A GENERAL CRITERION FOR THE EXISTENCE OF TRANSVERSALS

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[Received 19 August 1982]

ABSTRACT

We present a necessary and sufficient condition for a family of sets to possess a transversal. Its form follows that of P. Hall's theorem: a family has a transversal if and only if it does not contain one of a set of 'forbidden' substructures.

1. *Introduction*

Let $\mathscr{F} = \{S_i\colon i \in I\}$ be a family of subsets of a set S. A *transversal* of $\mathscr{F}$ is a set $\{x_i\colon i \in I\}$ such that $x_i \in S_i$ for every $i \in I$, and $x_i \neq x_j$ when $i \neq j$. In 1935, P. Hall [**5**] gave a criterion for deciding whether a finite family of sets possesses a transversal. Since then it has remained an open problem to do the same for infinite families. The first progress was made by M. Hall [**4**], who proved that P. Hall's criterion holds also for infinite families of finite sets. The problem was next solved for the case of countable families. Damerell and Milner [**3**] proved a criterion for deciding whether a countable family of sets has a transversal, and alternative criteria, also for countable families, were subsequently obtained by Podewski and Steffens [**9**] and by Nash-Williams [**8**]. Shelah [**10**] gave a criterion of an inductive nature, which, together with the criteria proved for countable families, solved the problem for families of countable sets.

In this paper we complete the solution of the problem, and give a general characterization of families which possess transversals. This result draws upon ideas from [**9**] and [**10**]. The usefulness of this characterization has already been shown in [**2**], where it was used to prove König's duality theorem for bipartite graphs of general cardinality.

2. *Notation and definitions*

The problem of characterizing the families of sets which possess transversals is known also as the 'marriage problem' because it can be re-phrased in the terminology of 'marriage in societies'. In this terminology the family of sets is replaced by a set M of 'men', the underlying set S by a set W of 'women', and the relation of an element belonging to a set by a relation K of 'knowing' between M and W. The notion of a transversal is transformed into that of monogamous marriage, in which all men are married to women they know. (Inequitably, it is not demanded that all women should marry!) We choose here to use this terminology, of which details—together with other required definitions—are given in the remainder of this section.

If F is a set of ordered pairs, a is any element, and A is any set, then $F\langle a\rangle$ denotes $\{y\colon (a, y) \in F\}$, $F(a)$ denotes the element of $F\langle a\rangle$ if $|F\langle a\rangle| = 1$, $F[A]$ denotes

Proc. London Math. Soc. (3), 47 (1983), 43–68.

$\bigcup\{F\langle a\rangle: a \in A\}$, $\operatorname{dom} F$ (the *domain* of F) denotes $\{a: F\langle a\rangle \neq \varnothing\}$, $F^{-1} = \{(y,x): (x,y) \in F\}$, $\operatorname{rge} F$ (the *range* of F) denotes $\operatorname{dom} F^{-1}$, and $F \restriction A$ denotes $F \cap (A \times \operatorname{rge} F)$ (i.e. the restriction of F to A). We say that F is a *function* if $|F\langle a\rangle| = 1$ for every $a \in \operatorname{dom} F$, and that F is *injective* if $|F^{-1}\langle x\rangle| = 1$ for every $x \in \operatorname{rge} F$.

A *society* is a triple (M, W, K) where M, W are disjoint sets and $K \subseteq M \times W$. Elements of M and W are *men* and *women* of the society respectively. A man m and woman w are said to *know* each other if $(m, w) \in K$. For a society Λ we denote by $M_\Lambda, W_\Lambda, K_\Lambda$ the sets such that $\Lambda = (M_\Lambda, W_\Lambda, K_\Lambda)$. The symbols M, W, K, when appearing without subscripts, will be understood to pertain to a given society denoted by Γ.

If $X \subseteq W$ then $D(X)$ denotes $\{a \in M: K\langle a\rangle \subseteq X\}$. If we want to specify the society Γ in which $D(X)$ is taken, we shall write $D_\Gamma(X)$ for $D(X)$. For $A \subseteq M$ and $X \subseteq W$, $\Gamma[A, X]$ denotes the society $(A, X, K \cap (A \times X))$, and $|\Gamma|$, $\Gamma - A$, $\Gamma - X$, and $\Gamma - A - X$ denote, respectively, $|M \cup W|$, $\Gamma[M \backslash A, W]$, $\Gamma[M, W \backslash X]$, and $\Gamma[M \backslash A, W \backslash X]$. A society Γ' is called a *subsociety* of Γ if $\Gamma' = \Gamma[A, X]$ for some $A \subseteq M$, $X \subseteq W$. We then write $\Gamma' \leqslant \Gamma$. The society $\Gamma - A - X$ is denoted in this case by Γ/Γ'. A subsociety Π of Γ is called *saturated* if $K_\Gamma[M_\Pi] \subseteq W_\Pi$. We write $\Pi \lhd \Gamma$ for 'Π is a saturated subsociety of Γ'. If $\bar{\Pi} = (\Pi_i: i \in I) = ((M_i, W_i, K_i): i \in I)$ is a family of subsocieties of Γ then the *union* $\bigcup \bar{\Pi}$, *intersection* $\bigcap \bar{\Pi}$, and *join* $\bigvee \bar{\Pi}$ of these subsocieties are the societies

$$(\bigcup M_i, \bigcup W_i, \bigcup K_i), \quad \Gamma[\bigcap M_i, \bigcap W_i] = (\bigcap M_i, \bigcap W_i, \bigcap K_i),$$

and

$$\Gamma[\bigcup M_i, \bigcup W_i]$$

respectively, where $\bigcup M_i$ means $\bigcup_{i \in I} M_i$ and $\bigcup W_i$, $\bigcup K_i$, $\bigcap M_i$, $\bigcap W_i$, $\bigcap K_i$ have similar meanings. If I has just two elements a, b then $\bigcup \bar{\Pi}$, $\bigcap \bar{\Pi}$, $\bigvee \bar{\Pi}$ may be denoted by $\Pi_a \cup \Pi_b$, $\Pi_a \cap \Pi_b$, $\Pi_a \vee \Pi_b$ respectively.

A society $(\varnothing, \{u\}, \varnothing)$ which contains a single woman and no man is called *maidenly* and is denoted by $\langle u\rangle$. The society $(\varnothing, \varnothing, \varnothing)$ will be said to be *empty*.

An *espousal* of Γ is an injective function E such that $\operatorname{dom} E = M$ and $E \subseteq K$. If Γ has an espousal then it is called *espousable*. A subset A of M is called *espousable* if $\Gamma[A, W]$ is espousable, and an espousal of $\Gamma[A, W]$ may also be called an *espousal of* A. A society Γ is called *critical* if it is espousable and $\operatorname{rge} E = W$ for every espousal E of Γ. A subset C of M is called *critical* if $\Gamma[C, K[C]]$ is critical.

Let Φ, Ψ be sets of ordinals. A function $f: \Phi \to \Psi$ is *ascending* if $f(\alpha) < f(\beta)$ whenever $\alpha, \beta \in \Phi$ and $\alpha < \beta$. The *order type* $\operatorname{ord} \Phi$ of Φ is the unique ordinal α such that there exists an ascending bijection of α onto Φ. A function $f: \Phi \to \Psi$ is *regressive* if $f(\alpha) < \alpha$ for every $\alpha \in \Phi \backslash \{0\}$.

Let κ be a regular uncountable cardinal. A subset Ω of κ is *closed* (in κ) if $\sup \Xi \in \Omega \cup \{\kappa\}$ for every non-empty subset Ξ of Ω, and is *unbounded* (in κ) if $\sup \Omega = \kappa$. A subset Φ of κ is *stationary* (or *κ-stationary*) if $\Phi \cap \Omega \neq \varnothing$ for every closed unbounded subset Ω of κ.

Given a set S and a cardinal λ, a *λ-enumeration* of S is a function f from $\lambda|S|$ into S such that $|f^{-1}\langle s\rangle| = \lambda$ for every $s \in S$. A 1-enumeration is simply called an *enumeration*.

In this paper, the word 'sequence' means 'transfinite sequence', i.e. a function whose domain is an ordinal number or, equivalently, a family of the form $(x_\alpha: \alpha < \zeta)$ indexed by the ordinals α less than some ordinal ζ. (These definitions are equivalent since we understand a 'family' $(x_i: i \in I)$ to be the same thing as the function $\{(i, x_i): i \in I\}$.) We shall call x_α the αth *term* of a sequence $(x_\alpha\colon \alpha < \zeta)$. If s denotes this sequence and $\theta \leqslant \zeta$ then s_θ will denote the sequence $(x_\alpha\colon \alpha < \theta)$, or, equivalently, $s \restriction \theta$.

A sequence of subsocieties of Γ may often be denoted by a Greek capital letter with a bar above it, and then the αth term of this sequence will be denoted by the same Greek capital letter, unbarred, with subscript α. For example, if $\bar{\Lambda}$ is a sequence of subsocieties of Γ then Λ_α is its αth term. Moreover, if $\bar{\Lambda} = (\Lambda_\alpha\colon \alpha < \zeta)$ and $\theta \leqslant \zeta$ then $\bar{\Lambda}_\theta$ will denote the sequence $(\bar{\Lambda})_\theta = (\Lambda_\alpha\colon \alpha < \theta)$. The sequence $\bar{\Lambda}$ will be called *non-descending* if $\Lambda_\alpha \leqslant \Lambda_\beta$ whenever $\alpha < \beta < \zeta$; and $\bar{\Lambda}$ will be called *continuous* if it is non-descending and $\bigcup \bar{\Lambda}_\theta = \Lambda_\theta$ for every limit ordinal $\theta < \zeta$. A *ζ-tower* in Γ is a continuous non-descending sequence $(\Pi_\alpha\colon \alpha \leqslant \zeta)$ of saturated subsocieties of Γ such that Π_0 is empty. A *ζ-ladder* in Γ is a sequence $\bar{\Lambda} = (\Lambda_\alpha\colon \alpha < \zeta)$ of subsocieties of Γ such that $\Lambda_\alpha \cap \Lambda_\beta$ is empty whenever $\alpha < \beta < \zeta$ and $\bigvee \bar{\Lambda}_\alpha \lhd \Gamma$ for every $\alpha \leqslant \zeta$. A sequence of subsocieties of Γ is a *tower* (*ladder*) if it is a ζ-tower (ζ-ladder) for some ordinal ζ.

If $\mathscr{T}$ is the set of towers in Γ and $\mathscr{L}$ is the set of ladders in Γ then there is an obvious bijection $l\colon \mathscr{T} \to \mathscr{L}$ such that

(i) if $\bar{\Pi}$ is a ζ-tower in Γ then $l(\bar{\Pi})$ is the ζ-ladder $(\Pi_{\alpha+1}/\Pi_\alpha\colon \alpha < \zeta)$,

(ii) if $\bar{\Lambda}$ is a ζ-ladder in Γ then $l^{-1}(\bar{\Lambda})$ is the ζ-tower $(\bigvee \bar{\Lambda}_\alpha\colon \alpha \leqslant \zeta)$.

We shall call $l(\bar{\Pi})$ the *ladder of* $\bar{\Pi}$.

The *deficiency* $\delta(\Gamma)$ of a society Γ is

$$\min\{|L|\colon L \subseteq M,\ \Gamma - L \text{ is espousable}\}.$$

If Γ is espousable, its *surplus* $\sigma(\Gamma)$ is

$$\sup\{|V|\colon V \subseteq W,\ \Gamma - V \text{ is espousable}\}.$$

(Theorem 6.2 below will show that this supremum is in fact a maximum.) We formally write $\sigma(\Gamma) = -\delta(\Gamma)$ if Γ is not espousable. The *essential size* $\varepsilon(\Gamma)$ of Γ is

$$\min\{|L|\colon L \subseteq M,\ \Gamma - L - V \text{ is critical for some } V \subseteq W\},$$

and $\nu(\Gamma)$ will denote

$$\min\{|V|\colon V \subseteq W,\ \Gamma - L - V \text{ is critical for some } L \subseteq M\}.$$

In this paper, small Greek letters will denote ordinals, and in particular $\kappa, \lambda, \mu, \nu$ will denote cardinals. The letters i, h, k will denote indices which will sometimes be non-negative integers. The letters $\Gamma, \Delta, \Lambda, \Pi$, and Σ will denote societies, while other capital Greek letters will denote sets or classes of ordinals. When considering a society $\Gamma = (M, W, K)$, we shall denote subsets of M and subsets of W by capital letters from the first and second halves of the alphabet respectively (possibly with subscripts, superscripts, etc.). The least cardinal greater than κ will be denoted by κ^+. A *κ-subset* of a set S is a subset of S with cardinality κ.

3. *Preliminary lemmas*

LEMMA 3.1. *If $\Delta \lhd \Pi \lhd \Gamma$ then $\Delta \lhd \Gamma$.*

LEMMA 3.2. *If $\Pi \lhd \Gamma$ and $\Lambda \lhd \Gamma/\Pi$ then $\Pi \vee \Lambda \lhd \Gamma$.*

LEMMA 3.3. *If Γ is critical, F is an espousal of Γ, and A is a subset of M, then $\Gamma[A, F[A]]$ is critical.*

LEMMA 3.4. *If* $\Delta_\alpha = \Gamma[M_\alpha, W_\alpha]$ *is a critical subsociety of* Γ *for every* $\alpha < \zeta$ *and* $M_\alpha \cap M_\beta = W_\alpha \cap W_\beta = K \cap (M_\alpha \times W_\beta) = \varnothing$ *whenever* $\alpha < \beta < \zeta$ *then*

$$\Delta = \bigvee \{\Delta_\alpha \colon \alpha < \zeta\}$$

is critical.

Proof. Each Δ_α has an espousal E_α, and $\bigcup \{E_\alpha \colon \alpha < \zeta\}$ is an espousal of Δ. Now let F be any espousal of Δ. Suppose that $\theta < \zeta$ and $F[M_\alpha] = W_\alpha$ for every $\alpha < \theta$. Then $F[M_\theta] \cap W_\alpha = F[M_\theta] \cap F[M_\alpha] = \varnothing$ for $\alpha < \theta$, and moreover, for $\alpha > \theta$, $F[M_\theta] \cap W_\alpha = \varnothing$ since $K \cap (M_\theta \times W_\alpha) = \varnothing$. Therefore $F[M_\theta] \subseteq W_\theta$, and so $F[M_\theta] = W_\theta$ since Δ_θ is critical. We have now shown by transfinite induction that $F[M_\theta] = W_\theta$ for every $\theta < \zeta$. Therefore $\operatorname{rge} F = W_\Delta$.

COROLLARY 3.4a. *If* Π *is a critical saturated subsociety of* Γ *and* $\Pi \leqslant \Sigma \leqslant \Gamma$ *and* Σ/Π *is critical, then* Σ *is critical.*

Proof. Since Π is saturated, $K \cap (M_\Pi \times W_{\Sigma/\Pi}) = \varnothing$ and hence $\Sigma = \Pi \vee (\Sigma/\Pi)$ is critical by Lemma 3.4.

COROLLARY 3.4b. *If* $\bar{\Lambda}$ *is a* ζ*-ladder in* Γ *then*
(i) $\varepsilon(\bigvee \bar{\Lambda}) \leqslant \sum_{\alpha<\zeta} \varepsilon(\Lambda_\alpha)$,
(ii) $\nu(\bigvee \bar{\Lambda}) \leqslant \sum_{\alpha<\zeta} \nu(\Lambda_\alpha)$.

Proof. For each $\alpha < \zeta$, choose $L_\alpha \subseteq M_{\Lambda_\alpha}$ and $V_\alpha \subseteq W_{\Lambda_\alpha}$ such that $\Lambda_\alpha - L_\alpha - V_\alpha$ is critical and $|L_\alpha| = \varepsilon(\Lambda_\alpha)$. Let $A_\alpha = M_{\Lambda_\alpha} \backslash L_\alpha$ and $X_\alpha = W_{\Lambda_\alpha} \backslash V_\alpha$. Then $\Gamma[A_\alpha, X_\alpha] = \Lambda_\alpha - L_\alpha - V_\alpha$, which is critical; and, if $\alpha < \beta < \zeta$, then $K \cap (A_\alpha \times X_\beta) = \varnothing$ since $\bigvee \bar{\Lambda}_{\alpha+1} \lhd \Gamma$. Therefore, by Lemma 3.4, $\bigvee \{\Gamma[A_\alpha, X_\alpha] \colon \alpha < \zeta\}$ is critical, that is, $(\bigvee \bar{\Lambda}) - \bigcup \{L_\alpha \colon \alpha < \zeta\} - \bigcup \{V_\alpha \colon \alpha < \zeta\}$ is critical, and so

$$\varepsilon(\bigvee \bar{\Lambda}) \leqslant |\bigcup \{L_\alpha \colon \alpha < \zeta\}| = \sum_{\alpha<\zeta} \varepsilon(\Lambda_\alpha).$$

A similar argument, with L_α, V_α chosen so that $|V_\alpha| = \nu(\Lambda_\alpha)$, proves (ii).

COROLLARY 3.4c. *If* $\Lambda \lhd \Gamma$ *then* $\varepsilon(\Gamma) \leqslant \varepsilon(\Lambda) + \varepsilon(\Gamma/\Lambda)$.

Proof. Take $\zeta = 2$, $\Lambda_0 = \Lambda$, $\Lambda_1 = \Gamma/\Lambda$ in Corollary 3.4b.

Much of the importance of critical societies derives from the next two lemmas.

LEMMA 3.5 [9, Lemma 1]. *Every critical subset of* M *is contained in a maximal critical set.*

If Π, Σ are critical saturated subsocieties of Γ, then it is easy to see that $\Pi \leqslant \Sigma$ if and only if $M_\Pi \subseteq M_\Sigma$. Hence from Lemma 3.5 there follows:

COROLLARY 3.5a. *Every critical saturated subsociety of* Γ *is contained in a maximal critical saturated subsociety of* Γ.

Since the empty subsociety of any society is obviously critical and saturated, there follows:

COROLLARY 3.5b. *Every society has a maximal critical saturated subsociety.*

LEMMA 3.6 [9, Lemmas 1, 2]. (i) *If Γ is espousable then there exists a greatest critical saturated subsociety of Γ.*

(ii) *Let Γ be espousable, Π be the greatest critical saturated subsociety of Γ, and $w \in W \setminus W_\Pi$. Then $\Gamma - \{w\}$ is espousable.*

LEMMA 3.7. *Let Γ be critical and E an espousal of Γ. Then there does not exist a sequence $(a_i\colon i < \omega)$ of distinct elements of M such that $E(a_{i+1}) \in K\langle a_i \rangle$ for every $i < \omega$.*

Proof. Suppose that $(a_i\colon i < \omega)$ is a sequence violating the lemma. Define then an espousal F of Γ by $F(a_i) = E(a_{i+1})$ for every $i < \omega$ and $F(b) = E(b)$ for every $b \in M \setminus \{a_i\colon i < \omega\}$. Then $E(a_0) \notin \operatorname{rge} F$, contradicting the fact that Γ is critical.

COROLLARY 3.7a. *If Γ is critical then $|\{m \in M\colon K\langle m \rangle = W\}| < \aleph_0$.*

Proof. If there exists a sequence $(a_i\colon i < \omega)$ of distinct elements of M such that $K\langle a_i \rangle = W$, then this sequence clearly violates Lemma 3.7.

Since the aim of this paper is to characterize espousable societies, and since our characterization will involve critical societies, it is worth while mentioning that the structure of critical societies is well understood: in [1, Corollary 2] critical societies are characterized in terms of the function 'q' of [8].

LEMMA 3.8. *Let J, L be subsets of M, and let P, N be subsets of W such that $\Gamma - J - P$ is critical and $\Gamma - L - N$ is espousable. Then $|J| + |N| \leqslant |P| + |L|$.*

Proof. Let E be an espousal of $\Gamma - J - P$ and let G be an espousal of $\Gamma - L - N$. Since $\Gamma - J - P$ is critical, $\operatorname{rge} E = W \setminus P$. Let $\mathcal{D}$ be a directed graph whose set of vertices is $M \cup W$ and whose set of edges is $E^{-1} \cup G$. Then each vertex of $\mathcal{D}$ has invalency 0 or 1 and outvalency 0 or 1, and hence each connected component of $\mathcal{D}$ is a directed path or a directed circuit. If $x \in J \cup N$ then x has invalency 0 and hence x is the initial vertex of a directed path $\mathcal{P}_x$ which is a component of $\mathcal{D}$. The vertices of $\mathcal{P}_x$ other than x and its terminal vertex (if such exists) have invalency 1 and outvalency 1, and so cannot belong to $J \cup P$. Therefore, by Lemma 3.7, $\mathcal{P}_x$ cannot be infinite, and so must have a terminal vertex $g(x)$. Since $g(x)$ has outvalency 0, it does not belong to $(\operatorname{rge} E) \cup (\operatorname{dom} G) = (W \setminus P) \cup (M \setminus L)$ and so must belong to $P \cup L$. Hence g is an injection from $J \cup N$ into $P \cup L$.

COROLLARY 3.8a. *If L is a subset of M and N is a subset of W such that $\Gamma - L - N$ is espousable then $\nu(\Gamma) + |L| \geqslant |N|$.*

Proof. There exist $J \subseteq M$ and $P \subseteq W$ such that $\nu(\Gamma) = |P|$ and $\Gamma - J - P$ is critical. Hence $\nu(\Gamma) + |L| = |P| + |L| \geqslant |N|$ by Lemma 3.8.

COROLLARY 3.8b. *If $J \subseteq M$, $P \subseteq W$, $\Gamma - J - P$ is critical, and $|P| < |J|$, then Γ is inespousable.*

The next two lemmas follow readily from the definitions of $\delta(\Gamma)$ and $\varepsilon(\Gamma)$:

LEMMA 3.9. $\delta(\Gamma) \leqslant \varepsilon(\Gamma)$.

LEMMA 3.10. *If* $\Pi \lhd \Gamma$ *then* $\delta(\Pi) \leqslant \delta(\Gamma)$.

LEMMA 3.11. *If* $\Delta \lhd \Gamma$ *then* $\nu(\Gamma/\Delta) \leqslant \nu(\Gamma)$.

Proof. Let Π be a critical subsociety of Γ such that $|W \setminus W_\Pi| = \nu(\Gamma)$, and let F be an espousal of Π. Let $\Sigma = \Gamma[F^{-1}[W_\Pi \setminus W_\Delta], W_\Pi \setminus W_\Delta]$. By Lemma 3.3, Σ is critical. Since Δ is saturated, $F^{-1}[W \setminus W_\Delta] \subseteq M \setminus M_\Delta$, and hence $\Sigma \leqslant \Gamma/\Delta$. Since $|W_{\Gamma/\Delta} \setminus W_\Sigma| \leqslant |W \setminus W_\Pi|$, the lemma follows.

The main property of stationary sets which we shall need is given by Fodor's lemma:

LEMMA 3.12 (see, for example [7, Theorem 22]). *Let κ be a regular uncountable cardinal. If Φ is κ-stationary and $f\colon \Phi \to \kappa$ is a regressive function then there exist a κ-stationary subset Ψ of Φ and an ordinal $\beta < \kappa$ such that $f[\Psi] = \{\beta\}$; in particular,* $|f^{-1}\langle\beta\rangle| = \kappa$.

4. *Impediments and obstructions*

Let Υ be the class of cardinals κ such that either $0 < \kappa < \aleph_0$ or κ is regular. We shall define by induction on κ what is meant by saying that a subsociety Π of Γ is a 'κ-impediment', for every $\kappa \in \Upsilon$. First, when $0 < \kappa \leqslant \aleph_0$, a subsociety Π of Γ is a *κ-impediment in* Γ if it is saturated and $\Pi - L$ is critical for some κ-subset L of M_Π. Now assume that κ is regular and uncountable and that 'μ-impediment' has been defined for every $\mu \in \Upsilon$ such that $\mu < \kappa$. Let $\bar{\Pi}$ be a κ-tower in Γ and $\bar{\Lambda}$ be its ladder. We say that $\bar{\Pi}$ is a *κ-fortress* if, for each $\alpha < \kappa$, Λ_α is

(1) a μ-impediment in Γ/Π_α for some $\mu \in \Upsilon \cap \kappa$, or
(2) critical, or
(3) maidenly.

We denote by $\Phi_k(\bar{\Pi})$ the set $\{\alpha < \kappa\colon \text{Case } (k) \text{ holds at } \alpha\}$ for $k = 1, 2, 3$. A κ-tower $\bar{\Pi}$ is *impeding* (in Γ) if it is a κ-fortress in Γ and $\Phi_1(\bar{\Pi})$ is κ-stationary. A subsociety Π of Γ is a *κ-impediment in* Γ if $\Pi = \bigcup \bar{\Pi}$ for some impeding κ-tower $\bar{\Pi}$ in Γ.

In fact, the notion of an 'impediment' can be replaced by a simpler one, that of an 'obstruction', to be defined below. The reason for the use of 'impediments' is their convenience in the proof of our main result (Theorem 5.1). It may, however, be worth while to try to find a direct proof of Theorem 5.1, using 'obstructions'.

We define the notion of a *κ-obstruction* by induction on κ, for every $\kappa \in \Upsilon$. For $0 < \kappa \leqslant \aleph_0$ a *κ-obstruction* in Γ is a κ-impediment in Γ. Suppose now that κ is regular and uncountable and that 'μ-obstruction' has been defined for every $\mu < \kappa$ in Υ. A κ-tower $\bar{\Sigma}$ in Γ, with $l(\bar{\Sigma}) = \bar{\Delta}$, will be said to be *obstructive* if

(a) for each $\alpha < \kappa$, Δ_α is either
  (1) a μ-obstruction in Γ/Σ_α for some $\mu < \kappa$ or
  (2) maidenly, and
(b) $\Psi(\bar{\Sigma}) = \{\alpha < \kappa\colon \text{Case (1) holds at } \alpha\}$ is κ-stationary.

A subsociety Σ of Γ is called a *κ-obstruction in* Γ if $\Sigma = \bigcup \bar{\Sigma}$ for some obstructive κ-tower $\bar{\Sigma}$ in Γ.

A subsociety of Γ is
(i) an *impediment* in Γ if it is a κ-impediment in Γ for some $\kappa \in \Upsilon$,
(ii) a $(>\kappa)$-*impediment* in Γ if it is a λ-impediment in Γ for some $\lambda \in \Upsilon$ such that $\lambda > \kappa$.

We shall say that Γ is *impeded* (κ-*impeded*, $(>\kappa)$-*impeded*) if some subsociety of Γ is an impediment (a κ-impediment, a $(>\kappa)$-impediment) in Γ, and *unimpeded* (κ-*unimpeded*, $(>\kappa)$-*unimpeded*) if not. The expressions $(\leqslant\kappa)$-*impediment*, $(<\kappa)$-*impediment*, $(<\kappa)$-*obstruction*, $(\leqslant\kappa)$-*impeded*, $(<\kappa)$-*impeded* are defined similarly.

Whenever we refer to κ-impediments or κ-impeded societies or κ-obstructions, it will be understood (whether or not this is explicitly mentioned) that κ denotes a member of Υ. Likewise, any reference to κ-fortresses or impeding κ-towers indicates that κ is assumed to be regular and uncountable.

LEMMA 4.1. *If Σ is a critical saturated subsociety of Γ and Π is a κ-obstruction in Γ/Σ then $\Sigma \vee \Pi$ is a κ-obstruction in Γ.*

Proof. Suppose first that $\kappa \leqslant \aleph_0$. Then $\Pi \lhd \Gamma/\Sigma$ and $\Pi - L$ is critical for some κ-subset L of M_Π. Since $\Sigma \lhd \Gamma$, it follows that $K \cap (M_\Sigma \times W_\Pi) = \varnothing$. Therefore $\Sigma \vee (\Pi - L)$ is critical by Lemma 3.4, that is, $(\Sigma \vee \Pi) - L$ is critical. Since $\Sigma \lhd \Gamma$ and $\Pi \lhd \Gamma/\Sigma$, it follows that $\Sigma \vee \Pi \lhd \Gamma$. Hence $\Sigma \vee \Pi$ is a κ-obstruction in Γ.

Now suppose that κ is regular and uncountable, and assume the inductive hypothesis that the assertion of Lemma 4.1 is true if κ is replaced by any element of $\Upsilon \cap \kappa$. Let Σ, Π be as in the statement of the lemma. Then $\Pi = \bigcup \bar{\Pi}$ for some obstructive κ-tower $\bar{\Pi}$ in Γ/Σ. This implies that $\Psi(\bar{\Pi})$ is κ-stationary and that Λ_α is maidenly or a $(<\kappa)$-obstruction in $(\Gamma/\Sigma)/\Pi_\alpha$ for each $\alpha < \kappa$, where $\bar{\Lambda} = l(\bar{\Pi})$. Let ξ be the least element of $\Psi(\bar{\Pi})$. Let the sequence $\bar{\Delta} = (\Delta_\alpha : \alpha \leqslant \kappa)$ be defined by letting $\Delta_\alpha = \Pi_\alpha$ for $\alpha \leqslant \xi$ and $\Delta_\alpha = \Sigma \vee \Pi_\alpha$ for $\xi < \alpha \leqslant \kappa$. Then $l(\bar{\Delta}) = (\Lambda'_\alpha : \alpha < \kappa)$, where $\Lambda'_\alpha = \Lambda_\alpha$ for $\alpha \neq \xi$ and $\Lambda'_\xi = \Sigma \vee \Lambda_\xi$.

Since $\bar{\Pi}$ is obstructive in Γ/Σ and ξ is the least element of $\Psi(\bar{\Pi})$, it follows that Λ_α is maidenly for each $\alpha < \xi$. Therefore Π_α has no men for $\alpha \leqslant \xi$ and so $\Delta_\alpha = \Pi_\alpha \lhd \Gamma$ trivially when $\alpha \leqslant \xi$. Moreover, $\Pi_\alpha \lhd \Gamma/\Sigma$ for each $\alpha \leqslant \kappa$ since $\bar{\Pi}$ is a tower in Γ/Σ, and by hypothesis $\Sigma \lhd \Gamma$: therefore $\Delta_\alpha = \Sigma \vee \Pi_\alpha \lhd \Gamma$ for $\xi < \alpha \leqslant \kappa$. Hence $\bar{\Delta}$ is a κ-tower in Γ.

Since $\xi \in \Psi(\bar{\Pi})$, it follows that Λ_ξ is a $(<\kappa)$-obstruction in $(\Gamma/\Sigma)/\Pi_\xi = (\Gamma/\Pi_\xi)/\Sigma$. Moreover, $\Sigma \lhd \Gamma$ and therefore $\Sigma \lhd \Gamma/\Pi_\xi$ also; and Σ is by hypothesis critical. Therefore, by our inductive hypothesis, $\Sigma \vee \Lambda_\xi$ is a $(<\kappa)$-obstruction in Γ/Π_ξ, that is, Λ'_ξ is a $(<\kappa)$-obstruction in Γ/Δ_ξ. If $\alpha \in \Psi(\bar{\Pi})$ and $\alpha \neq \xi$ then $\alpha > \xi$ and $\Lambda'_\alpha = \Lambda_\alpha$, which is a $(<\kappa)$-obstruction in $(\Gamma/\Sigma)/\Pi_\alpha = \Gamma/\Delta_\alpha$. Hence Λ'_α is a $(<\kappa)$-obstruction in Γ/Δ_α for every $\alpha \in \Psi(\bar{\Pi})$. Therefore $\Psi(\bar{\Pi}) \subseteq \Psi(\bar{\Delta})$, and therefore $\Psi(\bar{\Delta})$ is κ-stationary. If $\alpha < \kappa$ and Λ'_α is not a $(<\kappa)$-obstruction in Γ/Δ_α, then $\alpha \notin \Psi(\bar{\Delta})$ and therefore $\alpha \notin \Psi(\bar{\Pi})$ (since $\Psi(\bar{\Pi}) \subseteq \Psi(\bar{\Delta})$) and therefore Λ_α is maidenly (since $\bar{\Pi}$ is obstructive in Γ/Σ), that is, Λ'_α is maidenly (since $\alpha \notin \Psi(\bar{\Pi})$ and so $\alpha \neq \xi$ and $\Lambda'_\alpha = \Lambda_\alpha$). Hence Λ'_α is either a $(<\kappa)$-obstruction in Γ/Δ_α or maidenly for each $\alpha < \kappa$.

We have now proved that $\bar{\Delta}$ is obstructive in Γ. Therefore $\bigcup \bar{\Delta} = \Sigma \vee \Pi$ is a κ-obstruction in Γ.

LEMMA 4.2. *A subsociety* Π of Γ *is a κ-obstruction in Γ if and only if it is a κ-impediment in Γ.*

Proof. From the definition of a κ-obstruction and a κ-impediment it clearly follows by induction on κ that a κ-obstruction is a κ-impediment. We have therefore to show that if Π is a κ-impediment then it is a κ-obstruction. For $\kappa \leqslant \aleph_0$ this is the definition of 'κ-obstruction'. So assume that κ is regular and uncountable, and that the assertion is true for $(<\kappa)$-impediments. Let $\bar{\Pi}$ be an impeding κ-tower such that $\bigcup \bar{\Pi} = \Pi$, and let $\bar{\Lambda} = l(\bar{\Pi})$. We would like to transform $\bar{\Pi}$ to an obstructive κ-tower by 'getting rid' of the critical rungs Λ_α, for $\alpha \in \Phi_2(\bar{\Pi})$. Roughly speaking, this is done by 'pushing upwards' each such Λ_α, to join it to Λ_β, where $\beta = \min\{\gamma \in \Phi_1 \colon \gamma > \alpha\}$. A more formal definition is the following.

For $k = 1, 2, 3$ let $\Phi_k = \Phi_k(\bar{\Pi})$. Let f be the unique ascending bijection from $\Phi_1 \cup \Phi_3$ onto κ, and let $g\colon \Phi_2 \to \Phi_1$ be defined by $g(\alpha) = \min(\Phi_1 \setminus \alpha)$. We construct a sequence $\bar{\Delta} = (\Delta_\alpha \colon \alpha < \kappa)$ of subsocieties of Γ by defining $\Delta_{f(\alpha)}$ for every $\alpha \in \Phi_1 \cup \Phi_3$, according to the following rules:

Case (1): $\alpha \in \Phi_1$. Let $\Lambda_\alpha^* = \bigvee \{\Lambda_\beta \colon \beta \in g^{-1}\langle \alpha \rangle\}$ and let

$$\Delta_{f(\alpha)} = \Lambda_\alpha \vee \Lambda_\alpha^*. \tag{4.1}$$

Case (2): $\alpha \in \Phi_3$. Let $\Delta_{f(\alpha)} = \Lambda_\alpha$.

Now let $\bar{\Sigma}$ be the sequence $(\Sigma_\alpha \colon \alpha \leqslant \kappa)$, where $\Sigma_\alpha = \bigvee \bar{\Delta}_\alpha$ for each $\alpha \leqslant \kappa$. We show, in a sequence of assertions, that $\bar{\Sigma}$ is an obstructive κ-tower.

ASSERTION 1. *If $\alpha \in \Phi_1$ then $\Pi_\alpha = \Sigma_{f(\alpha)} \vee \Lambda_\alpha^*$ and $\Pi_{\alpha+1} = \Sigma_{f(\alpha)+1}$.*

Proof of Assertion 1. Clearly

$$\Sigma_{f(\alpha)} = \bigvee \{\Delta_{f(\beta)} \colon \beta < \alpha,\ \beta \in \Phi_1\} \vee \bigvee \{\Delta_{f(\beta)} \colon \beta < \alpha,\ \beta \in \Phi_3\}.$$

Hence, by the definition of $\Delta_{f(\beta)}$,

$$\Sigma_{f(\alpha)} = \bigvee \{\Lambda_\beta \colon \beta < \alpha,\ \beta \in \Phi_1\} \vee \bigvee \{\Lambda_\beta^* \colon \beta < \alpha,\ \beta \in \Phi_1\} \vee \bigvee \{\Lambda_\beta \colon \beta < \alpha,\ \beta \in \Phi_3\}.$$

But, by the definition of Λ_β^*,

$$\bigvee \{\Lambda_\beta^* \colon \beta \leqslant \alpha,\ \beta \in \Phi_1\} = \bigvee \{\Lambda_\beta \colon \beta < \alpha,\ \beta \in \Phi_2\}.$$

Hence

$$\Sigma_{f(\alpha)} \vee \Lambda_\alpha^* = \bigvee \{\Lambda_\beta \colon \beta < \alpha,\ \beta \in \Phi_1 \cup \Phi_2 \cup \Phi_3\} = \Pi_\alpha.$$

By (4.1),

$$\Sigma_{f(\alpha)+1} = \Sigma_{f(\alpha)} \vee \Delta_{f(\alpha)} = \Sigma_{f(\alpha)} \vee \Lambda_\alpha \vee \Lambda_\alpha^* = \Pi_\alpha \vee \Lambda_\alpha = \Pi_{\alpha+1}.$$

ASSERTION 2. *$\bar{\Sigma}$ is a tower.*

Proof of Assertion 2. This will be clearly shown if we prove that $\Delta_{f(\alpha)}$ is saturated in $\Gamma/\Sigma_{f(\alpha)}$ for every $\alpha \in \Phi_1 \cup \Phi_3$. For $\alpha \in \Phi_3$ this is trivially true because $\Delta_{f(\alpha)}$ has no men. For $\alpha \in \Phi_1$ it follows from the fact that, by Assertion 1, $\Sigma_{f(\alpha)} \vee \Delta_{f(\alpha)} = \Pi_{\alpha+1}$, which is saturated in Γ.

ASSERTION 3. *Let $\alpha \in \Phi_1$ and let Λ_α be a ρ-impediment in Γ/Π_α, where $\rho < \kappa$. Then $\Delta_{f(\alpha)}$ is a ρ-obstruction in $\Gamma/\Sigma_{f(\alpha)}$.*

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Proof of Assertion 3. By the inductive hypothesis, Λ_α is a ρ-obstruction in Γ/Π_α, that is (by Assertion 1), a ρ-obstruction in $(\Gamma/\Sigma_{f(\alpha)})/\Lambda_\alpha^*$. By Lemma 3.4, Λ_α^* is critical. Since $\bar{\Pi}$ is a tower, $K[M_{\Lambda_\beta}] \subset W_{\Pi_\alpha}$ for every $\beta < \alpha$, and hence using Assertion 1, we have $K[M_{\Lambda_\alpha^*}] \subset W_{\Pi_\alpha} = W_{\Sigma_{f(\alpha)}} \cup W_{\Lambda_\alpha^*}$. This implies that Λ_α^* is saturated in $\Gamma/\Sigma_{f(\alpha)}$. By (4.1), $\Delta_{f(\alpha)} = \Lambda_\alpha \vee \Lambda_\alpha^*$. Hence, by Lemma 4.1, $\Delta_{f(\alpha)}$ is a ρ-obstruction in $\Gamma/\Sigma_{f(\alpha)}$.

We can now complete the proof of the lemma. By Assertion 2, $\bar{\Sigma}$ is a κ-tower. By Assertion 3, $\bar{\Sigma}$ satisfies Condition (a) in the definition of a κ-obstruction, and $\Psi(\bar{\Sigma}) = f[\Phi_1]$. Clearly $f(\alpha) \leqslant \alpha$ for every $\alpha \in \Phi_1 \cup \Phi_3$, and since f is one-to-one it follows by Lemma 3.12 that $\{\alpha \in \Phi_1 \colon f(\alpha) < \alpha\}$ is not κ-stationary. Hence $\Phi_1' = \{\alpha \in \Phi_1 \colon f(\alpha) = \alpha\}$ is κ-stationary. But clearly $\Phi_1' \subseteq f[\Phi_1] = \Psi(\bar{\Sigma})$, and hence $\Psi(\bar{\Sigma})$ is κ-stationary, which shows that $\bar{\Sigma}$ is obstructive in Γ. Since $\bigcup \bar{\Sigma} = \Pi$ (this follows, for example, by Assertion 1), Π is a κ-obstruction in Γ.

LEMMA 4.3. *If Π is an impediment in Γ then $\Pi \lhd \Gamma$.*

Proof. This is part of the definition if Π is a $(\leqslant \aleph_0)$-impediment. If Π is a $(> \aleph_0)$-impediment then $\Pi = \bigcup \bar{\Pi}$ for some impeding tower $\bar{\Pi}$. Since $\bar{\Pi}$ is a tower, each term in it is saturated, and hence so is Π.

LEMMA 4.4. *Suppose that $\Pi \leqslant \Gamma$ and $\Pi \leqslant \Delta$ and $K_\Gamma\langle m\rangle = K_\Delta\langle m\rangle$ for every $m \in M_\Pi$. Then Π is a κ-impediment in Γ if and only if it is a κ-impediment in Δ.*

A formal proof can easily be given by transfinite induction on κ. In essence, Lemma 4.4 is true because the definition of Π being a κ-impediment in Γ depends only on the sets M_Π, W_Π, and $K_\Gamma\langle m\rangle$ $(m \in M_\Pi)$.

COROLLARY 4.4a. *If a saturated subsociety of Γ is κ-impeded, then so is Γ.*

LEMMA 4.5. *If $|D(\varnothing)| \geqslant \kappa$ and $\kappa \in \Upsilon$ then Γ is κ-impeded.*

Proof. Let e be an enumeration of a κ-subset H of $D(\varnothing)$. Then

$$\bar{\Pi} = ((e[\alpha], \varnothing, \varnothing) \colon \alpha \leqslant \kappa)$$

is a κ-tower in Γ whose ladder is

$$\bar{\Lambda} = ((\{e(\alpha)\}, \varnothing, \varnothing) \colon \alpha < \kappa).$$

If I, J are disjoint subsets of H and $0 < |I| \leqslant \aleph_0$, then $\Delta = (I, \varnothing, \varnothing)$ is a $|I|$-impediment in $\Gamma - J$ because $\Delta \lhd \Gamma - J$ and $\Delta - I$ is critical. Therefore $(H, \varnothing, \varnothing)$ is a κ-impediment in Γ if $0 < \kappa \leqslant \aleph_0$, and, moreover, $\Lambda_\alpha = (\{e(\alpha)\}, \varnothing, \varnothing)$ is a 1-impediment in $\Gamma - e[\alpha] = \Gamma/\Pi_\alpha$ for each $\alpha < \kappa$, so that, if $\kappa > \aleph_0$, then $\bar{\Pi}$ is impeding and $(H, \varnothing, \varnothing)$ is once again a κ-impediment in Γ.

COROLLARY 4.5a. *If $\Pi \leqslant \Gamma$ and Γ/Π is unimpeded and $m \in M \backslash M_\Pi$ then $K\langle m\rangle \nsubseteq W_\Pi$.*

Proof. By Lemma 4.5, $D_{\Gamma/\Pi}(\varnothing) = \varnothing$ and therefore $m \notin D_{\Gamma/\Pi}(\varnothing)$, that is, $\varnothing \neq K_{\Gamma/\Pi}\langle m\rangle = K\langle m\rangle \backslash W_\Pi$.

LEMMA 4.6. *Suppose that*
(i) *there is no pair* Δ, a *such that* Δ *is a critical saturated subsociety of* Γ *and* $a \in M \setminus M_\Delta$ *and* $K\langle a\rangle \subseteq W_\Delta$,
(ii) $\Gamma - A$ *is espousable for some countable subset* A *of* M.
Then Γ *is espousable.*

This lemma is essentially a combination of Theorem 1 and Lemma 14 of [1] (because, in the terminology of [1], Theorem 1 shows that 'q-admissibility' can be replaced by 'c-admissibility' in Lemma 14).

COROLLARY 4.6a. *If* $\Gamma - A$ *is espousable for some countable subset* A *of* M *and* Γ *is* 1-*unimpeded then* Γ *is espousable.*

Proof. If there were a critical saturated subsociety Δ of Γ and an $a \in M \setminus M_\Delta$ such that $K\langle a\rangle \subseteq W_\Delta$ then $\Gamma[\{a\} \cup M_\Delta, W_\Delta]$ would be a 1-impediment in Γ. Therefore there is no such pair Δ, a and so Γ is espousable by Lemma 4.6.

LEMMA 4.7. *Suppose that* $\Pi \lhd \Gamma$ *and* Γ/Π *is* κ-*impeded and* $\nu(\Pi) < \kappa$. *Then*
(i) Γ *is impeded,*
(ii) Γ *is* κ-*impeded if* $\kappa > \aleph_0$.

Proof. Choose $L \subseteq M_\Pi$ and $V \subseteq W_\Pi$ such that $|V| = \nu(\Pi) = \mu$ (say) and $\Pi - L - V$ is critical.

Suppose that $\kappa \leqslant \aleph_0$. Let Δ be a κ-impediment in Γ/Π. Then M_Δ has a κ-subset J such that $\Delta - J$ is critical. Since Π is saturated, $K \cap (M_{\Pi - L - V} \times W_{\Delta - J}) = \varnothing$ and therefore $(\Pi - L - V) \vee (\Delta - J)$ is critical by Lemma 3.4, that is, $\Sigma - J - V$ is critical, where $\Sigma = (\Pi \vee \Delta) - L$. Since $|J| = \kappa > \nu(\Pi) = |V|$, it follows that Σ is inespousable by Corollary 3.8b. Since $\Sigma - J - V$ is critical and therefore espousable, it follows that $\Sigma - J$ is espousable. Since $|J| = \kappa \leqslant \aleph_0$ and $\Sigma - J$ is espousable and Σ is inespousable, Σ is, by Corollary 4.6a, 1-impeded. Since $M_\Sigma \subseteq M_\Delta \cup M_\Pi$, $W_\Sigma = W_\Delta \cup W_\Pi$, $\Pi \lhd \Gamma$, and $\Delta \lhd \Gamma/\Pi$, it is clear that $\Sigma \lhd \Gamma$. Hence, by Corollary 4.4a, Γ is impeded.

Now suppose that $\kappa > \aleph_0$ (so that κ is regular and uncountable). Let $\bar{\Delta}$ be an impeding κ-tower in Γ/Π. Let e be an enumeration of V. Since $|V| = \nu(\Pi) = \mu < \kappa$ and $\Pi \lhd \Gamma$, it is easily verified that a κ-tower $\bar{\Delta}'$ in Γ is obtained by letting $\Delta'_\alpha = (\varnothing, e[\alpha], \varnothing)$ for $\alpha \leqslant \mu$, $\Delta'_{\mu+1} = \Pi - L$, $\Delta'_{\mu+1+\alpha} = (\Pi - L) \vee \Delta_\alpha$ for $\alpha < \kappa$. If $\bar{\Lambda}, \bar{\Lambda}'$ are the ladders of $\bar{\Delta}, \bar{\Delta}'$ respectively then, for each $\alpha < \kappa$,

$$\Lambda_\alpha = \Delta_{\alpha+1}/\Delta_\alpha = \Delta'_{\mu+1+\alpha+1}/\Delta'_{\mu+1+\alpha} = \Lambda'_{\mu+1+\alpha} \tag{4.2}$$

and $(\Gamma/\Pi)/\Delta_\alpha = (\Gamma/\Delta'_{\mu+1+\alpha}) - L$. Hence, if Λ_α is a $(<\kappa)$-impediment in $(\Gamma/\Pi)/\Delta_\alpha$ then $\Lambda'_{\mu+1+\alpha}$ is a $(<\kappa)$-impediment in $(\Gamma/\Delta'_{\mu+1+\alpha}) - L$ and therefore also, by Lemma 4.4, in $\Gamma/\Delta'_{\mu+1+\alpha}$. Moreover, if Λ_α is critical or maidenly, then so is $\Lambda'_{\mu+1+\alpha}$, by (4.2); and $\Lambda'_\alpha = \langle e(\alpha)\rangle$ is maidenly for $\alpha < \mu$ and $\Lambda'_\mu = \Pi - L - V$ is critical. From these remarks and the fact that $\bar{\Delta}$ is a κ-fortress in Γ/Π, it follows that $\bar{\Delta}'$ is a κ-fortress in Γ and that $\mu + 1 + \alpha \in \Phi_1(\bar{\Delta}')$ if $\alpha \in \Phi_1(\bar{\Delta})$. Since $\bar{\Delta}$ is impeding in Γ/Π, it follows that $\Phi_1(\bar{\Delta})$ is κ-stationary and therefore $\{\mu+1+\alpha\colon \alpha \in \Phi_1(\bar{\Delta})\}$ is κ-stationary and consequently $\Phi_1(\bar{\Delta}')$ is κ-stationary. Hence $\bar{\Delta}'$ is impeding in Γ and so Γ is κ-impeded.

COROLLARY 4.7a. *If* $\varepsilon(\Gamma) > \nu(\Gamma)$ *then* Γ *is impeded.*

Proof. Select $L \subseteq M$ and $V \subseteq W$ such that $\Gamma - L - V$ is critical and $|V| = \nu(\Gamma)$. Let $\Pi = \Gamma - L$. Since $\Gamma - L - V$ is critical, $|L| \geqslant \varepsilon(\Gamma) \geqslant \nu(\Gamma)^+$. Since $D_{\Gamma/\Pi}(\varnothing) = L$, it follows by Lemma 4.5 that Γ/Π is $\nu(\Gamma)^+$-impeded. Since $\Pi - V = \Gamma - L - V$ is critical, $\nu(\Pi) \leqslant |V| < \nu(\Gamma)^+$. Therefore, by Lemma 4.7, Γ is impeded.

Corollary 4.7b. *If $V \subseteq W$ and $|V| < \kappa$ and $\Gamma - V$ is κ-impeded then*
(i) *Γ is impeded,*
(ii) *Γ is κ-impeded if $\kappa > \aleph_0$.*

Proof. Take $\Pi = (\varnothing, V, \varnothing)$ in Lemma 4.7.

Corollary 4.7c. *If Π is a critical saturated subsociety of Γ and Γ/Π is impeded then Γ is impeded.*

Lemma 4.8. *If Π is a κ-impediment in Γ then $\varepsilon(\Pi) \leqslant \kappa$ and $\nu(\Pi) \leqslant \kappa$.*

Proof. If $\kappa \leqslant \aleph_0$ then $\Pi - L$ is critical for some κ-subset L of M_Π, that is, $\Pi - L - V$ is critical, where $V = \varnothing \subseteq W_\Pi$, and so $\varepsilon(\Pi) \leqslant |L| = \kappa$, $\nu(\Pi) \leqslant |V| = 0 < \kappa$. Now suppose that $\kappa > \aleph_0$, and assume that $\varepsilon(\Pi') \leqslant \mu$ and $\nu(\Pi') \leqslant \mu$ if $\mu < \kappa$ and Π' is a μ-impediment in a society. Then $\Pi = \bigcup \bar{\Pi}$ for some impeding κ-tower $\bar{\Pi}$. Let $l(\bar{\Pi}) = \bar{\Lambda}$. For each $\alpha < \kappa$, we have that Λ_α is a $(<\kappa)$-impediment in Γ/Π_α, in which case $\varepsilon(\Lambda_\alpha) < \kappa$ and $\nu(\Lambda_\alpha) < \kappa$ by our inductive hypothesis, or Λ_α is critical, in which case $\varepsilon(\Lambda_\alpha) = \nu(\Lambda_\alpha) = 0$, or Λ_α is maidenly, in which case $\varepsilon(\Lambda_\alpha) = 0$ and $\nu(\Lambda_\alpha) = 1$. So, by Corollary 3.4b, $\varepsilon(\Pi) = \varepsilon(\bigvee \bar{\Lambda} \leqslant \sum_{\alpha<\kappa} \varepsilon(\Lambda_\alpha) \leqslant \kappa$ and

$$\nu(\Pi) = \nu(\bigvee \bar{\Lambda}) \leqslant \textstyle\sum_{\alpha<\kappa} \nu(\Lambda_\alpha) \leqslant \kappa.$$

Corollary 4.8a. *If $\bar{\Pi}$ is a κ-fortress in Γ and $\theta < \kappa$ then $\varepsilon(\Pi_\theta) < \kappa$ and $\nu(\Pi_\theta) < \kappa$.*

Proof. Let $l(\bar{\Pi}) = \bar{\Lambda}$. For each $\alpha < \theta$, we have that Λ_α is a $(<\kappa)$-impediment in Γ/Π_α, in which case $\varepsilon(\Lambda_\alpha) < \kappa$ and $\nu(\Lambda_\alpha) < \kappa$ by Lemma 4.8, or Λ_α is critical, in which case $\varepsilon(\Lambda_\alpha) = \nu(\Lambda_\alpha) = 0$, or Λ_α is maidenly, in which case $\varepsilon(\Lambda_\alpha) = 0$ and $\nu(\Lambda_\alpha) = 1$. So, by Corollary 3.4b, $\varepsilon(\Pi_\theta) \leqslant \sum_{\alpha<\theta} \varepsilon(\Lambda_\alpha) < \kappa$ and $\nu(\Pi_\theta) \leqslant \sum_{\alpha<\theta} \nu(\Lambda_\alpha) < \kappa$.

Lemma 4.9. *If Π is a κ-impediment in Γ then $\delta(\Pi) \geqslant \kappa$. In particular, Π is inespousable.*

Proof. There exists a $\delta(\Pi)$-subset L of M_Π such that $\Pi - L$ is espousable. Let E be an espousal of $\Pi - L$. If $\kappa \leqslant \aleph_0$ then, by the definition of κ-impediment, M_Π has a κ-subset J such that $\Pi - J$ is critical; and $\delta(\Pi) = |L| \geqslant |J| = \kappa$ by Lemma 3.8. Now suppose that $\kappa > \aleph_0$, and assume the inductive hypothesis that $\delta(\Sigma) \geqslant \mu$ if Σ is a μ-impediment in a society and $\mu < \kappa$. By the definition of κ-impediment, $\Pi = \bigcup \bar{\Pi}$ for some impeding κ-tower $\bar{\Pi}$ in Γ. Let $l(\bar{\Pi}) = \bar{\Lambda}$ and let $\Lambda_\alpha = \Gamma[M_\alpha, W_\alpha]$ for $\alpha < \kappa$. Let Ψ be the set of those $\alpha < \kappa$ for which $L \cap M_\alpha \neq \varnothing$. If $\alpha \in \Phi_1(\bar{\Pi})$ then Λ_α is inespousable by the inductive hypothesis and so $E \upharpoonright M_\alpha$ is not an espousal of Λ_α. Moreover, $M_\alpha \subseteq \operatorname{dom} E$ if $\alpha \notin \Psi$. Hence, if $\alpha \in \Phi_1(\bar{\Pi}) \backslash \Psi$, we can select $m_\alpha \in M_\alpha$ such that $E(m_\alpha) \in W \backslash W_\alpha$ and therefore $E(m_\alpha) \in W_{f(\alpha)}$ for some $f(\alpha) < \alpha$ since $\Pi_{\alpha+1} \lhd \Gamma$. Suppose that $\delta(\Pi) < \kappa$. Then $|L| < \kappa$ and so $|\Psi| < \kappa$. Moreover, $\Phi_1(\bar{\Pi})$ is κ-stationary since $\bar{\Pi}$ is impeding, and therefore $\Phi_1(\bar{\Pi}) \backslash \Psi$ is

κ-stationary. Furthermore, $f\colon \Phi_1(\bar{\Pi})\backslash\Psi \to \kappa$ is a regressive function. Therefore, by Lemma 3.12, there exists $\beta < \kappa$ such that $|f^{-1}\langle\beta\rangle| = \kappa$. Let $\Delta = \Pi_{\beta+1}$ and $V = \{E(m_\alpha)\colon \alpha \in f^{-1}\langle\beta\rangle\}$. If $\alpha \in f^{-1}\langle\beta\rangle$ then $\alpha > f(\alpha) = \beta$ and so $m_\alpha \notin M_\Delta$ but $E(m_\alpha) \in W_{f(\alpha)} = W_\beta \subseteq W_\Delta$. Therefore $V \subseteq W_\Delta \backslash E[M_\Delta]$, and so $E \restriction M_\Delta$ is an espousal of $\Delta - (L \cap M_\Delta) - V$. Therefore, by Corollary 3.8a,

$$\nu(\Delta) + |L \cap M_\Delta| \geqslant |V| = |f^{-1}\langle\beta\rangle| = \kappa.$$

Since $\nu(\Delta) < \kappa$ by Corollary 4.8a and $|L \cap M_\Delta| \leqslant |L| < \kappa$, the assumption that $\delta(\Pi) < \kappa$ has led to a contradiction. Therefore $\delta(\Pi) \geqslant \kappa$.

COROLLARY 4.9a. *If Π is a κ-impediment in Γ then $\delta(\Pi) = \varepsilon(\Pi) = \kappa$.*

Proof. Lemmas 4.9, 3.9, and 4.8 yield $\kappa \leqslant \delta(\Pi) \leqslant \varepsilon(\Pi) \leqslant \kappa$.

COROLLARY 4.9b. *If Γ is κ-impeded then $\delta(\Gamma) \geqslant \kappa$. In particular, if Γ is espousable then it is unimpeded.*

Proof. This follows from Lemmas 4.9, 4.3, and 3.10.

5. *A criterion for espousability*

The main aim of this paper is to prove

THEOREM 5.1. *A society Γ is inespousable if and only if, for some $\kappa \in \Upsilon$, there exists a κ-obstruction in Γ.*

By Lemma 4.2, this theorem is equivalent to

THEOREM 5.1′. *A society is espousable if and only if it is unimpeded.*

We shall therefore prove Theorem 5.1′. This proof constitutes the remainder of § 5.

Corollary 4.9b shows that an espousable society is unimpeded. We have therefore to prove that Γ is espousable if it is unimpeded. This is done by induction on $\lambda = \varepsilon(\Gamma)$. (Induction on $\varepsilon(\Gamma)$ is appropriate since the addition or removal of a critical society does not affect the prospects of espousability: hence the name 'essential size'.)

If $\lambda \leqslant \aleph_0$ then, by Lemma 3.9, $\delta(\Gamma) \leqslant \aleph_0$, and then Corollary 4.6a shows that Γ is espousable if it is unimpeded. So assume that λ is uncountable and Γ is unimpeded, and assume that every unimpeded society of essential size less than λ is espousable. We consider separately the cases in which λ is regular and singular.

Case I: λ is regular

Select sets $L \subseteq M$ and $V \subseteq W$ such that $|L| = \lambda$ and $\Gamma - L - V$ is critical. Let F be an espousal of $\Gamma - L - V$.

For each subset A of M select an enumeration e_A of A. By induction on α we define for every $\alpha < \lambda$ a quintuple $(J_\alpha, s^\alpha, \varphi(\alpha), \psi(\alpha), \Lambda_\alpha)$, where

J_α is a subset of M,
s^α is a λ-enumeration of $L \cup \bigcup\{J_\theta\colon \theta < \alpha\}$,
$\varphi(\alpha)$, $\psi(\alpha)$ are ordinals,
Λ_α is a subsociety of Γ.

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Suppose that $\alpha < \lambda$ and that J_θ, s^θ, $\varphi(\theta)$, $\psi(\theta)$, and Λ_θ have been defined for $\theta < \alpha$. We write

$$\hat{J}_\alpha = \bigcup\{J_\theta\colon \theta < \alpha\}, \quad \hat{\varphi}(\alpha) = \sup\{\varphi(\theta)\colon \theta < \alpha\},$$
$$\hat{\psi}(\alpha) = \sup\{\psi(\theta)\colon \theta < \alpha\}, \quad \Pi_\alpha = \bigvee\{\Lambda_\theta\colon \theta < \alpha\},$$
$$\Gamma_\alpha = \Gamma/\Pi_\alpha = (M_\alpha, W_\alpha, K_\alpha).$$

We observe that if $\xi < \hat{\psi}(\alpha)$ then $\xi < \psi(\theta)$ for some $\theta < \alpha$ and so the least ordinal $\theta(\xi)$ such that $\xi < \psi(\theta(\xi))$ exists and is less than α. If $\hat{\psi}(\alpha) \geqslant \lambda$, let s^α be any λ-enumeration of $L \cup \hat{J}_\alpha$. If $\hat{\psi}(\alpha) < \lambda$, let s^α be a λ-enumeration of $L \cup \hat{J}_\alpha$ such that

$$s^\alpha(\xi) = s^{\theta(\xi)}(\xi) \quad \text{for every } \xi < \hat{\psi}(\alpha). \tag{5.1}$$

We define Λ_α and $\varphi(\alpha)$ by considering four cases:

(C_1) if Γ_α is impeded, let Λ_α be an impediment in Γ_α, and let $\varphi(\alpha) = \hat{\varphi}(\alpha)$;

(C_2) if Γ_α is unimpeded and contains a non-empty critical saturated subsociety, let Λ_α be a maximal critical saturated subsociety of Γ_α (which exists by Corollary 3.5b), and let $\varphi(\alpha) = \hat{\varphi}(\alpha)$;

(C_3) if Γ_α is unimpeded and has no non-empty critical saturated subsociety and $\Psi_\alpha = (s^\alpha)^{-1}[(L \cup \hat{J}_\alpha)\setminus M_{\Pi_\alpha}]\setminus\hat{\varphi}(\alpha) \neq \varnothing$, let $\varphi'(\alpha)$ be the least element of Ψ_α, let $\varphi(\alpha) = \varphi'(\alpha)+1$, and let $\Lambda_\alpha = \langle x\rangle$ for some $x \in K\langle s^\alpha(\varphi'(\alpha))\rangle \setminus W_{\Pi_\alpha}$ (such an x exists by Corollary 4.5a);

(C_4) if Γ_α is unimpeded and has no non-empty critical saturated subsociety and $\Psi_\alpha = \varnothing$, let $\Lambda_\alpha = (\varnothing, \varnothing, \varnothing)$ and $\varphi(\alpha) = \hat{\varphi}(\alpha)$.

Finally, in all cases, let $J_\alpha = F^{-1}[W_{\Lambda_\alpha}]\setminus(M_{\Pi_\alpha} \cup M_{\Lambda_\alpha})$, and

$$\psi(\alpha) = \max(\varphi(\alpha)+1, \tau(\alpha)),$$

where $\tau(\alpha)$ is the least ordinal such that $\operatorname{ord}(s^\alpha_{\tau(\alpha)})^{-1}\langle m\rangle \geqslant \alpha$ for every

$$m \in e_L[\alpha] \cup \bigcup\{e_{J_\theta}[\alpha]\colon \theta < \alpha\}.$$

For $i = 1, 2, 3, 4$, let Ξ_i be the set of those ordinals $\alpha < \lambda$ for which Case (C_i) occurs. Let $\bar{\Lambda}$ be the sequence $(\Lambda_\alpha\colon \alpha < \lambda)$, let $\Pi = \Pi_\lambda = \bigvee\bar{\Lambda}$, and let $\bar{\Pi}$ denote the sequence $(\Pi_\alpha\colon \alpha \leqslant \lambda)$.

Let us pause here to explain informally the meaning of some of the objects defined above, and, in passing, describe part of the strategy of proof. In constructing $\bar{\Lambda}$ our aim is to ensure that $\Gamma' = \Gamma[M\setminus F^{-1}[W\setminus W_\Pi], W_\Pi]$ has an espousal G. Once this is proved, $(F \restriction (F^{-1}[W\setminus W_\Pi])) \cup G$ is clearly an espousal of Γ. To achieve espousability of Γ', we endeavour to show, for every $m \in M\setminus F^{-1}[W\setminus W_\Pi]$, that either (a) $m \in M_\Pi$ or (b) $|\{\alpha\colon K\langle m\rangle \cap W_{\Lambda_\alpha} \neq \varnothing\}| = \lambda$. We form a list of men for whom one of these conditions has to be fulfilled, each such man being listed λ times. This list is given at the αth step by the λ-enumeration s^α, and it changes at each step. At the $(\alpha+1)$th step we add to it λ copies of $J_\alpha = F^{-1}[W_{\Lambda_\alpha}]\setminus M_{\Pi_{\alpha+1}}$: the elements of J_α are men who are in $M\setminus F^{-1}[W\setminus W_\Pi]$ and for whom (a) is not 'yet' fulfilled, in the sense that they are not in $M_{\Pi_{\alpha+1}}$. The ordinal $\psi(\alpha)$ tells us at which point in s^α we should start interspersing elements from J_α to form the next version $s^{\alpha+1}$ of our list of men.

The steps at which we try to achieve (a) and (b) are those indexed by ordinals in Ξ_3: there we have direct control on the choice of Λ_α. We then choose Λ_α as $\langle x\rangle$, where $x \in K\langle m\rangle\setminus W_{\Pi_\alpha}$, and m is the first man after the first $\hat{\varphi}(\alpha)$ terms in s^α for which $K\langle m\rangle \nsubseteq W_{\Pi_\alpha}$. This is one step towards fulfilling (b) for m. Thus $\varphi(\alpha)$ is a pointer which indicates the last man who was treated in this way. We choose $\psi(\alpha)$ large enough to ensure that any man m who does not satisfy (a) is encountered λ times in this process: it prevents the intervention of men from subsequent J_β's before m has been encountered 'enough' times.

Let us now return to the rigorous proof.

LEMMA 5.2. *$\bar{\Lambda}$ is a ladder in Γ and $\bar{\Pi}$ is a tower in Γ.*

Proof. Trivially $\Pi_0 \lhd \Gamma$. Suppose that $\theta \leqslant \lambda$ and that $\Pi_\alpha \lhd \Gamma$ for every $\alpha < \theta$. This clearly implies that $\bigcup \bar{\Pi}_\theta \lhd \Gamma$, so that $\Pi_\theta \lhd \Gamma$ if θ is a limit ordinal. If θ is a successor ordinal $\eta+1$ then $\Pi_\eta \lhd \Gamma$ by assumption and $\Lambda_\eta \lhd \Gamma_\eta = \Gamma/\Pi_\eta$ by (C_1) and Lemma 4.3 (if $\eta \in \Xi_1$) or by (C_2) or (C_3) or (C_4) (if $\eta \in \Xi_2 \cup \Xi_3 \cup \Xi_4$): this clearly implies that $\Pi_\eta \vee \Lambda_\eta \lhd \Gamma$, that is, $\Pi_\theta \lhd \Gamma$. We have thus shown by transfinite induction that $\Pi_\theta \lhd \Gamma$ for every $\theta \leqslant \lambda$, which proves the lemma.

LEMMA 5.3. *If $\alpha \in \Xi_1$ then Λ_α is a $(<\lambda)$-impediment in Γ_α.*

Proof. Suppose that the lemma is false, and let α be the least element of Ξ_1 such that Λ_α is a $(\geqslant \lambda)$-impediment in Γ_α. Then, for every $\theta \in \Xi_1$ such that $\theta < \alpha$, Λ_θ is a $(<\lambda)$-impediment in Γ_θ and therefore $v(\Lambda_\theta) < \lambda$ by Lemma 4.8. Moreover, $v(\Lambda_\theta)$ is obviously less than λ when $\theta \in \Xi_2 \cup \Xi_3 \cup \Xi_4$. Therefore $v(\Pi_\alpha) < \lambda$ by Lemma 5.2 and Corollary 3.4b. Furthermore, $\Pi_\alpha \lhd \Gamma$ by Lemma 5.2. Hence, by Lemma 4.7, Γ is impeded, contrary to one of our assumptions.

COROLLARY 5.3a. *If $\alpha < \lambda$ then $\varepsilon(\Lambda_\alpha)$, $v(\Lambda_\alpha)$, $\varepsilon(\Pi_\alpha)$, $v(\Pi_\alpha)$ are all less than λ.*

Proof. That $\varepsilon(\Lambda_\alpha) < \lambda$ and $v(\Lambda_\alpha) < \lambda$ is obvious if $\alpha \in \Xi_2 \cup \Xi_3 \cup \Xi_4$ and follows from Lemmas 5.3 and 4.8 if $\alpha \in \Xi_1$. From this and Lemma 5.2 and Corollary 3.4b, it follows that $\varepsilon(\Pi_\alpha) < \lambda$ and $v(\Pi_\alpha) < \lambda$.

LEMMA 5.4. *$|J_\alpha| \leqslant \lambda$ for every $\alpha < \lambda$.*

Proof. Since $\Pi_{\alpha+1} \lhd \Gamma$ by Lemma 5.2, $F \restriction (M_{\Pi_{\alpha+1}} \setminus L)$ is an espousal of $\Pi_{\alpha+1} - (M_{\Pi_{\alpha+1}} \cap L)$, and is therefore also, by the definition of J_α, an espousal of $\Pi_{\alpha+1} - (M_{\Pi_{\alpha+1}} \cap L) - F[J_\alpha]$. Since $v(\Pi_{\alpha+1}) < \lambda$ by Corollary 5.3a and $|M_{\Pi_{\alpha+1}} \cap L| \leqslant |L| = \lambda$, it follows by Corollary 3.8a that $|F[J_\alpha]| \leqslant \lambda$, and hence $|J_\alpha| \leqslant \lambda$.

COROLLARY 5.4a. $\operatorname{dom} s^\alpha = \lambda$ *for every $\alpha < \lambda$.*

From Corollary 5.4a and the definition of $\varphi(\alpha)$ follows immediately:

COROLLARY 5.4b. $\varphi(\alpha) < \lambda$ *for every $\alpha < \lambda$.*

COROLLARY 5.4c. *If $\alpha < \lambda$ then $\psi(\alpha) < \lambda$.*

Proof. If $m \in L \cup \hat{J}_\alpha$ then, by Corollary 5.4a, the least ordinal $\tau(\alpha, m)$ such that $\operatorname{ord}((s^\alpha_{\tau(\alpha,m)})^{-1}\langle m \rangle) \geqslant \alpha$ is less than λ. Therefore

$$\lambda > \sup\{\tau(\alpha, m) \colon m \in e_L[\alpha] \cup \bigcup \{e_{J_\theta}[\alpha] \colon \theta < \alpha\}\} = \tau(\alpha).$$

From this and Corollary 5.4b, it follows that $\psi(\alpha) < \lambda$.

Corollary 5.4c implies that $\hat{\psi}(\alpha) < \lambda$ for every $\alpha < \lambda$ and so yields the following further corollary.

COROLLARY 5.4d. *Statement* (5.1) *holds for every* $\alpha < \lambda$.

LEMMA 5.5. $\Xi_4 = \varnothing$.

Proof. Suppose that $\alpha \in \Xi_4$ for some $\alpha < \lambda$. Then $(s^\alpha)^{-1}[(L \cup \hat{J}_\alpha) \setminus M_{\Pi_\alpha}] \subseteq \hat{\varphi}(\alpha)$ and therefore, by Corollary 5.4b, $|(s^\alpha)^{-1}[(L \cup \hat{J}_\alpha) \setminus M_{\Pi_\alpha}]| < \lambda$. Since s^α is a λ-enumeration of $L \cup \hat{J}_\alpha$, this implies that

$$L \cup \hat{J}_\alpha \subseteq M_{\Pi_\alpha}. \tag{5.2}$$

Moreover,

$$F^{-1}[W_{\Lambda_\theta}] \subseteq J_\theta \cup M_{\Pi_\theta} \cup M_{\Lambda_\theta} \subseteq \hat{J}_\alpha \cup M_{\Pi_\alpha}$$

for each $\theta < \alpha$. Therefore $F^{-1}[W_{\Pi_\alpha}] \subseteq \hat{J}_\alpha \cup M_{\Pi_\alpha}$: from this and (5.2) it follows that $L \cup F^{-1}[W_{\Pi_\alpha}] \subseteq M_{\Pi_\alpha}$ and therefore $F \restriction M_\alpha$ is an espousal of Γ_α. Therefore, by Lemma 3.3, $(\Gamma - L - V)[M_\alpha, F[M_\alpha]]$ is critical and so $\varepsilon(\Gamma_\alpha) = 0$. By Corollary 5.3a, $\varepsilon(\Pi_\alpha) < \lambda$. By Lemma 5.2, $\Pi_\alpha \lhd \Gamma$. Hence, by Corollary 3.4c, $\varepsilon(\Gamma) \leqslant \varepsilon(\Pi_\alpha) + \varepsilon(\Gamma_\alpha) < \lambda$, contradicting the definition of λ.

LEMMA 5.6. *If* $\alpha \in \Xi_2$ *then* $\alpha + 1 \in \Xi_3$.

Proof. Since $\alpha \in \Xi_2$, Λ_α is a maximal critical saturated subsociety of Γ_α. By Lemma 5.5, $\alpha + 1 \in \Xi_1 \cup \Xi_2 \cup \Xi_3$. If $\alpha + 1 \in \Xi_1$ then $\Gamma_{\alpha+1} = \Gamma_\alpha / \Lambda_\alpha$ is impeded and therefore Γ_α is impeded by Corollary 4.7c, contradicting $\alpha \in \Xi_2$. Now suppose that $\alpha + 1 \in \Xi_2$. Then $\Lambda_{\alpha+1}$ is a non-empty critical saturated subsociety of $\Gamma_{\alpha+1} = \Gamma_\alpha / \Lambda_\alpha$. Therefore $\Lambda_\alpha \vee \Lambda_{\alpha+1} \lhd \Gamma_\alpha$ by Lemma 3.2 and, since Λ_α is a critical saturated subsociety of Γ_α and $\Lambda_{\alpha+1} = (\Lambda_\alpha \vee \Lambda_{\alpha+1})/\Lambda_\alpha$ is critical, it follows that $\Lambda_\alpha \vee \Lambda_{\alpha+1}$ is critical by Corollary 3.4a. This contradicts the maximality of Λ_α.

LEMMA 5.7. $\bar{\Pi}$ *is a* λ-*fortress in* Γ.

Proof. By Lemma 5.2, $\bar{\Pi}$ is a λ-tower in Γ. Suppose that $\alpha < \lambda$. By Lemma 5.5, $\alpha \in \Xi_1 \cup \Xi_2 \cup \Xi_3$. If $\alpha \in \Xi_1$ then Λ_α is a $(<\lambda)$-impediment in Γ/Π_α by Lemma 5.3. If $\alpha \in \Xi_2$ then Λ_α is critical and if $\alpha \in \Xi_3$ then Λ_α is maidenly.

LEMMA 5.8. Ξ_1 *is not* λ-*stationary.*

Proof. Since Γ is unimpeded, $\bar{\Pi}$ is not an impeding λ-tower in Γ and therefore, by Lemmas 5.3 and 5.7, Ξ_1 is not λ-stationary.

LEMMA 5.9. $|\Xi_3| = \lambda$.

Proof. By Lemma 5.8, $\sup(\lambda \setminus \Xi_1) = \lambda$ and so $|\lambda \setminus \Xi_1| = \lambda$, that is, by Lemma 5.5, $|\Xi_2 \cup \Xi_3| = \lambda$. Moreover, $|\Xi_3| \geqslant |\Xi_2|$ by Lemma 5.6. Hence $|\Xi_3| = \lambda$.

COROLLARY 5.9a. $\sup \varphi[\Xi_3] = \lambda$.

Proof. If $\alpha, \beta \in \Xi_3$ and $\alpha < \beta$ then, by (C_3), $\varphi(\beta) > \varphi'(\beta) \geqslant \hat{\varphi}(\beta) \geqslant \varphi(\alpha)$. The corollary follows from this observation and Lemma 5.9 and Corollary 5.4b.

Let $J = \bigcup \{J_\theta : \theta < \lambda\}$ and $I = (L \cup J) \setminus M_\Pi$.

LEMMA 5.10. $M \setminus F^{-1}[W \setminus W_\Pi] = M_\Pi \cup J \cup L$.

Proof. By Lemma 5.2, $\Pi \lhd \Gamma$ and therefore $F[M_\Pi] \subseteq W_\Pi$. Since $F[J_\alpha] \subseteq W_{\Lambda_\alpha}$ for each $\alpha < \lambda$ by the definition of J_α, it follows that $F[J] \subseteq W_\Pi$. Moreover, $F[L] = \varnothing$. Therefore $F[M_\Pi \cup J \cup L] \subseteq W_\Pi$, and hence $M_\Pi \cup J \cup L \subseteq M \setminus F^{-1}[W \setminus W_\Pi]$. Now suppose that $m \in M \setminus F^{-1}[W \setminus W_\Pi]$. If $m \notin L$ then $F(m)$ is defined and is in W_Π, and hence $F(m) \in W_{\Lambda_\alpha}$ for some $\alpha < \lambda$. This implies, by the definition of J_α, that $m \in M_{\Pi_{\alpha+1}} \cup J_\alpha \subseteq M_\Pi \cup J$. Hence $M \setminus F^{-1}[W \setminus W_\Pi] \subseteq M_\Pi \cup J \cup L$.

In defining s^α we observed that if $\xi < \hat{\psi}(\alpha)$, then the least ordinal $\theta(\xi)$ such that $\xi < \psi(\theta(\xi))$ exists. In fact, from Corollaries 5.4c and 5.9a and the fact that $\psi(\alpha) > \varphi(\alpha)$ for each $\alpha < \lambda$, it follows that $\sup \psi[\lambda] = \lambda$: therefore $\theta(\xi)$ exists and is less than λ for every $\xi < \lambda$. We can therefore define a sequence s of length λ by letting $s(\xi) = s^{\theta(\xi)}(\xi)$ for every $\xi < \lambda$.

LEMMA 5.11. *If $\alpha < \lambda$ then $s_{\psi(\alpha)} = s^\alpha_{\psi(\alpha)}$.*

Proof. Suppose that $\xi < \psi(\alpha)$. Then either $\xi < \hat{\psi}(\alpha)$ and consequently $s(\xi) = s^{\theta(\xi)}(\xi) = s^\alpha(\xi)$ by Corollary 5.4d, or $\hat{\psi}(\alpha) \leqslant \xi < \psi(\alpha)$, in which case $\theta(\xi) = \alpha$ and so $s^\alpha(\xi) = s^{\theta(\xi)}(\xi) = s(\xi)$.

LEMMA 5.12. *If $\chi \in s^{-1}[I]$ then $\chi = \varphi'(\alpha)$ for some $\alpha \in \Xi_3$.*

Proof. Let α be the least ordinal such that $\varphi(\alpha) > \chi$. (By Corollary 5.9a, α exists.) Then $\varphi(\alpha) > \chi \geqslant \hat{\varphi}(\alpha)$ and so $\alpha \in \Xi_3$ (because otherwise the definition of $\varphi(\alpha)$ would give $\varphi(\alpha) = \hat{\varphi}(\alpha)$). Therefore $\varphi(\alpha) = \varphi'(\alpha)+1$. Since $\chi < \varphi(\alpha) < \psi(\alpha)$, it follows by Lemma 5.11 that $s(\chi) = s^\alpha(\chi) \in \operatorname{rge} s^\alpha = L \cup \hat{J}_\alpha$. Since $s(\chi) \in I$ it follows that $s(\chi) \notin M_{\Pi_\alpha}$. Therefore $s^\alpha(\chi) = s(\chi) \in (L \cup \hat{J}_\alpha) \setminus M_{\Pi_\alpha}$. Moreover, $\chi \geqslant \hat{\varphi}(\alpha)$ and so $\chi \in \Psi_\alpha$ and therefore $\chi \geqslant \varphi'(\alpha)$. Since $\varphi'(\alpha)+1 = \varphi(\alpha) > \chi \geqslant \varphi'(\alpha)$, it follows that $\chi = \varphi'(\alpha)$.

LEMMA 5.13. *If $m \in I$ then $|\{\alpha \in \Xi_3 : K\langle m\rangle \cap W_{\Lambda_\alpha} \neq \varnothing\}| = \lambda$.*

Proof. Since $m \in I$ it follows that $m \in L \cup \hat{J}_\gamma$ for some $\gamma < \lambda$ and consequently $m \in e_L[\delta] \cup \bigcup \{e_{J_\theta}[\delta] : \theta < \gamma\}$ for some $\delta < \lambda$ by Lemma 5.4. If $\max\{\gamma, \delta\} < \zeta < \lambda$ then $m \in e_L[\zeta] \cup \bigcup \{e_{J_\theta}[\zeta] : \theta < \zeta\}$ and so

$$\zeta \leqslant \operatorname{ord}((s^\zeta_{\tau(\zeta)})^{-1}\langle m\rangle) \leqslant \operatorname{ord}((s^\zeta_{\psi(\zeta)})^{-1}\langle m\rangle).$$

Since this holds for every ζ such that $\max\{\gamma, \delta\} < \zeta < \lambda$, it follows that $\operatorname{ord}(s^{-1}\langle m\rangle) = \lambda$ by Lemma 5.11.

Let $\chi \in s^{-1}\langle m\rangle$. Then $\chi = \varphi'(\alpha)$ for some $\alpha \in \Xi_3$ by Lemma 5.12. Since $\varphi'(\alpha) < \varphi(\alpha) < \psi(\alpha)$, it follows from Lemma 5.11 that $s^\alpha(\varphi'(\alpha)) = s(\varphi'(\alpha)) = s(\chi) = m$. By ($C_3$), $\Lambda_\alpha = \langle x\rangle$ for some $x \in K\langle s^\alpha(\varphi'(\alpha))\rangle = K\langle m\rangle$ and therefore

$$K\langle m\rangle \cap W_{\Lambda_\alpha} \neq \varnothing.$$

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We have thus shown that $\operatorname{ord}(s^{-1}\langle m\rangle) = \lambda$ and that for each $\chi \in s^{-1}\langle m\rangle$ there exists $\alpha \in \Xi_3$ such that $\varphi'(\alpha) = \chi$ and $K\langle m\rangle \cap W_{\Lambda_\alpha} \neq \varnothing$. This proves the lemma.

By Lemma 5.8 there exists a closed unbounded subset Θ of λ such that $\Theta \cap \Xi_1 = \varnothing$, which implies that $\Theta \subseteq \Xi_2 \cup \Xi_3$ by Lemma 5.5. Since Γ is unimpeded, $0 \notin \Xi_1$ and hence we may assume that $0 \in \Theta$. Let $|I| = \mu$, and let $(m_\alpha\colon \alpha < \mu)$ be an enumeration of I. By Lemma 5.4, $\mu \leqslant \lambda$. We construct functions $f\colon \lambda \to \lambda$, $g\colon \mu \to \lambda$ as follows. Let $f(0) = 0$. If α is a limit ordinal and $f(\xi)$ has been defined for $\xi < \alpha$, let $f(\alpha) = \sup f[\alpha]$. Now let $f(\alpha)$ be defined for some $\alpha < \mu$. By Lemma 5.13 there exists an ordinal $g(\alpha)$ such that $f(\alpha) < g(\alpha) < \lambda$ and $K\langle m_\alpha\rangle \cap W_{\Lambda_{g(\alpha)}} \neq \varnothing$. Define $f(\alpha+1)$ to be the least element of Θ which is greater than $g(\alpha)$. If $\mu \leqslant \alpha < \lambda$ and $f(\alpha)$ has been defined, define $f(\alpha+1)$ to be the least element of Θ which is greater than $f(\alpha)$. For each $\alpha < \lambda$ define $\Delta_\alpha = \Pi_{f(\alpha+1)}/\Pi_{f(\alpha)}$.

LEMMA 5.14. $f[\lambda] \subseteq \Theta$.

Proof. This follows directly from the definition of f and from the fact that Θ is closed.

LEMMA 5.15. *If $\alpha < \lambda$ then Δ_α is unimpeded.*

Proof. By Lemma 5.14, $f(\alpha) \notin \Xi_1$ and so $\Gamma_{f(\alpha)}$ is unimpeded. By Lemma 5.2, $\Delta_\alpha \lhd \Gamma_{f(\alpha)}$. Hence Δ_α is unimpeded by Corollary 4.4a.

LEMMA 5.16. *Δ_α is espousable for every $\alpha < \lambda$.*

Proof. Let $f(\alpha+1) = f(\alpha)+\zeta$. It is easily inferred from Lemma 5.2 that $(\Lambda_{f(\alpha)+\rho}\colon \rho < \zeta)$ is a ladder in $\Gamma_{f(\alpha)}$. Therefore, by Corollaries 3.4b and 5.3a, $\lambda > \varepsilon(\bigvee\{\Lambda_{f(\alpha)+\rho}\colon \rho < \zeta\}) \geqslant \varepsilon(\Delta_\alpha)$. Since we are assuming that Theorem 5.1′ is true for societies whose essential size is less than λ, the lemma follows by Lemma 5.15.

For each $\alpha < \mu$ choose an element w_α of the non-empty set $K\langle m_\alpha\rangle \cap W_{\Lambda_{g(\alpha)}}$. Then $w_\alpha \in W_{\Delta_\alpha}$ since $f(\alpha) < g(\alpha) < f(\alpha+1)$.

LEMMA 5.17. *If $\alpha < \mu$ then $\Delta_\alpha - \{w_\alpha\}$ is espousable.*

Proof. By Lemmas 5.5 and 5.14, $f(\alpha) \in \Xi_2 \cup \Xi_3$. We define Λ to be $\Lambda_{f(\alpha)}$ if $f(\alpha) \in \Xi_2$ and to be the empty society if $f(\alpha) \in \Xi_3$. In both these cases, Λ is a maximal critical saturated subsociety of $\Gamma_{f(\alpha)}$. Since $\Lambda \leqslant \Delta_\alpha \lhd \Gamma_{f(\alpha)}$ by Lemma 5.2, it follows by Lemma 3.1 that Λ is also a maximal critical saturated subsociety of Δ_α. By Lemmas 5.16 and 3.6(i) it follows that Λ is the greatest critical saturated subsociety of Δ_α. Since $w_\alpha \in W_{\Lambda_{g(\alpha)}}$ and $g(\alpha) > f(\alpha)$ and $W_\Lambda \subseteq W_{\Lambda_{f(\alpha)}}$, it follows that $w_\alpha \notin W_\Lambda$. The lemma follows now by Lemmas 5.16 and 3.6(ii).

We can now conclude the proof of Theorem 5.1′ when λ is regular. Let H be an espousal of I given by $H(m_\alpha) = w_\alpha$. By Lemmas 5.16 and 5.17 we can choose an espousal E_α of Δ_α for every $\alpha < \lambda$ so that $w_\alpha \notin \operatorname{rge} E_\alpha$ when $\alpha < \mu$. Let $E = \bigcup\{E_\alpha\colon \alpha < \lambda\}$. Finally, let $G = H \cup E$. Then $\operatorname{rge} G \subseteq W_\Pi$ and, by Lemma 5.10, $\operatorname{dom} G = M \setminus F^{-1}[W \setminus W_\Pi]$. Hence $G \cup (F \restriction (F^{-1}[W \setminus W_\Pi]))$ is an espousal of Γ.

Case II: λ is singular

LEMMA 5.18. *If $\kappa < \lambda$, $\Delta \lhd \Pi$, Π is $(> \kappa)$-unimpeded, Π/Δ is unimpeded, and Δ is espousable, then Π is unimpeded.*

Proof. Suppose that there exists a μ-impediment Λ in Π. Then $\mu \leqslant \kappa$ since Π is $(> \kappa)$-unimpeded, and hence $\nu(\Lambda) \leqslant \kappa$, by Lemma 4.8. Moreover, $\Lambda \cap \Delta \lhd \Lambda$ since $\Delta \lhd \Pi$, and hence $\nu(\Lambda/\Lambda \cap \Delta) \leqslant \kappa$ by Lemma 3.11. By Lemma 4.3, $\Lambda \lhd \Pi$ and therefore $\Lambda/\Lambda \cap \Delta \lhd \Pi/\Delta$, which is unimpeded. Therefore $\Lambda/\Lambda \cap \Delta$ is unimpeded by Corollary 4.4a. By Corollary 4.7a, $\varepsilon(\Lambda/\Lambda \cap \Delta) \leqslant \kappa < \lambda$, and therefore, by the induction hypothesis on Theorem 5.1′, $\Lambda/\Lambda \cap \Delta$ has an espousal H. Since $\Lambda \lhd \Pi$ and Δ has an espousal G (say), it follows that $G \restriction M_{\Lambda \cap \Delta}$ is an espousal of $\Lambda \cap \Delta$ and therefore $H \cup (G \restriction M_{\Lambda \cap \Delta})$ is an espousal of Λ, which contradicts Lemma 4.9.

REMARK. In a sense, the above proof of the lemma is not satisfactory. One would like to prove a much stronger result:

LEMMA 5.18′. *If Π has unimpeded subsocieties Π_1, Π_2 such that $\Pi_1 \vee \Pi_2 = \Pi$ and $\Pi_1 \cap \Pi_2$ is empty, then Π is unimpeded.*

While Lemma 5.18′ is a straightforward corollary of Theorem 5.1 (see Corollary 6.1a below for a strengthening of it), the authors cannot produce a reasonably short direct proof for it. Compare with [1, Lemma 12], where an analogue of Lemma 5.18′ is proved in which the concept 'unimpeded' is replaced by '$(\leqslant \aleph_0)$-unimpeded'.

The next lemma is a crucial step in the proof for Case II. Its proof follows closely the proof for Case I. Certain steps in the proof will be virtually identical to their counterparts in Case I, and we shall state them as 'assertions', referring for proofs to the corresponding lemmas in the proof for Case I.

LEMMA 5.19. *If Σ is unimpeded, $B \subseteq M_\Sigma$, $X \subseteq W_\Sigma$, and $|B \cup X| < \lambda$, then there exists a saturated subsociety Δ of Σ such that $X \subseteq W_\Delta$, $\Sigma[M_\Delta \cup B, W_\Delta]$ is espousable, and Σ/Δ is unimpeded.*

Proof. Since $\Sigma - M_\Sigma$ is espousable, it follows by Corollary 4.6a that Σ is espousable if M_Σ is finite and so we can take $\Delta = \Sigma$ in this case. We may therefore assume that M_Σ is infinite. Let A be an infinite set such that $B \subseteq A \subseteq M_\Sigma$ and $|A| < \lambda$. Let $|A \cup X| = \kappa$ and $|X| = \nu$. Let $(x_\alpha\colon \alpha < \nu)$ be an enumeration of X and $s = (a_\alpha\colon \alpha < \kappa^+)$ be a κ^+-enumeration of A.

We construct a κ^+-tower $\bar{\Pi}$ in Σ by defining its ladder $\bar{\Lambda}$. The societies Λ_α are defined inductively, and along with them we define ordinals $\zeta(\alpha)$ less than κ^+. Let $\Lambda_\alpha = \langle x_\alpha \rangle$ for $\alpha < \nu$ and let $\zeta(\alpha) = 0$ for $\alpha \leqslant \nu$. If $\alpha < \kappa^+$ is a limit ordinal and $\zeta(\beta) < \kappa^+$ is defined for all $\beta < \alpha$, we let $\zeta(\alpha) = \sup\{\zeta(\beta)\colon \beta < \alpha\}$. Now let $\nu \leqslant \alpha < \kappa^+$ and suppose that $\zeta(\alpha)\,(< \kappa^+)$ and Λ_β for all $\beta < \alpha$ are defined. Let $\Pi_\alpha = \bigvee \{\Lambda_\beta\colon \beta < \alpha\}$ and $\Sigma_\alpha = \Sigma/\Pi_\alpha$. We define $\zeta(\alpha+1)$ and Λ_α by considering four cases:

(C_1) if Σ_α is impeded, let Λ_α be an impediment in Σ_α and $\zeta(\alpha+1) = \zeta(\alpha)$;

(C_2) if Σ_α is unimpeded and contains a non-empty critical saturated subsociety, let Λ_α then be a maximal such subsociety (which exists by Corollary 3.5b) and let $\zeta(\alpha+1) = \zeta(\alpha)$;

(C_3) if Σ_α is unimpeded and has no non-empty critical saturated subsociety, and $K_\Sigma[A] \nsubseteq W_{\Pi_\alpha}$, let θ be the least ordinal greater than or equal to $\zeta(\alpha)$ such that $K_\Sigma\langle a_\theta\rangle \nsubseteq W_{\Pi_\alpha}$, choose any element w of $K_\Sigma\langle a_\theta\rangle \setminus W_{\Pi_\alpha}$, and define $\Lambda_\alpha = \langle w\rangle$ and $\zeta(\alpha+1) = \theta+1$;

(C_4) if none of the conditions in (C_1), (C_2), (C_3) hold, define $\Lambda_\alpha = (\varnothing, \varnothing, \varnothing)$ and $\zeta(\alpha+1) = \zeta(\alpha)$.

Let $\Pi = \bigcup \bar{\Pi}$, and for $k = 1, 2, 3, 4$ let Ξ_k be the set of ordinals α ($\nu \leqslant \alpha < \kappa^+$) for which Case ($C_k$) occurs.

ASSERTION 5.19a (Lemma 5.2). *$\bar{\Lambda}$ is a ladder in Σ and $\bar{\Pi}$ is a tower in Σ.*

ASSERTION 5.19b. $X = W_{\Pi_\nu}$.

Proof. This follows directly from the definition of Λ_α for $\alpha < \nu$.

ASSERTION 5.19c (Lemma 5.3). *Λ_α is a ($<\kappa^+$)-impediment in Σ_α for every $\alpha \in \Xi_1$.*

From Assertion 5.19c and Lemma 4.8 it follows that $\nu(\Lambda_\alpha) < \kappa^+$ for every $\alpha \in \Xi_1$, and this clearly implies:

ASSERTION 5.19d. *$\nu(\Lambda_\alpha) < \kappa^+$ for every $\alpha < \kappa^+$.*

ASSERTION 5.19e. *If $\Xi_4 \neq \varnothing$ then setting $\Delta = \Pi$ satisfies the conditions in Lemma* 5.19.

Proof. Let α be an element of Ξ_4. Then, by (C_4), $\beta \in \Xi_4$ and Λ_β is empty for $\alpha \leqslant \beta < \kappa^+$, and hence $\Pi = \Pi_\alpha$. By Assertions 5.19a and 5.19d and Corollary 3.4b, $\kappa^+ > \nu(\Pi_\alpha) = \nu(\Pi)$. By Assertion 5.19a, $\Pi \lhd \Sigma$ and therefore Π is unimpeded by Corollary 4.4a. Therefore, by Corollary 4.7a, $\varepsilon(\Pi) \leqslant \nu(\Pi) < \kappa^+ < \lambda$, and hence by the inductive hypothesis on Theorem 5.1′, Π is espousable. Since $\alpha \in \Xi_4$ it follows that $K_\Sigma[A] \subseteq W_\Pi$ and $\Sigma/\Pi = \Sigma_\alpha$ is unimpeded. Since Σ/Π is unimpeded and $K_\Sigma[A] \subseteq W_\Pi$, it follows by Corollary 4.5a that $B \subseteq A \subseteq M_\Pi$ and so $\Sigma[M_\Pi \cup B, W_\Pi] = \Pi$ which is espousable. Moreover, $X = W_{\Pi_\nu} \subseteq W_{\Pi_\alpha} = W_\Pi$.

By Assertion 5.19e we may assume that

$$\Xi_4 = \varnothing. \tag{5.3}$$

By (5.3) and Assertions 5.19a and 5.19c, $\bar{\Pi}$ is a κ^+-fortress.

ASSERTION 5.19f (Lemma 5.8). *Ξ_1 is not κ^+-stationary.*

ASSERTION 5.19g (Lemma 5.9). $|\Xi_3| = \kappa^+$.

Let $I = A \setminus M_\Pi$. The following assertion is parallel to Lemma 5.13, but its proof is much easier:

ASSERTION 5.19h. *If $a \in I$ then $|\{\alpha < \kappa^+ : K_\Sigma\langle a\rangle \cap W_{\Lambda_\alpha} \neq \varnothing\}| = \kappa^+$.*

Proof. Let $\theta < \kappa^+$ be any ordinal such that $a_\theta = a$. Let $\delta = \sup\{\gamma\colon \zeta(\gamma) \leqslant \theta\}$. Since $\zeta(\alpha+1) > \zeta(\alpha)$ whenever $\alpha \in \Xi_3$, and $\zeta(\beta) \geqslant \zeta(\alpha)$ whenever $v \leqslant \alpha \leqslant \beta < \kappa^+$, it follows by Assertion 5.19g that $\delta < \kappa^+$ and $\sup \zeta[\kappa^+] = \kappa^+$. Since $\zeta(\alpha+1) > \zeta(\alpha)$ only when $\alpha \in \Xi_3$, it follows that $\delta \in \Xi_3$. If τ is the first ordinal such that $\zeta(\delta) \leqslant \tau < \kappa^+$ and $K_\Sigma\langle a_\tau\rangle \nsubseteq W_{\Pi_\delta}$ then, by (C_3), $\zeta(\delta+1) = \tau+1$, and consequently $\tau+1 > \theta$ by the definition of δ, that is, $\tau \geqslant \theta$. Since $a \notin M_\Pi$, clearly $a \notin M_{\Pi_\delta}$. If we had $K_\Sigma\langle a\rangle \subseteq W_{\Pi_\delta}$ then, by Corollary 4.5a, $\Sigma_\delta = \Sigma/\Pi_\delta$ would be impeded, contradicting the fact that $\delta \in \Xi_3$. Thus $K_\Sigma\langle a\rangle = K_\Sigma\langle a_\theta\rangle \nsubseteq W_{\Pi_\delta}$, and, furthermore, $\zeta(\delta) \leqslant \theta$ because $\zeta(\delta) = \sup\{\zeta(\gamma)\colon \gamma < \delta\}$ if δ is a limit ordinal. Therefore $\tau \leqslant \theta$ by the definition of τ, and hence $\tau = \theta$, and consequently $a_\tau = a$. By (C_3), $\Lambda_\delta = \langle w\rangle$ for some $w \in K_\Sigma\langle a_\tau\rangle = K_\Sigma\langle a\rangle$, and thus $K_\Sigma\langle a\rangle \cap W_{\Lambda_\delta} \neq \varnothing$. Since this argument holds for each θ for which $a_\theta = a$, the assertion follows (because $\sup \zeta[\kappa^+] = \kappa^+$ and so suitable choices of θ will yield κ^+ distinct values of δ).

By Assertion 5.19f there exists a closed unbounded subset Θ of κ^+ such that $\Theta \cap \Xi_1 = \varnothing$. Since Σ is unimpeded, $0 \notin \Xi_1$, and hence we may assume that $0 \in \Theta$.

Enumerate I as $\{m_\alpha\colon 1 \leqslant \alpha < \mu\}$, where $\mu = 1+|I| \leqslant 1+|A| \leqslant \kappa$. We construct functions $f\colon \kappa^+ \to \kappa^+$ and $g\colon \mu\backslash\{0\} \to \kappa^+$ as follows. Let $f(0) = 0$ and let $f(1)$ be an element of Θ such that $f(1) > v$. If α is a limit ordinal less than κ^+ and $f(\xi)$ has been defined for $\xi < \alpha$, let $f(\alpha) = \sup f[\alpha]$. Now suppose that $1 \leqslant \alpha < \mu$ and $f(\alpha)$ is defined. By Assertion 5.19h there exists $g(\alpha)$ such that $f(\alpha) < g(\alpha) < \kappa^+$ and $K\langle m_\alpha\rangle \cap W_{\Lambda_{g(\alpha)}} \neq \varnothing$. Define $f(\alpha+1)$ to be the least element of Θ which is greater than $g(\alpha)$. If $\mu \leqslant \alpha < \kappa^+$ and $f(\alpha)$ has been defined, define $f(\alpha+1)$ to be the least element of Θ which is greater than $f(\alpha)$. Let $\Delta_\alpha = \Pi_{f(\alpha+1)}/\Pi_{f(\alpha)}$ for each $\alpha < \kappa^+$.

ASSERTION 5.19i (Lemma 5.14). $f[\kappa^+] \subseteq \Theta$.

ASSERTION 5.19j (Lemma 5.16). *Δ_α is espousable for every $\alpha < \kappa^+$.*

For each $\alpha \in \mu\backslash\{0\}$ choose an element w_α of the non-empty set $K\langle m_\alpha\rangle \cap W_{\Lambda_{g(\alpha)}}$.

ASSERTION 5.19k (Lemma 5.17). *If $1 \leqslant \alpha < \mu$ then $\Delta_\alpha - \{w_\alpha\}$ is espousable.*

By Assertions 5.19j and 5.19k, we can select an espousal E_α of Δ_α for each $\alpha < \kappa^+$, so that $w_\alpha \notin \operatorname{rge} E_\alpha$ when $1 \leqslant \alpha < \mu$. Since $|A| \leqslant \kappa$, there is an ordinal η such that $\max(1, \mu) \leqslant \eta < \kappa^+$ and $A \cap M_\Pi \subseteq M_{\Pi_{f(\eta)}}$. Let $\Delta = \Pi_{f(\eta)}$. Then

$$M_\Delta \cup I = M_\Delta \cup (A\backslash M_\Pi) = M_\Delta \cup A,$$

since $A \cap M_\Pi \subseteq M_\Delta$. Since Θ is closed, $\bigcup\{\operatorname{dom} E_\alpha\colon \alpha < \eta\} = M_\Delta$. Let

$$E = \{(m_\alpha, w_\alpha)\colon 1 \leqslant \alpha < \mu\} \cup \bigcup\{E_\alpha\colon \alpha < \eta\}.$$

Then $\operatorname{dom} E = M_\Delta \cup I = M_\Delta \cup A$ and so E is an espousal of $\Sigma[M_\Delta \cup A, W_\Delta]$, and therefore $E \restriction (M_\Delta \cup B)$ is an espousal of $\Sigma[M_\Delta \cup B, W_\Delta]$. Moreover, $v < f(1) \leqslant f(\eta)$ and so $X \subseteq W_{\Pi_{f(\eta)}} = W_\Delta$ by Assertion 5.19b.

By Assertion 5.19i, $f(\eta) \notin \Xi_1$, and since $\Delta = \Pi_{f(\eta)}$ it follows that Σ/Δ is unimpeded. By Assertion 5.19a, Δ is saturated. Thus Δ satisfies all the requirements in the lemma.

LEMMA 5.20. *If $\aleph_0 \leqslant \kappa < \lambda$, $\Sigma' \leqslant \Sigma$, $|\Sigma'| \leqslant \kappa$, and Σ is unimpeded, then Σ has an espousable subsociety Σ'' such that $\Sigma' \leqslant \Sigma''$, $|\Sigma''| \leqslant \kappa$, and Σ/Σ'' is unimpeded.*

Proof. Write $\Sigma' = \Sigma[B, X]$. Let Δ be a subsociety of Σ as in the conclusion of Lemma 5.19, and let E be an espousal of $\Sigma[M_\Delta \cup B, W_\Delta]$. Let $M'' = B \cup E^{-1}[X]$, $W'' = X \cup E[B]$, $\Sigma'' = \Sigma[M'', W'']$, $\Lambda = \Sigma - M'' - W''$, and $\Pi = (\Sigma/\Delta) - (B \setminus M_\Delta)$. Then $E^{-1}[X] \subseteq \operatorname{dom} E = B \cup M_\Delta$ and therefore $M'' \cup M_\Delta = B \cup M_\Delta$, so that

$$M_\Pi = (M_\Sigma \setminus M_\Delta) \setminus (B \setminus M_\Delta) = M_\Sigma \setminus (B \cup M_\Delta) = M_\Sigma \setminus (M'' \cup M_\Delta) = M_\Lambda \setminus M_\Delta = M_{\Lambda/(\Lambda \cap \Delta)}.$$

Moreover, $X \subseteq W_\Delta$ and $E[B] \subseteq \operatorname{rge} E \subseteq W_\Delta$ and therefore $W'' \subseteq W_\Delta$, so that

$$W_\Pi = W_\Sigma \setminus W_\Delta = (W_\Sigma \setminus W'') \setminus (W_\Delta \setminus W'') = W_{\Lambda/(\Lambda \cap \Delta)}.$$

Therefore $\Lambda/(\Lambda \cap \Delta) = \Pi = (\Sigma/\Delta) - (B \setminus M_\Delta)$, which is unimpeded by Corollary 4.4a since Σ/Δ is unimpeded. If Λ was μ-impeded for some $\mu > \kappa$ then $\Sigma - W''$ would be μ-impeded, by Corollary 4.4a, and so (since $|W''| \leqslant \kappa < \mu$) Σ would be impeded by Corollary 4.7b. Therefore Λ is $(>\kappa)$-unimpeded. Moreover, since $\Delta \lhd \Sigma$ and $E^{-1}[W''] \subseteq M''$, it follows that $\Lambda \cap \Delta \lhd \Lambda$ and $E \restriction M_{\Lambda \cap \Delta}$ is an espousal of $\Lambda \cap \Delta$. Hence, by Lemma 5.18, Λ is unimpeded, that is, Σ/Σ'' is unimpeded. Moreover, $|\Sigma''| = |B \cup E^{-1}[X] \cup X \cup E[B]| \leqslant \kappa$ since $|X \cup B| = |\Sigma'| \leqslant \kappa$, and Σ'' is espousable since it has an espousal $E \restriction M''$.

Corollary 5.20a. *If $\aleph_0 \leqslant \kappa < \lambda$, $\Delta \leqslant \Pi' \leqslant \Pi$, Π/Δ is unimpeded, and $|\Pi'| \leqslant \kappa$, then Π has a subsociety Π'' such that $\Pi' \leqslant \Pi''$, Π''/Δ is espousable, $|\Pi''| \leqslant \kappa$, and Π/Π'' is unimpeded.*

Proof. Let $\Sigma = \Pi/\Delta$ and $\Sigma' = \Pi'/\Delta$. Then, by Lemma 5.20, Σ has an espousable subsociety Σ'' such that $\Sigma' \leqslant \Sigma''$, $|\Sigma''| \leqslant \kappa$, and Σ/Σ'' is unimpeded. If $\Pi'' = \Sigma'' \vee \Delta$ then $\Pi' = \Sigma' \vee \Delta \leqslant \Sigma'' \vee \Delta = \Pi''$ and $\Pi''/\Delta = (\Sigma'' \vee \Delta)/\Delta = \Sigma''$, which is espousable, and $|\Pi''| = |\Sigma''| + |\Delta| \leqslant |\Sigma''| + |\Pi'| \leqslant \kappa$ and

$$\Pi/\Pi'' = (\Sigma \vee \Delta)/(\Sigma'' \vee \Delta) = \Sigma/\Sigma'',$$

which is unimpeded.

Corollary 5.20a is the culmination of a first stage of the proof of Case II. It will be the only result from the first stage used in the second stage. This stage follows the proof of the compactness theorem for singular cardinals **[11]**. (See **[6]** for a shorter proof.)

Since we assume that $\varepsilon(\Gamma) = \lambda$, there exist $L \subseteq M$ and $V \subseteq W$ such that $|L| = \lambda$ and $\Gamma - L - V$ is critical, which implies that $\Gamma - L$ has an espousal F. Since λ is singular, we can write $\lambda = \sup\{\kappa_\alpha : \alpha < \mu\}$, where $\mu = \operatorname{cf}(\lambda) < \lambda$, and we may assume that the sequence $(\kappa_\alpha : \alpha < \mu)$ is continuous (that is, $\sup\{\kappa_\theta : \theta < \alpha\} = \kappa_\alpha$ for each limit ordinal $\alpha < \mu$) and ascending, and that $\kappa_\alpha > \mu$ for every $\alpha < \mu$. We can now write $L = \bigcup\{L_\alpha : \alpha < \mu\}$, where $|L_\alpha| = \kappa_\alpha$.

Definition. By an *admissible sequence* we shall mean a non-descending sequence $(\Delta_\alpha : \alpha < \mu)$ of subsocieties of Γ such that $|\Delta_\alpha| \leqslant \kappa_\alpha$ and Γ/Δ_α is unimpeded for each $\alpha < \mu$.

Lemma 5.21. *Let $(\Delta_\alpha : \alpha < \mu)$ be an admissible sequence. For each $\alpha < \mu$ let H_α be a subset of M such that $|H_\alpha| \leqslant \kappa_\alpha$. Then there exists an admissible sequence $(\Sigma_\alpha : \alpha < \mu)$ such that $\Delta_\alpha \leqslant \Sigma_\alpha$ and $H_\alpha \subseteq M_{\Sigma_\alpha}$ and $\Sigma_\alpha/\Delta_\alpha$ is espousable for each $\alpha < \mu$.*

Proof. Suppose that $\alpha < \mu$ and Σ_θ has been defined for every $\theta < \alpha$, and suppose that $|\Sigma_\theta| \leqslant \kappa_\theta$, $\Delta_\theta \leqslant \Sigma_\theta$, $H_\theta \subseteq M_{\Sigma_\theta}$, Γ/Σ_θ is unimpeded, and $\Sigma_\theta/\Delta_\theta$ is espousable for every $\theta < \alpha$. Let $A_\alpha = H_\alpha \cup \bigcup\{M_{\Sigma_\theta}: \theta < \alpha\}$ and $X_\alpha = \bigcup\{W_{\Sigma_\theta}: \theta < \alpha\}$. Then, by Corollary 5.20a (with Π, Δ, Π', κ replaced by Γ, Δ_α, $\Delta_\alpha \vee \Gamma[A_\alpha, X_\alpha]$, κ_α respectively), Γ has a subsociety Σ_α such that $\Delta_\alpha \vee \Gamma[A_\alpha, X_\alpha] \leqslant \Sigma_\alpha$, $\Sigma_\alpha/\Delta_\alpha$ is espousable, $|\Sigma_\alpha| \leqslant \kappa_\alpha$, and Γ/Σ_α is unimpeded. This defines Σ_α for $\alpha < \mu$ by induction on α; and the sequence $(\Sigma_\alpha: \alpha < \mu)$ is non-descending because $\Gamma[A_\alpha, X_\alpha] \leqslant \Sigma_\alpha$ for each $\alpha < \mu$.

We can now complete the proof of Theorem 5.1′ in the case in which λ is singular.

We make the convention that, when a society is denoted by Γ^k_α, its sets of men and women will be denoted by M^k_α, W^k_α respectively.

By taking Δ_α to be empty for each $\alpha < \mu$ and applying Lemma 5.21, we see that there exists an admissible sequence $(\Gamma^0_\alpha: \alpha < \mu)$ such that $L_\alpha \subseteq M^0_\alpha$ and Γ^0_α is espousable for each $\alpha < \mu$. Let G^0_α be an espousal of Γ^0_α and let e^0_α be an enumeration of M^0_α for each $\alpha < \mu$. By Lemma 5.21, there exists an admissible sequence $(\Gamma^1_\alpha: \alpha < \mu)$ such that $\Gamma^0_\alpha \leqslant \Gamma^1_\alpha$ and

$$(G^0_{\alpha+1} \cup F)^{-1}[W^0_\alpha] \cup \bigcup\{e^0_\theta[\kappa_\alpha]: \theta < \mu\} \subseteq M^1_\alpha$$

and $\Gamma^1_\alpha/\Gamma^0_\alpha$ is espousable for each $\alpha < \mu$. Let e^1_α be an enumeration of M^1_α and let G^1_α be an espousal of $\Gamma^1_\alpha/\Gamma^0_\alpha$ for each $\alpha < \mu$. By Lemma 5.21, there exists an admissible sequence $(\Gamma^2_\alpha: \alpha < \mu)$ such that $\Gamma^1_\alpha \leqslant \Gamma^2_\alpha$ and

$$(G^0_{\alpha+1} \cup G^1_{\alpha+1} \cup F)^{-1}[W^1_\alpha] \cup \bigcup\{e^1_\theta[\kappa_\alpha]: \theta < \mu\} \subseteq M^2_\alpha$$

and $\Gamma^2_\alpha/\Gamma^1_\alpha$ is espousable for each $\alpha < \mu$. Let e^2_α be an enumeration of M^2_α and let G^2_α be an espousal of $\Gamma^2_\alpha/\Gamma^1_\alpha$ for each $\alpha < \mu$. By Lemma 5.21, there exists an admissible sequence $(\Gamma^3_\alpha: \alpha < \mu)$ such that $\Gamma^2_\alpha \leqslant \Gamma^3_\alpha$ and

$$(G^0_{\alpha+1} \cup G^1_{\alpha+1} \cup G^2_{\alpha+1} \cup F)^{-1}[W^2_\alpha] \cup \bigcup\{e^2_\theta[\kappa_\alpha]: \theta < \mu\} \subseteq M^3_\alpha$$

and $\Gamma^3_\alpha/\Gamma^2_\alpha$ is espousable for each $\alpha < \mu$, etc.

By iterating this procedure, we define Γ^k_α, G^k_α, e^k_α for every non-negative integer k and every ordinal $\alpha < \mu$.

Let $\bigcup\{\Gamma^k_\alpha: k < \omega\} = \Gamma_\alpha = \Gamma[M_\alpha, W_\alpha]$ for each $\alpha < \mu$, and let

$$\Pi = \bigcup\{\Gamma_\alpha: \alpha < \mu\} = \bigcup\{\Gamma^k_\alpha: k < \omega, \alpha < \mu\}.$$

Since $F^{-1}[W^k_\alpha] \subseteq M^{k+1}_\alpha \subseteq M_\Pi$ whenever $k < \omega$ and $\alpha < \mu$, it follows that $F^{-1}[W_\Pi] \subseteq M_\Pi$ and therefore $F[M \setminus M_\Pi] \subseteq W \setminus W_\Pi$. Moreover, $L_\alpha \subseteq M^\alpha_0 \subseteq M_\Pi$ for each $\alpha < \mu$ and therefore $L \subseteq M_\Pi$, and therefore $M \setminus M_\Pi \subseteq M \setminus L = \operatorname{dom} F$. Hence $F \upharpoonright (M \setminus M_\Pi)$ is an espousal of Γ/Π.

For $\alpha < \mu$, let $G_{\alpha+1} = \bigcup\{G^k_{\alpha+1}: k < \omega\}$. Then $G_{\alpha+1}$ is an espousal of $\Gamma_{\alpha+1}$. Let $m \in M_{\alpha+1}$. Then $m \in M^k_{\alpha+1} \setminus M^{k-1}_{\alpha+1}$ and $G_{\alpha+1}(m) = G^k_{\alpha+1}(m) \in W^k_{\alpha+1} \setminus W^{k-1}_{\alpha+1}$ for some k (with the convention that $M^{k-1}_{\alpha+1}$ and $W^{k-1}_{\alpha+1}$ are both $\varnothing$ if $k = 0$). If $h < k$ then $W^h_\alpha \subseteq W^{k-1}_\alpha \subseteq W^{k-1}_{\alpha+1}$ and therefore $G_{\alpha+1}(m) \notin W^h_\alpha$. It follows that if $G_{\alpha+1}(m) \in W_\alpha$ then $G^k_{\alpha+1}(m) = G_{\alpha+1}(m) \in W^h_\alpha$ for some $h \geqslant k$ and so

$$m \in (G^0_{\alpha+1} \cup G^1_{\alpha+1} \cup \ldots \cup G^h_{\alpha+1})^{-1}[W^h_\alpha] \subseteq M^{h+1}_\alpha \subseteq M_\alpha.$$

Therefore $G_{\alpha+1} \upharpoonright (M_{\alpha+1} \setminus M_\alpha)$ is an espousal of $\Gamma_{\alpha+1}/\Gamma_\alpha$.

If α is a limit ordinal less than μ and $m \in M_\alpha$ then $m \in M^k_\alpha$ for some k and consequently $m \in e^k_\alpha[\kappa_\theta] \subseteq M^{k+1}_\theta \subseteq M_\theta$ for some $\theta < \alpha$. Therefore

$$M_\alpha = \bigcup\{M_\theta: \theta < \alpha\}$$

for every limit ordinal $\alpha < \mu$. From this and the fact that $G_{\alpha+1} \upharpoonright (M_{\alpha+1} \setminus M_\alpha)$ is an

espousal of $\Gamma_{\alpha+1}/\Gamma_\alpha$ for each $\alpha < \mu$, it follows that
$$G = \bigcup \{G_{\alpha+1} \restriction (M_{\alpha+1} \setminus M_\alpha) \colon \alpha < \mu\}$$
is an espousal of Π. We have also proved that $F \restriction (M \setminus M_\Pi)$ is an espousal of Γ/Π; and so $G \cup (F \restriction (M \setminus M_\Pi))$ is an espousal of Γ.

6. *Some applications*

A subsociety of Γ will be called

(i) an *obstruction* in Γ if it is a κ-obstruction in Γ for some $\kappa \in \Upsilon$.

(ii) a $(\leqslant \kappa)$-*obstruction* in Γ if it is a λ-obstruction in Γ for some $\lambda \in \Upsilon$ such that $\lambda \leqslant \kappa$,

(iii) a $(> \kappa)$-*obstruction* in Γ if it is a λ-obstruction in Γ for some $\lambda \in \Upsilon$ such that $\lambda > \kappa$.

We shall say that Γ is *obstructed* (κ-*obstructed*) if there is an obstruction (a κ-obstruction) in Γ.

It is possible to deduce from Theorem 5.1 a strengthening of itself. For any society Γ, define
$$\eta(\Gamma) = \sup\{\kappa \colon \Gamma \text{ is } \kappa\text{-obstructed}\}$$
(and $\eta(\Gamma) = 0$ if Γ is unobstructed).

THEOREM 6.1. $\delta(\Gamma) = \eta(\Gamma)$.

Proof. Let $\eta = \eta(\Gamma)$. By Lemma 4.2 and Corollary 4.9b, $\delta(\Gamma) \geqslant \eta$, and so it remains to be proved that $\delta(\Gamma) \leqslant \eta$. This follows from Theorem 5.1 if $\eta = 0$, and so we assume that $\eta > 0$.

Suppose first that η is finite. Then there exists an η-obstruction Π in Γ. It follows that $\Pi - L$ is critical for some η-subset L of M_Π. We will now show that Γ/Π is unobstructed. Suppose, to the contrary, that there is a κ-obstruction Σ in Γ/Π. If $\kappa \leqslant \aleph_0$, then $\Sigma - J$ is critical for some κ-subset J of M_Σ. By Lemma 3.4, $(\Pi - L) \vee (\Sigma - J) = (\Pi \vee \Sigma) - (L \cup J)$ is critical, and moreover $\Pi \vee \Sigma \lhd \Gamma$ since $\Pi \lhd \Gamma$ and $\Sigma \lhd \Gamma/\Pi$. Hence $\Pi \vee \Sigma$ is an $(\eta + \kappa)$-obstruction in Γ, contradicting the definition of η. If $\kappa > \aleph_0$ then $\Sigma = \bigcup \bar{\Sigma}$ for some obstructive κ-tower $\bar{\Sigma}$ in Γ/Π. Since $\Pi \lhd \Gamma$, a κ-tower $\bar{\Sigma}' = (\Sigma'_\alpha \colon \alpha < \kappa)$ in Γ is obtained by letting $\Sigma'_0 = (\varnothing, \varnothing, \varnothing)$ and $\Sigma'_{1+\alpha} = \Pi \vee \Sigma_\alpha$ for each $\alpha < \kappa$. From the facts that $\bar{\Sigma}$ is an obstructive κ-tower in Γ/Π and Π is an η-obstruction in Γ, it is easy to deduce that $\bar{\Sigma}'$ is an obstructive κ-tower in Γ and so $\bigcup \bar{\Sigma}' = \Pi \vee \Sigma$ is a κ-obstruction in Γ, contradicting the fact that $\eta < \aleph_0 < \kappa$. Thus the supposition that Γ/Π is obstructed leads to a contradiction. We conclude that Γ/Π is unobstructed and therefore, by Theorem 5.1, espousable. Moreover, $\Pi - L$ is espousable since it is critical. The union of an espousal of Γ/Π and one of $\Pi - L$ is an espousal of $\Gamma - L$, and therefore $\delta(\Gamma) \leqslant |L| = \eta$.

Assume now that η is infinite. We define a sequence $(\Lambda_\alpha \colon \alpha < \eta^+)$ of subsocieties of Γ by induction on α. Assume, for a given $\alpha < \eta^+$, that Λ_β has been defined for all $\beta < \alpha$. Let $\Pi_\alpha = \bigcup \{\Lambda_\beta \colon \beta < \alpha\}$ and $\Gamma_\alpha = \Gamma/\Pi_\alpha$. We consider the following cases:

(C_a) Γ_α is obstructed; let Λ_α be any obstruction in Γ_α;

(C_b) Γ_α is unobstructed and $W_{\Pi_\alpha} \neq W$; let $\Lambda_\alpha = \langle w \rangle$ for some $w \in W \setminus W_{\Pi_\alpha}$;

(C_c) Γ_α is unobstructed and $W_{\Pi_\alpha} = W$; let $\Lambda_\alpha = (\varnothing, \varnothing, \varnothing)$.

Then $\Lambda_\alpha \lhd \Gamma_\alpha$ for each $\alpha < \eta^+$: this is obvious in cases (C_b) and (C_c) and follows from Lemmas 4.2 and 4.3 in Case (C_a). Therefore $(\Lambda_\alpha \colon \alpha < \eta^+)$ is the ladder of a tower $\bar{\Pi} = (\Pi_\alpha \colon \alpha \leqslant \eta^+)$.

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Let us show first that if $\alpha < \eta^+$ and (C_a) holds then Λ_α is a $(\leqslant\eta)$-obstruction in Γ_α. Suppose that this is not the case, and let $\alpha < \eta^+$ be the least ordinal such that Λ_α is a $(>\eta)$-obstruction in Γ_α. Then, by Lemmas 4.2 and 4.8, $v(\Lambda_\theta) \leqslant \eta$ for each $\theta < \alpha$ and therefore, by Corollary 3.4b, $v(\Pi_\alpha) \leqslant \eta$. Hence, by Lemmas 4.7 and 4.2, there is a $(>\eta)$-obstruction in Γ, contradicting the definition of η. Hence Λ_α is a $(\leqslant \eta)$-obstruction in Γ in Case (C_a) and therefore $\varepsilon(\Lambda_\alpha) \leqslant \eta$ in this case by Lemmas 4.2 and 4.8. Obviously $\varepsilon(\Lambda_\alpha) = 0$ in Cases (C_b), (C_c). Hence $\varepsilon(\Pi_\alpha) \leqslant \eta$ for every $\alpha < \eta^+$ by Corollary 3.4b and consequently, by Lemma 3.9,

$$\delta(\Pi_\alpha) \leqslant \eta \quad \text{for every } \alpha < \eta^+. \tag{6.1}$$

If case (C_c) holds for some $\alpha < \eta^+$ then, since $W = W_{\Pi_\alpha}$ and $\Gamma_\alpha = (M \setminus M_{\Pi_\alpha}, \varnothing, \varnothing)$ is unobstructed, Lemmas 4.2 and 4.5 show that $M = M_{\Pi_\alpha}$, and thus $\Gamma = \Pi_\alpha$. This, by (6.1), implies that $\delta(\Gamma) \leqslant \eta$. So, we may assume that for each $\alpha < \eta^+$ either (C_a) or (C_b) holds. Since Λ_α is a $(\leqslant\eta)$-obstruction whenever (C_a) occurs and $\bar{\Pi}$ is not obstructive, there exists an $\alpha < \eta^+$ for which Case (C_b) holds. Then Γ_α is unobstructed, and hence, by Theorem 5.1, it is espousable. This implies that $\delta(\Gamma) \leqslant \delta(\Pi_\alpha)$, and hence, by (6.1), $\delta(\Gamma) \leqslant \eta$.

COROLLARY 6.1a. *Let* $\Gamma = \Gamma_1 \vee \Gamma_2$, *where* $\Gamma_1 \cap \Gamma_2 = (\varnothing, \varnothing, \varnothing)$. *Then* $\eta(\Gamma) \leqslant \eta(\Gamma_1) + \eta(\Gamma_2)$. *In particular, if* Γ_1 *and* Γ_2 *are unobstructed, then so is* Γ.

Proof. This follows from Theorem 6.1 and the easily proved fact that $\delta(\Gamma) \leqslant \delta(\Gamma_1) + \delta(\Gamma_2)$.

Let us repeat a question posed above: can Corollary 6.1a be proved directly from the definition of 'obstruction'?

THEOREM 6.2. *If* Γ *is espousable then*

(a) *there exists a subset* Z *of* W *such that* $\Gamma - Z$ *is espousable and* $|Z| = \sigma(\Gamma)$ *(that is, the 'supremum' in the definition of* $\sigma(\Gamma)$ *is a maximum),*

(b) $\sigma(\Gamma) = v(\Gamma)$.

Proof. Let $v = v(\Gamma)$. By Corollary 3.8a,

$$\sigma(\Gamma) \leqslant v. \tag{6.2}$$

Suppose first that v is finite. Then $\sigma(\Gamma)$ is finite by (6.2). Therefore (a) is clearly true and $\Gamma - Z$ is critical when Z is as in (a), so that $v \leqslant |Z| = \sigma(\Gamma)$, which, with (6.2), proves (b).

Now suppose that v is infinite. Let D be a set such that $|D| = v$ and $D \cap (M \cup W) = \varnothing$, and let $\Gamma' = (M \cup D, W, K \cup (D \times W))$.

Suppose that there is a κ-impediment Π in Γ'. Since Γ is espousable, Π is, by Corollary 4.9b, not a κ-impediment in Γ, and therefore $\Pi \not\leqslant \Gamma$ by Lemma 4.4 (with $\Delta = \Gamma'$). Since $\Pi \lhd \Gamma'$ and $\Pi \not\leqslant \Gamma$, it follows that $W_\Pi = W$. Since $\Gamma' - D = \Gamma$ is espousable, Corollary 4.9b gives $\kappa \leqslant \delta(\Gamma') \leqslant |D| = v$. Suppose first that $\kappa \leqslant \aleph_0$. Then $\Pi_1 = \Pi - J$ is critical for some κ-subset J of M_Π. Let G be an espousal of Π_1. Then $\Pi_1[M_{\Pi_1} \setminus D, G[M_{\Pi_1} \setminus D]]$ is critical by Lemma 3.3, that is (since $G[M_{\Pi_1}] = W_{\Pi_1} = W_\Pi = W$), $\Gamma - J^* - G[M_{\Pi_1} \cap D]$ is critical for some subset J^* of M. Since $|G[M_{\Pi_1} \cap D]| = |M_{\Pi_1} \cap D| < \aleph_0$ by Corollary 3.7a and v is infinite, this contradicts the definition of v. Now suppose that $\kappa > \aleph_0$. Then $\Pi = \bigcup \bar{\Pi}$ for some impeding κ-tower $\bar{\Pi}$ in Γ'. Since $\Pi \not\leqslant \Gamma$, there is an $\alpha < \kappa$ such that $\Pi_\alpha \not\leqslant \Gamma$. From this and the fact that $\Pi_\alpha \lhd \Gamma'$, it follows that $W_{\Pi_\alpha} = W$. If $l(\bar{\Pi}) = \bar{\Lambda}$, then $\nu(\Lambda_\theta) < \kappa$ for each $\theta < \kappa$ by Lemma 4.8, and therefore $\nu(\Pi_\alpha) < \kappa$ by Corollary 3.4b. Let L, V be subsets of $M_{\Pi_\alpha}, W_{\Pi_\alpha}$ $(= W)$ respectively such that $\Sigma = \Pi_\alpha - L - V$ is critical and $\nu(\Pi_\alpha) = |V|$. Let F be an espousal of Σ. Then $\Sigma[M_\Sigma \backslash D, F[M_\Sigma \backslash D]]$ is critical by Lemma 3.3, that is, $\Gamma - L^* - (V \cup F[M_\Sigma \cap D])$ is critical for some subset L^* of M. Since $|F[M_\Sigma \cap D]| = |M_\Sigma \cap D| < \aleph_0$ by Corollary 3.7a and $|V| = \nu(\Pi_\alpha) < \kappa \leqslant \nu$, this contradicts the definition of ν.

We conclude that Γ' is unimpeded, and so has an espousal E by Theorem 5.1′. Therefore $\Gamma - E[D]$ is espousable and consequently $\sigma(\Gamma) \geqslant |E[D]| = \nu$. From this and (6.2) it follows that $\sigma(\Gamma) = |E[D]| = \nu$, which proves (a) and (b) (taking $Z = E[D]$ in (a)).

Remark. It is mentioned in [9] that part (a) of this theorem has been proved by M. Ziegler in an unpublished paper: 'Cotransversals of infinite families'.

It has been mentioned above that the last stage of the proof of Theorem 5.1′ when λ is singular followed closely the proof of (the transversals case of) the compactness theorem for singular cardinals in [11]. In fact, Theorems 5.1 and 5.1′ have the character of a compactness result when λ is singular, a fact which can be made explicit in the following lemma.

Lemma 6.3. *If $\varepsilon(\Gamma)$ is singular then Γ is espousable if and only if every saturated subsociety Γ' of Γ with $\varepsilon(\Gamma') < \varepsilon(\Gamma)$ is espousable.*

Proof. If Γ is espousable then clearly every saturated subsociety of Γ is espousable. Conversely, if Γ is inespousable, then, by Theorem 5.1′, it contains a κ-impediment Π for some $\kappa \in \Upsilon$. By Corollary 4.9a, $\varepsilon(\Pi) = \kappa$. By Corollary 4.9b and Lemma 3.9, $\kappa \leqslant \delta(\Gamma) \leqslant \varepsilon(\Gamma)$. Therefore, since $\kappa \in \Upsilon$ and $\varepsilon(\Gamma)$ is singular, it follows that $\kappa < \varepsilon(\Gamma)$. Thus Π is (by Lemma 4.9) an inespousable saturated subsociety of Γ satisfying $\varepsilon(\Pi) < \varepsilon(\Gamma)$.

The compactness theorem for singular cardinals itself follows quite straightforwardly from Theorem 5.1 or Theorem 5.1′. In fact, we shall now deduce a slightly stronger version of it (Theorem 6.4) from Theorem 5.1′. (Theorem 6.4 can also be proved directly. It reduces to the compactness theorem for singular cardinals when $L = M$.)

Theorem 6.4. *Let λ be a singular cardinal. Assume that $M \backslash L$ is espousable for some λ-subset L of M and $\kappa < \lambda$ and $|K\langle m\rangle| \leqslant \kappa$ for every $m \in M$. Then Γ is espousable if and only if every subset M' of M with $|M'| < \lambda$ is espousable.*

Proof. Clearly, if Γ is espousable then any subset M' of M is espousable. So, assume that Γ is inespousable. By Theorem 5.1′ there exists a μ-impediment Π in Γ for some $\mu \in \Upsilon$. Since $\delta(\Gamma) \leqslant \lambda$, it follows from Corollary 4.9b that $\mu \leqslant \lambda$, and since μ is regular or countable, $\mu < \lambda$. By Corollary 4.9a, $\delta(\Pi) = \mu$, and so $\Pi - A$ has an espousal H for some μ-subset A of M_Π. We define a sequence $(B_k\colon k < \omega)$ of subsets of M_Π as follows: $B_0 = A$ and if B_k is defined then $B_{k+1} = B_k \cup H^{-1}[K[B_k]]$. Let $B = \bigcup \{B_k\colon k < \omega\}$. Since $|K\langle m\rangle| \leqslant \kappa$ for every $m \in M$, $|B_k| \leqslant \max(\kappa, \mu, \aleph_0)$ for every $k < \omega$, and hence $|B| < \lambda$. Suppose that B has an espousal G. By the definition of B, $K[B] \cap H[M_\Pi \backslash B] = \varnothing$, and hence $(\mathrm{rge}\, G) \cap (\mathrm{rge}\, H \upharpoonright (M_\Pi \backslash B)) = \varnothing$. This implies that $G \cup (H \upharpoonright (M_\Pi \backslash B))$ is an espousal of M_Π, which by Lemma 4.9 contradicts the assumption that Π is an impediment in Γ. Hence B must be inespousable, and so M has an inespousable subset M' $(= B)$ with $|M'| < \lambda$.

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