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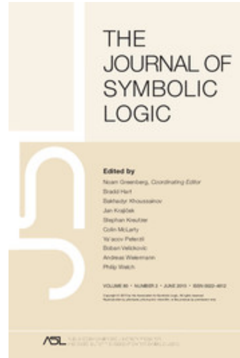

Constructing strongly equivalent nonisomorphic models for unsuperstable theories. Part B

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CONSTRUCTING STRONGLY EQUIVALENT NONISOMORPHIC MODELS FOR UNSUPERSTABLE THEORIES. PART B

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Abstract. We study how equivalent nonisomorphic models of unsuperstable theories can be. We measure the equivalence by Ehrenfeucht-Fraïssé games. This paper continues [HS].

§**1. Introduction.** In [HT] we started the studies of so-called strong nonstructure theorems. By a *strong nonstructure theorem* we mean a theorem which says that if a theory belongs to some class of theories then it has very equivalent nonisomorphic models. Usually the equivalence is measured by the length of the Ehrenfeucht-Fraïssé games (see Definition 2.2) in which $\exists$ has a winning strategy. These theorems are called nonstructure theorems because intuitively the models must be complicated if they are very equivalent but still nonisomorphic. Also structure theorems usually imply that a certain degree of equivalence gives isomorphism (see, for example, [Sh1] (Chapter XIII)).

In [HT] we studied mainly unstable theories. We also looked at unsuperstable theories, but we were not able to say much if the equivalence is measured by the length of the Ehrenfeucht-Fraïssé games in which $\exists$ has a winning strategy. In this paper we make a new attempt to study the unsuperstable case.

The main result of this paper is the following: if $\lambda = \mu^+$, $\mathrm{cf}(\mu) = \mu$, $\kappa = \mathrm{cf}(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$, $\mu^\kappa = \mu$ and T is an unsuperstable theory, $|T| \leq \lambda$ and $\kappa(T) > \kappa$, then there are models $\mathscr{A}, \mathscr{B} \models T$ of cardinality λ such that

$$\mathscr{A} \equiv^{\lambda}_{\mu\times\kappa} \mathscr{B} \quad \text{and} \quad \mathscr{A} \not\cong \mathscr{B}.$$

In [HS] we proved this theorem in a special case.

From Theorem 4.4 in [HS] we get the following theorem easily: Let T_c be the canonical example of unsuperstable theories i.e. $T_c = \mathrm{Th}(({}^{\omega}\omega, E_i)_{i<\omega})$, where $\eta E_i \xi$ iff for all $j \leq i$, $\eta(j) = \xi(j)$.

1.1. THEOREM ([HS]). *Let $\lambda = \mu^+$, and let I_0 and I_1 be models of T_c of cardinality λ. Assume $\lambda \in I[\lambda]$. Then*

$$I_0 \equiv^{\lambda}_{\mu\times\omega+2} I_1 \quad \Leftrightarrow \quad I_0 \cong I_1.$$

So the main result of §3 is essentially the best possible.

In the Introduction of [HT] there is more background for strong nonstructure theorems.

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§**2. Basic definitions.** In this section we define the basic concepts we shall use and construct two linear orders needed in §3.

2.1. DEFINITION. Let λ be a cardinal and α an ordinal. Let t be a tree (i.e. for all $x \in t$, the set $\{y \in t | \ y < x\}$ is well-ordered by the ordering of t). If $x, y \in t$ and $\{z \in t \mid z < x\} = \{z \in t \mid z < y\}$, then we write $x \sim y$, and the equivalence class of x for $\sim$ we denote by $[x]$. By a λ, α-tree t we mean a tree which satisfies:

(i) $|[x]| < \lambda$ for every $x \in t$;

(ii) there are no branches of length $\geq \alpha$ in t;

(iii) t has a unique root;

(iv) if $x, y \in t$, x and y have no immediate predecessors and $x \sim y$, then $x = y$.

Note that in a λ, α-tree each ascending sequence of a limit length has at most one supremum.

2.2. DEFINITION. Let t be a tree and κ a cardinal. The *Ehrenfeucht-Fraïssé game* of length t between models $\mathscr{A}$ and $\mathscr{B}$, $G_t^\kappa(\mathscr{A}, \mathscr{B})$, is the following. At each move α:

(i) player $\forall$ chooses $x_\alpha \in t$, $\kappa_\alpha < \kappa$ and either $a_\alpha^\beta \in \mathscr{A}$, $\beta < \kappa_\alpha$, or $b_\alpha^\beta \in \mathscr{B}$, $\beta < \kappa_\alpha$—we will denote this sequence by X_α;

(ii) if $\forall$ chose from $\mathscr{A}$ then $\exists$ chooses $b_\alpha^\beta \in \mathscr{B}$, $\beta < \kappa_\alpha$, else $\exists$ chooses $a_\alpha^\beta \in \mathscr{A}$, $\beta < \kappa_\alpha$—we will denote this sequence by Y_α.

$\forall$ must move so that $(x_\beta)_{\beta \leq \alpha}$ form a strictly increasing sequence in t. $\exists$ must move so that $\{(a_\gamma^\beta, b_\gamma^\beta) | \gamma \leq \alpha, \beta < \kappa_\gamma\}$ is a partial isomorphism from $\mathscr{A}$ to $\mathscr{B}$. The player who first has to break the rules loses.

We write $\mathscr{A} \equiv_t^\kappa \mathscr{B}$ if $\exists$ has a winning strategy for $G_t^\kappa(\mathscr{A}, \mathscr{B})$.

2.3. DEFINITION. Let t and t' be trees.

(i) If $x \in t$, then $\mathrm{pred}(x)$ denotes the sequence $(x_\alpha)_{\alpha < \beta}$ of the predecessors of x, excluding x itself, ordered by $<$. Alternatively, we consider $\mathrm{pred}(x)$ as a set. The notation $\mathrm{succ}(x)$ denotes the set of immediate successors of x. If $x, y \in t$ and there is z such that $x, y \in \mathrm{succ}(z)$, then we say that x and y are *brothers*.

(ii) By $t^{<\alpha}$ we mean the set

$$\{x \in t | \text{ the order type of } \mathrm{pred}(x) \text{ is } < \alpha\}.$$

Similarly we define $t^{\leq \alpha}$.

(iii) The sum $t \oplus t'$ is defined as the disjoint union of t and t', except that the roots are identified.

2.4. DEFINITION. Let ρ_i, $i < \alpha$, ρ and θ be linear orders.

(i) We define the ordering $\rho \times \theta$ as follows: the domain of $\rho \times \theta$ is $\{(x, y) | \ x \in \rho,\ y \in \theta\}$, and the ordering in $\rho \times \theta$ is defined by last differences, i.e., each point in θ is replaced by a copy of ρ.

(ii) We define the ordering $\rho + \theta$ as follows: The domain of $\rho + \theta$ is $(\{0\} \times \rho) \cup (\{1\} \times \theta)$ and the ordering in $\rho + \theta$ is defined by the first difference, i.e., $(i, x) < (j, y)$ iff $i < j$ or $i = j$ and $x < y$.

(iii) We define the ordering $\sum_{i<\alpha} \rho_i$ as follows: The domain of $\sum_{i<\alpha} \rho_i$ is $\{(i, x) | \ i \in \alpha, x \in \rho_i\}$ and the ordering in $\sum_{i<\alpha} \rho_i$ is defined by the first difference, i.e., $(i, x) < (j, y)$ iff $i < j$ or $i = j$ and $x < y$.

2.5. DEFINITION. We define *generalized Ehrenfeucht-Mostowski models* (E-M-models for short). Let K be a class of models which we call index models. In this definition the notation $\mathrm{tp}_{\mathrm{at}}(\overline{x}, A, \mathscr{A})$ means the atomic type of $\overline{x}$ over A in the model $\mathscr{A}$.

Let Φ be a function. We say that Φ is *proper for* K if there is a vocabulary τ_1 and for each $I \in K$ a model $\mathbf{M}_\mathrm{I}$ and tuples $\overline{a}_s$, $s \in I$, of elements of $\mathbf{M}_\mathrm{I}$, such that:

(i) each element in $\mathbf{M}_\mathrm{I}$ is an interpretation of some $\mu(\overline{a}_{\overline{s}})$, where μ is a τ_1-term;

(ii) $\mathrm{tp}_{\mathrm{at}}(\overline{a}_{\overline{s}}, , \mathbf{M}_\mathrm{I}) = \Phi(\mathrm{tp}_{\mathrm{at}}(\overline{s}, \emptyset, I))$.

Here $\overline{s} = (s_0, \ldots, s_n)$ denotes a tuple of elements of I and $\overline{a}_{\overline{s}}$ denotes $\overline{a}_{s_0} \frown \cdots \frown \overline{a}_{s_n}$.

Note that if $\mathbf{M}_\mathrm{I}$, $\overline{a}_s$, $s \in I$, and $\mathbf{M}'_\mathrm{I}$, $\overline{a}'_s$, $s \in I$, satisfy the above conditions, then there is a canonical isomorphism $\mathbf{M}_\mathrm{I} \cong \mathbf{M}'_\mathrm{I}$ which takes $\mu(\overline{a}_{\overline{s}})$ in $\mathbf{M}_\mathrm{I}$ to $\mu(\overline{a}'_{\overline{s}})$ in $\mathbf{M}'_\mathrm{I}$. Therefore we may assume below that $\mathbf{M}_\mathrm{I}$ and $\overline{a}_s$, $s \in I$, are unique for each I. We denote this unique $\mathbf{M}_\mathrm{I}$ by $\mathrm{EM}^1(I, \Phi)$ and call it an Ehrenfeucht-Mostowski model. The tuples $\overline{a}_s$, $s \in I$, are the generating elements of $\mathrm{EM}^1(I, \Phi)$, and the indexed set $(\overline{a}_s)_{s \in I}$ is the skeleton of $\mathrm{EM}^1(I, \Phi)$.

Note that if

$$\mathrm{tp}_{\mathrm{at}}(\overline{s}_1, \emptyset, I) = \mathrm{tp}_{\mathrm{at}}(\overline{s}_2, \emptyset, J),$$

then

$$\mathrm{tp}_{\mathrm{at}}(\overline{a}_{\overline{s}_1}, \emptyset, \mathrm{EM}^1(I, \Phi)) = \mathrm{tp}_{\mathrm{at}}(\overline{a}_{\overline{s}_2}, \emptyset, \mathrm{EM}^1(J, \Phi)).$$

2.6. Definition. Let θ be a linear order and κ an infinite regular cardinal. Let $K^\kappa_{\mathrm{tr}}(\theta)$ be the class of models of the form

$$I = (M, <, \ll, H, P_\alpha)_{\alpha \leq \kappa},$$

where $M \subseteq \theta^{\leq \kappa}$ and:

(i) M is closed under initial segments;

(ii) $<$ denotes the initial segment relation;

(iii) $H(\eta, \nu)$ is the maximal common initial segment of η and ν;

(iv) $P_\alpha = \{\eta \in M \mid \mathit{length}(\eta) = \alpha\}$;

(v) $\eta \ll \nu$ iff either $\eta < \nu$ or there is $n < \kappa$ such that $\eta(n) < \nu(n)$ and $\eta \restriction n = \nu \restriction n$.

Let $K^\kappa_{\mathrm{tr}} = \bigcup\{K^\kappa_{\mathrm{tr}}(\theta) \mid \theta \text{ a linear order }\}$.

If $I \in K^\kappa_{\mathrm{tr}}(\theta)$ and $\eta, \nu \in I$, we define $\eta <_s \nu$ iff η and ν are brothers and $\eta < \nu$. But we do not add $<_s$ to the vocabulary of I.

Thus the models in K^κ_{tr} are lexically ordered trees of height $\kappa + 1$ from which we have removed the relation $<_s$ and where we have added relations indicating the levels and a function giving the maximal common predecessor.

The following theorem gives us means to construct for T E-M-models such that the models of K^κ_{tr} act as index models. Furthermore the properties of the models of K^κ_{tr} are reflected to these E-M-models.

2.7. Theorem ([Sh1]). *Suppose $\tau \subseteq \tau_1$, T is a complete τ-theory, T_1 is a complete τ_1-theory with Skolem functions and $T \subseteq T_1$. Suppose further that T is unsuperstable, $\kappa(T) > \kappa$ and $\phi_n(\overline{x}, \overline{y}_n)$, $n < \kappa$, witness this. (The definition of witnessing is not needed in this paper. See* [Sh1].)

Then there is a function Φ, which is proper for K^κ_{tr}, such that for every $I \in K^\kappa_{\mathrm{tr}}$, $\mathrm{EM}^1(I, \Phi)$ is a τ_1-model of T_1, for all $\eta \in I$, $\overline{a}_\eta$ is finite and for $\eta, \xi \in P^I_n$, $\nu \in P^I_\kappa$,

(i) *if $I \models \eta < \nu$, then* $\mathrm{EM}^1(I, \Phi) \models \phi_n(\overline{a}_\nu, \overline{a}_\eta)$;

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(ii) *if η and ξ are brothers and $\eta < \nu$ then $\xi = \eta$ iff* $\mathrm{EM}^1(I,\Phi) \models \phi_n(\overline{a}_\xi, \overline{a}_\nu)$. □

Above $\phi_n(\overline{x}, \overline{y}_n)$ is a first-order τ-formula. We denote the reduct

$$\mathrm{EM}^1(I,\Phi) \restriction \tau$$

by $\mathrm{EM}(I,\Phi)$. In order to simplify the notation, instead of $\overline{a}_\eta$, we just write η. It will be clear from the context whether η means $\overline{a}_\eta$ or η.

Next we construct two linear orders needed in the next section. The first of these constructions is a modification of a linear order construction in [Hu] (Chapter 9).

2.8. Definition. Let γ be an ordinal closed under ordinal addition and let $\theta_\gamma = ({}^{<\omega}\gamma, <)$, where $<$ is defined by $x < y$ iff

(i) y is an initial segment of x

or

(ii) there is $n < \min\{\mathrm{length}(x), \mathrm{length}(y)\}$ such that $x \restriction n = y \restriction n$ and $x(n) < y(n)$.

2.9. Lemma. *Assume γ is an ordinal closed under ordinal addition. Let $x \in \theta_\gamma$, $\mathrm{length}(x) = n < \omega$ and $\alpha < \gamma$. Let A^α_x be the set of all elements y of θ_γ which satisfy*:

(i) *x is an initial segment of y (not necessarily proper)*;

(ii) *if $\mathrm{length}(y) > n$ then $y(n) \geq \alpha$.*

Then $(A^\alpha_x, < \restriction A^\alpha_x) \cong \theta_\gamma$.

Proof. Follows immediately from the definition of θ_γ. □

If $\alpha \leq \beta$ are ordinals then by $(\alpha, \beta]$ we mean the unique ordinal order isomorphic to

$$\{\delta \mid \alpha < \delta \leq \beta\} \cup \{\delta \mid \delta = \alpha \text{ and limit}\}$$

together with the natural ordering. Notice that if $(\alpha_i)_{i<\delta}$ is strictly increasing continuous sequence of ordinals, $\alpha_0 = 0$, $\beta = \sup_{i<\delta} \alpha_i$ and for all successor $i < \delta$, α_i is successor, then $\sum_{i<\delta}(\theta \times (\alpha_i, \alpha_{i+1}]) \cong \theta \times \beta$, for all linear orderings θ.

2.10. Lemma. *Let γ be an ordinal closed under ordinal addition and not a cardinal.*

(i) *Let $\alpha < \gamma$ be an ordinal. Then $\theta_\gamma \cong \theta_\gamma \times (\alpha + 1)$.*

(ii) *Let $\alpha < \beta < |\gamma|^+$. Then $\theta_\gamma \cong \theta_\gamma \times (\alpha, \beta]$.*

Proof. (i) For all $i < \alpha$ we let $x_i = (i)$. Then by the definition of θ_γ,

$$\theta_\gamma \cong \left(\sum_{i<\alpha} A^0_{x_i}\right) + A^\alpha_{()},$$

where by () we mean the empty sequence. By Lemma 2.9

$$\left(\sum_{i<\alpha} A^0_{x_i}\right) + A^\alpha_{()} \cong \theta_\gamma \times (\alpha + 1).$$

(ii) We prove this by induction on β. For $\beta = 1$ the claim follows from (i). Assume we have proved the claim for $\beta < \beta'$; we prove it for β'. If $\beta' = \delta + 1$, then by induction assumption $\theta_\gamma \cong \theta_\gamma \times (\alpha, \delta]$, and so

$$\theta_\gamma \times (\alpha, \delta + 1] \cong \theta_\gamma + \theta_\gamma \cong \theta_\gamma$$

by (i).

If β' is limit, then we choose a strictly increasing continuous sequence of ordinals $(\beta_i)_{i<\mathrm{cf}(\beta')}$ so that $\beta_0 = \alpha$, $\sup_{i<\mathrm{cf}(\beta')} \beta_i = \beta'$ and, for all successor $i < \mathrm{cf}(\beta')$, β_i is successor. Then

$$\theta_\gamma \times (\alpha, \beta'] \cong \sum_{i<\mathrm{cf}(\beta')} (\theta_\gamma \times (\beta_i, \beta_{i+1}]) + \theta_\gamma.$$

By the induction assumption

$$\sum_{i<\mathrm{cf}(\beta')} (\theta_\gamma \times (\beta_i, \beta_{i+1}]) + \theta_\gamma \cong \theta_\gamma \times (\mathrm{cf}(\beta') + 1).$$

Because γ is not a cardinal, $\mathrm{cf}(\beta') < \gamma$, and so by (i)

$$\theta_\gamma \times (\mathrm{cf}(\beta') + 1) \cong \theta_\gamma. \qquad \square$$

2.11. Corollary. *Let γ be an ordinal closed under ordinal addition and not a cardinal. If $\alpha < |\gamma|^+$ is a successor ordinal then, $\theta_\gamma \cong \theta_\gamma \times \alpha$.*

Proof. Follows immediately from Lemma 2.10 (ii). $\square$

2.12. Lemma. *Assume μ is a regular cardinal and $\lambda = \mu^+$. Then there are a linear order θ of power λ, a one-to-one and onto function $h : \theta \to \lambda \times \theta$, and order isomorphisms $g_\alpha : \theta \to \theta$ for $\alpha < \lambda$ such that the following hold:*

(i) *If $g_\alpha(x) = y$ then $x \neq y$ and either* (a) *$h(x) = (\alpha, y)$ or* (b) *$h(y) = (\alpha, x)$, but not both.*

(ii) *If $g_\alpha(x) = g_{\alpha'}(x)$ for some $x \in \theta$, then $\alpha = \alpha'$.*

(iii) *If $h(x) = (\alpha, y)$, then $g_\alpha(x) = y$ or $g_\alpha(y) = x$.*

Proof. Let the universe of θ be $\mu \times \lambda$. The ordering will be defined by induction. Let $f : \lambda \to \lambda \times \lambda$ be one-to-one and onto, and if $\alpha < \alpha'$, $f(\alpha) = (\beta, \gamma)$ and $f(\alpha') = (\beta', \gamma')$, then $\gamma < \gamma'$. This f is used only to guarantee that in the induction we pay attention to every $\beta < \lambda$ cofinally often.

By induction on $\alpha < \lambda$ we do the following: Let $f(\alpha) = (\beta, \gamma)$. We define $\theta^\alpha = (\mu \times (\alpha + 1), <^\alpha)$, $h^\alpha : \theta^\alpha \to \lambda \times \theta^\alpha$ and order isomorphisms (in the ordering $<^\alpha$) $g^\alpha_\beta : \theta^\alpha \to \theta^\alpha$ so that

(i) if $\alpha < \alpha'$, then $h^\alpha \subseteq h^{\alpha'}$ and $<^\alpha \subseteq <^{\alpha'}$,

(ii) if $\alpha < \alpha'$, $f(\alpha) = (\beta, \gamma)$ and $f(\alpha') = (\beta, \gamma')$, then $g^\alpha_\beta \subseteq g^{\alpha'}_\beta$,

(iii) if $g^\alpha_\beta(x) = y$, then $x \neq y$ and either (a) $h^\alpha(x) = (\beta, y)$ or (b) $h^\alpha(y) = (\beta, x)$, but not both.

The induction is easy since at each stage we have μ "new" elements to use: Let $B \subseteq \mu \times \alpha$ be the set of those element from $\mu \times \alpha$ which are not in the domain of any $g^{\alpha'}_\beta$ such that $\alpha' < \alpha$ and $f(\alpha') = (\beta, \gamma')$ for some γ'. (Notice that B is also the set of those element from $\mu \times \alpha$ which are not in the range of any $g^{\alpha'}_\beta$ such that $\alpha' < \alpha$ and $f(\alpha') = (\beta, \gamma')$ for some γ'.) Clearly if $B \neq \emptyset$, then $|B| = \mu$.

Let A_i, $i \in \mathbf{Z}$, be a partition of $\mu \times \{\alpha\}$ into sets of power μ. We first define g^α_β so that the following are true:

(a) g^α_β is one-to-one,

(b) if $B \neq \emptyset$, then $g^\alpha_\beta \restriction A_0$ is onto B; otherwise $g^\alpha_\beta \restriction A_0$ is onto A_{-1},

(c) if $B \neq \emptyset$, then $g^\alpha_\beta \restriction B$ is onto A_{-1},

(d) for all $i \neq 0$, $g^\alpha_\beta \restriction A_i$ is onto A_{i-1}.

By an easy induction on $|i| < \omega$ we can define $<^\alpha$ so that $<^{\alpha'} \subseteq <^\alpha$ for all $\alpha' < \alpha$ and g^α_β is an order isomorphism. We define the function $h^\alpha \restriction (\mu \times \{\alpha\})$ as follows:

(a) if $B = \emptyset$, then $h^\alpha(x) = (\beta, g^\alpha_\beta(x))$;

(b) if $B \neq \emptyset$ and $i \geq 0$ and $x \in A_i$, then $h^\alpha(x) = (\beta, g^\alpha_\beta(x))$;

(c) if $B \neq \emptyset$ and $i < 0$ and $x \in A_i$, then $h^\alpha(x) = (\beta, y)$ where $y \in A_{i+1}$ or B is the unique element such that $g^\alpha_\beta(y) = x$.

It is easy to see that (iii) above is satisfied.

We define $\theta = (\mu \times \lambda, <)$, where $< = \bigcup_{\alpha<\lambda} <^\alpha$, $h = \bigcup_{\alpha<\lambda} h^\alpha$ and for all $\beta < \lambda$ we let $g_\beta = \bigcup\{g^\alpha_\beta \mid \alpha < \lambda,\ f(\alpha) = (\beta, \gamma) \text{ for some } \gamma\}$. Clearly these satisfy (i). (ii) follows from the fact that if $g^\alpha_\beta(x) = y$ then either $x \in \mu \times \{\alpha\}$ and $y \in \mu \times (\alpha + 1)$ or $y \in \mu \times \{\alpha\}$ and $x \in \mu \times (\alpha + 1)$. (iii) follows immediately from the definition of h. □

§3. **On nonstructure of unsuperstable theories.** In this section we will prove the main theorem of this paper, i.e., Conclusion 3.19. The idea of the proof continues III, Claim 7.8 in [Sh2]. Throughout this section we assume that T is an unsuperstable theory, $|T| < \lambda$ and $\kappa(T) > \kappa$. The cardinal assumptions are: $\lambda = \mu^+$, $\mathrm{cf}(\mu) = \mu$, $\kappa = \mathrm{cf}(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$, $\mu^\kappa = \mu$.

If $i < \kappa$, we say that i is of type n, $n = 0, 1, 2$, if there are a limit ordinal $\alpha < \kappa$ and $k < \omega$ such that $i = \alpha + 3k + n$.

We define linear orderings θ_n, $n < 3$, as follows. Let $\theta_0 = \lambda$ and θ_1, h' and g_α, $\alpha < \lambda$, be as θ, h and g_α in Lemma 2.12. Let $\theta_2 = \theta_{\mu\times\omega} \times \lambda$, where $\theta_{\mu\times\omega}$ is as in Definition 2.8.

For $n < 2$, let J^-_n be the set of sequences η of length $< \kappa$ such that

(i) $\eta \neq ()$;

(ii) $\eta(0) = n$;

(iii) if $0 < i < \mathrm{length}(\eta)$ is of type $m < 3$, then $\eta(i) \in \theta_m$.

Let

$$f : (\lambda - \{0\}) \to \{(\eta, \xi) \in J^-_0 \times J^-_1 \mid \mathit{length}(\eta) = \mathit{length}(\xi) \text{ is of type } 1\}$$

be one-to-one and onto. Then we define $h : \theta_1 \to J^-_0 \cup J^-_1$ and order isomorphisms $g_{\eta,\xi} : \mathrm{succ}(\eta) \to \mathrm{succ}(\xi)$, for $(\eta, \xi) \in \mathrm{rng}(f)$, as follows:

(i) $g_{\eta,\xi}(\eta \frown (x)) = \xi \frown (g_\alpha(x))$, where α is the unique ordinal such that $f(\alpha) = (\eta, \xi)$;

(ii) Assume $h'(x) = (\alpha, y)$, $\alpha \neq 0$, and $f(\alpha) = (\eta, \xi)$. Then $h(x) = \xi \frown (y)$ if $g_\alpha(x) = y$; otherwise $h(x) = \eta \frown (y)$. If $h'(x) = (0, y)$, then $h(x) = (0)$ (here the idea is to define $h(x)$ so that $\mathrm{length}(h(x))$ is not of type 2).

3.1. Lemma. *Assume $\eta \in J^-_0$ and $\xi \in J^-_1$ are such that $m = \mathrm{length}(\eta) = \mathrm{length}(\xi)$ is of type 2. Let $m = n + 1$. If $g_{\eta,\xi}(\eta') = \xi'$ then either* (a) $h(\eta'(n)) = \xi'$ *or* (b) $h(\xi'(n)) = \eta'$, *but not both.*

Proof. We show first that either (a) or (b) holds. So we assume that (a) is not true and prove that (b) holds. Let $\eta'(n) = x$, $\xi'(n) = y$ and $f(\alpha) = (\eta, \xi)$. Now

$g_\alpha(x) = y$, $x \neq y$, and either $h'(x) = (\alpha, y)$ or $h'(y) = (\alpha, x)$. Because (a) is not true we have $h'(x) \neq (\alpha, y)$, and so $h'(y) = (\alpha, x)$. We have two cases:

(i) Case $y > x$: Because g_α is order-preserving, $g_\alpha(y) > y > x$. So $g_\alpha(y) \neq x$ and, by the definition of h, $h(y) = \eta \frown (x) = \eta'$.

(ii) Case $y < x$: As the case $y > x$.

Next we show that it is impossible that both (a) and (b) hold. For a contradiction assume that this is not the case. Then (a) implies that there is β such that $h'(x) = (\beta, y)$ and $g_\beta(x) = y$. On the other hand, (b) implies that there is γ such that $h'(y) = (\gamma, x)$ and $g_\gamma(y) \neq x$. By Lemma 2.12 (iii), $g_\gamma(x) = y$. By Lemma 2.12 (ii), $\beta = \gamma$. So $h'(y) = (\beta, x)$ and $h'(x) = (\beta, y)$, which contradicts Lemma 2.12 (i). □

For $n < 2$, let J_n^+ be the set of sequences η of length $\leq \kappa$ such that

(i) $\eta \neq ()$;

(ii) $\eta(0) = n$;

(iii) if $0 < i < \text{length}(\eta)$ is of type $m < 3$, then $\eta(i) \in \theta_m$.

Let $e : \theta_1 \to \lambda$ be one-one and onto. We define functions s and d as follows: if $i < \text{length}(\eta)$ is of type 0, then $d(\eta, i) = \eta(i)$ and $s(\eta, i) = \eta(i)$; if $i < \text{length}(\eta)$ is of type 1, then $d(\eta, i) = \eta(i)$ and $s(\eta, i) = e(\eta(i))$; and if $i < \text{length}(\eta)$ is of type 2 and $\eta(i) = (d, s)$, then $d(\eta, i) = d$ and $s(\eta, i) = s$.

For $n < 2$ and $\gamma < \lambda$, we define

$$J_n^+(\gamma) = \{\eta \in J_n^+ \mid \text{ for all } i < \text{length}(\eta),\ s(\eta, i) < \gamma\},$$

and $J_n^-(\gamma) = J_n^+(\gamma) \cap J_n^-$.

Let us fix $d \in \theta_1$ so that $h(d) = (0)$.

3.2. Definition. For all $\eta \in J_0^-$ and $\xi \in J_1^-$ such that $n = \text{length}(\eta) = \text{length}(\xi)$ is of type 1, let $\alpha(\eta, \xi)$ be the set of ordinals $\alpha < \lambda$ such that, for all $\eta' \in \text{succ}(\eta)$, $s(\eta', n) < \alpha$ iff $s(g_{\eta,\xi}(\eta'), n) < \alpha$ and $e(d) < \alpha$. Notice that $\alpha(\eta, \xi)$ is a closed and unbounded subset of λ. By $\alpha(\beta)$, $\beta < \lambda$, we mean

$$\text{Min} \bigcap \{\alpha(\eta, \xi) \mid \eta \in J_0^-(\beta),\ \xi \in J_1^-(\beta),\ \text{length}(\eta) = \text{length}(\xi) \text{ is of type } 1\}.$$

3.3. Definition. For all $\eta \in J_0^+$ and $\xi \in J_1^+$, we write $\eta R^- \xi$ and $\xi R^- \eta$ iff

(i) $\eta(j) = \xi(j)$ for all $0 < j < \min\{\text{length}(\eta), \text{length}(\xi)\}$ of type 0;

(ii) for all $j < \min\{\text{length}(\eta), \text{length}(\xi)\}$ of type 1,

$$\xi \restriction (j+1) = g_{\eta \restriction j, \xi \restriction j}(\eta \restriction (j+1)).$$

Let $\text{length}(\eta) = \text{length}(\xi) = j + 1$, j of type 1, and $\eta R^- \xi$. We write $\eta \to \xi$ if $h(\eta(j)) = \xi$. We write $\xi \to \eta$ if $h(\xi(j)) = \eta$.

3.4 Remark. If $\xi \to \eta$ and $\xi \to \eta'$, then $\eta = \eta'$, and if $\eta R^- \xi$, then $\eta \to \xi$ or $\xi \to \eta$, but not both.

3.5. Definition. Let $\eta \in J_0^+ - J_0^-$ and $\xi \in J_1^+ - J_1^-$. We write $\eta R \xi$ and $\xi R \eta$ iff

(i) $\eta R^- \xi$;

(ii) for every $j < \kappa$ of type 2, η and ξ satisfy the following: if $\eta \restriction j \to \xi \restriction j$, then $s(\eta, j) \leq s(\xi, j)$, and if $\xi \restriction j \to \eta \restriction j$, then $s(\xi, j) \leq s(\eta, j)$;

(iii) the set $W^{\kappa}_{\eta,\xi}$ is bounded in κ, where $W^{\kappa}_{\eta,\xi}$ is defined in the following way: Let $\eta \in J_0^+ - J_0^{<\delta}$ (see Definition 2.3 (ii)) and $\xi \in J_1^+ - J_1^{<\delta}$. Then

$$W^{\delta}_{\eta,\xi} = W^{\delta}_{\xi,\eta} = V^{\delta}_{\eta,\xi} \cup U^{\delta}_{\eta,\xi},$$

where

$$V^{\delta}_{\eta,\xi} = \{j < \delta \mid j \text{ is of type 2 and } \xi \restriction j \to \eta \restriction j \text{ and}$$

$$\operatorname{cf}(s(\eta, j)) = \mu \text{ and } s(\xi, j) = s(\eta, j)\}$$

and

$$U^{\delta}_{\eta,\xi} = \{j < \delta \mid j \text{ is of type 2 and } \eta \restriction j \to \xi \restriction j \text{ and}$$

$$\operatorname{cf}(s(\xi, j)) = \mu \text{ and } s(\eta, j) = s(\xi, j)\}.$$

Our next goal is to prove that if J_0 and J_1 are such that

(i) $J_n^- \subseteq J_n \subseteq J_n^+$, $n = 0, 1$, and

(ii) if $\eta \in J_0^+$, $\xi \in J_1^+$ and $\eta R \xi$, then $\eta \in J_0$ iff $\xi \in J_1$,

then $(J_0, <, <_s) \equiv^{\lambda}_{\mu\times\kappa} (J_1, <, <_s)$, where $<$ is the initial segment relation and $<_s$ is the union of natural orderings of $\operatorname{succ}(\eta)$ for all elements η of the model. From now on in this section we assume that J_0 and J_1 satisfy (i) and (ii) above.

The relation R is designed not only to guarantee the equivalence but also to make it possible to prove that the final models are not isomorphic. Here (iii) in the definition of R plays a vital role. The pressing-down elements η such that $\operatorname{cf}(s(\eta, i)) = \mu$, i of type 2, in (iii) prevent us from adding too many elements to $J_n - J_n^-$, $n < 2$.

For $n < 2$, we write $J_n(\gamma) = J_n^+(\gamma) \cap J_n$.

3.6. Definition. Let $\alpha < \kappa$. G_α is the family of all partial functions f satisfying the following six conditions:

(a) f is a partial isomorphism from J_0 to J_1.

(b) $\operatorname{dom}(f)$ and $\operatorname{rng}(f)$ are closed under initial segments and for some $\beta < \lambda$ they are included in $J_0(\beta)$ and $J_1(\beta)$, respectively.

(c) If $f(\eta) = \xi$, then $\eta R^- \xi$.

(d) If $\eta \in J_0^+$, $\xi \in J_1^+$, $f(\eta) = \xi$ and $j < \operatorname{length}(\eta)$ of type 2, then η and ξ satisfy the following: if $\eta \restriction j \to \xi \restriction j$, then $s(\eta, j) \leq s(\xi, j)$, and if $\xi \restriction j \to \eta \restriction j$, then $s(\xi, j) \leq s(\eta, j)$.

(e) Assume $\eta \in J_0^+ - J_0^{<\delta}$ and $\{\eta \restriction \gamma \mid \gamma < \delta\} \subseteq \operatorname{dom}(f)$, and let

$$\xi = \bigcup_{\gamma<\delta} f(\eta \restriction \gamma).$$

Then $W^{\delta}_{\eta,\xi}$ has order type $\leq \alpha$.

(f) If $\eta \in \mathrm{dom}(f)$ and length(η) is of type 2, then

$$\begin{aligned}\{i<\lambda \mid & \text{ for all } d \in \theta_2,\ \eta \frown ((d,i)) \in \mathrm{dom}(f)\} \\ &= \{i<\lambda \mid \text{ for some } d \in \theta_2,\ \eta \frown ((d,i)) \in \mathrm{dom}(f)\} \\ &= \{i<\lambda \mid \text{ for all } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in \mathrm{rng}(f)\} \\ &= \{i<\lambda \mid \text{ for some } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in \mathrm{rng}(f)\}\end{aligned}$$

is an ordinal.

We define $F_\alpha \subseteq G_\alpha$ by replacing (f) above by

(f') if $\eta \in \mathrm{dom}(f)$ and length(η) is of type 2 then

$$\begin{aligned}\{i<\lambda \mid & \text{ for all } d \in \theta_2,\ \eta \frown ((d,i)) \in \mathrm{dom}(f)\} \\ &= \{i<\lambda \mid \text{ for some } d \in \theta_2,\ \eta \frown ((d,i)) \in \mathrm{dom}(f)\} \\ &= \{i<\lambda \mid \text{ for all } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in \mathrm{rng}(f)\} \\ &= \{i<\lambda \mid \text{ for some } d \in \theta_2,\ f(\eta) \frown ((d,i)) \in \mathrm{rng}(f)\}\end{aligned}$$

is an ordinal and of cofinality $< \mu$.

The idea in this definition is roughly the following: If $f \in G_\alpha$ and $f(\eta) = \xi$, then $\eta R \xi$ and the order type of $W^\delta_{\eta,\xi}$ is $\le \alpha$. If $f \in F_\alpha$, then not only $f \in G_\alpha$ but f is such that for all small $A \subset J_0 \cup J_1$ we can find $g \supset f$ such that $A \subset \mathrm{dom}(g) \cup \mathrm{rng}(g)$ and $g \in F_\alpha$.

3.7. Definition. For $f, g \in G_\alpha$ we write $f \le g$ if $f \subseteq g$ and if $\gamma < \delta \le \kappa$, $\eta \in J_0^+ - J_0^{<\delta}$, $\eta \restriction \gamma \in \mathrm{dom}(f)$, $\eta \restriction (\gamma+1) \notin \mathrm{dom}(f)$, $\eta \restriction j \in \mathrm{dom}(g)$ for all $j < \delta$ and $\xi = \bigcup_{j<\delta} g(\eta \restriction j)$, then $W^\gamma_{\eta,\xi} = W^\delta_{\eta,\xi}$.

Notice that $f \le g$ is a transitive relation.

3.8 Remark. Let $f \in G_\alpha$. We define $\overline{f} \supseteq f$ by

$$\mathrm{dom}(\overline{f}) = \mathrm{dom}(f) \cup \{\eta \in J_0 \mid \eta \restriction \gamma \in \mathrm{dom}(f) \text{ for all } \gamma < \mathrm{length}(\eta) \text{ and } \mathrm{length}(\eta) \text{ is limit}\}$$

and if $\eta \in \mathrm{dom}(\overline{f}) - \mathrm{dom}(f)$, then

$$\overline{f}(\eta) = \bigcup_{\gamma < \mathrm{length}(\eta)} f(\eta \restriction \gamma).$$

If $f \in F_\alpha$, then $\overline{f} \in F_\alpha$, and if $f \in G_\alpha$, then $\overline{f} \in G_\alpha$.

3.9. Lemma. *Assume $\alpha < \kappa$, $\delta \le \mu$, $f_i \in F_\alpha$ for all $i < \delta$, and $f_i \le f_j$ for all $i < j < \delta$.*

(i) $\bigcup_{i<\delta} f_i \in G_\alpha$.

(ii) *If $\delta < \mu$, then $\bigcup_{i<\delta} f_i \in F_\alpha$, and $f_j \le \bigcup_{i<\delta} f_i$ for all $j \le \delta$.*

Proof. (i) We have to check that $f = \bigcup_{i<\delta} f_i$ satisfies (a)–(f) in Definition 3.6. Excluding perhaps (e), all of these are trivial.

Without loss of generality we may assume δ is a limit ordinal. So assume $\eta \in J_0^+ - J_0^{<\beta}$ and $\{\eta \restriction \gamma \mid \gamma < \beta\} \subseteq \mathrm{dom}(f)$, and let $\xi = \bigcup_{\gamma<\beta} f(\eta \restriction \gamma)$. We need to show that $W^\beta_{\eta,\xi} \le \alpha$.

If there is $i < \delta$ such that $\eta \restriction \gamma \in \mathrm{dom}(f_i)$ for all $\gamma < \beta$, then the claim follows immediately from the assumption $f_i \in F_\alpha$. Otherwise for all $\gamma < \beta$ we let $i_\gamma < \delta$ be the least ordinal such that $\eta \restriction \gamma \in \mathrm{dom}(f_{i_\gamma})$. Let $\gamma^* < \beta$ be the least ordinal such that $i_{\gamma^*+1} > i_{\gamma^*}$. Because $f_{i_\gamma} \in F_\alpha$ for all $\gamma < \beta$, we get that $W^\gamma_{\eta\restriction\gamma,\xi\restriction\gamma}$ has order type $\leq \alpha$. If $\gamma^* < \gamma' < \beta$, then $f_{i_{\gamma^*}} \leq f_{i_{\gamma'}}$, and so $W^{\gamma^*}_{\eta\restriction\gamma^*,\xi\restriction\gamma^*} = W^{\gamma'}_{\eta\restriction\gamma',\xi\restriction\gamma'}$. Because $W^\beta_{\eta,\xi} = \bigcup_{\gamma<\beta} W^\gamma_{\eta\restriction\gamma,\xi\restriction\gamma}$, we get $W^\beta_{\eta,\xi} \leq \alpha$.

(ii) As (i), just check the definitions. $\square$

3.10. LEMMA. *If $\delta < \kappa$, $f_i \in G_i$ for all $i < \delta$, and $f_i \subseteq f_j$ for all $i < j < \delta$, then $\bigcup_{i<\delta} f_i \in G_\delta$.*

PROOF. Follows immediately from the definitions. $\square$

3.11. LEMMA. *If $f \in F_\alpha$ and $A \subseteq J_0 \cup J_1$, $|A| < \lambda$, then there is $g \in F_\alpha$ such that $f \leq g$ and $A \subseteq \mathrm{dom}(g) \cup \mathrm{rng}(g)$.*

PROOF. We may assume that A is closed under initial segments. Let $A' = A \cap (J_0^- \cup J_1^-)$. We enumerate $A' = \{a_i \mid 0 < i < \mu\}$ so that if a_i is an initial segment of a_j, then $i < j$. Let $\gamma < \lambda$ be such that $A \cup \mathrm{dom}(f) \cup \mathrm{rng}(f) \subseteq J_0(\gamma) \cup J_1(\gamma)$. By induction on $i < \mu$, we define functions g_i.

If $i = 0$, we define $g_i = \overline{f \cup \{((0),(1))\}}$.

If $i < \mu$ is limit, we define $g_i = \overline{\bigcup_{j<i} g_j}$.

If $i = j+1$, then there are two different cases. For simplicity we assume $a_i \in J_0$.

(i) $n = \mathrm{length}(a_i)$ is of type 0 or 1: Then we choose g_i to be such that

(a) $g_j \leq g_i$;

(b) $g_i \in F_\alpha$;

(c) if $\xi \in \mathrm{dom}(g_i) - \mathrm{dom}(g_j)$, then $\xi \in \mathrm{succ}(a_i)$;

(d) if $\xi \in \mathrm{succ}(a_i)$ and $s(\xi, n) < \gamma$, then $\xi \in \mathrm{dom}(g_i)$;

(e) if $\xi \in \mathrm{succ}(g_j(a_i))$ and $s(\xi, n) < \gamma$, then $\xi \in \mathrm{rng}(g_i)$.

Trivially, such a g_i exists.

(ii) $n = \mathrm{length}(a_j)$ is of type 2: Then we choose g_i to be such that (a)–(c) above and (d′)–(f′) below are satisfied.

Let $\beta = \sup\{i+1 < \lambda \mid \text{for all } d \in \theta_2,\ a_i \frown ((d,i)) \in \mathrm{dom}(g_j)\}$.

(d′) if $\xi \in \mathrm{succ}(a_i)$, then $s(\xi, n) < \gamma + 2$ iff $\xi \in \mathrm{dom}(g_i)$;

(e′) if $\xi \in \mathrm{succ}(g_j(a_i))$, then $s(\xi, n) < \gamma + 2$ iff $\xi \in \mathrm{rng}(g_i)$;

(f′) $g_i \restriction \{\eta \in \mathrm{succ}(a_i) \mid \beta \leq s(\eta, n) < \gamma + 1\}$ is an order isomorphism to $\{\eta \in \mathrm{succ}(g_j(a_i)) \mid \beta \leq s(\eta, n) < \beta + 1\}$ and $g_i \restriction \{\eta \in \mathrm{succ}(a_i) \mid \gamma + 1 \leq s(\eta, n) < \gamma + 2\}$ is an order isomorphism to $\{\eta \in \mathrm{succ}(g_j(a_i)) \mid \beta + 1 \leq s(\eta, n) < \gamma + 2\}$.

By Corollary 2.11 it is easy to satisfy (d′)–(f′). Because $g_j \in F_\alpha$, it follows that $\mathrm{cf}(\beta) < \mu$ and we do not have problems with (a) and (b). So there is g_i satisfying (a)–(c) and (d′)–(f′).

Finally we define $g = \overline{\bigcup_{i<\mu} g_i}$. It is easy to see that g is as desired (notice that $f \leq g$ follows from the construction, not from Lemma 3.9). $\square$

3.12. LEMMA. *If $f \in G_\alpha$ and $A \subseteq J_0 \cup J_1$, $|A| < \lambda$, then there is $g \in F_{\alpha+1}$ such that $f \subseteq g$ and $A \subseteq \mathrm{dom}(g) \cup \mathrm{rng}(g)$.*

PROOF. Essentially as the proof of Lemma 3.11. $\square$

3.13. THEOREM. *If J_0 and J_1 are such that*

(i) $J_n^- \subseteq J_n \subseteq J_n^+$, $n = 0, 1$ and

(ii) *if* $\eta R\xi$, $\eta \in J_0^+$ *and* $\xi \in J_1^+$, *then* $\eta \in J_0$ *iff* $\xi \in J_1$,
then $(J_0, <, <_s) \equiv^{\lambda}_{\mu\times\kappa} (J_1, <, <_s)$.

PROOF. Because $\emptyset \in F_0$, the theorem follows from the previous lemmas. □

3.14. COROLLARY. *If* J_0 *and* J_1 *are as above and* Φ *is proper for* T, *then*

$$\mathrm{EM}(J_0, \Phi) \equiv^{\lambda}_{\mu\times\kappa} \mathrm{EM}(J_1, \Phi).$$

PROOF. Follows immediately from the definition of E-M-models and Theorem 3.13. □

In the rest of this section we show that there are trees J_0 and J_1 which satisfy the assumptions of Corollary 3.14 and

$$\mathrm{EM}(J_0, \Phi) \not\cong \mathrm{EM}(J_1, \Phi).$$

3.15. LEMMA (Claim 7.8B in [Sh2]). *There are closed increasing cofinal sequences* $(\alpha_i)_{i<\kappa}$ *in* α, $\alpha < \lambda$ *and* $\mathrm{cf}(\alpha) = \kappa$, *such that if* i *is successor then* $\mathrm{cf}(\alpha_i) = \mu$ *and for all cub* $A \subseteq \lambda$ *the set*

$$\{\alpha < \lambda |\ \mathrm{cf}(\alpha) = \kappa \textit{ and } \{\alpha_i |\ i < \kappa\} \subseteq A \cap \alpha\ \}$$

is stationary.

We define $J_0 - J_0^-$ and $J_1 - J_1^-$ by using Lemma 3.15. For all $\alpha < \lambda$ we define I_0^α and I_1^α. Let $I_0^0 = J_0^-$ and $I_1^0 = J_1^-$. If $0 < \alpha < \lambda$, then $\mathrm{cf}(\alpha) = \kappa$, and there are sequence $(\beta_i)_{i<\kappa}$ and an $\eta \in J_0^+ - J_0^-$ such that

(i) $(\beta_i)_{i<\kappa}$ is properly increasing and cofinal in α;
(ii) for all $i < \kappa$, $\mathrm{cf}(\beta_{i+1}) = \mu$, $\beta_{i+1} > \alpha(\beta_i)$, and $\beta_i \in \{\alpha_i |\ i < \kappa\}$;
(iii) for all $0 < i < \kappa$ of type 0 or 2, $s(\eta, i) = \beta_i$;
(iv) for all $i < \kappa$ of type 1, $\eta(i) = d$;

then we choose some such η, let it be η_α, and define I_0^α and I_1^α to be the least sets such that

(i) $\{\eta_\alpha\} \cup \bigcup_{\beta<\alpha} I_0^\beta \subseteq I_0^\alpha$ and $\bigcup_{\beta<\alpha} I_1^\beta \subseteq I_1^\alpha$, and
(ii) $I_0^\alpha \cup I_1^\alpha$ is closed under R.

Otherwise we let $I_0^\alpha = \bigcup_{\beta<\alpha} I_0^\beta$ and $I_1^\alpha = \bigcup_{\beta<\alpha} I_1^\beta$. Finally we define $J_0 = \bigcup_{\alpha<\lambda} I_0^\alpha$ and $J_1 = \bigcup_{\alpha<\lambda} I_1^\alpha$.

3.16. LEMMA. *For all* $\alpha < \lambda$ *and* $\eta \in (J_0 \cup J_1) - (J_0^- \cup J_1^-)$, *the following are equivalent*:

(i) $\eta \in (I_0^\alpha \cup I_1^\alpha) - (\bigcup_{\beta<\alpha} I_0^\beta \cup \bigcup_{\beta<\alpha} I_1^\beta)$.
(ii) $\sup\{s(\eta, i) |\ i < \kappa\} = \alpha$.

PROOF. By the construction it is enough to show that (i) implies (ii). So assume (i). Because of levels of type 0, it is enough to show that $s(\eta, i) < \beta_{i+1}$ for all $i < \kappa$. We prove this by induction on $i < \kappa$. If i is of type 0, the claim is clear. If i is of type 1, this follows from $\beta_{i+1} > \alpha(\beta_i)$ and $e(d) < \alpha(\beta_i)$ together with the induction assumption. For i of type 2, $i = j + 1$, it is enough to show that $s(\eta_\alpha, i) \geq s(\eta, i)$. This follows easily from the fact that $\eta_\alpha(j) = d$ and $\mathrm{length}(h(d)) \neq i$. □

3.17. DEFINITION. Let $g : \mathrm{EM}(J_0, \Phi) \to \mathrm{EM}(J_1, \Phi)$ be an isomorphism. We say that $\alpha < \lambda$ is *g-saturated* iff for all $\eta \in J_0$ and $\xi_0, \ldots, \xi_n \in J_1$ the following holds: if

(i) $\mathrm{length}(\eta) = l + 1$ and for all $i < l$, $s(\eta, i) < \alpha$;
(ii) for all $k \leq n$ and $i < \mathrm{length}(\xi_k)$, $s(\xi_k, i) < \alpha$;

(iii) $g(\eta) = t(\delta_0, ..., \delta_m)$ for some term t and $\delta_0, ..., \delta_m \in J_1$,
then there are $\eta' \in J_0$ and $\delta'_0, ..., \delta'_n \in J_1$ such that

(a) $g(\eta') = t(\delta'_0, ..., \delta'_m)$;

(b) $\text{length}(\eta') = l + 1$ and $\eta' \restriction l = \eta \restriction l$;

(c) $s(\eta', l) < \alpha$;

(d) the basic type of $(\xi_0, ..., \xi_n, \delta_0, ..., \delta_m)$ in $(J_1, <, \ll, H, P_j)$ is the same as the basic type of $(\xi_0, ..., \xi_n, \delta'_0, ..., \delta'_m)$.

Notice that for all isomorphisms $g : \text{EM}(J_0, \Phi) \to \text{EM}(J_1, \Phi)$ the set of g-saturated ordinals is unbounded in λ and closed under increasing sequences of length $\alpha < \lambda$ if $\text{cf}(\alpha) > \kappa$.

3.18. LEMMA. *Let Φ be proper for T. Then*

$$\text{EM}(J_0, \Phi) \not\cong \text{EM}(J_1, \Phi).$$

PROOF. We write $\mathscr{A}_\gamma$ for the submodel of $\text{EM}(J_0, \Phi)$ generated (in the extended language) by $J_0(\gamma)$. Similarly, we write $\mathscr{B}_\gamma$ for the submodel of $\text{EM}(J_1, \Phi)$ generated by $J_1(\gamma)$. Let g be a one-to-one function from $\text{EM}(J_0, \Phi)$ onto $\text{EM}(J_1, \Phi)$. We say that g is closed in γ if $\mathscr{A}_\gamma \cup \mathscr{B}_\gamma$ is closed under g and g^{-1}.

For a contradiction we assume that g is an isomorphism from $\text{EM}(J_0, \Phi)$ to $\text{EM}(J_1, \Phi)$. By Lemma 3.15 we choose $\alpha < \lambda$ to be such that

(i) $\text{cf}(\alpha) = \kappa$, and, for all $i < \kappa$, g is closed in α_i and $\text{cf}(\alpha_{i+1}) = \mu$ and α_{i+1} is g-saturated;

(ii) there are a sequence $(\beta_i)_{i<\kappa}$ and an $\eta = \eta_\alpha \in J_0 - J_0^-$ satisfying (i)–(iv) in the definition of $(J_0 - J_0^-) \cup (J_1 - J_1^-)$.

Let $g(\eta) = t(\xi_0, ..., \xi_n)$, $\xi_0, ..., \xi_n \in J_1$. Now for all $k \leq n$, either $\xi_k \in J_1(\beta_i)$ for some $i < \kappa$, or there is $j < \kappa$ such that $s(\xi_k, j) \geq \alpha$ or $\text{length}(\xi_k) = \kappa$, $\sup\{s(\xi_k, j) | \ j < \kappa\} = \alpha$ and, for all $j < \kappa$, $s(\xi_k, j) < \alpha$. By Lemma 3.16, in the last case ξ_k has been put to J_1 at stage α.

We choose $i < \kappa$ so that

(a) i is of type 2 and > 2;

(b) for all $k < l \leq n$, $\xi_k \restriction i \neq \xi_l \restriction i$;

(c) for all $k \leq n$, if $\text{length}(\xi_k) = \kappa$, $\sup\{s(\xi_k, j) | \ j < \kappa\} = \alpha$ and for all $j < \kappa$, $s(\xi_k, j) < \alpha$, then there are $\rho_0, ..., \rho_r \in J_0 \cup J_1$ such that

(i) $\rho_o = \eta$ and $\rho_r = \xi_k$;

(ii) if $p < r$, then $\rho_p R \rho_{p+1}$;

(iii) if $p < r$, then $W^\kappa_{\rho_p, \rho_{p+1}} \subseteq i$;

(iv) for all $p < q \leq r$, $\rho_p \restriction i \neq \rho_q \restriction i$;

(d) for all $k \leq n$, if $\xi_k \in J_1(\beta_j)$ for some $j < \kappa$, then $\xi_k \in J_1(\beta_i)$;

(e) for all $k \leq n$, if $s(\xi_k, j) \geq \alpha$ for some $j < \kappa$, then $\xi_k \restriction j_k \in J_1(\beta_i)$ and $j_k < i$, where $j_k = \min\{j < i | \ s(\xi_k, j) \geq \alpha\}$.

Let $l \leq l' \leq n + 1$ be such that $\xi_k \in J_1(\beta_i)$ iff $k < l$, $\text{length}(\xi_k) = \kappa$, $\sup\{s(\xi_k, j) | \ j < \kappa\} = \alpha$ and, for all $j < \kappa$, $s(\xi_k, j) < \alpha$ iff $l \leq k < l'$ and $\xi_k \restriction i \notin J_1(\alpha)$ iff $l' \leq k \leq n$. (Of course we may assume that we have ordered $\xi_0, ..., \xi_m$ so that l and l' exist.) If $l \leq k < l'$, then there are $\rho_0, ..., \rho_r \in J_1 \cup J_0$ satisfying (c)(i)–(c)(iv) above. By the choice of $\eta(i-1)$ we have $\rho_p \restriction i \leftarrow \rho_{p+1} \restriction i$ for all $p < r$, and so $\xi_k \restriction (i+1) \in J_1(\beta_i)$. For all $k \leq n$ we define ξ'_k as follows:

(α) if $k < l$, then $\xi'_k = \xi_k$;

(β) if $l \leq k < l'$, then $\xi'_k = \xi_k \restriction (i+1)$;

(γ) if $l' \leq k \leq n$, then $\xi'_k = \xi_k \restriction j_k$.

Let $g(\eta \restriction (i+1)) = u(\delta_0, ..., \delta_m)$, u a term and $\delta_0, ..., \delta_m \in J_1(\beta_{i+1})$. Because β_i is g-saturated there is $\eta' \in J_0(\beta_i)$ and $\delta'_0, ..., \delta'_m \in J_1(\beta_i)$ such that

(a) $g(\eta') = u(\delta'_0, ..., \delta'_m)$;

(b) $\mathrm{length}(\eta') = i+1$ and $\eta' \restriction i = \eta \restriction i$;

(c) the basic type of $(\xi'_0, ..., \xi'_n, \delta_0, ..., \delta_m)$ in $(J_1, <, \ll, H, P_j)$ is the same as the basic type of $(\xi'_0, ..., \xi'_n, \delta'_0, ..., \delta'_m)$.

Because $s(\xi_k, i+1) \geq \beta_{i+1}$ for all $l \leq k < l'$ and $s(\xi_k, j_k) > \beta_{i+1}$ for all $l' \leq k \leq n$, it is easy to see that the basic type of $(\xi_0, ..., \xi_n, \delta_0, ..., \delta_m)$ in $(J_1, <, \ll, H, P_j)$ is the same as the basic type of $(\xi_0, ..., \xi_n, \delta'_0, ..., \delta'_m)$.

Let ϕ_n, $n < \kappa$, be as in Theorem 2.7. Then

$$\mathrm{EM}^1(J_1, \Phi) \models \phi_{i+1}(u(\delta'_0, ..., \delta'_m), t(\xi_0, ..., \xi_n)).$$

So $\eta' \neq \eta \restriction (i+1)$, $\eta' \restriction i = \eta \restriction i$, and

$$\mathrm{EM}^1(J_0, \Phi) \models \phi_{i+1}(\eta', \eta).$$

This is impossible by Theorem 2.7 (ii). $\square$

Conclusion 3.19. Let $\lambda = \mu^+$, $\mathrm{cf}(\mu) = \mu$, $\kappa = \mathrm{cf}(\kappa) < \mu$, $\lambda^{<\kappa} = \lambda$, and $\mu^\kappa = \mu$. Assume T is an unsuperstable theory, $|T| \leq \lambda$, and $\kappa(T) > \kappa$. Then there are models $\mathscr{A}$, $\mathscr{B} \models T$ of cardinality λ such that

$$\mathscr{A} \equiv^\lambda_{\mu \times \kappa} \mathscr{B} \quad \text{and} \quad \mathscr{A} \not\cong \mathscr{B}.$$

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