$, $i < \alpha$, $b, b' \in B^*_\xi$, $c, c' \in B^*_\xi$,*

$$F_i(a_1, \ldots, a_{m(i)-2}, b, c) = F_i(a_1, \ldots, a_{m(i)-2}, b', c') = F_i(a_1, \ldots, a_{m(i)-2}, b, a^*_\eta).$$

REMARK. We can replace in (4), $\operatorname{cf} \lambda^*_\xi$ by λ^*_ξ, if we strengthen the last hypothesis (on P_ξ) by weakening "$|C| = \lambda_\xi$". We can also generalize the lemma for λ_ξ singular cardinals (even for $\xi \in S$) but the gain is marginal and it seems not to be worth complicating the conditions. Always the proof is without significant change. We can also replace $|B_\xi| \le \kappa(\xi)$ by $|B_\xi| < \kappa(\xi)$ or by $|B_\xi| = \kappa(\xi)$ without appropriate changes.

PROOF. Similar to 1.1 [Sh 1]. For any $B \subseteq A$, $a \in A$, let $\mathrm{tp}(a, B) = \{F_i(x, b_1 \cdots) = c: F_i(a, b_1 \cdots) = c, c \in \chi(i), b_1 \cdots \in B, i < \alpha\}$. Clearly $|\{\mathrm{tp}(a, B): a \in A\}| \le \chi^{|B|}$ (except when χ and B are finite, and then $\{\mathrm{tp}(a, B): a \in A\}$ is finite when $\alpha < \omega$), w.l.o.g. each A_ξ has order type λ_ξ. We define by induction on $\xi < \theta$, $a_\xi^* \in A_\xi$ such that if $\chi^{\Sigma_{\eta<\xi}\kappa(\eta)+|\alpha|} < \lambda_\xi$: (see the first two sentences in the lemma)

(*) If $B_\eta \subseteq A_\eta$; $|B_\eta| \le \kappa(\eta)$; $\eta \in S \Rightarrow B_\eta^* < a_\eta^*$ for $\eta < \xi$, *then*

$$\left|\left\{a' \in A_\xi: \mathrm{tp}\left(a', \bigcup_{\eta<\xi} B_\eta^*\right) = \mathrm{tp}\left(a_\xi^*, \bigcup_{\eta<\xi} B_\eta^*\right) \text{ and } \xi \in S \Rightarrow a' < a_\xi^*\right\}\right| = \lambda_\xi^*.$$

Otherwise choose a_ξ^* such that $|\{a \in A_\xi: a < a_\xi\}| = \lambda_\xi^*$ or $\xi \notin S$.

Now we define inductively $B_\xi^* \subseteq A_\xi$, $|B_\xi^*| \le \kappa(\xi)$, $\xi \in S \Rightarrow B_\xi < a_\xi^*$. Suppose we have defined for each $\eta < \xi$. If the hypothesis of (3) fails let $C_\xi = A_\xi$ when $\xi \notin S$, and $C_\xi = \{a \in A_\xi: a < a_\xi^*\}$ when $\xi \in S$. If the hypothesis of (3) holds let:

$$C_\xi = \left\{a' \in A_\xi: \mathrm{tp}\left(a_\xi^*, \bigcup_{\eta<\xi} B_\eta^*\right) = \mathrm{tp}\left(a', \bigcup_{\eta<\xi} B_\eta^*\right), \quad \text{and} \quad \xi \in S \Rightarrow a' < a_\xi^*\right\}$$

so $|C_\xi| = \lambda_\xi^*$. If $\chi^{|\theta|} + \chi^{\Sigma_{\eta<\xi}\kappa(\eta)} + \chi^{|\alpha|} < \mathrm{cf}\,\lambda_\xi^*$ there is $C'_\xi \subseteq C_\xi$, $|C'_\xi| = \lambda_\xi^*$, such that for all $a \in C'_\xi$, $\mathrm{tp}(a, \bigcup_{\eta<\xi} B_\eta^* \cup \{a_\eta^*: \eta < \theta\})$ is the same; otherwise $C'_\xi = C_\xi$. Now choose $B_\xi^* \subseteq C'_\xi$, $|B_\xi^*| \le \kappa_\xi$, such that $P_\xi(\langle B_\eta^*: \eta \le \xi\rangle, \langle a_\eta^*: \eta < \theta\rangle)$.

LEMMA 3. *Suppose* $(\aleph_0 \le)\ \lambda_0 < \cdots < \lambda_{\theta-1}$, $\theta < \omega$, $\kappa(l) < \lambda_l$, $\prod_{m<l} \lambda_m^{\kappa(m)} < \lambda_l$ *and* $\chi^{\Sigma_{m<l}\kappa(m)+|\alpha|} < \lambda_l$, *and* λ_l *is regular and* F_i $(i < \alpha \le \omega)$ *is an* $m(i)$*-place function from* $\bigcup_l A_l$ *into* $\chi(i)$, $\chi = \sum_{i<\alpha}\chi(i)$. *Suppose further that for* $l < \theta$, $a_n^* \in A_n$ $(n < \theta)$, $B_n \subseteq A_n$, $|B_n| \le \kappa(l)$ $(n < l)$, $C \subseteq A_l$, $|C| = \lambda_l$ *there is* $B_l \subseteq A_l$, $|B_l| \le \kappa(l)$ *and* $P_l(\langle B_n: n \le l\rangle, \langle a_n: n < \theta\rangle)$.

Then there are $a_l^* \in A_l$, $B_l^* \subseteq A_l$ *such that*

(1) $P_l(\langle B_n^*: n \le l\rangle, \langle a_n^*: n < \theta\rangle)$,

(2) $|B_l^*| \le \kappa(l)$,

(3) *if* $\chi^{\Sigma_{n<l}\kappa(n)+|\alpha|+|\theta|} < \lambda_l$ *then for all* $a_1, \ldots, a_{l-1} \in \bigcup_{n<l} B_n^*$, $i < \alpha$, $b \in B_l$,

$$F_i(a_1, \ldots, a_{l-1}; a_l^*; a_{l+1}^*, \ldots, a_{\theta+1}^*) = F_i(a_1, \ldots, a_{l-1}; b, a_{l+1}^*, a_{l+2}^*, \ldots, a_{\theta+1}^*).$$

PROOF. Similar to (2) but we define a_l^* by downward induction on l.

Now we return to canonization lemmas.

Claim 4. (1) If for each $\xi < \theta$, $\kappa(\xi) + \chi < \lambda_\xi$ *or* $\chi < \mathrm{cf}\,\lambda_\xi$, $\lambda_\xi = \kappa(\xi)$ *or* $\kappa(\xi) \le 1$, then $\langle\lambda_\xi: \xi < \theta\rangle$ has a $\langle\kappa(\xi): \xi < \theta\rangle$-canonical form for $\{\langle 1\rangle_\chi^1\}$.

(2) In Definition 1 if $\Gamma = \bigcup_{\rho<\rho(0)} \Gamma_\rho$, $\Gamma_\rho = \{\bar r(\rho)^{l(\rho)}_{\chi(\rho,i)}: i < \alpha_\rho\}$, $\chi(\rho, i) > 0$ let $\chi(\rho) = \prod_{i<\alpha_\rho} \chi(\rho, i)$, $\Gamma^* = \{\bar r(\rho)^{l(\rho)}_{\chi(\rho)}: \rho < \rho(0)\}$. Then the truth of the statement there does not change if we replace Γ by Γ^*.

(3) $\langle\lambda_\xi: \xi < \theta\rangle$ has $\langle\kappa(\xi): \xi < \theta\rangle$-canonical form for Γ, *iff* $\langle\lambda_\xi : \xi < \theta, \kappa(\xi) \ne 0\rangle$ has $\langle\kappa(\xi): \xi < \theta, \kappa(\xi) \ne 0\rangle$-canonical form for Γ.

(4) Define λ'_ξ as λ_ξ if $\kappa(\xi) \ne 1$ and as 1 if $\kappa(\xi) = 1$ then: $\langle\lambda_\xi: \xi < \theta\rangle$ has $\langle\kappa(\xi): \xi < \theta\rangle$-canonical form for Γ iff $\langle\lambda'_\xi: \xi < \theta\rangle$ has $\langle\kappa(\xi): \xi < \theta\rangle$-canonical form.

We add if $\langle\lambda_\xi: \xi < \theta, \kappa(\xi) \ne 1\rangle$ has $\langle\kappa(\xi): \xi < \theta, \kappa(\xi) \ne 1\rangle$-canonical form for Γ then $\langle \lambda_\xi \colon \xi < \theta \rangle$ has $\langle \kappa(\xi) \colon \xi < \theta \rangle$-canonical form for Γ when e.g. $\bar{r}^l_\chi \in \Gamma$ implies $\chi^{\|\theta\|} = \chi$.

(5) If for some ξ, $\kappa(\xi) > \lambda_\xi$ then $\langle \lambda_\xi \colon \xi < \theta \rangle$ does not have $\langle \kappa(\xi) \colon \xi < \theta \rangle$-canonical form.

(6) Suppose $\langle \lambda_\xi \colon \xi < \theta \rangle$ has $\langle \kappa(\xi) \colon \xi < \theta \rangle$-canonical form for $\Gamma = \{\bar{r}(i)^{l(i)}_{\chi(i)} \colon i < \alpha\}$. If $\lambda'_\xi \geq \lambda_\xi$, $\kappa'(\xi) \leq \kappa(\xi)$ (for $\xi < \theta$) and $0 < \mu(i) \leq \chi(i)$, $m(i) \leq l(i)$ then $\langle \lambda'_\xi \colon \xi < \theta \rangle$ has $\langle \kappa'(\xi) \colon \xi < \theta \rangle$-canonical form for $\{\bar{r}(i)^{m(i)}_{\mu(i)} \colon i < \alpha,\ \mu(i) > 1\}$. If $h \colon \theta' \to \theta$ is strictly increasing then $\langle \lambda_{h(\xi)} \colon \xi < \theta' \rangle$ has $\langle \kappa(h(\xi)) \colon \xi < \theta' \rangle$-canonical form for Γ.

REMARK. In 4(5) there are also monotonicity properties on $\bar{r}$.

PROOF. Immediate.

Convention. In Definition 1 we always assume λ_ξ is nondecreasing, $\chi(i) > 1$, $\lambda_\xi \geq \kappa(\xi) \geq 2$ (but we concentrate on $\kappa(\xi)$ infinite).

The Composition Claim 5. Suppose $\langle \lambda^{l+1}_\xi \colon \xi < \theta \rangle$ has a $\langle \lambda^l_\xi \colon \xi < \theta \rangle$-canonical form for Γ_{l+1} for $l = 1, 2$. *Then* $\langle \lambda^3_\xi \colon \xi < \theta \rangle$ has a $\langle \lambda^1_\xi \colon \xi < \theta \rangle$-canonical form for Γ_1 where

$$\begin{aligned} \Gamma_1 = \{&(n_1; \ldots; n_k; n_{k+1}; \ldots; n_m)^{p+q}_\chi \colon (n_1; \ldots; n_k; \ldots; n_m)^p_\chi \in \Gamma_3, \\ &p = s + n_{k+1} + \cdots + n_m,\ 0 \leq s < n_k,\ (n_1; \ldots; n_{k-1}; n_k - s)^q_{\chi(1)} \in \Gamma_2 \text{ where} \\ &\chi(1) = \chi^{|\theta|} \text{ or } \chi + |\theta| < \aleph_0, \\ &\chi(1) = \chi^{|\theta|^p}, \text{ or } k = m, s > 0, \chi(1) = \chi\}. \end{aligned}$$

PROOF. Trivial.

Now from 2 and 3 we can get canonization theorems, and use 5 to get more.

THE CANONIZATION LEMMA 6. *Suppose λ_ξ $(\xi < \theta)$ is a nondecreasing sequence of regular cardinals, $S \subseteq \theta$,* let

$$\lambda^0_\xi = \begin{cases} \lambda^+_\xi, & \xi \notin S, \\ \lambda_\xi, & \xi \in S, \end{cases}$$

and suppose

(1) *for every $\lambda^1_\xi < \lambda^0_\xi$ and ρ, $\prod_{\xi<\rho} (\lambda^1_\xi)^{\kappa(\xi)} < \lambda_\rho$,*

(2) $\aleph_0 \leq \kappa(\xi)$, $\kappa(\xi) < \lambda^0_\xi$.

Then $\langle \lambda_\xi \colon \xi < \theta \rangle$ has $\langle \kappa(\xi) \colon \xi < \theta \rangle$-canonical form for Γ, when Γ consists of elements of the following cases, for finitely many distinct χ's only:

(A) $\bar{r}{}^\frown\langle 1 \rangle^1_\chi$ *when (for every ζ)* $(\chi^{\Sigma\{\kappa(\xi) : \xi<\zeta\}})^+ < \lambda^0_\zeta$,

(B) $\bar{r}^1_\chi$ *when* $[(\chi^{1+\Sigma\{\kappa(\xi):\xi<\zeta\}})^{<\kappa(\zeta)}]^+ < \lambda^0_\zeta$,

(C) $\bar{r}{}^\frown\langle 1 \rangle^2_\chi$ *when* $[(\chi^{|\theta|+\Sigma\{\kappa(\xi):\xi<\zeta\}})^{<\kappa(\zeta)}]^+ < \lambda^0_\zeta$,

(D) $\langle 1; 1 \rangle^2_\chi$ *when* $(\chi^{|\theta|})^+ < \lambda^0_0$,

(E) $\langle n \rangle^1_\chi$ *when* $[\chi^{<\kappa(\zeta)}]^+ < \lambda^0_\zeta$,

(F) $\langle n; 1 \rangle^2_\chi$ *when* $[\chi^{|\theta|}]^+ + [\chi^{<\kappa(\zeta)}]^+ < \lambda^0_\zeta$,

(G) $\bar{r}{}^\frown\langle 1; 1 \rangle^2_\chi$ *when* $(\chi^{\Sigma\{\kappa(\xi):\xi<\zeta\}+|\theta|})^+ < \lambda^0_\zeta$.

REMARK. We can in (B), (E) add "or λ^0_ζ is the successor of a weakly compact cardinal $> \sum_{\xi<\zeta} \kappa(\xi)$" and in (C), (F) "or λ^0_ξ is the successor of a weakly compact cardinal $> |\theta| + \sum_{\xi<\zeta} \kappa(\xi)$".

PROOF. For simplicity we assume λ_ξ is a successor cardinal for $\xi \in S$. So $\lambda_\xi = (\lambda^*_\xi)^+$ for $\xi \in S$, and $\lambda^*_\xi = \lambda_\xi$ for $\xi < \theta$, $\xi \notin S$. So we are given sets A_ξ $(\xi < \theta)$, a wellordering of $A = \bigcup_{\xi<\theta} A_\xi$, and $A_\xi < A_\zeta$ for $\xi < \zeta$. $\Gamma = \{\bar{r}(i)^{l(i)}_{\chi(i)}: i < \alpha\}$, and f_i an $n(\bar{r}(i))$-function from A to $\chi(i)$. Clearly $\alpha < \omega_1$.

First assume Γ comes from cases (A), (D), (G) at most. Then we define the P_ξ: $P_\xi(\langle B^*_\eta: \eta \le \xi\rangle, \langle a^*_\eta: \eta < \theta\rangle)$ holds iff the following conditions hold:

(i) $B^*_\eta \subseteq A_\eta$, $|B^*_\eta| = \kappa(\eta)$ and $a^*_\eta \in A_\eta$,

(ii) if the assumption of case (D) holds, f_i is two-place, then for every $\eta < \theta$ and $b, b' \in B^*_\xi$, $f_i(b, a^*_\eta) = f_i(b', a^*_\eta)$.

It is easy to check the assumptions of Lemma 2 hold; moreover (*) there is $C' \subseteq C$, $|C'| = \lambda^*_\xi$, such that $B^*_\xi \subseteq C'$, $|B^*_\xi| = \kappa(\xi) \Rightarrow P_\xi(\langle B^*_\eta: \eta \le \xi\rangle, \langle a^*_\eta: \eta < \theta\rangle)$.

So we have B^*_ξ, a^*_ξ $(\xi < \theta)$ satisfying (1)–(4) from Lemma 2. Now we have to check that $\langle B^*_\xi: \xi < \theta\rangle$ satisfies the required conclusion of Definition 1. Now $|B^*_\xi| = \kappa(\xi)$ by the definition of P_ξ, part (i); if $\bar{r}(i)^{l(i)}_{\eta(i)}$ is from case A [(D)] [(G)]] then by (3) of Lemma 2, [(ii) of the definition of P_ξ] [(4) of Lemma 2] the canonization requirement on f_i in Definition 1 is satisfied.

Now suppose some members of Γ come from case (E) and possibly (A), (D), (G). In the above we change the definition of P_ξ by adding

(iii) if $b, b', a_1, \dots, a_n \in B^*_\xi$, $a_1 < a_2 < \cdots < a_n$, $a_n < b$, $a_n < b'$, $i < \alpha$, then $f_i(a_1, \dots, a_n, b) = f_i(a_1, \dots, a_n, b')$.

By (*) above, given $C' \subseteq A_\xi$, $|C'| = \lambda^*_\xi$, it suffices to find $B^*_\xi \subseteq C'$, $|B^*_\xi| = \kappa(\xi)$ satisfying (iii). But by the assumption of (E), $\chi^{<\kappa(\xi)} < \lambda^*_\xi$, so by the known facts on existence end-homogeneous sets, it exists (we may have $\aleph_0$ functions to consider so if $\kappa(\xi) > \aleph_0$ we can replace χ by $\chi^{\aleph_0}$, and if $\kappa(\xi) = \aleph_0$, in choosing the nth elements of B^*_ξ we have to consider only functions with $\le n$ places, which are finite).

If case (F) is represented, we should replace (ii) by

(ii)′ if $b, b', a_1, \dots, a_n \in B^*_\xi$, $\eta < \theta$, $a_1 < a_2 < \cdots < a_n$, $a_n < b$, $a_n < b'$, $i < \alpha$ then $f_i(a_1, \dots, a_n, b, a^*_\eta) = f_i(a_1, \dots, a_n, b', a^*_\eta)$.

The proof is as before.

We leave cases (B), (C) to the reader.

We can have exact conclusions from 5 and 6 just by induction, but we state just two conclusions necessary in the applications.

Conclusion 7. (1) If $\beth_{r-1}(\kappa(\xi))$ is strictly increasing for $\xi < \theta$, and $|\theta| \le \kappa(0)$, *then* $\langle \beth_{r-1}(\kappa(\zeta))^+: \xi < \theta\rangle$ has a $\langle \kappa(\xi)^+: \xi < \theta\rangle$-canonical form for $\Gamma_r = \{\bar{r}^{r-1}_\chi: \chi = 2^{\kappa(0)}, \text{any } \bar{r}\} \cup \{\bar{r}^r_{\kappa(0)}: n(\bar{r}) = r\}$.

(2) In (1) if $\chi^{|\theta|} \le \kappa(0)$, we can add to Γ_r for $r = 2$, $\{\bar{r}\char`\^\langle 1\rangle^2_\chi: \text{any } \bar{r}\}$. For bigger r's use Claim 5. (We can replace $\kappa(\xi)^+$ by $\kappa(\xi)$ with appropriate changes.)

Proof. Easy by Lemma 6; (1) is proved using 5. More exactly, we should collect by induction on r all "good" $\bar{r}^l_{\kappa(0)}$, $\bar{r}^l_{2^{\kappa(0)}}$. The point is that if we look for the history needed for $\langle r\rangle^r_{\kappa(0)}$ the number of colors do not increase. For other $\bar{r}_{\kappa(0)}$ at least for one "point" by Lemma 6(C) we gain one, so the increase to $\kappa(0)^{|\theta|} \le 2^{\kappa(\xi)}$ in the number of colors is not important. Notice we can add to Γ_r even $\bar{r}\char`\^(2;2;\dots;2;1)^{2r-2}_{2^{\kappa(0)}}$ ($r - 2$ two's).[2]

Conclusion 8. (1) $\langle((2^\kappa)^{++})_\kappa\rangle$ has a $\langle(\kappa)_\kappa\rangle$-canonical form for $\{\bar{r}\char`\^\langle 1\rangle^2_{2^\kappa}: \text{any } \bar{r}\}$ when $\langle(\lambda)_\mu\rangle$ means $\langle\lambda, \lambda, \dots\rangle_{i<\mu}$.

[2]For more detailed proof see *Added in proof.*

(2) Suppose $\kappa(l)$, r_l $(l \leq n)$ are such that

(A) at least one of the following holds:

(α) $\chi \leq 2^\kappa$, $|\theta| \leq \kappa$, $r_1 = 2$, $\kappa(1) = (2^\kappa)^{++}$,

(β) $\chi^{|\theta|} \leq \kappa$, $r_1 = 2$, $\kappa(1) = (2^\kappa)^+$,

(γ) $\chi^{|\theta|} < \text{cf}\ \kappa$, $r_1 = 1$, $\kappa(1) = \kappa$;

(B) for each $l \geq 1$ at least one of the following holds:

(a) $r_{l+1} = r_l + 2$, $\kappa(l+1) = (2^{\kappa(l)})^{++}$,

(b) $r_{l+1} = r_l + 1$, $\kappa(l+1) = (2^{<\kappa(l)})^+$.

Then $\langle(\kappa(n))_\theta\rangle$ has $\langle(\kappa)_\theta\rangle$-canonical form for $\langle 1; (1)_{r_n}\rangle_\chi^{r_n}$.

(3) In (2) we can add $\{\bar{r}^\frown((1)_{r_n-2})_\chi^{r_n-1} : \text{any } \bar{r}\}$ even if in (2) we change r_1 to (3) but then omit the first conclusion.

(4) For $n = 1$ in $\beta[\gamma]$, $\chi < \kappa[\text{cf}\ \chi < \text{cf}\ \kappa]$ suffice.

PROOF. Trivial by Lemma 6. Note that for β, if $\chi^{|\theta|} \leq \kappa$, $\kappa(1) = (2^\kappa)^+$, F a 2-place function on $\bigcup_{\xi<\theta} A_\xi$, ... let $A_\xi = \{a_\xi^\alpha : \alpha < (2^\kappa)^+\}$ and define F^* on $[(2^\kappa)^+]^2$:

$$F^*(\alpha, \beta) = \{(\xi, \zeta, c): F(a_\xi^\alpha, a_\zeta^\beta) = c;\ \xi, \zeta < \theta\}.$$

Let $\{\alpha_i : i \leq \kappa^+\}$ be such that $F^*(\alpha_i, \alpha_j) = F^*(\alpha_i, \alpha_{\kappa^+})$ for $i < j < \kappa^+$ (exists as $(2^\kappa)^+ \to \langle\kappa^+\rangle_\chi$). Now we have to replace κ^+ by κ for fixing the orders; so let $B_\xi = \{a_\xi^{\alpha_i} : \kappa(\xi) \leq i < \kappa(\xi + 1)\}$. For $l + 1$, case (b), we need not do this, for later we shall fix the order by the coloring of pairs. This explains the difference between (β) and (b).

Conclusion 9. Suppose $\lambda_0 < \cdots < \lambda_{\theta-1}$, $\theta < \omega$, $\kappa(l) < \lambda_l$, $\prod_{m<l} \lambda_m^{\kappa(m)} < \lambda_l$. Then $\langle \lambda_l : l < \theta\rangle$ has $\langle\kappa(l) : l < \theta\rangle$-canonization form for $\{\bar{r}^\frown\langle(1)_i\rangle_\chi^i : \text{any } i, \bar{r}\}$ and if $(\chi^{1+\sum_{m<l}\kappa(m)})^{<\kappa(l)} < \lambda_l$ then also for $\{\bar{r}^\frown\langle(1)_i\rangle_\chi^{i+1} : \text{any } i, \bar{r}\}$.

PROOF. Easy by Lemma 3.

Conclusion 10. When $2^\chi \leq 2^\kappa$

$$\begin{pmatrix} (2^\kappa)^+ \\ (2^\kappa)^{++} \\ (2^\kappa)^{+3} \\ \vdots \\ (2^\kappa)^{+n} \end{pmatrix} \longrightarrow \begin{pmatrix} \kappa \\ \kappa \\ \vdots \\ \kappa \\ (2^\kappa)^{+n} \end{pmatrix} 1, 1, 1, 1, \ldots, 1$$

PROOF. By (9).

Applications.

Conclusion 11. Suppose X is a Hausdorff space $(\forall\lambda < \kappa) 2^{2^\lambda} < |X|$, κ singular (or a successor) then X has a discrete subspace of card κ.

PROOF. The new case is κ singular, 2^{2^λ} $(\lambda < \kappa)$ not eventually constant. Choose $\lambda_\alpha = \lambda(\alpha) < \kappa$ such that $2^{2^{\lambda(\alpha)}}$ $(\alpha < \text{cf}\ \kappa)$ is strictly increasing, $\kappa = \sum_\alpha \lambda_\alpha$, $\text{cf}\ \kappa < \lambda(\alpha)$. Choose pairwise disjoint $A_\alpha \subseteq X$, $|A_\alpha| = (2^{2^{\lambda(\alpha)}})^+$. For each $x, y \in X$, $U_{x,y}$, $U_{y,x}$ will be disjoint neighborhoods of x and y. Let $f(x_1x_2x_3) = \{\sigma : \sigma \text{ a permutation of } \{1, 2, 3\},\ x_{\sigma(1)} \in U_{x_{\sigma(2)}, x_{\sigma(3)}}\}$. The number of colors is finite so as $\langle(2^{2^{\lambda(\alpha)}})^+ : \alpha < \text{cf}\ \kappa\rangle$ has $\langle\lambda_\alpha^+ : \alpha < \text{cf}\ \kappa\rangle$ canonical form for $\{\langle 2;1\rangle_{\aleph_0}^3, \langle 1;\ 2\rangle_{\aleph_0}^3, \langle 3\rangle_{\aleph_0}^3\}$ [by 7] let $B_\alpha \subseteq A_\alpha$, $|B_\alpha| = \lambda_\alpha^+$ exemplify it, $B_\alpha = \{b_\alpha^i : i < \lambda_\alpha^+\}$. So $B^* = \{b_\alpha^i : i < \lambda_\alpha^+, i \text{ odd}, \alpha < \text{cf}\ \kappa\}$ is a subspace of card κ and $U_{b_\alpha^i, b_\alpha^{i-1}} \cap U_{b_\alpha^i, b_\alpha^{i+1}}$ show b_α^i is isolated in B^*.

DEFINITION 12. (1) $(\kappa, r, \lambda) \to \mu$ if for any r-place function f, from κ to $[\kappa]^{<\lambda}$, there is a free set $B \subseteq \kappa$, $|B| = \mu$.

(2) $(\kappa, r) \to \mu$ if when f is an r-place function from κ to $[\kappa]^{\leq \mu}$, $|A_\xi| = \kappa$. $\kappa = \bigcup_{\xi<\mu} A_\xi$ (A_ξ pairwise disjoint) then there are $a_\xi \in A_\xi$ such that $\{a_\xi : \xi < \mu\}$ is free (for f).

LEMMA 13. *(1) $(\kappa, r) \to \mu$ implies $(\kappa, r, \mu^+) \to \mu$.*

(2) $(\mu^{++}, 1) \to \mu$.

(3) If $\langle(\kappa)_\mu\rangle$ has $\langle(\mu^+)_\mu\rangle$-canonical form for $((1)_{r+1})_{r-2}^{r-1}$ then $(\kappa, r) \to \mu$.

PROOF. Easy; (1) directly; (2) letting $A_\xi = \{a^i_\xi : i < \mu^{++}\}$, $g(\xi) = \bigcup\{\zeta : (\exists\, i, j) a^i_\zeta \in f(a^j_\xi)\}$ applying the well-known theorem of Hajnal [H] on μ^{++}; (3) directly.[3]

Now by 8(3) $\Rightarrow$ 13(3) $\Rightarrow$ 13(1) we get result on $(\kappa, r, \mu^+) \to \mu$ where we get κ from μ by $\approx r/2$ exponentiations and $2r$ successor operations (if in 8(3) we use case (a) only). In the book [EHMR] $r - 1$ exponentiations are needed. If sometimes G.C.H. holds (b) of 8.(2)(B) helps. For $r = 2$ the book gets a better result but otherwise, when not always G.C.H. holds, we seem to gain. Also the theorem on $\exp_r \kappa_\xi$ strictly increasing $(\sum_\xi \exp_r \kappa(\xi), r, \lambda) \to \sum_\xi \kappa_\xi$ can be strictly improved.

Unfortunately it is not clear whether this result is near the best possible (provable from ZFC). This is connected to:

Question. Suppose $2^\theta < \kappa_0$, $2^{\kappa(i)} > (2^{\sum_{j<i}\kappa(j)})^{+\alpha}$.

Does $\langle(2^{\kappa(i)})^{+\alpha}; i < \theta\rangle$ have a $\langle\kappa(i) : i < \theta\rangle$-canonical form for $\langle 1, 1, 1, 1, 1, \rangle^1_{\aleph_0}$?

Added in proof

PROOF OF CONCLUSION 7. For a sequence $\bar{r}$ define by induction on i a decreasing sequence $k(i, \bar{r}) \leq n(\bar{r})$: $k(0, \bar{r}) = n(\bar{r})$. If $k(i, \bar{r})$ is defined then there are two possibilities. If $1 + \sum_{l=1}^{p} r_l < k(i, \bar{r}) \leq \sum_{l=1}^{p+1} r_l$ for some p then define $k(i + 1, \bar{r}) = k(i, \bar{r}) - 1$; otherwise, if $k(i, \bar{r}) = 1 + \sum_{l=1}^{p} r_l$ then $k(i + 1, \bar{r}) = k(i, \bar{r}) - 2$ ($k(i, \bar{r})$ measures the progress in canonicity after i exponents). Now prove by induction on r that $\langle \beth_{r-1}(\kappa(\xi))^+ : \xi < \theta\rangle$ has $\langle\kappa(\xi)^+ : \xi < \theta\rangle$ canonical form for $\Gamma_r^{\kappa(0)}$ when

$$\Gamma_r^{\kappa(0)} = \Gamma_{r,1}^{\kappa(0)} \cup \Gamma_{r,2}^{\kappa(0)} \cup \Gamma_{r,3}^{\kappa(0)} \cup \Gamma_{r,4}^{\kappa(0)}.$$

The definitions are

$\Gamma^\kappa_{r,1} = \{\bar{r}^m_\kappa : m \leq m\{r, r_{k(\bar{r})}\}\}$,

$\Gamma^\kappa_{r,2} = \{\bar{r}^m_\chi : \chi^{|\theta|} = \chi \leq k, m = n(\bar{r}) - k(r, \bar{r})\}$,

$\Gamma^\kappa_{r,3} = \{\bar{r}^m_\chi : \chi = 2^\kappa, m = n(\bar{r}) - (k(r, \bar{r}) + 1)\}$,

$\Gamma^\kappa_{r,4} = \{\bar{r}^m_\chi : \bar{r}^{m-1}_\chi \in \Gamma^\kappa_{r,1} \cup \Gamma^\kappa_{r,2} \cup \Gamma^\kappa_{r,3}$ or $m = 1, m = n(\bar{r})\}$.

For $r = 1$, easy.

For $r + 1$ first apply the induction hypothesis for: $\langle \beth_{n-1}(2^{\kappa(\xi)})^+ : \xi < \theta\rangle$ has $\langle(2^{\kappa(\xi)})^+ : \xi < \theta\rangle$-canonical form for $\Gamma_r^{2^{\kappa(0)}}$; we know that $\langle(2^{\kappa(\xi)})^+ : \xi < \theta\rangle$ has $\langle\kappa(\xi)^+ : \xi < \theta\rangle$-canonical form for $\Gamma_1^{\kappa(0)}$.

Now we want to add the two facts together using the Composition Claim 5. Check according to the set $\bar{r}^m_\chi$ belongs:

(A) $\bar{r}^m_{\kappa(0)} \in \Gamma^{\kappa(0)}_{r+1,1}$, $\bar{r}^{m-1}_{\kappa(0)} \in \Gamma^{2^{\kappa(0)}}_{r,1}$ $\langle r_1; \cdots r_{n-1}, r_n - m + 1\rangle^1_{\kappa(0)} \in \Gamma^{\kappa(0)}_{1,1}$.

(B) If $\bar{r}^m_\chi \in \Gamma^{\kappa(0)}_{r+1,2}$ then according to the cases in $k(m, \bar{r})$ definition.

(C) $\bar{r}^m_\chi \in \Gamma^{\kappa(0)}_{r+1,3}$ then $\bar{r}^m_\chi \in \Gamma^{2^{\kappa(0)}}_{r,2}$.

(D) $\bar{r}^m_\chi \in \Gamma^{\kappa(0)}_{r+1,4}$ not important.

[3] For more detailed proof see *Added in proof.*

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Note that in case (C) is the main idea, $\Gamma^{\kappa}_{r,3}$ is not autonomous.

REMARK. When $\langle \beth_{r-1}(\kappa(\xi)) : \xi < \theta \rangle$ is not increasing then our notation becomes complicated therefore we do not phrase the general case, but Conclusion 8 is the most interesting case.

PROOF OF LEMMA 13(3). Let f be a given r-place function and we shall find a free set B for it.

Divide κ into μ subsets pairwise disjoint each of cardinality κ; denote them by A_i, $i < \mu$ (remember $\mu \leq \kappa$). Define a function G, G: $[\kappa]^{r+1} \to (r+1)$ such that $G(a_1, \ldots, a_{r+1})$ is the minimal l, $1 \leq l \leq r+1$, $a_l \in f(a_1, \ldots, a_{l-1}, a_{l+1}, \ldots, a_{r+1})$ and 0 (zero) if there is no such.

By the canonicity assumption for G and $\langle A_i : i < \mu \rangle$ there are $B_i \subseteq A_i$, $|B_i| = \mu^+$ which exemplify our assumption; choose $a_i \in B_i$ and define $B = \{a_i : i < \kappa\}$. Now we shall prove that B is a free set: $B \subseteq \lambda$; clear because $B \subseteq \bigcup_i A_i \subseteq \lambda$. $|B| = \kappa$; because $a_i \in A_i$, and the A_i are pairwise disjoint, so $a_i = a_j$ for $i \neq j$.

It suffices to prove $G(a_{i_1}, \cdots, a_{i_r}) = 0$ for $i_1 < \cdots i_{r+1} < \kappa$. If it is l, $1 \leq l \leq r+1$, then by the choice of $\langle B_j : j \leq \mu \rangle$ for every $a' \in B_{i_l}$, $G(a_{i_l}, \ldots, a_{i_{l-1}}, a', a_{i_{l+1}}, \ldots, a_{r+1}) = l$, so $a' \in f(a_{i_1}, \ldots, a_{i_{l-1}}, a_{i_{l+1}}, \ldots, a_{i_{r+1}})$ hence $B_{i_l} \subseteq f(a_{i_1}, \ldots, a_{i_{l-1}}, a_{i_{l+1}}, \ldots, a_{i_{r+1}})$ a contradiction to $|B_{i_l}| = \mu^+$, $|f(\cdots)| \leq \mu$.

REMARK. If $|f(\cdots)| \leq 1$, $\langle (2)_\mu \rangle$-canonical form suffice, and the number of colors can be reduced to $r+1$ but there is no real difference.

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